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GROUND STATE SOLUTIONS OF THE SCHRÖDINGER–POISSON–SLATER EQUATION WITH DOUBLE CRITICAL EXPONENTS

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Abstract: This paper is concerned with the Schrödinger–Poisson–Slater (SPS) equation with double critical exponents. Such exponents appear in the Coulomb–Sobolev inequality, one being the Sobolev exponent and the other being called the Coulomb exponent here. We study the existence of nontrivial solutions of the SPS equation. This can be done by solving a variational problem with lack of compactness which is caused by these two critical exponents. Although the concentration compactness principle can be used to deal with the lack of compactness caused by the Sobolev exponent, it seems difficult to handle the other lack of compactness caused by the Coulomb exponent. Here we employ the Nehari–Pohozaev manifolds instead of the direct argument of concentration compactness on the Coulomb–Sobolev space to overcome this difficulty and prove that the equation possesses ground state solutions in these manifolds.

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1. Introduction

In this paper, we study the existence of positive solutions of the following Schrödinger–Poisson–Slater (SPS) equation:

$$-\Delta u+(|x|^{\alpha-n}*u^2)u=\mu|u|^{q-2}u+|u|^{2^*-2}u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n\geq 3$, $\alpha\in(0,n)$, $2^*=\frac{2n}{n-2}$, $q:=\frac{8+2\alpha}{2+\alpha}$, and $\mu>0$. Here we call 2^* and q the Sobolev exponent and the Coulomb exponent respectively. In equation (1.1), $|x|^{\alpha-n}*u^2$ is the repulsive Coulomb potential, which implies the Coulomb–Sobolev space is a suitable work space (cf. [**14**]). The Coulomb–Sobolev space is

$$X^{1,\alpha}:=\{v\in\mathcal{D}^{1,2}(\mathbb{R}^n);\, L(v)<\infty\},$$

with the norm $\|u\|:=\|u\|_{X^{1,\alpha}}=(\|\nabla u\|_2^2+[L(u)]^{\frac12})^{\frac12}$ (cf. [**19**]), where

$$L(v):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{v^2(x)v^2(y)}{|x-y|^{n-\alpha}}\,dx\,dy$$

is the Coulomb energy of the wave. It is known that each solution of (1.1) is a critical point of the energy functional $J\colon X^{1,\alpha}\to\mathbb{R}$, given by

$$J(u)=\frac12\int_{\mathbb{R}^n}|\nabla u|^2\,dx+\frac14\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{u^2(x)u^2(y)}{|x-y|^{n-\alpha}}\,dx\,dy-\frac{\mu}{q}\int_{\mathbb{R}^n}|u|^q\,dx-\frac{1}{2^*}\int_{\mathbb{R}^n}|u|^{2^*}\,dx.$$

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Here 2^* and q are two critical end points of the admission interval of p in the Coulomb–Sobolev inequality

$$\|\phi\|_p^p \leq C\|\nabla\phi\|_2^{\frac{p(n+\alpha)-4n}{4+\alpha-n}}[L(\phi)]^{\frac{2n-p(n-2)}{2(4+\alpha-n)}} \tag{1.2}$$

with $n \neq 4+\alpha$, where

$$\begin{cases} p \in \left[\dfrac{2(4+\alpha)}{2+\alpha}, \infty\right), & n=2, \\ p \in \left[\dfrac{2(4+\alpha)}{2+\alpha}, \dfrac{2n}{n-2}\right], & 3 \leq n < 4+\alpha, \\ p \in \left[\dfrac{2n}{n-2}, \dfrac{2(4+\alpha)}{2+\alpha}\right], & n > 4+\alpha. \end{cases} \tag{1.3}$$

The best constant of (1.2) is helpful to estimate the lower bound of the Coulomb energy (cf. [**4**] and [**13**]). To obtain the best constant, one can consider the minimization problem

$$\inf_{\phi\neq 0} \frac{\|\nabla\phi\|_2^{\frac{p(n+\alpha)-4n}{4+\alpha-n}}[L(\phi)]^{\frac{2n-p(n-2)}{2(4+\alpha-n)}}}{\|\phi\|_p^p}. \tag{1.4}$$

In [**2**] and [**3**], the authors proved that (1.4) is attained under the assumption (1.3). The Euler–Lagrange equation is

$$-\Delta u + (|x|^{\alpha-n} * u^2)u = \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^n, \tag{1.5}$$

where $n \geq 2$, $\alpha \in (0,n)$, and $\mu > 0$ is the so-called Slater constant.

Systems (1.1) and (1.5) appear in various physical frameworks, such as plasma physics, semiconductor physics, and the Hartree–Fock theory (cf. [**5, 13, 16**] and the references therein). There is a series of analytical results on the Schrödinger–Poisson systems in the literature (see [**1, 8, 9, 20, 21, 24**] and many others).

When $n=3$, $\alpha=2$ (now $2^*=6$ and $q=3$), Ianni and Ruiz ([**11**]) studied the following version of the Schrödinger–Poisson–Slater equation:

$$-\Delta u + \left(u^2 * \frac{1}{4\pi|x|}\right)u = \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

With the help of the 'monotonicity trick', a positive ground state solution was obtained when $3 < p < 6$. When $p=3$ (i.e., p is equal to the Coulomb exponent q), (1.6) has no solution if μ is suitably small. A natural question is whether (1.6) has a nontrivial solution for large μ. In addition, Ruiz ([**19**]) investigated the existence of ground state solutions with the radial structure when $\frac{18}{7} < p < 3$. Now, the Coulomb exponent is $\bar{q} = 18/7$ instead of $q=3$. Following the ideas in [**18**] and [**22**], the authors of paper [**12**] studied the higher-dimensional version of the Schrödinger–Poisson–Slater equation (1.5) where p belongs to the intervals in (1.3). Under the assumption $q < p < 2^*$ when $n < 4+\alpha$, or $2^* < p < q$ when $n > 4+\alpha$, they obtained a ground state solution of the Nehari–Pohozaev type.

In 2019, Liu, Zhang, and Huang studied the following equation of the Schrödinger–Poisson–Slater type with the Sobolev exponent and the subcritical exponent ([**17**]):

$$-\Delta u + \left(u^2 * \frac{1}{|4\pi x|}\right)u = \mu|u|^{p-2}u + |u|^4 u \quad \text{in } \mathbb{R}^3, \tag{1.7}$$

where $\mu > 0$. The main results are listed as follows. When $p \in (3,6)$, they obtained the existence of positive ground state solutions. They also studied the existence of radial solutions. When $p = 3$, they obtained the existence of radial solutions in radial space $X^{1,2}_{\text{rad}}$ by the constrained minimization method provided μ is suitably large. When $p \in (18/7, 3)$ and $\mu \in (0, \mu^*)$ with some $\mu^* > 0$, they proved the existence of radial solutions. The analogous results were generalized to the equation with the fractional Laplacian (cf. [**10**]).

When $p = 3$ and μ is suitably small, it is unknown whether (1.7) has nontrivial solutions without the radial structure. This is the main motivation of this paper. In addition, we are concerned with the existence of nontrivial solutions (which are not limited to radial structure) of higher-dimensional equation (1.1), because there are many differences between the case of $3 \le n < 4 + \alpha$ and the case of $n > 4 + \alpha$.

Now our main results in this paper are stated as follows. Set

$$\mathcal{M}_\pm := \{u \in X^{1,\alpha}\backslash\{0\} : I_\pm(u) = 0\}, \tag{1.8}$$

where

$$I_\pm(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{4}L(u) - \frac{1}{q}\mu\int_{\mathbb{R}^n} |u|^q\,dx \mp \frac{2^* - nb}{q - nb}\frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx,$$

with $b = \frac{2}{2+\alpha}$.

Theorem 1.1. *Let $n \ge 4$, $\alpha \in (0, n)$, $2^* = \frac{2n}{n-2}$, $q := \frac{8+2\alpha}{2+\alpha}$, and let $\mu > 0$ be suitably small. Then*

(i) (1.1) *has a ground state solution in $\mathcal{M}_+$ when $4 \le n < 4 + \alpha$,*
(ii) (1.1) *has a ground state solution in $\mathcal{M}_-$ when $n > 4 + \alpha$.*

In addition, these ground state solutions are the $L^{2^*}(\mathbb{R}^n)$-limit of some minimizing sequence of J in $\mathcal{M}_\pm$.

Clearly, the following two embedding results

$$X^{1,\alpha} \hookrightarrow L^{2^*}(\mathbb{R}^n), \quad X^{1,\alpha} \hookrightarrow L^q(\mathbb{R}^n)$$

are not compact. Applying the concentration compactness principle, we can only obtain the strong convergence of minimizing sequence $\{u_m\}$ in $L^{2^*}_{\text{loc}}(\mathbb{R}^n)$ when $3 \le n < 4+\alpha$. Here the best Sobolev constant S comes into play to estimate the threshold value of J, because the test function can be chosen as the extremal function of the Sobolev inequality. Now, $q < 2^*$ and hence the $L^q_{\text{loc}}(\mathbb{R}^n)$ convergence of u_m also holds to ensure the limit of $J(u_m)$ makes sense. When $n > 4 + \alpha$ (which implies $q > 2^*$), $L^q_{\text{loc}}(\mathbb{R}^n)$ convergence of u_m is not natural any more if we use S to estimate the threshold value of J. The best constant of the Coulomb–Sobolev inequality may be suitable to estimate the threshold value of J. However, we do not know whether the extremal functions exist or not when p in (1.2) is equal to 2^* or q (cf. Theorem 2.2 in [**2**]). Therefore, it seems difficult to take the test functions to estimate the threshold value of J. Consequently, it is not easy to prove that the limit of the minimizing sequence of J is the critical point in $X^{1,\alpha}$. In this paper, we adopt a new approach to take place of the direct argument of the concentration compactness on $X^{1,\alpha}$. Here, the Pohozaev-type identity $\mathcal{P}(u) = 0$ plays a key role, where

$$\mathcal{P}(u) := \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{n+\alpha}{4}L(u) - \mu\frac{n}{q}\int_{\mathbb{R}^n} |u|^q\,dx - \frac{n}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx.$$

We will apply this Pohozaev-type identity to construct suitable constraint manifolds to find the nontrivial solutions of (1.1).

Finally, we consider ground states in the radial space. Set

$$X^{1,\alpha}_{\mathrm{rad}} = \{v \in X^{1,\alpha};\ v \text{ is a radially symmetric function}\}$$

with the same norm from $X^{1,\alpha}$. Define

$$\widetilde{\mathcal{M}}_{\pm} := \{u \in X^{1,\alpha}_{\mathrm{rad}} \backslash \{0\} : I_{\pm}(u) = 0\}.$$

Now, the embedding

$$X^{1,\alpha} \hookrightarrow L^{\frac{8+2\alpha}{2+\alpha}}(\mathbb{R}^n) \tag{1.9}$$

is compact if $\alpha \in (1, n)$ (cf. Theorem 1.5 in [**3**]). So we can prove the following theorem.

Theorem 1.2. *Let $n \geq 4$, $\alpha \in (0, n)$, $2^* = \frac{2n}{n-2}$, $q := \frac{8+2\alpha}{2+\alpha}$, and let $\mu > 0$ be suitably small. Then the conclusions (i)-(ii) of Theorem 1.1 still hold when $\mathcal{M}_{\pm}$ is replaced with $\widetilde{\mathcal{M}}_{\pm}$. Furthermore, if $\alpha \in (1, n)$, those ground state solutions are the $X^{1,\alpha}_{\mathrm{rad}}$-limits of some minimizing sequence of J in $\widetilde{\mathcal{M}}_{\pm}$.*

Remark 1.3. When $\mu = 0$, equation (1.1) has no nontrivial solution. Indeed, supposing that u is a solution of equation (1.1) with $n \neq 4 + \alpha$, there holds $\langle J'(u), u\rangle = 0$. Combining with $\mathcal{P}(u) = 0$ yields

$$\begin{cases} \dfrac{n-2}{2}\displaystyle\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \dfrac{n+\alpha}{4} L(u) - \dfrac{n}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx = 0, \\ \displaystyle\int_{\mathbb{R}^n} |\nabla u|^2\,dx + L(u) - \int_{\mathbb{R}^n} |u|^{2^*}\,dx = 0. \end{cases}$$

Consequently,

$$\left(\frac{n+\alpha}{4} - \frac{n-2}{2}\right) L(u) = \frac{4+\alpha-n}{4} L(u) = 0.$$

This implies that $u = 0$.

Remark 1.4. When $\mu > 0$, it seems difficult to prove the $X^{1,\alpha}$-convergence of (PS)-sequences because of $q = \frac{8+2\alpha}{2+\alpha} < 4$, where 4 is the order of $L(u)$. In addition, when $\alpha = 2$, $n = 3$, and $p = q(= 3)$, the authors of [**17**] pointed out that the perturbation method and the monotonicity trick technique are invalid even though μ is suitably large. So they studied the existence of (1.1) in the radial space $X^{1,\alpha}_{\mathrm{rad}}$. Here Theorem 1.1 shows that we can find a nontrivial solution of (1.1) in $X^{1,\alpha}$ when $p = q$. Although we do not know whether this solution is the $X^{1,\alpha}$-limit of the (PS)-sequence, Theorem 1.2 shows that the radial solution is.

Remark 1.5. According to Section 1.4 in [**3**], when $\alpha \in (1, n)$, one end point of the admissible interval of p in (1.3) changes from q to $\bar{q} := 2(5n - 4 - \alpha)/(3n - 4 + \alpha)$. (In particular, $\bar{q} = 18/7$ when $n = 3$ and $\alpha = 2$.) Thus, the new admissible interval becomes larger and q belongs to the new interval. Equation (1.1) is not the double critical problem any more. The argument in this paper may be helpful to understand the existence of the new double critical problem (1.1) where q is replaced with $\bar{q}$ in the radial space.

2. Nehari–Pohozaev manifold

In this section, we follow the ideas in [**18**] and [**22**] to introduce some properties of the Nehari–Pohozaev manifold related to the functional J.

We first establish the following result.

Lemma 2.1. *The functional J is unbounded from below.*

Proof: Let $u \in X^{1,\alpha}$, and $u_t^{\pm} = t^{\pm 1} u(t^{\pm b} x)$, where $b = \frac{2}{2+\alpha}$ and $t > 0$.

By the standard scaling we have

$$\int_{\mathbb{R}^n} |\nabla u_t^{\pm}|^2 \, dx = t^{\pm q \mp nb} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx, \quad L(u_t^{\pm}) = t^{\pm q \mp nb} L(u),$$

and

$$\int_{\mathbb{R}^n} |u_t^{\pm}|^q \, dx = t^{\pm q \mp nb} \int_{\mathbb{R}^n} |u|^q \, dx, \quad \int_{\mathbb{R}^n} |u_t^{\pm}|^{2^*} \, dx = t^{\pm 2^* \mp nb} \int_{\mathbb{R}^n} |u|^{2^*} \, dx.$$

Hence,

$$\begin{aligned} J(u_t^{\pm}) &= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_t^{\pm}|^2 \, dx + \frac{1}{4} L(u_t^{\pm}) - \frac{\mu}{q} \int_{\mathbb{R}^n} |u_t^{\pm}|^q \, dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |u_t^{\pm}|^{2^*} \, dx \\ &= \frac{t^{\pm q \mp nb}}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \frac{t^{\pm q \mp nb}}{4} L(u) - \frac{\mu t^{\pm q \mp nb}}{q} \int_{\mathbb{R}^n} |u|^q \, dx \\ &\qquad - \frac{t^{\pm 2^* \mp nb}}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} \, dx. \end{aligned}$$

When $3 \le n < 4 + \alpha$, we see $2^* > q > nb$. Therefore, $J(u_t^+) \to -\infty$ as $t \to +\infty$. When $n > 4 + \alpha$, we see $2^* < q < nb$. Therefore, $J(u_t^-) \to -\infty$ as $t \to +\infty$. □

This result implies that we have to search for the ground state under some constraint conditions. So we will introduce a restriction manifold.

For introducing the suitable constraint conditions, we observe the following result.

Lemma 2.2. *Set*

$$\varphi^{\pm}(t) := t^{\pm q \mp nb} \left[\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \frac{1}{4} L(u) - \frac{\mu}{q} \int_{\mathbb{R}^n} |u|^q \, dx \right] - \frac{t^{\pm 2^* \mp nb}}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} \, dx,$$

where $t \ge 0$, $u \in \mathcal{M}_+$ when $3 \le n < 4 + \alpha$, and $u \in \mathcal{M}_-$ when $n > 4 + \alpha$. Then both φ^+ with $3 \le n < 4 + \alpha$ and φ^- with $n > 4 + \alpha$ have their unique critical points, corresponding to their maximum.

Proof: Since $u \in \mathcal{M}_+$ when $3 \le n < 4 + \alpha$ and $u \in \mathcal{M}_-$ when $n > 4 + \alpha$, we have

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \frac{1}{4} L(u) - \frac{\mu}{q} \int_{\mathbb{R}^n} |u|^q \, dx = \frac{2^* - nb}{q - nb} \frac{1}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} \, dx > 0.$$

Therefore, when $3 \le n < 4 + \alpha$,

$$\varphi^+(t) \to 0^+ \text{ as } t \to 0^+, \quad \text{and} \quad \varphi^+(t) \to -\infty \text{ as } t \to +\infty.$$

Similarly, when $n > 4 + \alpha$,

$$\varphi^-(t) \to 0^+ \text{ as } t \to 0^+, \quad \text{and} \quad \varphi^-(t) \to -\infty \text{ as } t \to +\infty.$$

Noting $\varphi(0) = 0$, we see that φ^+ has a unique positive critical point when $3 \le n < 4 + \alpha$. Similarly, φ^- has a unique positive critical point when $n > 4 + \alpha$. The proof of Lemma 2.2 is completed. □

Write $u_t^{\pm} = t^{\pm 1} u(t^{\pm b} x)$, with $b = \frac{2}{2+\alpha}$ and $t > 0$. By the proof of Lemma 2.1, there holds

$$\varphi^{\pm}(t) = J(u_t^{\pm}).$$

By Lemma 2.2, if u is a critical point of J on $\mathcal{M}_{\pm}$, the maxima of $\varphi^+(t)$ with $3 \le n < 4 + \alpha$ and $\varphi^-(t)$ with $n < 4 + \alpha$ should be achieved at $t = 1$ and $[\varphi^{\pm}]'(1) = 0$. Noting

$$I_{\pm}(u) = (q - nb)^{-1} [\varphi^{\pm}]'(1),$$

we know that $\mathcal{M}_{\pm}$ defined in (1.8) makes sense.

Obviously, $\mathcal{M}_\pm \neq \emptyset$. Indeed, for any given $v \neq 0$, the proof of Lemma 2.2 shows that there exists $t > 0$ such that $v_t \in \mathcal{M}_\pm$. Moreover, the curve $\Gamma = \{u_t\}_{t\in\mathbb{R}}$ intersects with manifolds $\mathcal{M}_\pm$ (where $\mathcal{M}_\pm$ are C^1-manifolds (see Lemma 2.3)), and $J|_\Gamma$ attains its maximum along Γ at the point u. If u is a mountain pass type solution of (1.1), it is natural to look for the minimizers of J in $\mathcal{M}_\pm$. In addition, if u is a critical point of J, it satisfies the Pohozaev identity $\mathcal{P}(u) = 0$. By a simple calculation, we get

$$(4+\alpha-n)bI_\pm(u) = \pm\langle J'(u), u\rangle \mp b\mathcal{P}(u) \tag{2.1}$$

with $b = \frac{2}{\alpha+2}$. Therefore, $\mathcal{M}_\pm$ are called the Nehari–Pohozaev manifolds. In this paper, we will look for nontrivial solutions of (1.1) in the Nehari–Pohozaev manifolds $\mathcal{M}_+$ with $3 \le n < 4+\alpha$ and $\mathcal{M}_-$ with $n > 4+\alpha$.

Next, we have the following result.

Lemma 2.3. *When $3 \le n < 4+\alpha$, $\mathcal{M}_+$ is a C^1-manifold, and every critical point of J in $\mathcal{M}_+$ is a critical point of J in $X^{1,\alpha}$. When $n > 4+\alpha$, if we replace $\mathcal{M}_+$ with $\mathcal{M}_-$, the conclusion above still holds.*

Proof: We proceed by three steps.

Step 1. We claim $J > 0$.

In fact, for any $u \in \mathcal{M}_+$ with $3 \le n < 4+\alpha$, there holds

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{4}L(u) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx \\ &= \frac{1}{2^*}\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*}\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx \\ &= \frac{1}{2^*}\frac{2^*-q}{q-nb}\int_{\mathbb{R}^n} |u|^{2^*}\,dx > 0. \end{aligned} \tag{2.2}$$

Similarly, for any $u \in \mathcal{M}_-$ with $n > 4+\alpha$, there holds

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{4}L(u) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx \\ &= \frac{1}{2^*}\frac{-2^*+nb}{-q+nb}\int_{\mathbb{R}^n} |u|^{2^*}\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u|^{2^*}\,dx \\ &= \frac{1}{2^*}\frac{q-2^*}{nb-q}\int_{\mathbb{R}^n} |u|^{2^*}\,dx > 0. \end{aligned} \tag{2.3}$$

Step 2. We claim that $\mathcal{M}_+$ with $3 \le n < 4+\alpha$ and $\mathcal{M}_-$ with $n > 4+\alpha$ are C^1-manifolds.

In fact, by the implicit function theorem, it is sufficient to prove that $[I_\pm]'(u) \neq 0$ for any $u \in \mathcal{M}_+$ with $3 \le n < 4+\alpha$ and $u \in \mathcal{M}_-$ with $n > 4+\alpha$. We prove it by argument of contradiction. Specifically, suppose that $[I_\pm]'(u) = 0$ for some $u_+ \in \mathcal{M}_+$ with $3 \le n < 4+\alpha$ or $u_- \in \mathcal{M}_-$ with $n > 4+\alpha$. Thus, in a weak sense there holds

$$-\Delta u_\pm + (|x|^{\alpha-n} * u_\pm^2)u_\pm = \mu|u_\pm|^{q-2}u_\pm \pm \frac{2^*-nb}{q-nb}|u_\pm|^{2^*-2}u_\pm. \tag{2.4}$$

Multiplying (2.4) by $u_\pm$ and integrating, we have

$$\int_{\mathbb{R}^n} |\nabla u_\pm|^2\,dx + L(u_\pm) - \mu\int_{\mathbb{R}^n} |u_\pm|^q\,dx \mp \frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n} |u_\pm|^{2^*}\,dx = 0. \tag{2.5}$$

The Pohozaev identity corresponding to equation (2.4) is

$$\begin{aligned}\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla u_\pm|^2\,dx+\frac{n+\alpha}{4}L(u_\pm)-\mu\frac{n}{q}\int_{\mathbb{R}^n}|u_\pm|^q\,dx&\\ \mp\frac{n}{2^*}\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n}|u_\pm|^{2^*}\,dx=0.&\end{aligned}\tag{2.6}$$

It follows from $I_\pm(u_\pm)=0$ (which is implied by $u_\pm\in\mathcal{M}_\pm$) that

$$\frac{n}{2}\int_{\mathbb{R}^n}|\nabla u_\pm|^2\,dx+\frac{n}{4}L(u_\pm)-\mu\frac{n}{q}\int_{\mathbb{R}^n}|u_\pm|^q\,dx\mp\frac{n}{2^*}\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n}|u_\pm|^{2^*}\,dx=0.$$

Therefore, by (2.6),

$$\int_{\mathbb{R}^n}|\nabla u_\pm|^2\,dx=\frac{\alpha}{4}L(u_\pm).\tag{2.7}$$

Multiplying (2.5) by q^{-1} and applying $I_\pm(u_\pm)=0$, we have

$$\left(\frac{1}{2}-\frac{1}{q}\right)\int_{\mathbb{R}^n}|\nabla u_\pm|^2\,dx+\left(\frac{1}{4}-\frac{1}{q}\right)L(u_\pm)=\pm\left(\frac{1}{2^*}-\frac{1}{q}\right)\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n}|u_\pm|^{2^*}\,dx.$$

Inserting (2.7) into the equation above, we obtain

$$\frac{q(2+\alpha)-2\alpha-8}{8q}L(u_\pm)=\pm\left(\frac{1}{2^*}-\frac{1}{q}\right)\frac{2^*-nb}{q-nb}\int_{\mathbb{R}^n}|u_\pm|^{2^*}\,dx.\tag{2.8}$$

In view of $q=\frac{8+2\alpha}{2+\alpha}$, there hold $\frac{q(2+\alpha)-2\alpha-8}{8q}=0$ and $\left(\frac{1}{2^*}-\frac{1}{q}\right)\frac{2^*-nb}{q-nb}\neq 0$. Therefore, (2.8) leads to a contradiction. Thus, $\mathcal{M}_\pm$ are C^1-manifolds.

Step 3. We claim that the critical points of J in $\mathcal{M}_+$ with $3\le n<4+\alpha$ and in $\mathcal{M}_-$ with $n>4+\alpha$ are critical points of J in $X^{1,\alpha}$.

(1) Case of $3\le n<4+\alpha$.

Assume that u is a critical point of J in $\mathcal{M}_+$; there exists a Lagrange multiplier λ such that

$$J'(u)=\lambda(I_+)'(u).$$

It can be written, in a weak sense, as

$$\begin{aligned}&-\Delta u+(|x|^{\alpha-n}*u^2)u-\mu|u|^{q-2}u-|u|^{2^*-2}u\\ &\qquad=\lambda\left[-\Delta u+(|x|^{\alpha-n}*u^2)u-\mu|u|^{q-2}u-\frac{2^*-nb}{q-nb}|u|^{2^*-2}u\right].\end{aligned}$$

That is,

$$-(1-\lambda)\Delta u+(1-\lambda)(|x|^{\alpha-n}*u^2)u=(1-\lambda)\mu|u|^{q-2}u+\left(1-\frac{2^*-nb}{q-nb}\lambda\right)|u|^{2^*-2}u.\tag{2.9}$$

It remains to prove $\lambda=0$.

Denote

$$\mathcal{T}=\int_{\mathbb{R}^n}|\nabla u|^2\,dx,\quad \mathcal{B}=\mu\int_{\mathbb{R}^n}|u|^q\,dx,\quad \mathcal{C}=\int_{\mathbb{R}^n}|u|^{2^*}\,dx.$$

Clearly, $\lambda \neq 1$ (otherwise, (2.9) implies $u \equiv 0$). We can establish the following system:

$$\begin{cases} I_+(u) = \frac{1}{2}\mathcal{T} + \frac{1}{4}L(u) - \frac{1}{q}\mathcal{B} - \frac{1}{2^*}\frac{2^*-nb}{q-nb}\mathcal{C} = 0, \\ \mathcal{T} + L(u) - \mathcal{B} - \frac{1}{1-\lambda}\left(1 - \frac{2^*-nb}{q-nb}\lambda\right)\mathcal{C} = 0, \\ \frac{n-2}{2}\mathcal{T} + \frac{n+\alpha}{4}L(u) - \frac{n}{q}\mathcal{B} - \frac{1}{1-\lambda}\left(1 - \frac{2^*-nb}{q-nb}\lambda\right)\frac{n}{2^*}\mathcal{C} = 0, \end{cases} \tag{2.10}$$

where the second equation follows by multiplying (2.9) by u and integrating, and the third one is the Pohozaev identity corresponding to equation (2.9).

Combining the first and the third equations in (2.10), we have

$$\mathcal{T} = \frac{\alpha}{4}L(u) + \frac{n}{2^*}\left[\frac{2^*-nb}{q-nb} - \frac{1}{1-\lambda} + \frac{2^*-nb}{q-nb}\frac{\lambda}{1-\lambda}\right]\mathcal{C}. \tag{2.11}$$

The second equation in (2.10) can be rewritten as

$$\frac{1}{q}\mathcal{T} + \frac{1}{q}L(u) - \frac{1}{q}\mu\int_{\mathbb{R}^n}|u|^q\,dx - \frac{1}{1-\lambda}\left(1 - \frac{2^*-nb}{q-nb}\lambda\right)\frac{1}{q}\mathcal{C} = 0. \tag{2.12}$$

It follows from (2.12) and the first equation in (2.10) that

$$\frac{q-2}{2q}\mathcal{T} + \frac{q-4}{4q}L(u) = F(u), \tag{2.13}$$

where

$$F(u) = \left[\frac{1}{2^*}\frac{2^*-nb}{q-nb} - \frac{1}{1-\lambda}\left(1 - \frac{2^*-nb}{q-nb}\lambda\right)\frac{1}{q}\right]\mathcal{C}.$$

Thus by (2.11) and (2.13) we see that

$$\frac{q(2+\alpha)-2\alpha-8}{8q}L(u) = F(u) - G(u), \tag{2.14}$$

where

$$G(u) = \frac{q-2}{2q}\frac{n}{2^*}\left(\frac{2^*-nb}{q-nb} - \frac{1}{1-\lambda} + \frac{2^*-nb}{q-nb}\frac{\lambda}{1-\lambda}\right)\mathcal{C}.$$

In view of $q = \frac{8+2\alpha}{2+\alpha}$, it follows from (2.14) that

$$F(u) = G(u).$$

Specifically,

$$\frac{1}{2^*}\frac{2^*-nb}{q-nb} - \frac{1}{1-\lambda}\left(1 - \frac{2^*-nb}{q-nb}\lambda\right)\frac{1}{q} = \frac{q-2}{2q}\frac{n}{2^*}\left(\frac{2^*-nb}{q-nb} - \frac{1}{1-\lambda} + \frac{2^*-nb}{q-nb}\frac{\lambda}{1-\lambda}\right).$$

This is equivalent to

$$\frac{1}{2^*}\frac{2^*-nb}{q-nb}(1-\lambda) - \frac{1}{q}\left(1 - \frac{2^*-nb}{q-nb}\lambda\right) = \frac{q-2}{2q}\frac{n}{2^*}\left(\frac{2^*-nb}{q-nb}(1-\lambda) - 1 + \frac{2^*-nb}{q-nb}\lambda\right).$$

To solve λ, we rewrite the result above as

$$\left(\frac{1}{q} - \frac{1}{2^*}\right)\frac{2^*-nb}{q-nb}\lambda + K_{q,n,b} = 0, \tag{2.15}$$

where

$$K_{q,n,b} = \frac{1}{2^*}\frac{2^*-nb}{q-nb} - \frac{1}{q} - \frac{n}{2^*}\frac{q-2}{2q}\frac{2^*-nb}{q-nb} + \frac{q-2}{2q}\frac{n}{2^*}.$$

By computing, we have

$$
\begin{aligned}
K_{q,n,b} &= \frac{1}{2^*}\frac{2^*-nb}{q-nb} - \frac{1}{q} - \frac{n}{2^*}\frac{q-2}{2q}\frac{2^*-nb}{q-nb} + \frac{q-2}{2q}\frac{n}{2^*} \\
&= \frac{1}{2^*}\frac{2^*-nb}{q-nb}\left(1-\frac{n(q-2)}{2q}\right) + \frac{n(q-2)-2\cdot 2^*}{2\cdot 2^* q} \\
&= \frac{1}{2^*}\frac{2^*-nb}{q-nb}\left(1-\frac{n(q-2)}{2q} + \frac{n(q-2)-2\cdot 2^*}{2q}\frac{q-nb}{2^*-nb}\right) \\
&= \frac{1}{2^*}\frac{2^*-nb}{q-nb}\frac{2q(2^*-nb)-n(q-2)(2^*-nb)+(q-nb)[n(q-2)-2\cdot 2^*]}{2q(2^*-nb)} \\
&= \frac{1}{2^*}\frac{2^*-nb}{q-nb}\frac{-2qnb+(q-2^*)(nq-2n)+2\cdot 2^* nb}{2q(2^*-nb)} \\
&= \frac{n}{2^*}\frac{2^*-nb}{q-nb}\frac{-2qb+(q-2^*)(q-2)+2\cdot 2^* b}{2q(2^*-nb)} \\
&= \frac{n}{2^*}\frac{2^*-nb}{q-nb}\frac{2b(2^*-q)+(q-2^*)(q-2)}{2q(2^*-nb)} \\
&= \frac{n}{2^*}\frac{2^*-nb}{q-nb}\frac{(2^*-q)(2b-q+2)}{2q(2^*-nb)} \\
&= 0,
\end{aligned}
$$

where we use the fact $2b-q+2 = \frac{4}{2+\alpha} + 2 - \frac{8+2\alpha}{2+\alpha} = 0$. Consequently, it follows from (2.15) that

$$\lambda \equiv 0.$$

Therefore, we conclude that $J'(u) = 0$ for $n \geq 3$, i.e., u is a critical point of J.

(2) Case of $n > 4+\alpha$.

Assume that u is a critical point of J in $\mathcal{M}_-$. There exists the Lagrange multiplier λ such that $J'(u) = \lambda(I_-)'(u)$, which implies

$$-(1-\lambda)\Delta u + (1-\lambda)(|x|^{\alpha-n} * u^2)u = \mu(1-\lambda)|u|^{q-2}u + \left(1 - \frac{-2^*+nb}{-q+nb}\lambda\right)|u|^{2^*-2}u.$$

We see that $\lambda \neq 1$, and it remains to prove $\lambda = 0$.

By the same derivation as in (2.10), we can see that

$$
\begin{cases}
\dfrac{1}{2}\mathcal{T} + \dfrac{1}{4}L(u) - \dfrac{1}{q}\mathcal{B} - \dfrac{1}{2^*}\dfrac{-2^*+nb}{-q+nb}\mathcal{C} = 0, \\[2ex]
\mathcal{T} + L(u) - \mathcal{B} - \dfrac{1}{1-\lambda}\left(1 - \dfrac{-2^*+nb}{-q+nb}\lambda\right)\mathcal{C} = 0, \\[2ex]
\dfrac{n-2}{2}\mathcal{T} + \dfrac{n+\alpha}{4}L(u) - \dfrac{n}{q}\mathcal{B} - \dfrac{1}{1-\lambda}\left(1 - \dfrac{-2^*+nb}{-q+nb}\lambda\right)\dfrac{n}{2^*}\mathcal{C} = 0.
\end{cases}
\tag{2.16}
$$

It follows from the first and the second equations in (2.16) that

$$\frac{1}{4}L(u) = \left(\frac{1}{2} - \frac{1}{q}\right)\mathcal{B} + \left[\frac{1}{2(1-\lambda)}\left(1 - \frac{-2^*+nb}{-q+nb}\lambda\right) - \frac{1}{2^*}\frac{-2^*+nb}{-q+nb}\right]\mathcal{C}. \tag{2.17}$$

Applying the third and the second equations in (2.16), we have

$$\begin{aligned}\frac{n-4-\alpha}{4}L(u) &= \frac{n-2}{2}\mathcal{B}-\frac{n}{q}\mathcal{B}\\ &\quad+\frac{n-2}{2(1-\lambda)}\left(1-\frac{-2^*+nb}{-q+nb}\lambda\right)\mathcal{C}-\frac{1}{1-\lambda}\left(1-\frac{-2^*+nb}{-q+nb}\lambda\right)\frac{n}{2^*}\mathcal{C}\\ &= \frac{(n-2)q-2n}{2q}\mathcal{B}.\end{aligned}$$

It follows from this result and (2.17) that

$$\left[\frac{1}{2(1-\lambda)}\left(1-\frac{-2^*+nb}{-q+nb}\lambda\right)-\frac{1}{2^*}\frac{-2^*+nb}{-q+nb}\right]\mathcal{C}=l\left[\frac{1}{q}-\frac{1}{2}+\frac{1}{n-4-\alpha}\frac{(n-2)q-2n}{2q}\right]\mathcal{B}.$$

That is,

$$\begin{aligned}&\left[\frac{1}{2}\left(1-\frac{-2^*+nb}{-q+nb}\lambda\right)-\frac{1}{2^*}\frac{-2^*+nb}{-q+nb}(1-\lambda)\right]\mathcal{C}\\ &\qquad=\left[\frac{1}{q}-\frac{1}{2}+\frac{1}{n-4-\alpha}\frac{(n-2)q-2n}{2q}\right](1-\lambda)\mathcal{B}.\end{aligned}\tag{2.18}$$

In view of $\frac{-2^*+nb}{-q+nb}=\frac{n}{n-2}$, we have

$$\begin{aligned}\frac{1}{2}\left(1-\frac{-2^*+nb}{-q+nb}\lambda\right)-\frac{1}{2^*}\frac{-2^*+nb}{-q+nb}(1-\lambda) &= \frac{1}{2}\left(1-\frac{n}{n-2}\lambda\right)-\frac{1}{2^*}\frac{n}{n-2}(1-\lambda)\\ &= \frac{1}{2}\left(1-\frac{n}{n-2}\lambda\right)-\frac{1}{2}(1-\lambda)\\ &= \frac{1}{2}\left(1-\frac{n}{n-2}\right)\lambda\\ &= \frac{1}{n-2}\lambda.\end{aligned}$$

On the other hand, noting $(n-2)q-2n=\frac{(8+2\alpha)(n-2)}{2+\alpha}-2n=\frac{4(n-4-\alpha)}{2+\alpha}$, we get

$$\frac{1}{q}-\frac{1}{2}+\frac{1}{n-4-\alpha}\frac{(n-2)q-2n}{2q}=\frac{2+\alpha}{8+2\alpha}-\frac{1}{2}+\frac{2}{8+2\alpha}=0.$$

From the information above and (2.18), we obtain $\frac{1}{n-2}\lambda\mathcal{C}=0$. Consequently, we conclude that

$$\lambda\equiv 0.$$

Therefore, we obtain $J'(u)=0$, i.e., u is a critical point of J. The proof is completed. □

Remark 2.4. Comparing with Step 4 in the proof of Lemma 5 in [**12**], we find (2.10) is more complicated than (12) in [**12**], because the functional $I_\pm(u)$ contains four terms. From the algebraic structure of (2.10), it seems difficult to determine the value of λ. Thanks to the fact that q is the Coulomb exponent, we can also deduce that $\lambda=0$ here.

Remark 2.5. All the results above are still true if we replace $\mathcal{M}_\pm$ with $\widetilde{\mathcal{M}}_\pm$.

3. Minimizing sequence

Hereafter, $u \in \mathcal{M}_+$ means $u \in \mathcal{M}_+$ with $3 \le n < 4+\alpha$, and $u \in \mathcal{M}_-$ means $u \in \mathcal{M}_-$ with $n > 4+\alpha$.

By Step 1 in the proof of Lemma 2.3, we can find a minimizing sequence $\{u_m\}$ of J in $\mathcal{M}_\pm$. Specifically,

$$J(u_m) \to \inf_{\mathcal{M}_\pm} J \quad \text{as } m \to \infty. \tag{3.1}$$

Therefore, using the same calculations as in (2.2) and (2.3), and noting $\frac{1}{2^*}\frac{2^*-q}{q-nb} = \frac{1}{n}$, we can deduce from $u_m \in \mathcal{M}_\pm$ that

$$\frac{1}{n}\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx \to \inf_{\mathcal{M}_\pm} J \quad (m \to \infty). \tag{3.2}$$

Lemma 3.1. *Let $n \ge 3$ and $\alpha \in (0,n)$. Assume that $\{u_m\}$ is the minimizing sequence of J in $\mathcal{M}_\pm$. Then, for suitably small $\mu > 0$, we can find a subsequence of $\{u_m\}$ denoted by itself such that $\{u_m\}$ is bounded in $X^{1,\alpha}$.*

Proof: Clearly, $u_m \in \mathcal{M}_\pm$ implies $I_\pm(u_m) = 0$. In particular,

$$\frac{1}{2}\|\nabla u_m\|_2^2 + \frac{1}{4}L(u_m) - \frac{\mu}{q}\|u_m\|_q^q - \frac{1}{2^*}\frac{2^*-nb}{q-nb}\|u_m\|_{2^*}^{2^*} = 0.$$

Combining with (3.2) yields

$$\frac{1}{2}\|\nabla u_m\|_2^2 + \frac{1}{4}L(u_m) \le \frac{\mu}{q}\|u_m\|_q^q + C.$$

Applying (1.2) and the Young inequality, and noting that μ is suitably small, we have

$$\|u_m\|_{X^{1,\alpha}} \le C_*. \tag{3.3}$$

Here $C_* > 0$ is an absolute constant (independent of m). By (1.2), from (3.3) it follows that

$$\|\nabla u_m\|_2^2 + L(u_m) + \|u_m\|_q^q + \|u_m\|_{2^*}^{2^*} \le C. \tag{3.4}$$

□

By Lemma 3.1 and the Ekeland variational principle (see Theorem 8.5 in [**23**]), there exist a subsequence of $\{u_m\}$ (denoted by itself) and $\{\lambda_m\} \subset \mathbb{R}$ such that

$$J'(u_m) - \lambda_m I'(u_m) \to 0 \quad \text{as } m \to \infty. \tag{3.5}$$

Obviously, $\Phi_m(u_m) := J(u_m) - \lambda_m I(u_m) \to \inf_{\mathcal{M}_\pm} J$ as $m \to \infty$. Therefore, $\{u_m\}$ is a bounded $(\mathrm{PS})_c$-sequence of Φ_m.

Lemma 3.2. *The infimum of J on the constraints $\mathcal{M}_\pm$ is strictly positive, and the Lagrange multiplier $\lambda_m \ne 1$ in (3.5) for all m.*

Proof: We first claim that the infimum of J on the constraints $\mathcal{M}_\pm$ is strictly positive. Indeed, since $0 \notin \mathcal{M}_\pm$, and $u_m \in \mathcal{M}_\pm$, by the Coulomb–Sobolev, the Sobolev and the Young inequalities, we have

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac{1}{4}L(u_m)&\le\frac{\mu}{q}\int_{\mathbb{R}^n}|u_m|^q\,dx+\frac{2^*-nb}{q-nb}\frac{1}{2^*}\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx\\
&\le C_q\mu\Big(\int_{\mathbb{R}^2}|\nabla u_m|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}[L(u_m)]^{\frac{2}{2+\alpha}}+C\Big(\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx\Big)^{\frac{2^*}{2}}\\
&\le C_1\mu\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+C_2\mu L(u_m)+C\Big(\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx\Big)^{\frac{2^*}{2}},
\end{aligned}$$

where C_1, C_2, and C are positive constants. Then

$$\Big(\frac{1}{2}-C_1\mu\Big)\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\Big(\frac{1}{4}-C_2\mu\Big)L(u_m)\le C\Big(\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx\Big)^{\frac{2^*}{2}}.$$

Since μ is small enough and $u_m \not\equiv 0$, from the above inequality we deduce

$$\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx\ge c>0. \tag{3.6}$$

The above inequalities also imply that

$$\begin{aligned}
0<c\Big(\frac{1}{2}-C_1\mu\Big)&<\Big(\frac{1}{2}-C_1\mu\Big)\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx\\
&\le\Big(\frac{1}{2}-C_1\mu\Big)\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\Big(\frac{1}{4}-C_2\mu\Big)L(u_m)\\
&\le\frac{2^*-nb}{q-nb}\frac{1}{2^*}\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx\\
&\to\frac{n}{2}\inf_{\mathcal{M}_\pm}J.
\end{aligned}$$

Therefore, $\inf_{\mathcal{M}_\pm} J > 0$.

In the following we prove $\lambda_m \neq 1$. It follows from (3.5) that, when $m\to\infty$,

$$\begin{aligned}
&(1-\lambda_m)\|\nabla u_m\|_2^2+(1-\lambda_m)L(u_m)-\mu(1-\lambda_m)\|u_m\|_q^q\\
&\qquad-\Big(1-\frac{2^*-nb}{q-nb}\lambda_m\Big)\|u_m\|_{2^*}^{2^*}=o(1).
\end{aligned}\tag{3.7}$$

If $\lambda_m = 1$, then

$$0<\frac{(2^*-q)n}{q-nb}\inf_{\mathcal{M}_\pm}J\to\frac{2^*-q}{q-nb}\|u_m\|_{2^*}^{2^*}=o(1).$$

This is a contradiction. Now, (3.7) can be written as

$$\|\nabla u_m\|_2^2+L(u_m)-\mu\|u_m\|_q^q-\frac{1}{1-\lambda_m}\Big(1-\frac{2^*-nb}{q-nb}\lambda_m\Big)\|u_m\|_{2^*}^{2^*}\to 0. \tag{3.8}$$

It follows from (3.4) and (3.8) that

$$\theta_m:=\frac{1}{1-\lambda_m}\Big(1-\frac{2^*-nb}{q-nb}\lambda_m\Big)\Big(=\frac{1-n\lambda_m/(n-2)}{1-\lambda_m}\Big)$$

is bounded and positive because μ is small. Thus, after passing to a subsequence of m, we have that $\theta_*=\lim_{m\to\infty}\theta_m$ is nonnegative. □

Remark 3.3. By (3.5), it is natural to verify the following Pohozaev-type result:

$$\begin{aligned}\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac{n+\alpha}{4}L(u_m)-\mu\frac{n}{q}\int_{\mathbb{R}^n}|u_m|^q\,dx\\ -\frac{n-2}{2}\theta_*\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx=o(1).\end{aligned}\tag{3.9}$$

Since the value of the Lagrange multiplier is unknown, (3.9) seems difficult to be proved when $3\le n<4+\alpha$. In order to overcome these difficulties, we consider two cases: $\theta_*\ge 1$ and $0\le\theta_*<1$.

3.1. Case $\boldsymbol{\theta_*\ge 1}$.

Lemma 3.4. *Let $3\le n<4+\alpha$ and $\alpha\in(0,n)$. If $\theta_*\ge 1$ and $\inf_{\mathcal{M}_+}J<\frac{1}{n}S^{\frac{n}{2}}$, then the sequence $\{u_m\}$ satisfies* (3.9)*. Here S is the best constant in the Sobolev inequality.*

Proof: Since $\{u_m\}$ is bounded in $X^{1,\alpha}$, by (1.2) we know that $\{u_m\}$ is also bounded in $L^{2^*}(\mathbb{R}^n)$ and in $L^q(\mathbb{R}^n)$, which implies that $\{|u_m|^{2^*-1}\}$ and $\{|u_m|^{q-1}\}$ are bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n)$ and $L^{\frac{q}{q-1}}(\mathbb{R}^n)$ respectively. Therefore, we can find a subsequence of u_m denoted by itself such that

$$\begin{aligned}u_m&\rightharpoonup v^0 &&\text{weakly in } X^{1,\alpha},\\ |u_m|^{2^*-2}u_m&\rightharpoonup |v^0|^{2^*-2}v^0 &&\text{weakly in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n),\\ |u_m|^{q-2}u_m&\rightharpoonup |v^0|^{q-2}v^0 &&\text{weakly in } L^{\frac{q}{q-1}}(\mathbb{R}^n),\end{aligned}$$

when $m\to\infty$. It follows from (3.5) that

$$\begin{aligned}\int_{\mathbb{R}^n}\nabla v^0\nabla\varphi\,dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[v^0(x)]^2v^0(y)\varphi(y)}{|x-y|^{n-\alpha}}\,dx\,dy\\ -\mu\int_{\mathbb{R}^n}|v^0|^{q-2}v^0\varphi\,dx-\theta_*\int_{\mathbb{R}^n}|v^0|^{2^*-2}v^0\varphi\,dx=0\end{aligned}$$

for $\varphi\in C_0^\infty(\mathbb{R}^n)$. Then v^0 is a critical point for Φ_* ($\Phi_*(u)=J(u)-\theta_*I(u)$). That is, v^0 is a solution of the equation

$$-\Delta u+(|x|^{\alpha-n}*u^2)u=\mu|u|^{q-2}u+\theta_*|u|^{2^*-2}u,\tag{3.10}$$

and

$$\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla v^0|^2\,dx+\frac{n+\alpha}{4}L(v^0)-\mu\frac{n}{q}\int_{\mathbb{R}^n}|v^0|^q\,dx-\frac{n-2}{2}\theta_*\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx=0.\tag{3.11}$$

Step 1. Write

$$u_m^1:=u_m-v^0.$$

By Lemma 3.1, $\{u_m\}$ is bounded in $X^{1,\alpha}$. Therefore, $\{u_m^1\}$ is also bounded in $X^{1,\alpha}$. By (1.2), $\{u_m^1\}$ is also bounded in $L^{2^*}(\mathbb{R}^n)$ and in $L^q(\mathbb{R}^n)$. This implies that $\{|u_m^1|^{2^*-1}\}$ and $\{|u_m^1|^{q-1}\}$ are bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n)$ and $L^{\frac{q}{q-1}}(\mathbb{R}^n)$ respectively. Therefore, we can find a subsequence of u_m^1 denoted by itself such that as $m\to\infty$,

$$\begin{cases}u_m^1\rightharpoonup 0 & \text{weakly in } X^{1,\alpha},\\ |u_m^1|^{2^*-2}u_m^1\rightharpoonup 0 & \text{weakly in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n),\\ |u_m^1|^{q-2}u_m^1\rightharpoonup 0 & \text{weakly in } L^{\frac{q}{q-1}}(\mathbb{R}^n).\end{cases}$$

Therefore, using (3.5), when $m\to\infty$, we have

$$\int_{\mathbb{R}^n}\nabla u_m^1\nabla\varphi\,dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u_m^1(x)]^2u_m^1(y)\varphi(y)}{|x-y|^{n-\alpha}}\,dx\,dy \\ -\mu\int_{\mathbb{R}^n}|u_m^1|^{q-2}u_m^1\varphi\,dx-\theta_m\int_{\mathbb{R}^n}|u_m^1|^{2^*-2}u_m^1\varphi\,dx=o(1)$$

for $\varphi\in C_0^\infty(\mathbb{R}^n)$. Therefore,

$$\Phi_m'(u_m^1)\to 0\quad(m\to\infty). \tag{3.12}$$

According to the Brézis–Lieb lemma (cf. [**6**]), when $m\to\infty$ we have

$$\int_{\mathbb{R}^n}|u_m^1|^q\,dx=\int_{\mathbb{R}^n}|u_m|^q\,dx-\int_{\mathbb{R}^n}|v^0|^q\,dx+o(1), \tag{3.13}$$

$$\int_{\mathbb{R}^n}|\nabla u_m^1|^2\,dx=\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx-\int_{\mathbb{R}^n}|\nabla v^0|^2\,dx+o(1), \tag{3.14}$$

$$\int_{\mathbb{R}^n}|u_m^1|^{2^*}\,dx=\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx-\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx+o(1). \tag{3.15}$$

In addition, by the nonlocal Brézis–Lieb lemma (cf. Lemma 2.2 in [**2**]), we get

$$L(u_m^1)=L(u_m)-L(v^0)+o(1) \tag{3.16}$$

when $m\to\infty$. Hence, from the above information, we obtain

$$\Phi_m(u_m^1)=\Phi_m(u_m)-\Phi_*(v^0)+o(1)\to\inf_{\mathcal{M}_+}J-\Phi_*(v^0)\quad(m\to\infty). \tag{3.17}$$

Step 2. If $u_m^1\to 0$ $(m\to\infty)$ in $X^{1,\alpha}$, we are done.

In order to illustrate the conclusion, when $u_m^1\to 0$ $(m\to\infty)$ in $X^{1,\alpha}$, by (1.2),

$$\int_{\mathbb{R}^n}|u_m^1|^q\,dx\to 0,\quad\int_{\mathbb{R}^n}|u_m^1|^{2^*}\,dx\to 0,\quad\text{as } m\to\infty.$$

By (3.11), we have that when $m\to\infty$,

$$\left[\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac{n+\alpha}{4}L(u_m)-\frac{n}{q}\mu\int_{\mathbb{R}^n}|u_m|^q\,dx-\frac{n-2}{2}\theta_m\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx\right] \\ \to\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla v^0|^2\,dx+\frac{n+\alpha}{4}L(v^0)-\frac{n}{q}\mu\int_{\mathbb{R}^n}|v^0|^q\,dx \\ -\frac{n-2}{2}\theta_*\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx=0.$$

Step 3. If $u_m^1 \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, the argument is divided into two cases:

Case 1. $\displaystyle\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^q \, dx = \int_{\mathbb{R}^n} |v^0|^q \, dx;$

Case 2. $\displaystyle\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^q \, dx \neq \int_{\mathbb{R}^n} |v^0|^q \, dx.$

In Case 1, by the condition $u_m \in \mathcal{M}_+$, we get

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2 \, dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m|^q \, dx - \frac{1}{2}\int_{\mathbb{R}^n} |u_m|^{2^*} \, dx = 0.$$

When $v^0 = 0$, according to (3.13)–(3.16), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 \, dx + \frac{1}{4}L(u_m^1) - \frac{1}{2}\int_{\mathbb{R}^n} |u_m^1|^{2^*} \, dx = o(1).$$

Consequently,

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 \, dx \geq S^{\frac{n}{2}} \quad \text{or} \quad \int_{\mathbb{R}^n} |\nabla u_m^1|^2 \, dx = o(1).$$

If the former holds, by (3.13)–(3.16) again, we obtain that when $m \to \infty$,

$$\begin{aligned}
\frac{1}{n}S^{\frac{n}{2}} &> \inf_{\mathcal{M}_+} J = J(u_m) + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2 \, dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^0|^q \, dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m|^{2^*} \, dx + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 \, dx + \frac{1}{4}L(u_m^1) - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*} \, dx + J(v^0) + o(1) \\
&= \frac{1}{n}\int_{\mathbb{R}^n} |\nabla u_m^1|^2 \, dx + \frac{1}{2n}L(u_m^1) + o(1) \\
&\geq \frac{1}{n}S^{\frac{n}{2}}.
\end{aligned}$$

This contradiction implies that

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2 \, dx \to 0, \quad \int_{\mathbb{R}^n} |u_m^1|^{2^*} \, dx \to 0, \quad L(u_m^1) \to 0 \quad (m \to \infty).$$

Therefore, (3.9) holds true from (3.11).

When $v^0 \neq 0$, noting that v^0 is a solution of (3.10), we know

$$\begin{cases}
\dfrac{n-2}{2}\displaystyle\int_{\mathbb{R}^n} |\nabla v^0|^2 \, dx + \dfrac{n+\alpha}{4}L(v^0) - \mu\dfrac{n}{q}\int_{\mathbb{R}^n} |v^0|^q \, dx - \dfrac{n-2}{2}\theta_*\int_{\mathbb{R}^n} |v^0|^{2^*} \, dx = 0, \\
\displaystyle\int_{\mathbb{R}^n} |\nabla v^0|^2 \, dx + L(v^0) - \mu\int_{\mathbb{R}^n} |v^0|^q \, dx - \theta_*\int_{\mathbb{R}^n} |v^0|^{2^*} \, dx = 0,
\end{cases}$$

and hence

$$\begin{cases}
\dfrac{4+\alpha-n}{4}L(v^0) = \mu\dfrac{2n-q(n-2)}{2q}\displaystyle\int_{\mathbb{R}^n} |v^0|^q \, dx, \\
\dfrac{n-4-\alpha}{4}\displaystyle\int_{\mathbb{R}^n} |\nabla v^0|^2 \, dx = \mu\dfrac{4n-q(n+\alpha)}{4q}\int_{\mathbb{R}^n} |v^0|^q \, dx + \dfrac{n-4-\alpha}{4}\theta_*\int_{\mathbb{R}^n} |v^0|^{2^*} \, dx.
\end{cases}$$

Therefore,

$$\begin{aligned}
J(v^0) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^0|^2\,dx + \frac{1}{4}L(v^0) - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^0|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx \\
&= \frac{1}{2}\mu\frac{q(n+\alpha)-4n}{q(4+\alpha-n)}\int_{\mathbb{R}^n} |v^0|^q\,dx + \mu\frac{2n-q(n-2)-4n}{2q(4+\alpha-n)}\int_{\mathbb{R}^n} |v^0|^q\,dx \\
&\quad - \mu\frac{1}{q}\int_{\mathbb{R}^n} |v^0|^q\,dx + \frac{1}{2}\theta_*\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx \\
&= \frac{1}{2}\theta_*\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx \quad (\text{by } \theta_* \geq 1) \\
&> 0.
\end{aligned}$$

In addition, note that $u_m \in \mathcal{M}_+$ and by (3.13)–(3.16), we get

$$\begin{aligned}
0 &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m|^q\,dx - \frac{1}{2}\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{4}L(u_m^1) - \frac{1}{2}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^0|^2\,dx + \frac{1}{4}L(v^0) - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^0|^q\,dx - \frac{1}{2}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx + o(1),
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx &= \int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{2}L(u_m^1) + 2J(v^0) - \frac{2}{n}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx \\
&= \int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{2}L(u_m^1) + 2\left(\frac{1}{2}\theta_* - \frac{1}{2^*}\right)\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx - \frac{2}{n}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx \\
&= \int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{2}L(u_m^1) + 2(\theta_* - 1)\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx.
\end{aligned}$$

Since $\theta_* \geq 1$, we get

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx \geq S^{\frac{n}{2}} \quad \text{or} \quad \int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx = o(1).$$

If the former holds, applying the above information, we obtain

$$\begin{aligned}\inf_{\mathcal{M}_+} J &= J(u_m) + o(1)\\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^0|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx + o(1)\\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{4}L(u_m^1) - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx + J(v^0) + o(1)\\ &= \frac{1}{n}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{2n}L(u_m^1) + \frac{2}{n}J(v^0) + \frac{2}{n2^*}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx + o(1)\\ &\geq \frac{1}{n}S^{\frac{n}{2}},\end{aligned}$$

which implies that

$$u_m^1 \to 0 \quad \text{in } X^{1,\alpha} \quad (m \to \infty).$$

Therefore, (3.9) holds true.

Step 4. In Case 2, since $X^{1,\alpha} \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^n)$ (now $q < 2^*$ for $n < 4+\alpha$) is compact, there exist $\delta_1 > 0$, $\{\xi_m^1\} \subset \mathbb{R}^n$, such that

$$\int_{B_1} |u_m^1(x+\xi_m^1)|^q\,dx \geq \delta_1 > 0. \tag{3.18}$$

According to (3.18), we have $|\xi_m^1| \to +\infty$ $(m \to \infty)$.

Write $v_m^1 := u_m^1(\cdot + \xi_m^1)$. Obviously, (3.12) and (3.17) show that $\{v_m^1\}$ is a bounded (PS)-sequence at level $\inf_{\mathcal{M}_+} J - \Phi_*(v^0)$. Up to a subsequence, we may assume that $v_m^1 \rightharpoonup v^1$ $(m \to \infty)$ in $X^{1,\alpha}$. Similar as in Step 1, we also see that v^1 is a solution of (3.10), and hence

$$\Phi_*'(v^1) = 0.$$

By (3.18) we have that

$$v^1 \neq 0,$$

and the Pohozaev identity of (3.10) implies

$$\frac{n-2}{2}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |\nabla v^i|^2\,dx + \frac{n+\alpha}{4}\sum_{i=0}^{1} L(v^i) - \mu\frac{n}{q}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^q\,dx - \frac{n-2}{2}\theta_*\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^{2^*}\,dx = 0.$$

Step 5. Define

$$u_m^2 := u_m^1 - v^1(\cdot - \xi_m^1).$$

Then $u_m^2 \rightharpoonup 0$ $(m \to \infty)$ in $X^{1,\alpha}$. Arguing as in Step 1, we obtain that when $m \to \infty$,

$$\begin{cases} \mathcal{F}[u_m^2] = \mathcal{F}[u_m^1] - \mathcal{F}[v^1] + o(1) = \mathcal{F}[u_m] - \mathcal{F}[v^0] - \mathcal{F}[v^1] + o(1),\\ \Phi_n(u_m^2) = \Phi_n(u_m^1) - \Phi_n(v^1) = \Phi_n(u_m) - \Phi_*(v^0) - \Phi_*(v^1) + o(1),\\ \Phi_n'(u_m^2) \to 0. \end{cases}$$

Here $\mathcal{F}(u) = \|\nabla u\|_2^2 + L(u)$.

When $u_m^2 \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, we are done. When $u_m^2 \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, as in the argument of Step 3, if $u_m^2 \to 0$ $(m \to \infty)$ in $L^q(\mathbb{R}^n)$, as $v^1 \neq 0$, we still have for $i = 0, 1$

$$
\begin{aligned}
J(v^i) &= \frac{1}{2}\theta_* \int_{\mathbb{R}^n} |v^i|^{2^*}\, dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |v^i|^{2^*}\, dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^n} |v^i|^{2^*}\, dx \quad (\text{by } \theta_* \geq 1) \\
&> 0,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\, dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m|^q\, dx - \frac{1}{2}\int_{\mathbb{R}^n} |u_m|^{2^*}\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\, dx + \frac{1}{4}L(u_m^1) - \frac{1}{2}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\, dx - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m^1|^q\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^0|^2\, dx + \frac{1}{4}L(v^0) - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^0|^q\, dx - \frac{1}{2}\int_{\mathbb{R}^n} |v^0|^{2^*}\, dx + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^2|^2\, dx + \frac{1}{4}L(u_m^2) - \frac{1}{2}\int_{\mathbb{R}^n} |u_m^2|^{2^*}\, dx + \frac{1}{2}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |\nabla v^i|^2\, dx \\
&\quad + \frac{1}{4}\sum_{i=0}^{1} L(v^i) - \frac{\mu}{q}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^q\, dx - \frac{1}{2}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^{2^*}\, dx + o(1),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^n} |u_m^2|^{2^*}\, dx &= \int_{\mathbb{R}^n} |\nabla u_m^2|^2\, dx + \frac{1}{2}L(u_m^2) + 2\sum_{i=0}^{1} J(v^i) - \frac{2}{n}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^{2^*}\, dx \\
&= \int_{\mathbb{R}^n} |\nabla u_m^2|^2\, dx + \frac{1}{2}L(u_m^2) \\
&\quad + 2\left(\frac{1}{2}\theta_* - \frac{1}{2^*}\right)\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^{2^*}\, dx - \frac{2}{n}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^{2^*}\, dx \\
&= \int_{\mathbb{R}^n} |\nabla u_m^2|^2\, dx + \frac{1}{2}L(u_m^2) + 2(\theta_* - 1)\int_{\mathbb{R}^n}\sum_{i=0}^{1} |v^i|^{2^*}\, dx.
\end{aligned}
$$

Since $\theta_* \geq 1$, we get

$$
\int_{\mathbb{R}^n} |\nabla u_m^2|^2\, dx \geq S^{\frac{n}{2}} \quad \text{or} \quad \int_{\mathbb{R}^n} |\nabla u_m^2|^2\, dx = o(1).
$$

If the former holds, applying the above information, we obtain

$$\begin{aligned}
\inf_{\mathcal{M}_+} J &= J(u_m) + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{4}L(u_m^1) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m^1|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx + J(v^0) + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^2|^2\,dx + \frac{1}{4}L(u_m^2) - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^1|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^2|^{2^*}\,dx + J(v^0) + o(1) \\
&= \frac{1}{n}\int_{\mathbb{R}^n} |\nabla u_m^2|^2\,dx + \frac{1}{2n}L(u_m^2) + \frac{2}{n}\sum_{i=0}^{1} J(v^i) + \frac{2}{n2^*}\sum_{i=0}^{1}\int_{\mathbb{R}^n} |v^i|^{2^*}\,dx + o(1) \\
&\geq \frac{1}{n}S^{\frac{n}{2}},
\end{aligned}$$

which implies that

$$u_m^2 \to 0 \quad \text{in } X^{1,\alpha} \quad (m \to \infty).$$

Therefore, (3.9) holds true.

If $u_m^2 \not\to 0$ $(m \to \infty)$ in $L^q(\mathbb{R}^n)$, we may assume the existence of $\{\xi_m^2\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u_m^2(x + \xi_m^2)|^q\,dx \geq \delta_2 \quad \text{for some } \delta_2 > 0.$$

Since $u_m^2 \rightharpoonup 0$ $(m \to \infty)$ and $u_m^2(\cdot + \xi_m^1) \rightharpoonup 0$ $(m \to \infty)$ in $X^{1,\alpha}$, we can deduce that

$$|\xi_m^2| \to +\infty, \quad |\xi_m^2 - \xi_m^1| \to +\infty \quad (m \to \infty).$$

Therefore, up to a subsequence, we may assume that $u_m^2(\cdot + \xi_m^2) \rightharpoonup v^2$ $(m \to \infty)$ in $X^{1,\alpha}$, and v^2 is a nontrivial solution of (3.10), which implies

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla v^2|^2\,dx + \frac{n+\alpha}{4}L(v^2) - \mu\frac{n}{q}\int_{\mathbb{R}^n} |v^2|^q\,dx - \frac{n-2}{2}\theta_*\int_{\mathbb{R}^n} |v^2|^{2^*}\,dx = 0.$$

We now define

$$u_m^3 := u_m^2 - v^2(\cdot - \xi_m^2).$$

Iterating by the procedure above we construct sequences $\{u_m^j\}_j$ and $\{\xi_m^j\}_j$ in the following way:

$$u_m^{j+1} := u_m^j - v^j(\cdot - \xi_m^j),$$

$$\mathcal{F}[u_m^j] = \mathcal{F}[u_m] - \sum_{i=0}^{j-1}\mathcal{F}[v^i] + o(1) \quad (m \to \infty), \tag{3.19}$$

$$\Phi_n(u_m^j) = J(u_m) - \sum_{i=0}^{j-1}\Phi_*(v^i) + o(1) \quad (m \to \infty),$$

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla v^i|^2\,dx + \frac{n+\alpha}{4}L(v^i) - \mu\frac{n}{q}\int_{\mathbb{R}^n} |v^i|^q\,dx - \frac{n-2}{2}\theta_*\int_{\mathbb{R}^n} |v^i|^{2^*}\,dx = 0. \tag{3.20}$$

Since $\{u_m\}$ is bounded in $X^{1,\alpha}$, $\mathcal{F}[u_m]$ is also bounded. And note that

$$\begin{cases} \frac{4+\alpha-n}{4}L(v^i)=\mu\frac{2n-q(n-2)}{2q}\int_{\mathbb{R}^n}|v^i|^q\,dx, \\ \frac{n-4-\alpha}{4}\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx=\mu\frac{4n-q(n+\alpha)}{4q}\int_{\mathbb{R}^n}v^i|^q\,dx+\frac{n-4-\alpha}{4}\theta_*\int_{\mathbb{R}^n}|v^i|^{2^*}\,dx. \end{cases} \tag{3.21}$$

Therefore, by (1.2), we have

$$\begin{aligned} \frac{q(4+\alpha-n)}{2[2n-q(n-2)]}L(v^i)&=\mu\int_{\mathbb{R}^n}|v^i|^q\,dx \\ &\le \mu C\Big(\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}[L(v^i)]^{\frac{2}{2+\alpha}}. \end{aligned}$$

When $i>0$, $v^i\neq 1$, then

$$L(v^i)\le C\mu^{\frac{2+\alpha}{2}}\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx.$$

Applying (3.21), one has

$$\begin{aligned} \int_{\mathbb{R}^n}|\nabla v^i|^2\,dx &= CL(v^i)+\theta_*\int_{\mathbb{R}^n}|v^i|^{2^*}\,dx \\ &\le C\mu^{\frac{2+\alpha}{2}}\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx+\int_{\mathbb{R}^n}|v^i|^{2^*}\,dx \\ &\le C\mu^{\frac{2+\alpha}{2}}\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx+C\Big(\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx\Big)^{\frac{2^*}{2}}. \end{aligned}$$

Since μ is small enough, we have

$$\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx\ge C>0,$$

where C is independent of i. Consequently, we have $\mathcal{F}[v^i]\ge C>0$. This implies that the iteration must stop at some k. Otherwise, it contradicts (3.19) and the boundedness of $\mathcal{F}[u_m]$. Specifically, for some k, $u_m^k\to 0$ $(m\to\infty)$ in $X^{1,\alpha}$. Consequently, it follows from (3.20) that

$$\begin{aligned} \lim_{m\to\infty}\Big[\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac{n+\alpha}{4}L(u_m)-\mu\frac{n}{q}\int_{\mathbb{R}^n}|u_m|^q\,dx-\frac{n-2}{2}\theta_*\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx\Big] \\ =\frac{n-2}{2}\sum_{i=0}^k\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx+\frac{n+\alpha}{4}\sum_{i=0}^k L(v^i)-\mu\frac{n}{q}\sum_{i=0}^k\int_{\mathbb{R}^n}|v^i|^q\,dx \\ -\frac{n-2}{2}\theta_*\sum_{i=0}^k\int_{\mathbb{R}^n}|v^i|^{2^*}\,dx=0. \end{aligned}$$

The proof is complete. □

3.2. Case $\theta_* \in [0,1)$. First, we prove the following result.

Theorem 3.5. *Assume that $\theta_* \in [0,1)$ and $\{u_m\}$ is the* $(\mathrm{PS})_c$*-sequence of Φ_m. If $n \geq 4$, $\alpha \in (0,n)$, and $\mu > 0$ is suitably small, we can find a subsequence of u_m denoted by itself such that*

$$\lim_{m\to\infty} u_m = v^0 \quad \text{in } L^{2^*}_{\mathrm{loc}}(\mathbb{R}^n).$$

This result is a corollary of Lemmas 3.6 and 3.10.

Lemma 3.6. *Let $n \geq 3$, $\alpha \in (0,n)$, and $\theta_* \in [0,1)$. Assume that $\{u_m\} \subset X^{1,\alpha}$ is a bounded* $(\mathrm{PS})_c$*-sequence of Φ_m. We can find a subsequence of u_m denoted by itself and $u_* \in X^{1,\alpha}$ such that*

$$\lim_{m\to\infty} u_m = u_* \quad \text{in } L^{2^*}_{\mathrm{loc}}(\mathbb{R}^n), \tag{3.22}$$

as long as either

(i) *$n > 4+\alpha$, or*

(ii) *$3 \leq n < 4+\alpha$, and $\{u_m\}$ is a* $(\mathrm{PS})_c$*-sequence of Φ_m with $c < \frac{1}{n}S^{\frac{n}{2}}$, where S is the best Sobolev constant.*

Proof: When $n > 4+\alpha$, take $\theta \in (0,1)$ such that $\frac{2^*-\theta}{1-\theta} \in (2^*, q)$. Since $\|u_m\|_{X^{1,\alpha}}$ is bounded, $\|u_m\|_{\frac{2^*-\theta}{1-\theta}}$ is also bounded. In addition, we can find a subsequence denoted by itself and $u_* \in X^{1,\alpha}$ such that u_m weakly converges to u_* in $X^{1,\alpha}$. Let Ω be an arbitrary compact subset of $\mathbb{R}^n$. Since the embedding operator from $X^{1,\alpha}$ to $L^1(\Omega)$ is compact (cf. Lemma 6.1 in [**3**]), there is a subsequence of u_m denoted by itself such that u_m converges to u_* in $L^1(\Omega)$. By the Hölder inequality,

$$\int_\Omega |u_m - u_*|^{2^*}\,dx \leq \left(\int_\Omega |u_m - u_*|\,dx\right)^{\theta}\left(\int_\Omega |u_m - u_*|^{\frac{2^*-\theta}{1-\theta}}\,dx\right)^{1-\theta} \to 0 \tag{3.23}$$

when $m \to \infty$. Specifically, (3.22) is true.

For $3 \leq n < 4+\alpha$, and since $\{u_m\} \in \mathcal{M}_\pm$ is the $(\mathrm{PS})_c$-sequence of Φ_m, when $m \to \infty$, there holds

$$\begin{aligned} c &= J(u_m) - I(u_m) + o(1) \\ &= \frac{1}{n}\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx + o(1). \end{aligned} \tag{3.24}$$

Since $\{u_m\} \subset X^{1,\alpha}$ is bounded, $\{u_m\}$ is also bounded in $L^{2^*}(\mathbb{R}^n)$. Therefore, we can find $u_* \in X^{1,\alpha}$ and a subsequence of u_m denoted by itself, such that as $m \to \infty$,

$$u_m \rightharpoonup u_* \quad \text{weakly in } L^{2^*}(\mathbb{R}^n), \tag{3.25}$$

and

$$|\nabla u_m|^2 \rightharpoonup |\nabla u_*|^2 + \tilde{\mu}, \quad |u_m|^{2^*} \rightharpoonup |u_*|^{2^*} + \nu \tag{3.26}$$

weakly in the Radon measure space. Using the concentration compactness principle due to Lions (see Lemma I.1 in [**15**]), we get the existence of a set, at most countable Λ, a sequence $\{x_i\} \subset \mathbb{R}^n$, and $\{\mu_i\}_{i\in\Lambda}, \{\nu_i\}_{i\in\Lambda} \subset [0,\infty)$ such that

$$\nu = \sum_{i\in\Lambda} \nu_i \delta_{x_i}, \quad \tilde{\mu} \geq \sum_{i\in\Lambda} \mu_i \delta_{x_i}, \quad \text{and} \quad \mu_i \geq S\nu_i^{\frac{2}{2^*}}, \quad \forall\, i \in \Lambda, \tag{3.27}$$

where δ_{x_i} is the Dirac mass centred at $x_i \in \mathbb{R}^n$.

Take $\zeta \in C_0^\infty(\mathbb{R}^n, [0,1])$ such that $\zeta = 1$ on $B_1(0)$, $\zeta = 0$ on $\mathbb{R}^n \backslash B_2(0)$, and $|\nabla \zeta|_\infty \leq 2$. Define

$$\zeta_\varepsilon(x) = \zeta\left(\frac{x - x_i}{\varepsilon}\right), \quad \varepsilon > 0.$$

We claim that for every $i \in \Lambda$, $\nu_i < S^{\frac{n}{2}}$. Indeed, if $\nu_{i_0} \geq S^{\frac{n}{2}}$ for some $i_0 \in \Lambda$, by (3.24), one has

$$c \geq \frac{1}{n} \int_{\mathbb{R}^n} |u_m|^{2^*} \zeta_\varepsilon \, dx + o(1).$$

Then, passing to the limit $m \to \infty$ and using (3.26) and (3.27), we deduce that

$$c \geq \frac{1}{n}\left(\int_{\mathbb{R}^n} |u_*|^{2^*} \zeta_\varepsilon \, dx + \int_{\mathbb{R}^n} \sum_{i \in \Lambda} \nu_i \zeta_\varepsilon \, d\delta_{x_i}\right) \geq \frac{1}{n}\nu_{i_0} \geq \frac{1}{n} S^{\frac{n}{2}},$$

which contradicts $c < \frac{1}{n} S^{n/2}$. Thus,

$$\nu_i < S^{\frac{n}{2}}, \quad \forall\, i. \tag{3.28}$$

On the other hand, (3.5) implies $\langle \Phi_n'(u_m), \zeta_\varepsilon u_m \rangle \to 0$ $(m \to \infty)$. Specifically, when $m \to \infty$,

$$\begin{aligned} o(1) &= \langle \Phi_n'(u_m), \zeta_\varepsilon u_m \rangle \\ &= \int_{\mathbb{R}^n} u_m \nabla u_m \nabla \zeta_\varepsilon \, dx + \int_{\mathbb{R}^n} |\nabla u_m|^2 \zeta_\varepsilon \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u_m^2(x) u_m^2(y) \zeta_\varepsilon(y)}{|x-y|^{n-\alpha}} \, dx \, dy \\ &\quad - \mu \int_{\mathbb{R}^n} |u_m|^q \zeta_\varepsilon \, dx - \theta_* \int_{\mathbb{R}^n} |u_m|^{2^*} \zeta_\varepsilon \, dx \\ &\geq \int_{\mathbb{R}^n} u_m \nabla u_m \nabla \zeta_\varepsilon \, dx + \int_{\mathbb{R}^n} |\nabla u_m|^2 \zeta_\varepsilon \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u_m^2(x) u_m^2(y) \zeta_\varepsilon(y)}{|x-y|^{n-\alpha}} \, dx \, dy \\ &\quad - \mu \int_{\mathbb{R}^n} |u_m|^q \zeta_\varepsilon \, dx - \int_{\mathbb{R}^n} |u_m|^{2^*} \zeta_\varepsilon \, dx. \end{aligned} \tag{3.29}$$

By the same derivation as in (3.23), we can find a subsequence of u_m denoted by itself such that u_m converges to u_* in $L^2_{\text{loc}}(\mathbb{R}^n)$ and in $L^q_{\text{loc}}(\mathbb{R}^n)$. Therefore, there holds

$$\begin{aligned} \left|\int_{\mathbb{R}^n} u_m \nabla u_m \nabla \zeta_\varepsilon \, dx\right| &\leq \frac{C}{\varepsilon}\left(\int_{B_{2\varepsilon}(x_i)} |\nabla u_m|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{B_{2\varepsilon}(x_i)} |u_m|^2 \, dx\right)^{\frac{1}{2}} \\ &\leq \frac{C}{\varepsilon}\left(\int_{B_{2\varepsilon}(x_i)} |u_m|^2 \, dx\right)^{\frac{1}{2}} \\ &\leq \frac{C}{\varepsilon}\left(\int_{B_{2\varepsilon}(x_i)} |u_*|^2 \, dx\right)^{\frac{1}{2}} \quad (m \to \infty) \\ &\leq C\left(\int_{B_{2\varepsilon}(x_i)} |u_*|^{2^*} \, dx\right)^{\frac{1}{2^*}} \to 0 \quad (\varepsilon \to 0). \end{aligned}$$

In addition,

$$\lim_{\varepsilon \to 0}\left(\lim_{m \to \infty} \int_{\mathbb{R}^n} |u_m|^q \zeta_\varepsilon \, dx\right) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |u_*|^q \zeta_\varepsilon \, dx = 0.$$

Moreover, by (3.26) we have

$$\lim_{\varepsilon\to0}\lim_{m\to\infty}\int_{\mathbb{R}^n}|\nabla u_m|^2\zeta_\varepsilon\,dx\ge\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\zeta_\varepsilon\,d\tilde{\mu},$$

$$\lim_{\varepsilon\to0}\lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m|^{2^*}\zeta_\varepsilon\,dx=\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\zeta_\varepsilon\,d\nu.$$

Inserting these results into (3.29), we deduce

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\zeta_\varepsilon\,d\nu\ge\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\zeta_\varepsilon\,d\tilde{\mu}.$$

Specifically, $\nu_i\ge\mu_i$. And by (3.27), we infer that

$$\nu_i\ge S^{\frac{n}{2}}.$$

This of course contradicts (3.28) and hence $\Lambda=\emptyset$. Consequently, (3.26) implies that $\|u_m\|^{2^*}_{L^{2^*}(\Omega)}$ converges to $\|u_*\|^{2^*}_{L^{2^*}(\Omega)}$, where Ω is an arbitrary compact subset of $\mathbb{R}^n$. Combining with (3.25), we obtain (3.22). □

Lemma 3.7. *Let $n\ge3$ and $\alpha\in(0,n)$. Assume either $n>4+\alpha$, or else $\theta_*\in[0,1)$, and $\inf_{\mathcal{M}_+}J<\frac{1}{n}S^{\frac{n}{2}}$. Then (3.9) holds for $u_m\in\mathcal{M}_\pm$.*

Proof: Step 1. It is similar to the proofs of Steps 1 and 2 in Lemma 3.4 that (3.9) holds. Now, when $u_m^1\not\to0$ $(m\to\infty)$ in $X^{1,\alpha}$, the argument is divided into two cases:

Case 1. $\displaystyle\lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx=\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx;$

Case 2. $\displaystyle\lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx\ne\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx.$

In Case 1, by the condition $u_m\in\mathcal{M}_\pm$, there holds

$$\frac12\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac14L(u_m)-\frac{\mu}{q}\int_{\mathbb{R}^n}|u_m|^q\,dx-\frac12\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx=0.$$

When $v^0=0$, we have

$$\frac12\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac14L(u_m)-\frac{\mu}{q}\int_{\mathbb{R}^n}|u_m|^q\,dx=0.$$

By the Coulomb–Sobolev and the Young inequalities, we obtain

$$\left(\frac12-C\mu\right)\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\left(\frac14-C\mu\right)L(u_m)\le0.$$

Since μ is suitably small, we have $u_m\to v^0$ in $X^{1,\alpha}$. Then we are done.

When $v^0\ne0$, we know

$$\begin{cases}\displaystyle\frac12\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\frac14L(u_m)-\mu\frac1q\int_{\mathbb{R}^n}|u_m|^q\,dx-\frac12\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx=o(1),\\[2ex] \displaystyle\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+L(u_m)-\mu\int_{\mathbb{R}^n}|u_m|^q\,dx-\theta_*\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx=o(1),\end{cases}$$

and hence

$$(1-\theta_*)\int_{\mathbb{R}^n}|\nabla u_m|^2\,dx+\left(1-\frac12\theta_*\right)L(u_m)=\mu\frac1q(1-2\theta_*)\int_{\mathbb{R}^n}|u_m|^q\,dx+o(1).$$

Since μ is small, when $0 \le \theta_* < 1$, we deduce $u_m \to v^0$ in $X^{1,\alpha}$. When $n > 4+\alpha$, by (3.13)–(3.16), we also have $u_m \to v^0$ in $X^{1,\alpha}$ as $m \to \infty$ provided μ is suitably small. Therefore, our conclusion is true.

Step 2. In Case 2, since $X^{1,\alpha} \hookrightarrow L^{2^*}_{\mathrm{loc}}(\mathbb{R}^n)$ (for $n < 4+\alpha$) is compact by Lemma 3.6, there exist $\delta_1 > 0$, $\{\xi_m^1\} \subset \mathbb{R}^n$, such that

$$\int_{B_1} |u_m^1(x+\xi_m^1)|^{2^*}\,dx \ge \delta_1 > 0.$$

This implies that $|\xi_m^1| \to +\infty$ $(m \to \infty)$.

Write $v_m^1 := u_m^1(\cdot + \xi_m^1)$. Since $\{v_m^1\}$ is a bounded (PS)-sequence at level $\inf_{\mathcal{M}_\pm} J - \Phi_*(v^0)$, we may assume, after passing to a subsequence, that $v_m^1 \rightharpoonup v^1 \ne 0$ $(m \to \infty)$ in $X^{1,\alpha}$. Note that v^1 is a solution of (3.10), and hence

$$\frac{n-2}{2}\sum_{i=0}^{1}\int_{\mathbb{R}^n}|\nabla v^i|^2\,dx + \frac{n+\alpha}{4}\sum_{i=0}^{1}L(v^i) - \mu\frac{n}{q}\sum_{i=0}^{1}\int_{\mathbb{R}^n}|v^i|^q\,dx - \frac{n-2}{2}\theta_*\sum_{i=0}^{1}\int_{\mathbb{R}^n}|v^i|^{2^*}\,dx = 0.$$

Step 3. Define

$$u_m^2 := u_m^1 - v^1(\cdot - \xi_m^1).$$

Then $u_m^2 \rightharpoonup 0$ $(m \to \infty)$ in $X^{1,\alpha}$. When $u_m^2 \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, we are done. When $u_m^2 \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, as in the argument of Step 1, if $u_m^2 \to 0$ $(m \to \infty)$ in $L^{2^*}(\mathbb{R}^n)$, as $v^1 \ne 0$, we have for $i = 0, 1$

$$\begin{aligned}
o(1) &= \int_{\mathbb{R}^n}|\nabla u_m|^2\,dx + L(u_m) - \mu\int_{\mathbb{R}^n}|u_m|^q\,dx - \theta_*\int_{\mathbb{R}^n}|u_m|^{2^*}\,dx \\
&= \int_{\mathbb{R}^n}|\nabla u_m^1|^2\,dx + L(u_m^1) - \mu\int_{\mathbb{R}^n}|u_m^1|^q\,dx - \theta_*\int_{\mathbb{R}^n}|u_m^1|^{2^*}\,dx \\
&\quad + \int_{\mathbb{R}^n}|\nabla v^0|^2\,dx + L(v^0) - \mu\int_{\mathbb{R}^n}|v^0|^q\,dx - \theta_*\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx + o(1) \\
&= \int_{\mathbb{R}^n}|\nabla u_m^1|^2\,dx + L(u_m^1) - \mu\int_{\mathbb{R}^n}|u_m^1|^q\,dx - \theta_*\int_{\mathbb{R}^n}|u_m^1|^{2^*}\,dx + o(1) \\
&\qquad (v^0 \text{ solves } (3.10)) \\
&= \int_{\mathbb{R}^n}|\nabla u_m^2|^2\,dx + L(u_m^2) - \mu\int_{\mathbb{R}^n}|u_m^2|^q\,dx - \theta_*\int_{\mathbb{R}^n}|v^1|^{2^*}\,dx \\
&\quad + \int_{\mathbb{R}^n}|\nabla v^1|^2\,dx + L(v^1) - \mu\int_{\mathbb{R}^n}|v^1|^q\,dx + o(1) \\
&= \int_{\mathbb{R}^n}|\nabla u_m^2|^2\,dx + L(u_m^2) - \mu\int_{\mathbb{R}^n}|u_m^2|^q\,dx + o(1) \quad (v^1 \text{ solves } (3.10)).
\end{aligned}$$

By the Coulomb–Sobolev and the Young inequalities again, when μ is suitably small, one has

$$u_m^2 \to 0 \quad \text{in } X^{1,\alpha} \quad (m \to \infty).$$

Therefore, (3.9) holds true.

If $u_m^2 \not\to 0$ $(m \to \infty)$ in $L^{2^*}(\mathbb{R}^n)$, we may assume the existence of $\{\xi_m^2\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u_m^2(x+\xi_m^2)|^{2^*}\,dx \geq \delta_2 \quad \text{for some } \delta_2 > 0.$$

Since $u_m^2 \rightharpoonup 0$ $(m \to \infty)$ and $u_m^2(\cdot + \xi_m^1) \rightharpoonup 0$ $(m \to \infty)$ in $X^{1,\alpha}$, and

$$|\xi_m^2| \to +\infty, \quad |\xi_m^2 - \xi_m^1| \to +\infty \quad (m \to \infty),$$

we may assume, after passing to a subsequence, that $u_m^2(\cdot + \xi_m^2) \rightharpoonup v^2$ $(m \to \infty)$ in $X^{1,\alpha}$, and v^2 is a nontrivial solution of (3.10), which implies

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla v^2|^2\,dx + \frac{n+\alpha}{4}L(v^2) - \mu\frac{n}{q}\int_{\mathbb{R}^n} |v^2|^q\,dx - \frac{n-2}{2}\theta_*\int_{\mathbb{R}^n} |v^2|^{2^*}\,dx = 0.$$

We now define

$$u_m^3 := u_m^2 - v^2(\cdot - \xi_m^2).$$

Iterating by the procedure above we construct sequences $\{u_m^j\}_j$ and $\{\xi_m^j\}_j$ in the following way:

$$u_m^{j+1} := u_m^j - v^j(\cdot - \xi_m^j).$$

Similarly as in Step 5 in Lemma 3.4, we also have

$$\begin{aligned}\lim_{m\to\infty}\Big[\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + \frac{n+\alpha}{4}L(u_m) - \mu\frac{n}{q}\int_{\mathbb{R}^n} |u_m|^q\,dx - \frac{n-2}{2}\theta_*\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx\Big] \\ = \frac{n-2}{2}\sum_{i=0}^{k}\int_{\mathbb{R}^n} |\nabla v^i|^2\,dx + \frac{n+\alpha}{4}\sum_{i=0}^{k}L(v^i) - \mu\frac{n}{q}\sum_{i=0}^{k}\int_{\mathbb{R}^n} |v^i|^q\,dx \\ - \frac{n-2}{2}\theta_*\sum_{i=0}^{k}\int_{\mathbb{R}^n} |v^i|^{2^*}\,dx = 0.\end{aligned}$$

The proof is complete. □

3.3. Solution of (1.1).

Remark 3.8. By Lemmas 3.4, 3.7, and 3.10, we see that $\{u_m\}$ satisfies (3.9). Similarly as in the proof of Step 3 in Lemma 2.3, from (3.8) we can deduce

$$\lim_{m\to\infty} \lambda_m = 0.$$

Thus,

$$J'(u_m) \to 0 \quad \text{as } m \to \infty. \tag{3.30}$$

Combining with (3.1), we see that $\{u_m\} \subset \mathcal{M}_\pm$ is a (PS)-sequence of J.

Noting $u_m \in \mathcal{M}_\pm$, by (3.30) we get

$$\mathcal{P}(u_m) \to 0 \quad \text{as } m \to \infty. \tag{3.31}$$

Lemma 3.9. *Let $n \geq 4$ and $\alpha \in (0,n)$. Assume that $\{u_m\}$ is the minimizing sequence of J in $\mathcal{M}_\pm$. Then for suitably small $\mu > 0$, we can find a subsequence of $\{u_m\}$ denoted by itself such that*

$$\lim_{m\to\infty} u_m = v^0 \quad \textit{weakly in } X^{1,\alpha}.$$

Here v^0 solves (1.1) *and satisfies $I_\pm(v^0) = 0$.*

Proof: Since $\{u_m\}$ is bounded in $X^{1,\alpha}$, by (1.2) we know that $\{u_m\}$ is also bounded in $L^{2^*}(\mathbb{R}^n)$ and in $L^q(\mathbb{R}^n)$, which implies that $\{|u_m|^{2^*-1}\}$ and $\{|u_m|^{q-1}\}$ are bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n)$ and $L^{\frac{q}{q-1}}(\mathbb{R}^n)$ respectively. Therefore, we can find a subsequence of u_m denoted by itself such that

$$\begin{aligned} u_m &\rightharpoonup v^0 && \text{weakly in } X^{1,\alpha}, \\ |u_m|^{2^*-2}u_m &\rightharpoonup |v^0|^{2^*-2}v^0 && \text{weakly in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n), \\ |u_m|^{q-2}u_m &\rightharpoonup |v^0|^{q-2}v^0 && \text{weakly in } L^{\frac{q}{q-1}}(\mathbb{R}^n), \end{aligned} \tag{3.32}$$

when $m \to \infty$. It follows from (3.30) that

$$\int_{\mathbb{R}^n} \nabla v^0 \nabla \varphi \, dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[v^0(x)]^2 v^0(y)\varphi(y)}{|x-y|^{n-\alpha}} \, dx\, dy - \mu \int_{\mathbb{R}^n} |v^0|^{q-2}v^0\varphi \, dx - \int_{\mathbb{R}^n} |v^0|^{2^*-2}v^0\varphi \, dx = 0$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then v^0 is a critical point for J. Therefore,

$$J'(v^0) = 0 \quad \text{and} \quad \mathcal{P}(v^0) = 0, \tag{3.33}$$

which, together with (2.1), implies $I_\pm(v^0) = 0$. □

Lemma 3.9 shows that v^0 solves (1.1). In Section 4 we prove that (1.1) has a ground state solution in $\mathcal{M}_\pm$. The concentration compactness principle comes into play, and we need the convergence of u_m in $L^{2^*}_{\text{loc}}$.

3.4. The threshold value. In this subsection, we mainly estimate the threshold value of J. Denote

$$\Psi(x) = \frac{[n(n-2)]^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}, \quad \Psi_\varepsilon(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{(\varepsilon^2+|x|^2)^{\frac{n-2}{2}}}, \quad x \in \mathbb{R}^n, \, \varepsilon > 0.$$

Ψ (and Ψ_ε) satisfies the limit equation

$$\Delta\Psi + \Psi^{2^*-1} = 0, \quad \Psi > 0 \text{ in } \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} |\nabla \Psi|^2 \, dx = \int_{\mathbb{R}^n} \Psi^{2^*} \, dx = S^{\frac{n}{2}}.$$

Choose $\eta \in C_0^\infty(\mathbb{R}^n, [0,1])$, satisfying $\eta(x) = 1$ for $x \in B_\delta(x_0)$ and $\eta(x) = 0$ for $x \notin B_{2\delta}(x_0)$. Denote $u_\varepsilon = \Psi_\varepsilon \eta$.

Lemma 3.10. *We have* $\sup_{t\geq 0} J(tu_\varepsilon(t^b x)) < \frac{1}{n}S^{\frac{n}{2}}$ *(*$b = \frac{2}{2+\alpha}$*) for suitably small* $\varepsilon > 0$*, if either*

(i) $3 < n < 4+\alpha$*, or*

(ii) $n = 3$ *with* $\mu = \mu(\varepsilon)$ *sufficiently large.*

Proof: From Lemma 1.1 in [**7**], we have

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\,dx = \int_{\mathbb{R}^n} |\nabla \Psi|^2\,dx + O(\varepsilon^{n-2}) = S^{\frac{n}{2}} + O(\varepsilon^{n-2}), \\ \displaystyle\int_{\mathbb{R}^n} u_\varepsilon^{2^*}\,dx = \int_{\mathbb{R}^n} \Psi^{2^*}\,dx + O(\varepsilon^n) = S^{\frac{n}{2}} + O(\varepsilon^n), \end{cases}$$

and

$$\int_{\mathbb{R}^n} u_\varepsilon^\sigma\,dx = \begin{cases} c\varepsilon^{\frac{(n-2)\sigma}{2}}, & 1 < \sigma < \dfrac{n}{n-2}, \\ c\varepsilon^{\frac{n}{2}}|\ln\varepsilon|, & \sigma = \dfrac{n}{n-2}, \\ c\varepsilon^{n-\frac{(n-2)\sigma}{2}}, & \dfrac{n}{n-2} < \sigma < 2^*. \end{cases} \tag{3.34}$$

Since $\lim_{t\to 0^+} J(tu_\varepsilon(t^b x)) = 0$ and $\lim_{t\to+\infty} J(tu_\varepsilon(t^b x)) \to -\infty$ as $t \to \infty$, there exists a $T_\varepsilon > 0$ such that $\sup_{t\geq 0} J(tu_\varepsilon(t^b x)) = J(T_\varepsilon u_\varepsilon(T_\varepsilon x))$. Moreover, we can obtain that there exist $t_1, t_2 > 0$ (independent of ε, μ), such that

$$t_1 \leq T_\varepsilon \leq t_2 < +\infty.$$

By the Hardy–Littlewood–Sobolev inequality,

$$L(u) \leq C\|u\|^4_{L^{\frac{4n}{n+\alpha}}(\mathbb{R}^n)}.$$

Consequently,

$$\begin{aligned} \sup_{t\geq 0} J(tu_\varepsilon(t^b x)) &\leq \sup_{t\geq 0}\left\{\frac{t^{q-nb}}{2}S^{\frac{n}{2}} - \frac{t^{2^*-nb}}{2^*}S^{\frac{n}{2}}\right\} \\ &\quad + \frac{t_2^{q-nb}}{4}L(u_\varepsilon) - \frac{\mu t_1^{q-nb}}{q}\int_{\mathbb{R}^n} u_\varepsilon^q\,dx + O(\varepsilon^{n-2}) \\ &\leq \frac{1}{n}S^{\frac{n}{2}} + O(\varepsilon^{n-2}) + C\left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{4n}{n+\alpha}}\,dx\right)^{\frac{n+\alpha}{n}} - C\mu\int_{\mathbb{R}^n} u_\varepsilon^q\,dx. \end{aligned} \tag{3.35}$$

(i) When $4 \leq n < 4+\alpha$, there holds $\frac{8+2\alpha}{2+\alpha} - \frac{n}{n-2} = \frac{6n+n\alpha-16-4\alpha}{(2+\alpha)(n-2)} > 0$. Then, it follows from (3.34) that

$$\int_{\mathbb{R}^n} u_\varepsilon^{\frac{8+2\alpha}{2+\alpha}}\,dx = C\varepsilon^{n-\frac{(n-2)}{2}\frac{8+2\alpha}{2+\alpha}} = C\varepsilon^{\frac{2(4+\alpha-n)}{2+\alpha}}.$$

On the other hand, since $n \geq 4$ and $\alpha < n$, we have $\frac{4n}{n+\alpha} - \frac{n}{n-2} = \frac{n(3n-8-\alpha)}{(n+\alpha)(n-2)} > 0$. Thus

$$\left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{4n}{n+\alpha}}\,dx\right)^{\frac{n+\alpha}{n}} = C(\varepsilon^{n-\frac{2n(n-2)}{n+\alpha}})^{\frac{n+\alpha}{n}} = C\varepsilon^{4+\alpha-n}.$$

Therefore, noting $4 \leq n < 4+\alpha$, which implies $\frac{2(4+\alpha-n)}{2+\alpha} < n-2$ and $\frac{2(4+\alpha-n)}{2+\alpha} < 4+\alpha-n$, we deduce that

$$\begin{aligned} \sup_{t\geq 0} J(tu_\varepsilon(t^b x)) &= \frac{1}{n}S^{\frac{n}{2}} + O(\varepsilon^{n-2}) + C\left(\int_{\mathbb{R}^n} u_\varepsilon^{\frac{4n}{n+\alpha}}\,dx\right)^{\frac{n+\alpha}{n}} - C\mu\int_{\mathbb{R}^n} u_\varepsilon^q\,dx \\ &\leq \frac{1}{n}S^{\frac{n}{2}} + O(\varepsilon^{n-2}) + C\varepsilon^{4+\alpha-n} - C\mu\varepsilon^{\frac{2(4+\alpha-n)}{2+\alpha}} < \frac{1}{n}S^{\frac{n}{2}} \end{aligned}$$

for ε suitably small.

(ii) When $n=3$. It follows from (3.34) that

$$\left(\int_{\mathbb{R}^3} u_\varepsilon^{\frac{12}{3+\alpha}}\,dx\right)^{\frac{3+\alpha}{3}} \le C\varepsilon^2 + C\varepsilon^{1+\alpha},$$

and

$$\int_{\mathbb{R}^3} u_\varepsilon^q\,dx = \int_{\mathbb{R}^3} u_\varepsilon^{\frac{8+2\alpha}{2+\alpha}}\,dx = \begin{cases} C\varepsilon^{\frac{q}{2}}, & 2<\alpha<3,\\ C\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, & \alpha=2,\\ C\varepsilon^{3-\frac{q}{2}}, & 0<\alpha<2. \end{cases} \tag{3.36}$$

It follows from (3.35) and (3.36) that

$$\begin{aligned}\sup_{t\ge 0} J(tu_\varepsilon(t^b x)) &= \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon) + C\left(\int_{\mathbb{R}^3} u_\varepsilon^{\frac{12}{3+\alpha}}\,dx\right)^{\frac{3+\alpha}{3}} - C\mu\int_{\mathbb{R}^3} u_\varepsilon^q\,dx\\ &\le \frac{1}{3}S^{\frac{3}{2}} + C(\varepsilon+\varepsilon^2+\varepsilon^{1+\alpha}) - C\mu(\varepsilon^{\frac{q}{2}} + \varepsilon^{\frac{3}{2}}|\ln\varepsilon| + \varepsilon^{3-\frac{q}{2}}).\end{aligned}$$

Since $2<q<4$, we have $\frac{q}{2}>1$ and $3-\frac{q}{2}>1$. Consequently, for any $\varepsilon\in(0,1)$, we can find large μ_* such that

$$\sup_{t\ge 0} J(tu_\varepsilon(t^b x)) < \frac{1}{3}S^{\frac{3}{2}}$$

provided $\mu\ge\mu_*$. The proof is complete. □

Remark 3.11. When $n=3$, Lemma 3.10 needs that $\mu>0$ be suitably large. Now, it seems difficult to obtain the boundedness of $\|u_m\|_{X^{1,\alpha}}$ (Lemma 3.1 requires that μ be suitably small). Thus, Lemma 3.6 cannot be applied to prove Theorem 3.5 when $n=3$.

4. Ground state solution

In this section, we look for the ground state solution on the Nehari–Pohozaev manifold $\mathcal{M}_\pm$.

Theorem 4.1. *Let $n\ge 4$, $\alpha\in(0,n)$, $2^*=\frac{2n}{n-2}$, $q:=\frac{8+2\alpha}{2+\alpha}$, and let $\mu>0$ be suitably small. Then*

(i) (1.1) *has a ground state solution in $\mathcal{M}_+$ when $4\le n<4+\alpha$,*
(ii) (1.1) *has a ground state solution in $\mathcal{M}_-$ when $n>4+\alpha$.*

Proof: Assume that $\{u_m\}$ is the minimizing sequence of J in $\mathcal{M}_\pm$, and v^0 is the weak limit of u_m in Lemma 3.9. Since $I_\pm(v^0)=0$, by the same calculations as in (2.2) and (2.3), we also get

$$J(v^0) = \frac{1}{2^*}\frac{2^*-q}{q-nb}\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx = \frac{1}{n}\int_{\mathbb{R}^n}|v^0|^{2^*}\,dx. \tag{4.1}$$

Step 1. Write

$$u_m^1 := u_m - v^0.$$

By Lemma 3.1, $\{u_m\}$ is bounded in $X^{1,\alpha}$. Therefore, $\{u_m^1\}$ is also bounded in $X^{1,\alpha}$. By (1.2), $\{u_m^1\}$ is also bounded in $L^{2^*}(\mathbb{R}^n)$ and in $L^q(\mathbb{R}^n)$. This implies that $\{|u_m^1|^{2^*-1}\}$

and $\{|u_m^1|^{q-1}\}$ are bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n)$ and $L^{\frac{q}{q-1}}(\mathbb{R}^n)$ respectively. Therefore, by (3.32), we can find a subsequence of u_m^1 denoted by itself such that as $m \to \infty$,

$$\begin{cases} u_m^1 \rightharpoonup 0 & \text{weakly in } X^{1,\alpha}, \\ |u_m^1|^{2^*-2} u_m^1 \rightharpoonup 0 & \text{weakly in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^n), \\ |u_m^1|^{q-2} u_m^1 \rightharpoonup 0 & \text{weakly in } L^{\frac{q}{q-1}}(\mathbb{R}^n). \end{cases}$$

Therefore, when $m \to \infty$,

$$\int_{\mathbb{R}^n} \nabla u_m^1 \nabla \varphi \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u_m^1(x)]^2 u_m^1(y) \varphi(y)}{|x-y|^{n-\alpha}} \, dx \, dy - \mu \int_{\mathbb{R}^n} |u_m^1|^{q-2} u_m^1 \varphi \, dx - \int_{\mathbb{R}^n} |u_m^1|^{2^*-2} u_m^1 \varphi \, dx = o(1)$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Therefore,

$$J'(u_m^1) \to 0 \quad (m \to \infty). \tag{4.2}$$

According to the Brézis–Lieb lemma again, when $m \to \infty$, we obtain

$$J(u_m^1) = J(u_m) - J(v^0) + o(1) \to \inf_{\mathcal{M}_\pm} J - J(v^0) \quad (m \to \infty). \tag{4.3}$$

In addition, from (3.13)–(3.16), we obtain $\mathcal{P}(u_m) = \mathcal{P}(u_m^1) + \mathcal{P}(v^0) + o(1)$ when $m \to \infty$. In view of (3.31) and (3.33), we have

$$\mathcal{P}(u_m^1) \to 0 \quad (m \to \infty). \tag{4.4}$$

Step 2. If $u_m^1 \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, we are done.

In fact, by (1.2),

$$\int_{\mathbb{R}^n} |u_m^1|^q \, dx \to 0, \quad \int_{\mathbb{R}^n} |u_m^1|^{2^*} \, dx \to 0, \quad \text{as } m \to \infty.$$

Specifically, $J(u_m^1) \to 0$ $(m \to \infty)$. By (4.3) and Lemma 3.2, we have

$$J(v^0) = \inf_{\mathcal{M}_\pm} J > 0.$$

This and (4.1) imply $v^0 \neq 0$. By Lemma 3.9 we see that $v^0 \in \mathcal{M}_\pm$, and hence v^0 is a ground state solution of (1.1).

Step 3. If $u_m^1 \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, the argument is divided into two cases:

Case 1. $\displaystyle\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^{2^*} \, dx = \int_{\mathbb{R}^n} |v^0|^{2^*} \, dx;$

Case 2. $\displaystyle\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^{2^*} \, dx \neq \int_{\mathbb{R}^n} |v^0|^{2^*} \, dx.$

In Case 1, $\inf_{\mathcal{M}_\pm} J > 0$. Indeed, if the infimum $\inf_{\mathcal{M}_\pm} J = 0$, then by Lemma 2.3, Step 1, one obtains

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m|^{2^*} \, dx = \int_{\mathbb{R}^n} |v^0|^{2^*} \, dx = 0.$$

By the conditions $u_m \in \mathcal{M}_\pm$ and u_m satisfying the Pohozaev identity, we get

$$\begin{cases} \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + \frac{1}{4}L(u_m) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m|^q\,dx = o(1), \\ \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + \frac{n+\alpha}{4}L(u_m) - \frac{\mu n}{q}\int_{\mathbb{R}^n} |u_m|^q\,dx = o(1). \end{cases} \tag{4.5}$$

Consequently,

$$\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx = \frac{\alpha}{2}L(u_m) + o(1).$$

This, together with (4.5) and the Coulomb–Sobolev inequality, yields

$$\begin{aligned} \frac{1+\alpha}{4}L(u_m) &= \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m|^q\,dx + o(1) \\ &\le C_q\mu\left(\int_{\mathbb{R}^2} |\nabla u_m|^2\,dx\right)^{\frac{\alpha}{2+\alpha}}[L(u_m)]^{\frac{2}{2+\alpha}} + o(1) \\ &= C_q\mu\left[\frac{\alpha}{2}L(u_m) + o(1)\right]^{\frac{\alpha}{2+\alpha}}[L(u_m)]^{\frac{2}{2+\alpha}} + o(1) \\ &= C_q'\mu L(u_m) + o(1). \end{aligned}$$

Since μ is small enough, the above inequality implies that

$$L(u_m) \to 0, \quad \int_{\mathbb{R}^n} |\nabla u_m|^2\,dx \to 0, \quad \text{as } m \to \infty.$$

This contradicts (3.6). Therefore, $\inf_{\mathcal{M}_\pm} J > 0$.

In the following we prove v^0 is a ground state solution of (1.1) in Case 1. When $m \to \infty$, it follows from (4.4) that

$$\frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{n+\alpha}{4}L(u_m^1) - \mu\frac{n}{q}\int_{\mathbb{R}^n} |u_m^1|^q\,dx = o(1). \tag{4.6}$$

In addition, from $\langle J'(u_m), u_m\rangle \to 0$ $(m \to \infty)$ (implied by (3.30)) it follows that

$$\int_{\mathbb{R}^n} |\nabla u_m|^2\,dx + L(u_m) - \mu\int_{\mathbb{R}^n} |u_m|^q\,dx - \int_{\mathbb{R}^n} |u_m|^{2^*}\,dx = o(1).$$

Thus, in view of (3.13)–(3.16), when $m \to \infty$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \int_{\mathbb{R}^n} |\nabla v^0|^2\,dx + L(u_m^1) + L(v^0) \\ &\qquad - \mu\int_{\mathbb{R}^n} |u_m^1|^q\,dx - \mu\int_{\mathbb{R}^n} |v^0|^q\,dx - \int_{\mathbb{R}^n} |v^0|^{2^*}\,dx = o(1). \end{aligned}$$

Combining this with $\langle J'(v^0), v^0\rangle = 0$ (implied by(3.33)), we conclude that

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + L(u_m^1) - \mu\int_{\mathbb{R}^n} |u_m^1|^q\,dx = o(1) \quad (m \to \infty). \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\left(\frac{n-2}{2} - \frac{n}{q}\right)\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \left(\frac{n+\alpha}{4} - \frac{n}{q}\right)L(u_m^1) = o(1) \quad (m \to \infty).$$

Specifically,

$$L(u_m^1) = \frac{2[(n-2)p - 2n]}{4n - (n+\alpha)q}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + o(1) \quad (m \to \infty). \tag{4.8}$$

By (3.13)–(3.16), we obtain that when $m \to \infty$,

$$\begin{aligned}\inf_{\mathcal{M}_\pm} J &= J(u_m) + o(1)\\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{4}L(u_m^1) - \frac{\mu}{q}\int_{\mathbb{R}^n} |u_m^1|^q\,dx + J(v^0) + o(1).\end{aligned}$$

Combining with (4.7) and (4.8) yields

$$\begin{aligned}0 < \inf_{\mathcal{M}_\pm} J &= \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \left(\frac{1}{4} - \frac{1}{q}\right)L(u_m^1) + J(v^0) + o(1)\\ &= \left(\frac{q-2}{2q} - \frac{4-q}{4q}\cdot\frac{2[(n-2)q-2n]}{4n-(n+\alpha)q}\right)\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + J(v^0) + o(1)\\ &= J(v^0) \quad (m \to \infty).\end{aligned} \tag{4.9}$$

This implies that $v^0 \neq 0$, and hence v^0 is a ground state solution of (1.1).

Step 4. In Case 2, there exist $\delta_1 > 0$, $\{\xi_m^1\} \subset \mathbb{R}^n$, such that

$$\int_{B_1} |u_m^1(x + \xi_m^1)|^{2^*}\,dx \geq \delta_1 > 0. \tag{4.10}$$

According to Theorem 3.5, we have $|\xi_m^1| \to +\infty$ $(m \to \infty)$.

Write $v_m^1 := u_m^1(\cdot + \xi_m^1)$. Obviously, (4.3) and (4.2) show that $\{v_m^1\}$ is a bounded (PS)-sequence at level $\inf_{\mathcal{M}_\pm} J - J(v^0)$. Up to a subsequence, we may assume that $v_m^1 \rightharpoonup v^1$ $(m \to \infty)$ in $X^{1,\alpha}$. Similarly as in the proof of Lemma 3.9, we also see that v^1 is a solution of (1.1), and hence

$$J'(v^1) = 0 \quad \text{and} \quad \mathcal{P}(v^1) = 0. \tag{4.11}$$

By (2.1), $I_\pm(v^1) = 0$. By (4.10) we have that

$$v^1 \neq 0, \tag{4.12}$$

which, together with $I_\pm(v^1) = 0$, implies

$$v^1 \in \mathcal{M}_\pm, \tag{4.13}$$

and hence $\inf_{\mathcal{M}_\pm} J > 0$.

Step 5. Define

$$u_m^2 := u_m^1 - v^1(\cdot - \xi_m^1).$$

Then $u_m^2 \rightharpoonup 0$ $(m \to \infty)$ in $X^{1,\alpha}$. Arguing as in Step 1, we obtain that when $m \to \infty$,

$$\begin{cases} \mathcal{F}[u_m^2] = \mathcal{F}[u_m^1] - \mathcal{F}[v^1] + o(1) = \mathcal{F}[u_m] - \mathcal{F}[v^0] - \mathcal{F}[v^1] + o(1),\\ J(u_m^2) = J(u_m^1) - J(v^1) = J(u_m) - J(v^0) - J(v^1) + o(1),\\ J'(u_m^2) \to 0,\\ \mathcal{P}(u_m^2) \to 0. \end{cases} \tag{4.14}$$

Here $\mathcal{F}(u) = \|\nabla u\|_2^2 + L(u)$.

Clearly, (4.2) and (4.3) show that u_m^1 is a $(\text{PS})_c$-sequence. Here $c = \inf_{\mathcal{M}_\pm} J - J(v^0) \leq \inf_{\mathcal{M}_\pm} J$ by virtue of $J(v^0) \geq 0$ (implied by (4.1)). According to Lemma 3.6,

$$u_m^2 \to 0 \quad \text{in } L^{2^*}_{\text{loc}}(\mathbb{R}^n). \tag{4.15}$$

When $u_m^2 \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, we are done. In fact, similarly as in the proof of Lemma 3.9, we can obtain that v^1 is a solution of (1.1). In view of (4.12), v^1 is also nontrivial, and hence $v^1 \in \mathcal{M}_\pm$. On the other hand, the second result of (4.14) and $J(u_m^2) \to 0$ $(m \to \infty)$ show that $J(v^0) + J(v^1) = \inf_{\mathcal{M}_\pm} J$. Noting $J(v^0) \geq 0$ (implied by (4.1)) and (4.13), we obtain $J(v^0) = 0$ and $J(v^1) = \inf_{\mathcal{M}_\pm} J$.

When $u_m^2 \not\to 0$ $(m \to \infty)$ in $X^{1,\alpha}$, as in the argument of Step 3, if $u_m^2 \to 0$ $(m \to \infty)$ in $L^{2^*}(\mathbb{R}^n)$, $J(v^0) + J(v^1) = \inf_{\mathcal{M}_\pm} J$ still holds by the same derivation as in (4.9). Therefore, we are done by an analogous argument above. If $u_m^2 \not\to 0$ $(m \to \infty)$ in $L^{2^*}(\mathbb{R}^n)$, we may assume the existence of $\{\xi_m^2\} \subset \mathbb{R}^n$ such that

$$\int_{B_1} |u_m^2(x + \xi_m^2)|^{2^*} \, dx \geq \delta_2 \quad \text{for some } \delta_2 > 0.$$

Since $u_m^2 \rightharpoonup 0$ $(m \to \infty)$ and $u_m^2(\cdot + \xi_m^1) \rightharpoonup 0$ $(m \to \infty)$ in $X^{1,\alpha}$, by (4.15) we deduce that

$$|\xi_m^2| \to +\infty, \quad |\xi_m^2 - \xi_m^1| \to +\infty \quad (m \to \infty).$$

Therefore, up to a subsequence, we may assume that $u_m^2(\cdot + \xi_m^2) \rightharpoonup v^2$ $(m \to \infty)$ in $X^{1,\alpha}$, and v^2 is a nontrivial solution of (1.1) (which implies $v^2 \in \mathcal{M}_\pm$). We now define

$$u_m^3 := u_m^2 - v^2(\cdot - \xi_m^2).$$

Iterating by the procedure above we construct sequences $\{u_m^j\}_j$ and $\{\xi_m^j\}_j$ in the following way:

$$u_m^{j+1} := u_m^j - v^j(\cdot - \xi_m^j),$$

$$\mathcal{F}[u_m^j] = \mathcal{F}[u_m] - \sum_{i=0}^{j-1} \mathcal{F}[v^i] + o(1) \quad (m \to \infty), \tag{4.16}$$

$$J(u_m^j) = J(u_m) - \sum_{i=0}^{j-1} J(v^i) + o(1) \quad (m \to \infty), \tag{4.17}$$

$$J'(v^i) = 0, \quad \text{for } i \geq 0. \tag{4.18}$$

Since $\{u_m\}$ is bounded in $X^{1,\alpha}$, $\mathcal{F}[u_m]$ is also bounded. And (4.18) implies $v^i \in \mathcal{M}_\pm$ for every $i \geq 1$. Therefore, when $3 \leq n < 4 + \alpha$, we have

$$\begin{aligned}\frac{1}{2}\mathcal{F}[v^i] &\geq \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^i|^2 \, dx + \frac{1}{4}L(v^i) = \frac{\mu}{q}\int_{\mathbb{R}^n} |v^i|^q \, dx + \frac{1}{2^*}\frac{2^* - nb}{q - nb}\int_{\mathbb{R}^n} |v^i|^{2^*} \, dx \\ &> \frac{1}{2^*}\frac{2^* - nb}{q - nb}\int_{\mathbb{R}^n} |v^i|^{2^*} \, dx = \frac{1}{2}\int_{\mathbb{R}^n} |v^i|^{2^*} \, dx \geq \frac{n}{2} \inf_{u \in \mathcal{M}_+} J.\end{aligned}$$

Similarly, when $n > 4+\alpha$, we also have $\mathcal{F}[v^i] > n \inf_{u \in \mathcal{M}_-} J$. Thus, $\mathcal{F}[v^i] > n \inf_{\mathcal{M}_\pm} J$ for $i = 1, 2, \ldots$ This implies that the iteration must stop at some k. Otherwise, it contradicts (4.16) and the boundedness of $\mathcal{F}[u_m]$. Specifically, for some k, $u_m^k \to 0$ $(m \to \infty)$ in $X^{1,\alpha}$. Consequently, $J(u_m^k) \to 0$ when $m \to \infty$. Letting $m \to \infty$ in (4.17) with $j = k$ yields

$$\sum_{i=0}^{k} J(v^i) = \inf_{u \in \mathcal{M}_\pm} J. \tag{4.19}$$

In view of $v^i \in \mathcal{M}_\pm$ for $i \geq 1$, we obtain

$$J(v^i) \geq \inf_{u\in\mathcal{M}_\pm} J, \quad i \geq 1.$$

Combining this result and (4.19) with $J(v^0) \geq 0$ (implied by (4.1)), we can see $v^0 \neq 0$ and $k = 0$, or $v^0 = 0$ and $k = 1$. In the first case, $u_m(\cdot + \xi_m^1) \to v^0(\cdot)$ $(m \to \infty)$ in $X^{1,\alpha}$ and v^0 is a solution of equation (1.1) with $J(v^0) = \inf_{u\in\mathcal{M}_\pm} J$, and so v^0 is a ground state solution of (1.1). In the latter, $u_m(\cdot + \xi_m^1) \to v^1(\cdot)$ in $X^{1,\alpha}$ as $m \to \infty$ and v^1 is a ground state solution of equation (1.1) with $J(v^1) = \inf_{u\in\mathcal{M}_\pm} J$. The proof is complete. □

Remark 4.2. The conclusions in Theorem 4.1 still hold if we replace $\mathcal{M}_\pm$ with $\widetilde{\mathcal{M}}_\pm$. In particular, we can find either v^0 or v^1 is the ground state solution of (1.1) in $\widetilde{\mathcal{M}}_\pm$.

5. Convergence relation

In this section, we investigate the convergence relation between the ground state solutions and the minimizing sequence under the assumptions in Theorems 1.1 and 1.2.

Theorem 5.1. *Let $n \geq 4$ and $\alpha \in (0, n)$. The ground state solution v^0 (or v^1) is the $L^{2^*}(\mathbb{R}^n)$-limit of some subsequence of the minimizing sequence u_m of J in $\mathcal{M}_\pm$.*

Proof: Since u_m is the minimizing sequence, similarly as in the derivation of (3.24), we get

$$\begin{aligned}\inf_{\mathcal{M}_\pm} J &= J(u_m) - \beta\langle J'(u_m), u_m\rangle - \theta\mathcal{P}(u_m) + o(1)\\ &= \frac{1}{n}\int_{\mathbb{R}^n} |u_m|^{2^*}\,dx + o(1)\end{aligned}$$

when $m \to \infty$. Applying (3.15) and (4.1), we obtain from the result above that

$$\begin{aligned}\inf_{\mathcal{M}_\pm} J &= \frac{1}{n}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx + \frac{1}{n}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx + o(1)\\ &= \frac{1}{n}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx + J(v^0) + o(1) \quad (m \to \infty).\end{aligned} \tag{5.1}$$

When $v^0 \in \mathcal{M}_\pm$ is the ground state solution, the result above shows

$$\lim_{m\to\infty} \frac{1}{n}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx + J(v^0) = \inf_{\mathcal{M}_\pm} J \leq J(v^0).$$

This implies that

$$\lim_{m\to\infty} \frac{1}{n}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx = 0.$$

Thus, u_m converges to v^0 in $L^{2^*}(\mathbb{R}^n)$.

When $v^1 \in \mathcal{M}_\pm$ is the ground state solution, we know that the Brézis–Lieb type results (3.13)–(3.16) still hold if we replace u_m^1, u_m, and v^0 with u_m^2, u_m^1, and v^1 respectively. By the same derivation as in (2.2) and (2.3), we have

$$J(v^1) = \frac{1}{n}\int_{\mathbb{R}^n} |v^1|^{2^*}\,dx.$$

Therefore, by the Brézis–Lieb type results, from (5.1) we have

$$\inf_{\mathcal{M}_\pm} J = \frac{1}{n}\int_{\mathbb{R}^n} |u_m^2|^{2^*}\,dx + J(v^1) + J(v^0) + o(1). \tag{5.2}$$

In view of $v^1 \in \mathcal{M}_\pm$ and $J(v^0) \geq 0$ (implied by (4.1)), the result above implies that

$$\frac{1}{n}\int_{\mathbb{R}^n} |u_m^2|^{2^*}\,dx \to 0 \quad (m \to \infty), \tag{5.3}$$

and $J(v^0) = 0$. In view of (4.1), we see that $v^0 = 0$ a.e. on $\mathbb{R}^n$. Thus, (5.3) implies that u_m converges to v^1 in $L^{2^*}(\mathbb{R}^n)$. Theorem 5.1 is proved. □

Remark 4.2 shows that (1.1) has ground state solution v^0 or v^1 in $\widetilde{\mathcal{M}}_\pm$.

Theorem 5.2. *Assume $n \geq 4$ and $\alpha \in (1, n)$. Then the ground state solution v^0 (or v^1) is the $X^{1,\alpha}_{\mathrm{rad}}$-limit of some subsequence of the minimizing sequence u_m of J in $\widetilde{\mathcal{M}}_\pm$.*

Proof: According to Remark 2.5, the results in Section 2 still hold for $\widetilde{\mathcal{M}}_\pm$. By Step 1 in the proof of Lemma 2.3, J is bounded from below on $\widetilde{\mathcal{M}}_\pm$. By the same argument as in the proof of Lemmas 3.1 and 3.2, we know that $\{u_m\}$ is a bounded (PS)-sequence of J in $X^{1,\alpha}_{\mathrm{rad}}$. That is, when $m \to \infty$,

$$J'(u_m) \to 0, \quad J(u_m) \to \inf_{\widetilde{\mathcal{M}}_\pm} J. \tag{5.4}$$

Since $\{u_m\}$ is bounded in $X^{1,\alpha}_{\mathrm{rad}}$, when $m \to \infty$, we see that, up to a subsequence, $u_m \rightharpoonup v^0$ weakly in $X^{1,\alpha}_{\mathrm{rad}}$, and hence

$$u_m \to v^0 \quad \text{in } L^q(\mathbb{R}^n) \tag{5.5}$$

because the embedding (1.9) with $\alpha \in (1, n)$ is compact. By the same derivation as in (3.33), there hold

$$J'(v^0) = 0 \quad \text{and} \quad \mathcal{P}(v^0) = 0. \tag{5.6}$$

Set $u_m^1 = u_m - v^0$. From (5.5) it follows that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} |u_m^1|^q\,dx = 0. \tag{5.7}$$

Therefore, using $\langle J'(u_m), u_m\rangle \to 0$ $(m \to \infty)$ (which is implied by (5.4)) and (3.13)–(3.16), we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \int_{\mathbb{R}^n} |\nabla v^0|^2\,dx + L(u_m^1) + L(v^0)\\ &\qquad - \mu\int_{\mathbb{R}^n} |v^0|^q\,dx - \int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx - \int_{\mathbb{R}^n} |v^0|^{2^*}\,dx = o(1) \quad (m \to \infty).\end{aligned} \tag{5.8}$$

Combining this with $\langle J'(v^0), v^0\rangle = 0$ (see (5.6)), we obtain

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + L(u_m^1) - \int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx = o(1) \quad (m \to \infty). \tag{5.9}$$

It follows from (5.9) and (4.4) that

$$L(u_m^1) = o(1) \quad (m \to \infty). \tag{5.10}$$

Inserting (5.10) into (5.9), we obtain

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx - \int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx = o(1) \quad (m \to \infty). \tag{5.11}$$

When $m \to \infty$, using (3.13)–(3.16), (5.7), and (5.10), we deduce from (5.4) that

$$\begin{aligned}\inf_{\widetilde{\mathcal{M}}^{\pm}} J &= J(u_m) + o(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v^0|^2\,dx + \frac{1}{4}L(v^0) \\ &\quad - \frac{\mu}{q}\int_{\mathbb{R}^n} |v^0|^q\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |v^0|^{2^*}\,dx + o(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m^1|^{2^*}\,dx + J(v^0) + o(1).\end{aligned}$$

Thus, by (5.11), it follows that

$$\inf_{\widetilde{\mathcal{M}}^{\pm}} J = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx + J(v^0) + o(1).$$

When $v^0 \in \widetilde{\mathcal{M}}_{\pm}$ is a ground state solution, we know that $J(v^0) \geq \inf_{\widetilde{\mathcal{M}}_{\pm}} J$. From the information above, it follows that

$$\int_{\mathbb{R}^n} |\nabla u_m^1|^2\,dx = o(1) \quad (m \to \infty).$$

Combining this with (5.10), we conclude that when $m \to \infty$,

$$u_m \to v^0 \quad \text{in } X^{1,\alpha}_{\mathrm{rad}}.$$

When $v^1 \in \widetilde{\mathcal{M}}_{\pm}$ is a ground state solution, we can find a subsequence of u_m denoted by itself such that $\|u_m^2\|_q^q \to 0$ $(m \to \infty)$ by the boundedness of $\|u_m\|_{X^{1,\alpha}}$ and the compactness of the embedding (1.9). By the same derivation as in (5.8), when $m \to \infty$, from $\langle J'(u_m^1), u_m^1\rangle \to 0$ (implied by (4.2)) we can see that

$$\|\nabla u_m^2\|_2^2 + L(u_m^2) - \|u_m^2\|_{2^*}^{2^*} \to \langle J'(v^1), v^1\rangle \quad (m \to \infty).$$

In view of $J'(v^1) = 0$ (implied by (4.11)), there holds

$$\|\nabla u_m^2\|_2^2 + L(u_m^2) - \|u_m^2\|_{2^*}^{2^*} \to 0 \quad (m \to \infty). \tag{5.12}$$

By the fourth result in (4.14) we know that

$$\frac{n-2}{2}[\|\nabla u_m^2\|_2^2 - \|u_m^2\|_{2^*}^{2^*}] + \frac{n+\alpha}{4}L(u_m^2) = o(1) \quad (m \to \infty).$$

Combining with (5.12) yields

$$L(u_m^2) \to 0 \quad (m \to \infty). \tag{5.13}$$

Inserting this into (5.12) we get

$$\|\nabla u_m^2\|_2^2 - \|u_m^2\|_{2^*}^{2^*} \to 0 \quad (m \to \infty).$$

This result, together with (5.2), implies

$$\inf_{\widetilde{\mathcal{M}}_{\pm}} J = \frac{1}{n}\|\nabla u_m^2\|_2^2 + J(v^1) + J(v^0) + o(1).$$

Noting $v^1 \in \widetilde{\mathcal{M}}_{\pm}$ and $J(v^0) \geq 0$, we have

$$\frac{1}{n}\|\nabla u_m^2\|_2^2 = o(1) \quad (m \to \infty), \tag{5.14}$$

and $J(v^0) = 0$. By (4.1) we have $v^0 = 0$ a.e. on $\mathbb{R}^n$. Therefore, (5.14) implies that $\nabla u_m \to \nabla v^1$ in $L^2(\mathbb{R}^n)$ when $m \to \infty$. Combining with (5.13), $u_m \to v^1$ in $X^{1,\alpha}_{\mathrm{rad}}$. Theorem 5.2 is proved. □

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