Differential forms as a unifying force for geometric structures

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Differential geometry seminar

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Erich Kähler's top three **geometric structures**:

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(Smooth category, M manifold)

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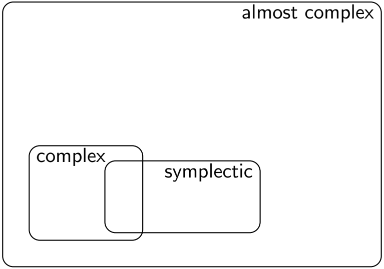

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(same)	$\sigma \in \Omega^1(M)$, $Y \in \mathfrak{X}(M)$
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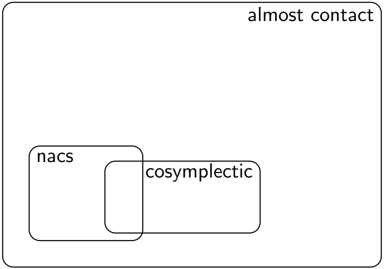

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"**Differential forms** as a unifying force for…

→ Can we do anything about complex?

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$K \subset \wedge^\bullet T^*_{\mathbb{C}}M$ (canonical bundle of J) whose **local** sections $\zeta \in \Gamma(K \setminus \{0\})$ satisfy:

- ζ decomposable
- $\zeta \wedge \bar{\zeta}$ volume
- $d\zeta = \bar{\partial} f \wedge \zeta$ for some f

(think of $\zeta = dz_1 \wedge \ldots \wedge dz_m$)

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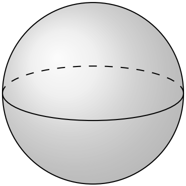

dz on $\mathbb{C}$,
$d(1/z)$ on $\mathbb{C}^* \cup \{\infty\}$

differ on $\mathbb{C}^*$ by
$dz = -z^2 d(1/z)$

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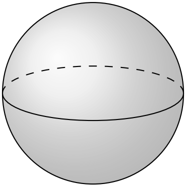

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$\rightarrow$ Can we unify ω and K in some sense?

A recipe for generalized geometry (à la Hitchin, Gualtieri...)

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Quadratic form on $TM \oplus T^*M$ given by $Q(X + \alpha) = \alpha(X)$

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$$(X + \alpha) \cdot \rho = \iota_X \rho + \alpha \wedge \rho$$

($\wedge^\bullet T^*_{\mathbb{C}} M \simeq$ the spinor representation)

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($\leftrightarrow \mathrm{Ann}(\rho) \subseteq T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ Lagrangian) $B,\omega \in \wedge^2$, $\theta_j \in \wedge^1_{\mathbb{C}}$

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$+$

Weakening of $d\rho = 0 \to d\rho = v\cdot\rho$ for $v = X+\alpha$
(or of $d\zeta = \bar\partial f\wedge\zeta$)

$\updownarrow$

$\Gamma(\mathrm{Ann}\,\varphi)$ involutive for **Dorfman bracket**

$$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - \iota_Y d\alpha$$

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Definition: a **generalized complex structure** is locally given by:

$\rho \in \Omega^\bullet_{\mathbb{C}}$ pure	shape
$(\rho, \overline{\rho}) \sim$ volume	non-degeneracy
$d\rho = v \cdot \rho$	integrability

(the local forms coincide pointwise up to $\mathbb{C}^*$)

The type and ~~another example~~ **the example**

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Pointwise: symplectic subspace
with r-dim complex transversal.
Definition of **type**: r.

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$d\rho = dz = (-\frac{\partial}{\partial w} + 0) \cdot \rho$

Some considerations

$$\rho \sim e^{B+i\omega} \wedge \theta_1 \wedge \ldots \wedge \theta_r$$
$$(\rho, \overline{\rho}) = \text{vol}$$
$$d\rho = v \cdot \rho$$

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$$(\rho, \overline{\rho}) = \text{vol}$$
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- $e^B \wedge$ is a symmetry for B closed (a B-field). E.g., $e^{B+i\omega} \cong e^{i\omega}$
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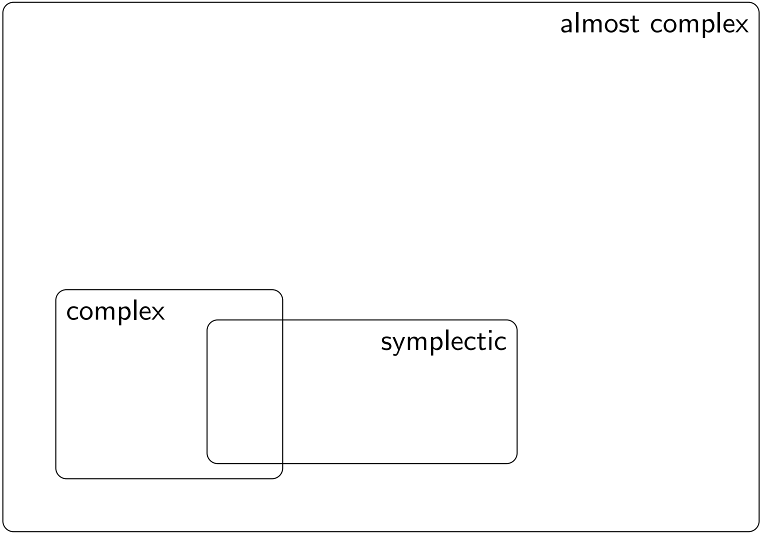
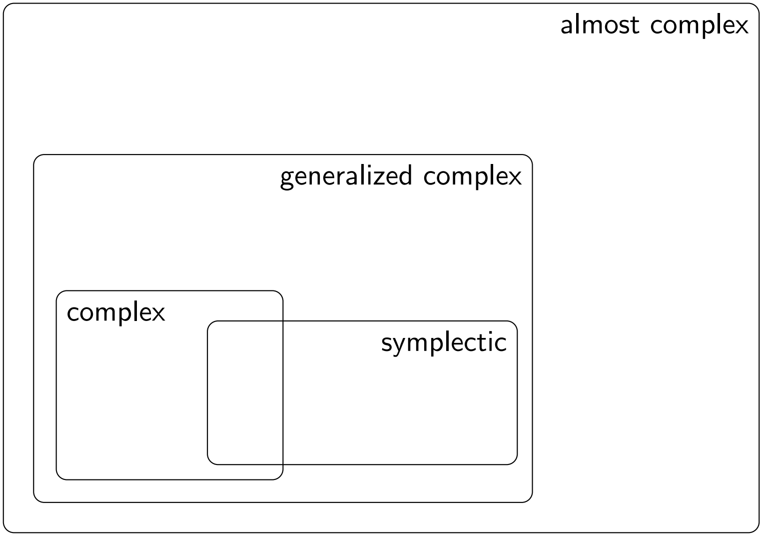
Within generalized complex structures

The interior of the curve is B-equivalent to symplectic structures.

Examples coming from hyperKähler or holomorphic symplectic structures.


almost complex
complex
symplectic


almost complex
generalized complex
complex
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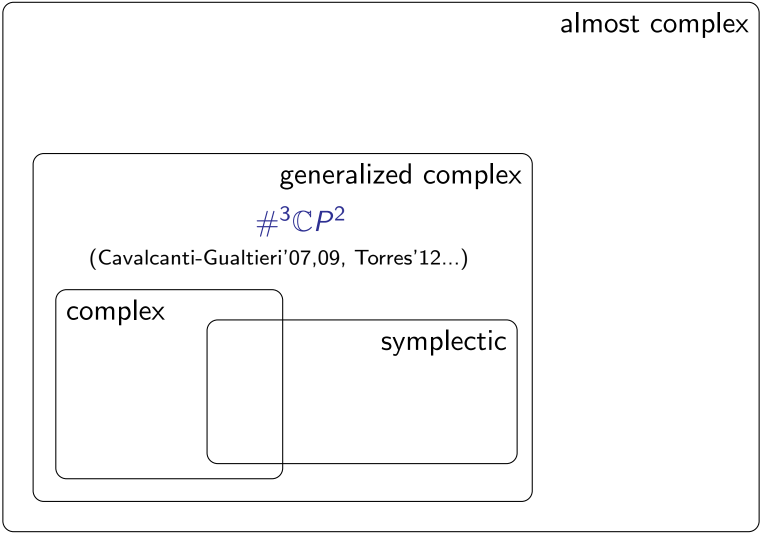
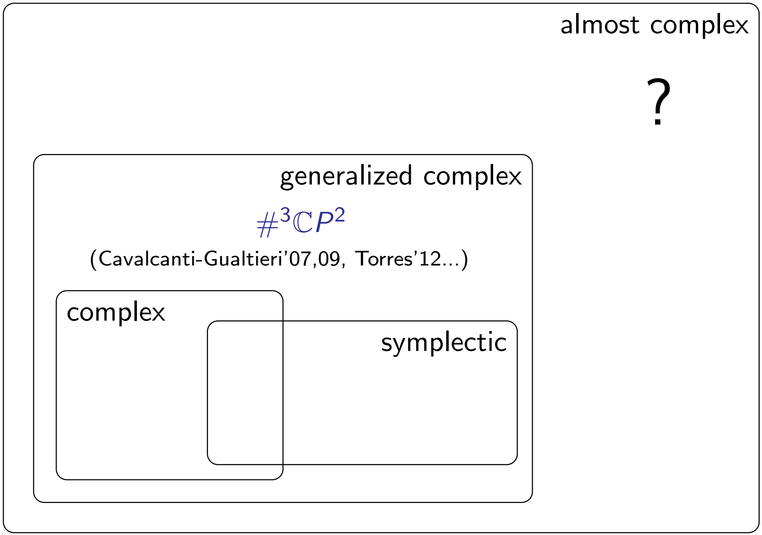


almost complex
generalized complex
$\#^3\mathbb{C}P^2$
(Cavalcanti-Gualtieri'07,09, Torres'12...)
complex
symplectic


almost complex
?
generalized complex
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So far:

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	symmetric	endomorphism	skew-symmetric
even			
odd			

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Can we go beyond generalized complex structures?:

A natural variation of the recipe

Quadratic form on $TM \oplus T^*M$ given by $Q(X+\alpha) = \alpha(X)$

$\mathrm{Cl}_{\mathbb{C}}(TM \oplus T^*M)$-module structure on $\wedge^\bullet T^*_{\mathbb{C}}M$

$$(X+\alpha)\cdot\rho = \iota_X\rho + \alpha\wedge\rho$$

($\wedge^\bullet T^*_{\mathbb{C}}M \simeq$ the spinor representation)

Pure spinors are pointwise $\sim e^{B+i\omega}\theta_1\wedge\ldots\wedge\theta_r$

($\leftrightarrow$ $\mathrm{Ann}(\rho)$ Lagrangian) $B,\omega\in\wedge^2$, $\theta_j\in\wedge^1_{\mathbb{C}}$

Chevalley pairing on spinors $(\rho,\psi) = (\rho^T\wedge\psi)_{top}$

($\wedge^{top}T^*_{\mathbb{C}}M$-valued)

$+$

Weakening of $d\rho = 0 \to d\varphi = v\cdot\varphi$ for $v = X+\alpha$

(or of $d\zeta = \bar{\partial}f\wedge\zeta$)

$\updownarrow$

$\Gamma(\mathrm{Ann}\,\varphi)$ involutive for **Dorfman bracket**

$$[X+\alpha, Y+\beta] = [X,Y] + L_X\beta - \iota_Y d\alpha$$

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Idea: a different generalized geometry

A generalized complex structure is locally given by:

$$\rho \in \Omega^{\bullet}_{\mathbb{C}} \text{ pure}$$
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Example: **any usual generalized complex is B_n-generalized.**

But they also exist in odd dimensions

- Cosymplectic structure: $\omega \in \Omega^2_{cl}$, $\sigma \in \Omega^1_{cl}$ such that $\sigma \wedge \omega^m$ volume

$$\rho = e^{i\sigma + i\omega} = 1 + i\sigma + i\omega - \sigma \wedge \omega \ldots$$

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- **Type-change example**: on $\mathbb{C} \times \mathbb{R}$ with coordinates (z, t),

$$\rho = z + dz + i\, dz \wedge dt$$

So far from the distance:

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even				
odd				

B_n-generalized complex structures

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Do B_n-generalized complex structures reach further?:

Some considerations

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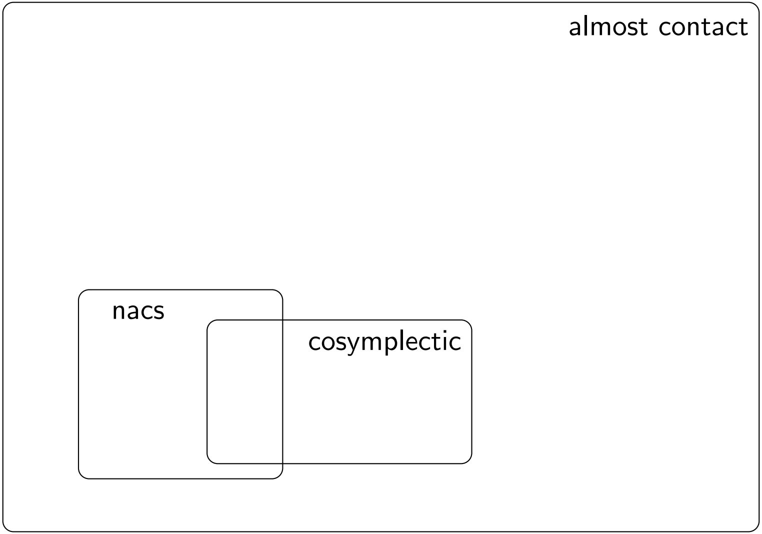
- Type change already possible for $\dim M = 2, 3 \ldots$.


almost contact
nacs
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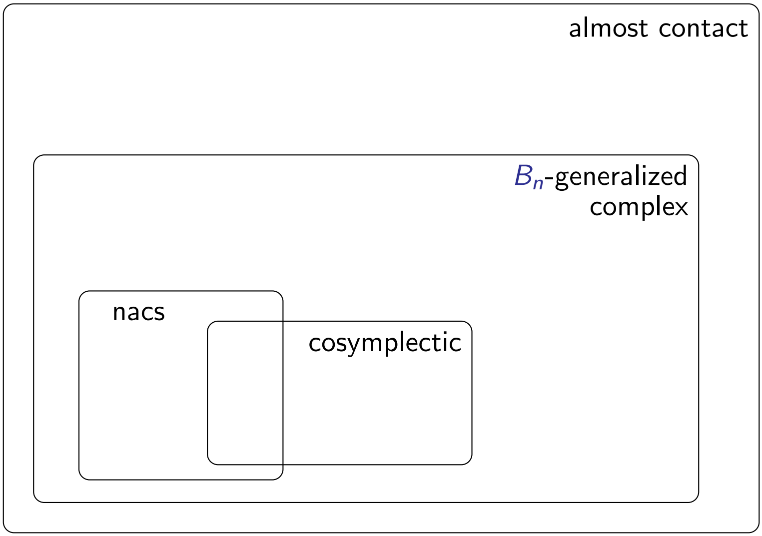
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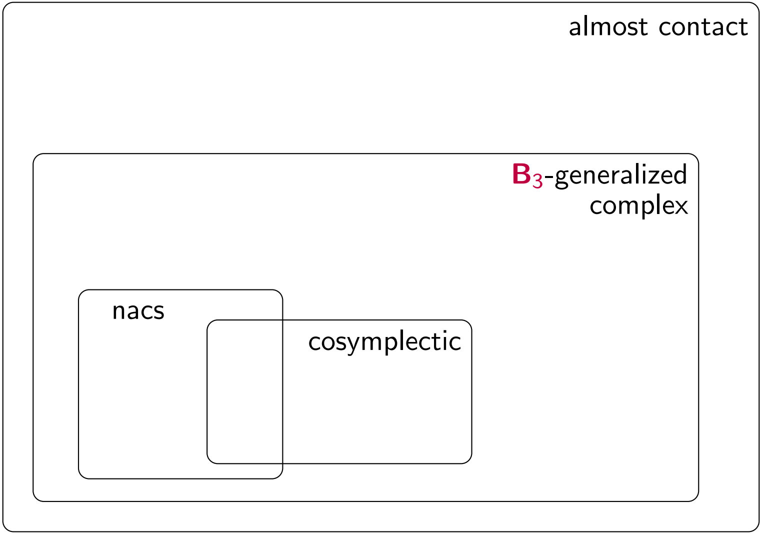
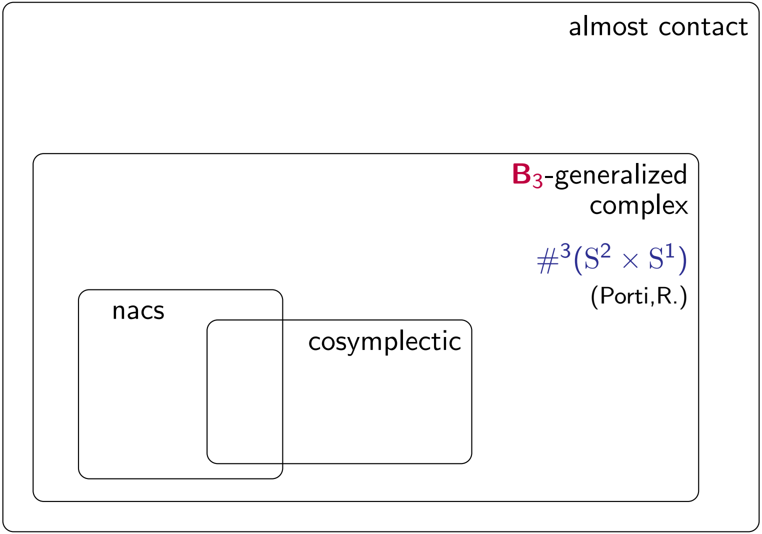


Joint work with J. Porti



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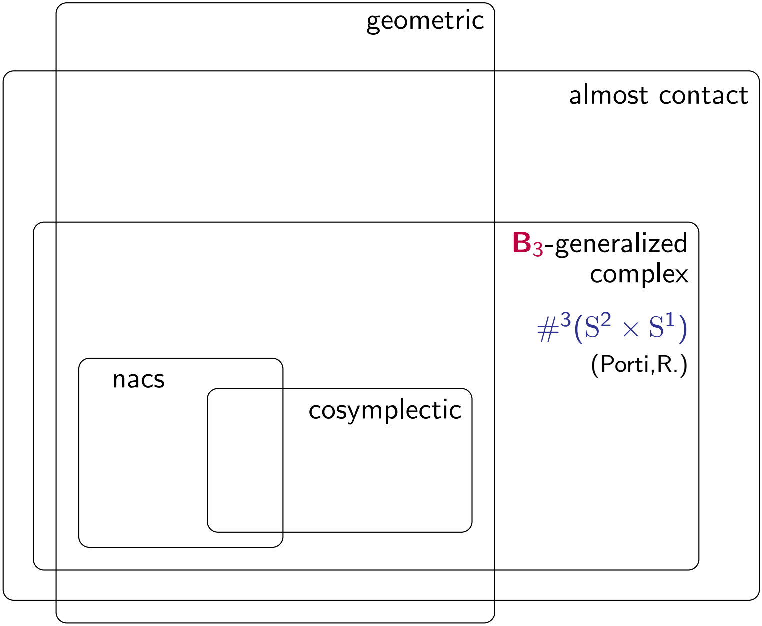
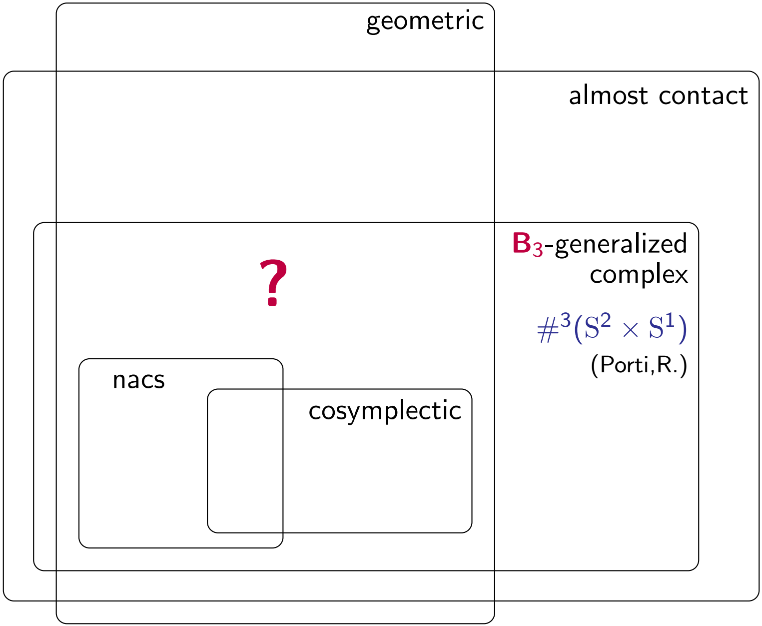


Joint work with J. Porti



Joint work with J. Porti

Main result

Mixing well:

- family of geometric generalized surgeries (topologically Dehn twists)
- open-book decomposition with connected binding, Moser's argument, transitiviy of symplectomorphisms, Dacorogna-Moser theorem

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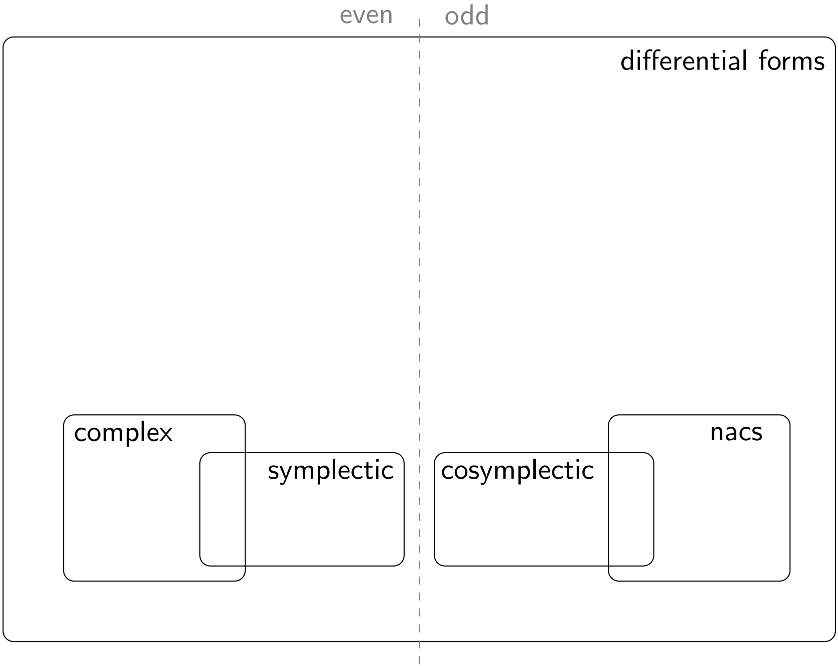
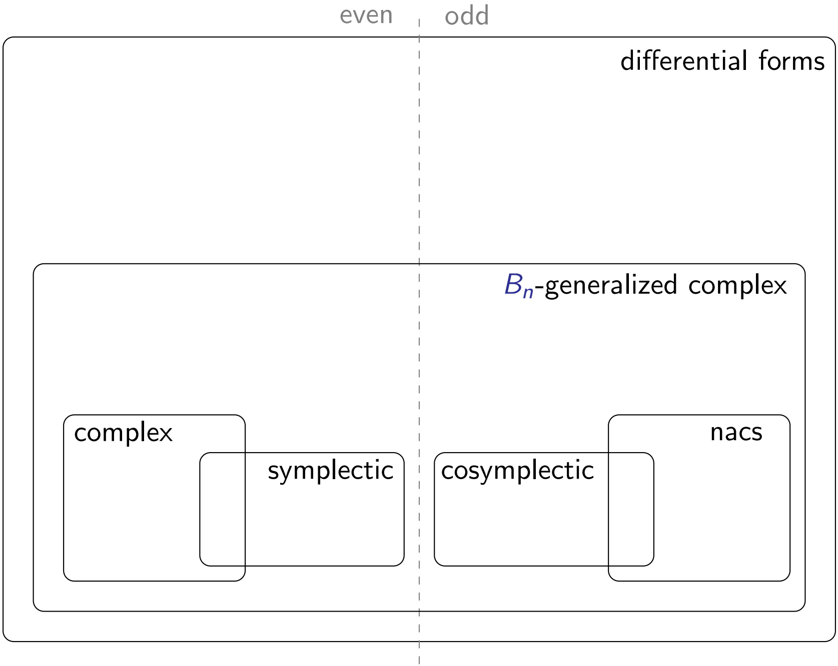
Theorem (Porti,R.)

Any closed oriented 3-manifold admits a B_3-generalized complex structure, which is moreover stable.

New geometric structures on 3-manifolds: surgery and generalized geometry

arXiv:2402.12471


even
odd
differential forms
complex
symplectic
cosymplectic
nacs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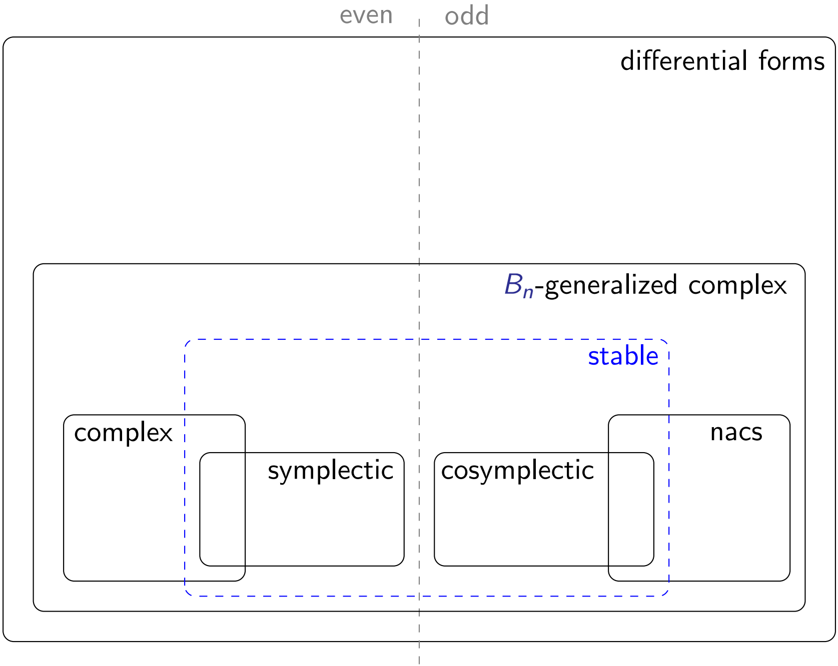
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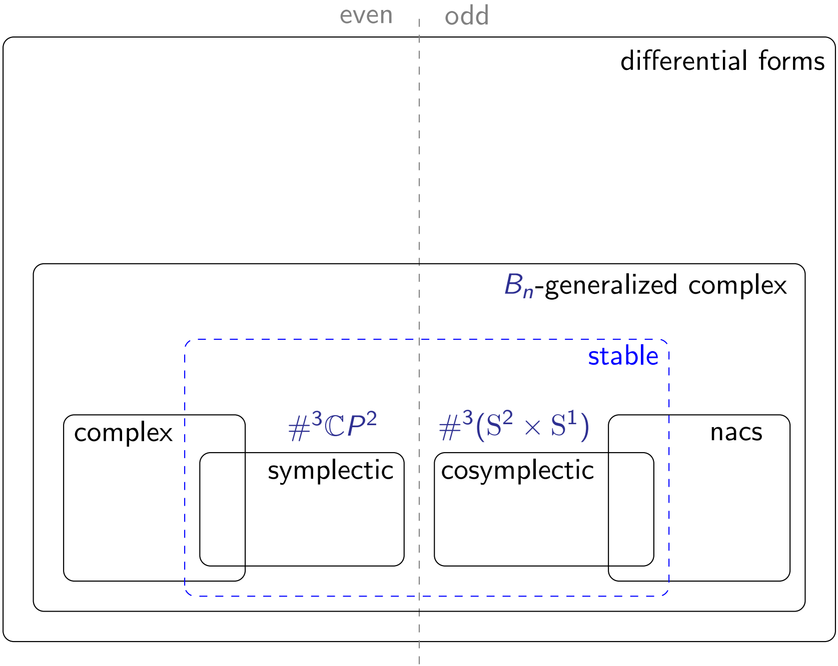
even
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$\#^3\mathbb{C}P^2$
$\#^3(S^2 \times S^1)$
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Local normal models in even dimensions

Complex: $dz_1 \wedge \ldots \wedge dz_m$ Symplectic: $\omega = dp_1 \wedge dq_1 + \ldots + dp_m \wedge dq_m$

Proposition (Cavalcanti-Gualtieri'18)

Around the change of type, stable generalized complex structures look like

$$(z_1 + dz_1 \wedge dz_2) \wedge e^{i\omega}$$

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Local models in odd dimensions? (Say, 3, $\rho = \rho_0 + \rho_1 + \rho_2 + \rho_3$)

Definition (Porti-R.)

Around type change, take γ a meridian curve and T a concentric torus.

$$\lambda := \frac{1}{2\pi i} \int_\gamma \iota^*(\rho_1/\rho_0), \qquad \mu := \frac{1}{4\pi^2 i} \int_T \iota^*(\rho_2/\rho_0).$$

Local invariant: λ **Semilocal invariant** (compact case): $arg(\mu)$

Example: $z + \lambda dz + \mu dz \wedge dt$

Condition $\mathrm{Im}(\lambda\overline{\mu}) \neq 0$. **Prop**: stable type-change never a single circle

A type-change example

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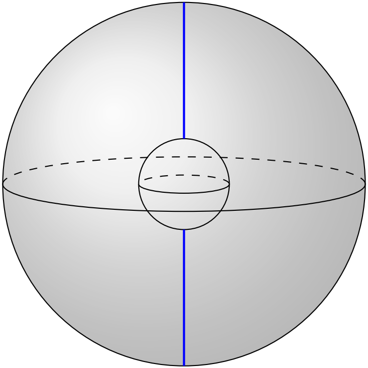

$S^2 \times S^1$,
$\lambda, \mu \in \mathbb{C}^\times$ such that
$\mathrm{Im}(\lambda\overline{\mu}) \neq 0$:

On $\mathbb{C} \times S^1$,
$\rho = z + \lambda dz + \mu dz \wedge dt$.

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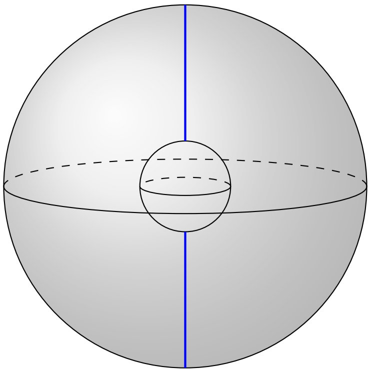

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$\rho = \frac{1}{z} - \lambda d\left(\frac{1}{z}\right) - \mu\, d\left(\frac{1}{z}\right) \wedge dt$.

They differ by z^2 on $\mathbb{C}^* \times S^1$.

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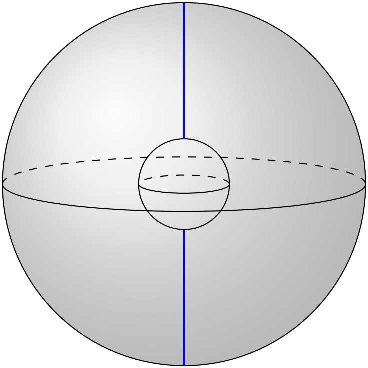

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What is the meaning of λ and μ?

Meaning of invariants (work in progress)

Have in mind $z + \lambda dz + \mu dz \wedge dt$ on $\mathrm{S}^2 \times \mathrm{S}^1$:

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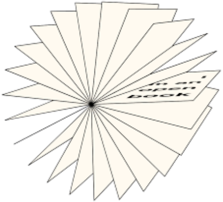

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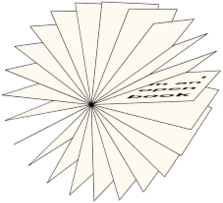

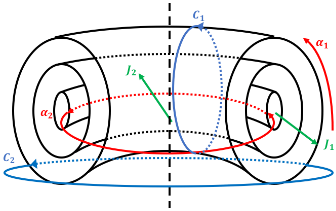

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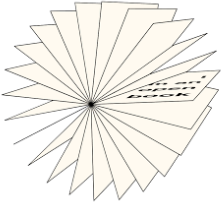

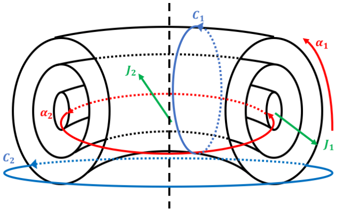

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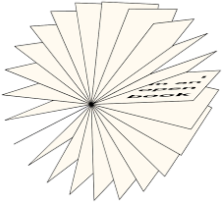

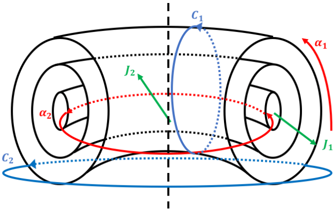

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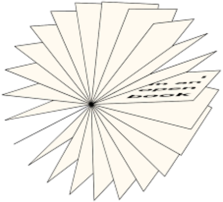
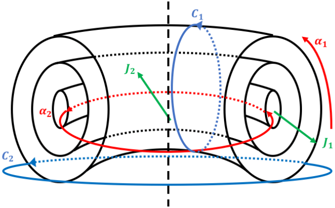
$arg(\mu)$ gives an invariant related to symplectic structure of leaves?

Meaning of invariants (work in progress)





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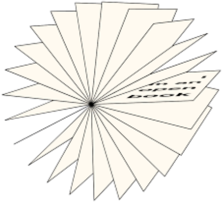

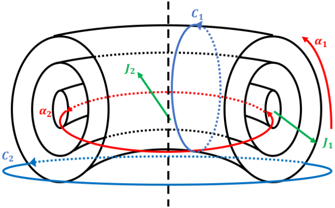

Just as Dirac and generalized complex structures give a smooth structure to foliations and geometric structures on and transverse to them...

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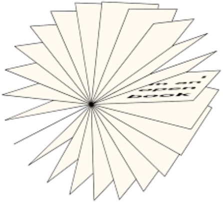

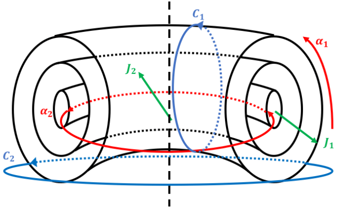

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...B_3-generalized complex structures do it for open-book decompositions, genus 1 Heegard splittings and related foliations (λ tells which!).

Take-home message: B_n-generalized complex structures

	symmetric	endomorphism	skew-symmetric	
even				
odd				

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$d\rho = v \cdot \rho$

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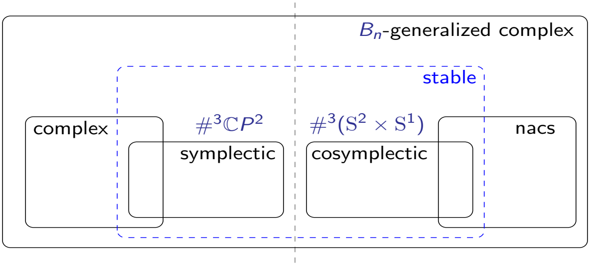

Complaints department:

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It all comes from one choice:

Quadratic form on $TM \oplus T^*M$ given by $Q(X + \alpha) = \alpha(X)$
that is,
symmetric pairing $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y))$

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But we could have done:

skew-symmetric pairing $\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) - \alpha(Y))$

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Beyond the canonical symmetric pairing in generalized geometry

Roberto Rubio
joint with Filip Moučka
(who typed most of these slides)

UAB
Universitat Autònoma de Barcelona

Symplectic geometry seminar

impa
Instituto de Matemática Pura e Aplicada

Rio de Janeiro, 9th April 2025

Thank you for your attention!

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Cofinanciado por la Unión Europea
AGENCIA ESTATAL DE INVESTIGACIÓN

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