Symmetric Poisson geometry, totally geodesic foliations and Jacobi-Jordan algebras

Roberto Rubio
(j. w. Filip Moučka)

UAB
Universitat Autònoma de Barcelona

Jilin University Colloquium
(Sino-Russian Mathematics Centre)

吉林大学
JILIN UNIVERSITY

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Recalling Poisson geometry (19th cent.)

Dynamics on $\mathbb{R}^{2n}$ with position-momentum coordinates (q_i, p_i) and Hamiltonian $H \in \mathcal{C}^\infty(\mathbb{R}^{2n})$: the trajectory $(\mathbf{q_i}, \mathbf{p_i})$ satisfies

$$\mathbf{q_i'} = \frac{\partial H}{p_i}, \qquad \mathbf{p_i'} = -\frac{\partial H}{q_i}.$$

($F = m\mathbf{q}''$ is recovered for $H = \frac{p^2}{2m} + V(q)$, with mass m, potential V.)

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Poisson: it can be rewritten using the bracket, for $f, \mathrm{g} \in \mathcal{C}^\infty(\mathbb{R}^{2n})$,

$$\{f, \mathrm{g}\} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial \mathrm{g}}{\partial q_i} - \frac{\partial \mathrm{g}}{\partial p_i}\frac{\partial f}{\partial q_i} \right)$$

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Moral: for $H \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, the vector field $\{H, \ \}$ determines the dynamics.

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Definition

A Poisson structure is a bivector field $\pi \in \mathfrak{X}^2(M)$ such that $[\pi, \pi] = 0$.

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$[\pi, \pi] = 0$ is equivalent to $(\star)$ being an algebra morphism

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Poisson structures in geometry and algebra

$\mathrm{Im}(\pi : T^*M \to TM) \subset TM$
is an integrable singular distribution

For $\alpha, \beta \in T^*M$,

$$\omega(\pi(\alpha), \pi(\beta)) := \pi(\alpha, \beta)$$

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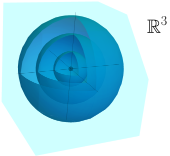

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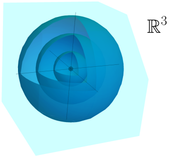

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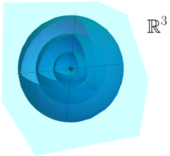

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linear Poisson $\longleftrightarrow$ Lie algebra
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Given, for $\iota : V \cong \mathcal{C}^\infty_{lin}(V^*)$, by
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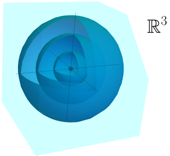

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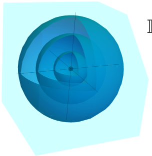

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What geometry is encoded by symmetric bivector fields?

A non-degenerate symmetric bivector field $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M) := \Gamma(\mathrm{Sym}^2\, TM)$
gives a (pseudo-)Riemannian metric $g := \vartheta^{-1}$.

Integrability condition for a symmetric bivector field ϑ

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We need some symmetric tools!

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Moučka, R., *Symmetric Cartan calculus, the Patterson-Walker metric and symmetric cohomology*. arXiv:2501.12442.

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We generate symmetric Cartan calculus by

$$L^s_X := [\iota_X, \nabla_s], \qquad \iota_{[X,Y]_s} := [L^s_X, \iota_Y].$$

Compare with: $L_X = [\iota_X, d]_g$ and $\iota_{[X,Y]} = [L_X, \iota_Y]_\mathfrak{g}$.

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and extend it to the **symmetric Schouten bracket** $[\ ,\ ]_s$ on $\mathfrak{X}^\bullet_{\mathsf{sym}}(M)$.

Compare with: $[X,Y] = XY - YX = \nabla_X Y - \nabla_Y X \rightsquigarrow$ Schouten bracket on $\mathfrak{X}^\bullet(M)$.

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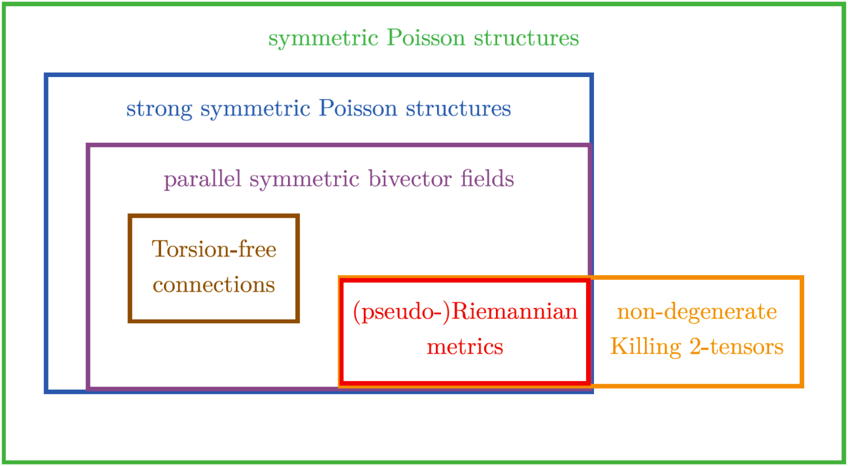
∇ is the Levi-Civita connection of g.

(all information contained in ϑ)

Examples


symmetric Poisson structures
strong symmetric Poisson structures
parallel symmetric bivector fields
Torsion-free connections
(pseudo-)Riemannian metrics
non-degenerate Killing 2-tensors

The characteristic distribution and module

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This motivates an a priori intermediate class

$$\left\{ \begin{array}{c} \text{symmetric} \\ \text{Poisson structures} \\ [\vartheta, \vartheta]_s = 0 \end{array} \right\} \supsetneq \left\{ \begin{array}{c} \textbf{involutive symmetric} \\ \textbf{Poisson structures} \\ [\mathcal{F}_\vartheta, \mathcal{F}_\vartheta] \subseteq \mathcal{F}_\vartheta \end{array} \right\} \underset{?}{\supsetneq} \left\{ \begin{array}{c} \text{strong symmetric} \\ \text{Poisson structures} \\ \nabla_{\vartheta(\,)}\vartheta = 0 \end{array} \right\}.$$

Low-hanging fruits and first questions

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When ϑ is **regular** ($\vartheta : T^*M \to TM$ of constant rank):

$$\mathrm{Im}\vartheta \text{ \textbf{geodesically invariant for} } \nabla$$

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Is there any extra structure on the leaves/distribution?

What happens in the singular case?

What happens in the non-involutive case?

Extra structure at each $m \in M$:

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Characteristic metric on $\mathrm{Im}\vartheta_m$, for $\alpha, \beta \in T^*M$,

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Still a question: **what happens in general?**

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Inspiration to define the notion of locally geodesically invariant, if the speed of a geodesic is in $\mathrm{Im}\vartheta$, all its speeds are locally in $\mathrm{Im}\vartheta$, and prove:

> **Theorem 2**. The characteristic distribution of a symmetric Poisson structure (ϑ, ∇) is locally geodesically invariant.

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Q2: are there singular non-involutive Poisson structures?

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Let me show you.

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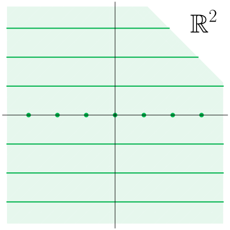
2	$y\,\partial_x \otimes \partial_x$		
3			



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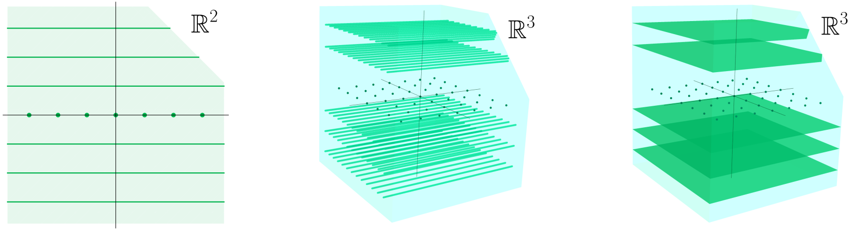

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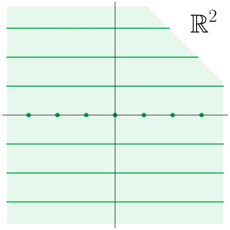

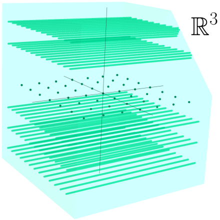

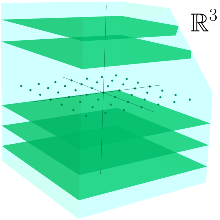

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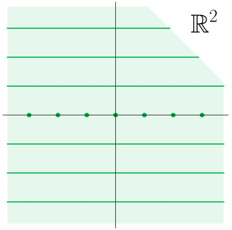

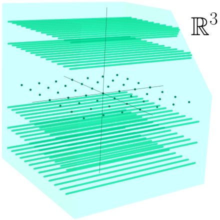

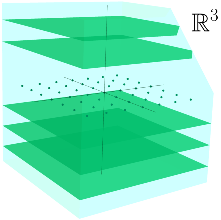

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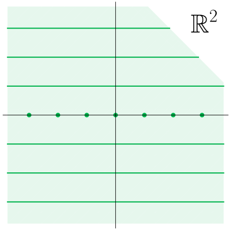

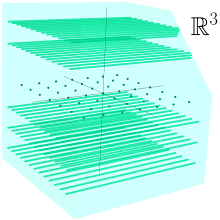

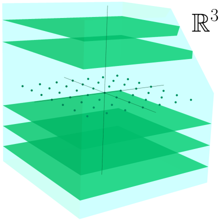

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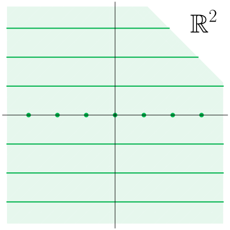

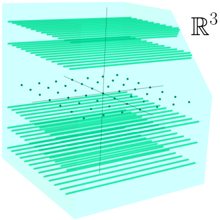

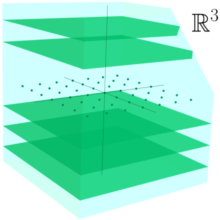

Geometric invariants can be defined for Jacobi-Jordan algebras.

It all looks very flat, but...

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[BF'14]: Unique non-associative Jacobi-Jordan algebra for $\dim V = 5$.

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Dimension 5 and 6 answer our two high-hanging questions.

A few open questions

Does there exist a symmetric Poisson structure whose characteristic distribution is locally geodesically invariant but not geodesically invariant?

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What about (pseudo-)Riemannian realizations?

To know more, and references,

Moučka, R. *Symmetric Poisson geometry, totally geodesic foliations and Jacobi-Jordan algebras*, arXiv:2508.15890.

but before we finish...

Beyond symmetric Poisson

We mentioned ‘Poisson is generalized by Dirac’.

Is symmetric Poisson generalized by something?

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Actually, I told you the story backwards, it all started there!

Symmetric Poisson structures are examples of C_n-Dirac structures, the generalized geometry coming from the skew-symmetric pairing.

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I hope I can tell you about it next time!

謝謝

Thank you for your attention!

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