Publ. Mat. **69** (2025), 429–444
DOI: 10.5565/PUBLMAT6922509

ON THE INDECOMPOSABLE INVOLUTIVE SOLUTIONS OF THE YANG–BAXTER EQUATION OF FINITE PRIMITIVE LEVEL

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Abstract: In this paper, we study the class of indecomposable involutive solutions of the Yang–Baxter equation of finite primitive level, recently introduced by Cedó and Okniński in [**13**]. We give a group-theoretic characterization of these solutions by means of displacement groups, and we apply this result to compute and enumerate those having small size. For some classes of indecomposable involutive solutions recently studied in the literature, we compute the exact value of the primitive level. Some relationships with other families of solutions are also discussed. Finally, following [**13**, Question 3.2], we provide a complete description of those having primitive level 2 by left braces.

2020 Mathematics Subject Classification: 16T25, 81R50, 20N02.

Key words: imprimitive group, Yang–Baxter equation, brace, cycle set.

Introduction

The quantum Yang–Baxter equation has been of interest ever since a paper of Yang [**34**], where it appears for the first time. Given a vector space V, a map $R\colon V\otimes V\to V\otimes V$ is said to be a *solution of the quantum Yang–Baxter equation* if

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12},$$

where $R_{ij}\colon V\otimes V\otimes V\to V\otimes V\otimes V$ is the map acting as R on the (i,j) tensor factor and as the identity on the remaining factor. Finding all the solutions of the quantum Yang–Baxter equation seems to be very hard, and it is still an open problem. In that regard, Drinfeld ([**17**]) suggested the study of an easier case, i.e., the solutions of the quantum Yang–Baxter equation that are induced by the linear extension of a map $\mathcal{R}\colon X\times X\to X\times X$, where X is a basis for V. A function $\mathcal{R}$ of this type is called a *set-theoretic solution of the quantum Yang–Baxter equation*. In recent years, several authors have studied these solutions using an equivalent formulation. Specifically, a map $r\colon X\times X\to X\times X$ is said to be a *set-theoretic solution of the Yang–Baxter equation* if

$$r_1r_2r_1=r_2r_1r_2,$$

where $r_1:=r\times \mathrm{id}_X$ and $r_2:=\mathrm{id}_X\times r$. It is easy to see that if $\tau\colon X\times X\to X\times X$ is the twist map, then a function $\mathcal{R}\colon X\times X\to X\times X$ is a set-theoretic solution of the quantum Yang–Baxter equation if and only if the map $r:=\tau\mathcal{R}$ is a set-theoretic solution of the Yang–Baxter equation. Now, let $\lambda_x\colon X\to X$ and $\rho_y\colon X\to X$ be maps such that $r(x,y)=(\lambda_x(y),\rho_y(x))$ for all $x,y\in X$. A set-theoretic solution of the Yang–Baxter equation (X,r), which we will simply call *solution*, is said to be a left (right) non-degenerate if $\lambda_x\in \mathrm{Sym}(X)$ ($\rho_x\in\mathrm{Sym}(X)$) for every $x\in X$ and *non-degenerate* if it is left and right non-degenerate. By seminal papers of Etingof, Schedler, and Soloviev [**19**] and Gateva-Ivanova and Van den Bergh [**20**], involutive

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solutions, i.e., those such that $r^2 = \mathrm{id}_X$, have received a lot of attention. In this context, various methods to construct new involutive set-theoretic solutions have been provided (see for example [**26, 32**]). A first attempt, made in [**19**], is based on the notion of a *retraction*: starting from a solution (X, r), it allows one to construct a new involutive solution, indicated by $\mathrm{Ret}(X, r)$, identifying two elements x, y whenever $\lambda_x = \lambda_y$. If the retraction of an involutive solution (X, r) does not provide a new solution, i.e., if $(X, r) = \mathrm{Ret}(X, r)$, then (X, r) is called *irretractable*; while if the retraction process of (X, r) stabilizes to a singleton, then the solution is called *multipermutation*, and the number of retractions-iterations is called the *multipermutation level*. Roughly speaking, the multipermutation level measures how far a solution is from being *trivial*, i.e., a solution for which $\lambda_x = \lambda_y$ for all $x, y \in X$. Particular attention has been devoted to the class of *indecomposable* involutive solutions, since every indecomposable involutive solution can be constructed by dynamical extension of a simple solution (see [**7**, Proposition 2]), and moreover, these solutions carry information on every involutive solution that is not necessarily indecomposable, as all the involutive solutions are constructed from the indecomposable ones (see [**19**, Section 2]). A successful strategy consists of studying these solutions by associating to them various algebraic structures, such as cycle sets, biracks, structure monoids and groups (see, for example, [**16, 22, 28**]). In that regard, in 2007 Rump ([**26**]) introduced an algebraic structure called the *left brace*. Recall that a left brace is a set A with two operations $+$ and $\circ$ such that $(A, +)$ is an abelian group, $(A, \circ)$ is a group, and

$$a \circ (b + c) + a = a \circ b + a \circ c$$

for every $a, b, c \in A$. As shown in [**26**, Section 1], left braces provide involutive solutions. Conversely, every involutive solution can be constructed from a left brace (see [**2**] for more details) for which the multiplicative group coincides with a standard permutation group called the *associated permutation group*. In particular, an arbitrary indecomposable solution (X, r) can be recovered using a suitable left brace B and a core-free subgroup H of $(B, \circ)$, identifying X with the left cosets B/H.

In recent years, left braces have been used systematically to give structural results on indecomposable multipermutation solutions, as done for example in [**12, 23, 29**]. Much less is known about indecomposable solutions that are not multipermutation; the only notable results were recently given in [**6, 13, 14**] for the family of the simple ones. Moreover, up to now, no analogue of the multipermutation level has existed to measure in some way the complexity of an indecomposable solution that is not a multipermutation solution. To partially make up for these discrepancies, in this paper we change our point of view, studying the notion of an indecomposable solution of *finite primitive level*. It was introduced for the first time in [**13**], after the main result of [**12**] in which it was shown that, apart from the finite indecomposable involutive solutions of prime size, all the indecomposable involutive solutions have an imprimitive associated permutation group. This implicitly suggested a new perspective to study finite indecomposable solutions, focusing on imprimitive block systems. Actually, the family of indecomposable involutive solutions of finite primitive level is an unexplored topic, except for a recent paper of the author [**9**], where a generalization on the non-involutive case is also given. The first main result of the paper provides a group-theoretic characterization of indecomposable solutions of finite primitive level by means of a standard subgroup of the associated permutation group called the *displacement group*, already considered in [**21**] to study multipermutation solutions and in [**5**] to study latin solutions. We note that this result, even if it is proved by means of left braces and cycle sets, allows us to detect all the indecomposable involutive solutions of finite primitive level simply by focusing on the action of the displacement groups. As an application of this fact, we find all the indecomposable involutive solutions of finite primitive level among those having size ≤ 9: we summarize the computation in Section 3. Here, we also exhibit several examples of indecomposable involutive solutions of finite primitive level, and we compute the value of the primitive level for some families of solutions. In this context, the links with other classes of solutions, such as latin solutions and soluble solutions, are also discussed. Moreover, we consider the cycle decomposition of the maps λ_x of these solutions, and we take advantage of our result to give a partial answer to [**24**, Question 3.16], providing a decomposability criterion for multipermutation solutions. Similarly to the approach used for multipermutation solutions, for which several authors have provided a nice description of those having low multipermutation level (see for example [**10, 22**]), in the last part of the paper we focus on indecomposable solutions having primitive level 2. In particular, following [**13**, Question 3.2], we provide a left brace-theoretic description of these solutions, and we illustrate it by an example.

1. Basic definitions and results

In this section, we give the preliminaries involving cycle sets and braces used throughout the paper.

1.1. Solutions of the Yang–Baxter equation and cycle sets. In [**25**] Rump found a one-to-one correspondence between solutions and an algebraic structure with a single binary operation, which he called *non-degenerate cycle sets.* To illustrate this correspondence, let us firstly recall the following definition.

Definition 1 ([**25**, p. 45])**.** A pair $(X, \cdot)$ is said to be a *cycle set* if each left multiplication $\sigma_x\colon X \to X$, $y \mapsto x \cdot y$, is bijective and

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

holds for all $x, y, z \in X$. Moreover, a cycle set $(X, \cdot)$ is called *non-degenerate* if the squaring map $\mathfrak{q}\colon X \to X$, $x \mapsto x \cdot x$, is bijective.

Example 2. The simplest example of a cycle set is that given by $X := \mathbb{Z}/n\mathbb{Z}$ and $x \cdot y := \alpha(y)$, for an arbitrary number n and a permutation α of $\mathrm{Sym}(X)$. We will call these cycle sets *trivial*.

Cycle sets are useful to construct new solutions of the Yang–Baxter equation.

Proposition 3 ([**25**, Propositions 1-2])**.** *Let $(X, \cdot)$ be a cycle set. Then the pair (X, r), where $r(x, y) := (\sigma_x^{-1}(y), \sigma_x^{-1}(y) \cdot x)$, for all $x, y \in X$, is an involutive left non-degenerate solution of the Yang–Baxter equation, which we call the associated solution to $(X, \cdot)$. Moreover, this correspondence is one-to-one.*

Convention. From now on, every cycle set will be non-degenerate. All the results will be given in the language of cycle sets, but they can be translated by Proposition 3.

A useful tool to construct new cycle sets from a given one, introduced in [**19**], is the so-called *retract relation*. Specifically, in [**25**] Rump showed that the binary relation $\sim_\sigma$ on X given by

$$x \sim_\sigma y\colon \iff \sigma_x = \sigma_y$$

for all $x, y \in X$ is a *congruence* of $(X, \cdot)$, i.e., an equivalence relation for which $x \sim_\sigma y$ and $x' \sim_\sigma y'$ imply $x \cdot x' \sim_\sigma y \cdot y'$, for all $x, x', y, y' \in X$. In [**19**] (and independently in [**25**]) it was shown that the quotient $X/\sim_\sigma$, which we denote by $\mathrm{Ret}(X)$, is a cycle set, that we will call a *retraction* of $(X,\cdot)$. An important class of cycle sets is given by those having finite multipermutation level.

Definition 4. A cycle set X is said to be of *multipermutation level* n if n is the minimal non-negative integer such that $|\mathrm{Ret}^n(X)| = 1$, where $\mathrm{Ret}^n(X)$ is the cycle set defined inductively by $\mathrm{Ret}^0(X) = X$ and $\mathrm{Ret}^n(X) = \mathrm{Ret}(\mathrm{Ret}^{n-1}(X))$, for all positive integers n.

We can define the notion of a cycle set homomorphism in a classical way.

Definition 5. Let X, Y be cycle sets. A map $p\colon X \to Y$ is said to be a *homomorphism* between X and Y if $p(x\cdot y) = p(x)\cdot p(y)$ for all $x, y \in X$. A surjective homomorphism is called an *epimorphism*, while a bijective homomorphism is said to be an *isomorphism*.

Two standard permutation groups related to a cycle set X are the one generated by the set $\{\sigma_x \mid x \in X\}$, called the *associated permutation group of* X and indicated by $\mathcal{G}(X)$, and the one generated by the set $\{\sigma_x\sigma_y^{-1} \mid x, y \in X\}$, called the *displacement group of* X and indicated by $\mathrm{Dis}(X)$.

In this context, our attention will be focused on indecomposable cycle sets.

Definition 6. A cycle set $(X,\cdot)$ is said to be *indecomposable* if the permutation group $\mathcal{G}(X)$ acts transitively on X.

Following the paper by Vendramin [**32**], if X is a cycle set, S a set, and $\alpha\colon X \times X \times S \to \mathrm{Sym}(S)$, $\alpha(x,y,s) \mapsto \alpha_{(x,y)}(s,-)$, a function such that

$$\alpha_{(x\cdot y, x\cdot z)}(\alpha_{(x,y)}(r,s), \alpha_{(x,z)}(r,t)) = \alpha_{(y\cdot x, y\cdot z)}(\alpha_{(y,x)}(s,r), \alpha_{(y,z)}(s,t)),$$

for all $x, y, z \in X$ and $r, s, t \in S$, then α is said to be a *dynamical cocycle* and the operation $\cdot$ given by

$$(x,s)\cdot(y,t) := (x\cdot y, \alpha_{(x,y)}(s,t))$$

for all $x, y \in X$ and $s, t \in S$ makes $X \times S$ into a cycle set which we denote by $X \times_\alpha S$ and we call a *dynamical extension* of X by α. A dynamical extension $X \times_\alpha S$ is called *indecomposable* if $X \times_\alpha S$ is an indecomposable cycle set: by [**7**, Theorem 7], this happens if and only if X is an indecomposable cycle set and the subgroup of $\mathcal{G}(X \times S)$ generated by $\{h \mid \forall s \in S, h(y,s) \in \{y\} \times S\}$ acts transitively on $\{y\} \times S$, for some $y \in X$. By results contained in [**7, 32**], the following corollary, which is of crucial importance for this paper, follows.

Corollary 7 (Theorem 7 of [**7**] and Theorem 2.12 of [**32**])**.** *Let X be an indecomposable cycle set, Y a cycle set, and $p\colon X \to Y$ a cycle set epimorphism. Then, there exist a set S and a dynamical cocycle α such that X is isomorphic to $Y \times_\alpha S$.*

1.2. Indecomposable cycle sets and left braces. First we introduce the following definition that, as observed in [**11**], is equivalent to the original introduced by Rump in [**26**].

Definition 8 ([**11**, Definition 1])**.** A set B endowed with two operations $+$ and $\circ$ is said to be a *left brace* if $(B,+)$ is an abelian group, $(B,\circ)$ a group, and

$$a\circ(b+c) + a = a\circ b + a\circ c$$

holds for all $a, b, c \in B$.

Examples 9. (1) If X is a cycle set, then one can show that the free abelian group $\mathbb{Z}^X$ gives rise to a left brace $(\mathbb{Z}^X, +, \circ)$, where $(\mathbb{Z}^X, \circ)$ is the group having X as a generating set and $x \circ y = \sigma_x^{-1}(y) \circ (\sigma_x^{-1}(y) \cdot x)$, where $x, y \in X$, as relations.

(2) If X is a cycle set, the associated permutation group $\mathcal{G}(X)$ gives rise to a left brace $(\mathcal{G}(X), +, \circ)$, where $\circ$ is the usual composition in $\mathcal{G}(X)$ (see, for example, [**3**, Section 2] for more details). From now on, we will refer to $(\mathcal{G}(X), +, \circ)$ as the *permutation left brace*.

(3) If $(B, +)$ is an abelian group, then the operation $\circ$ given by $a \circ b := a + b$ give rise to a left brace which we will call *trivial*.

If $(B_1, +, \circ)$ and $(B_2, +', \circ')$ are left braces, a homomorphism ψ between B_1 and B_2 is a function from B_1 to B_2 such that $\psi(a+b) = \psi(a) +' \psi(b)$ and $\psi(a \circ b) = \psi(a) \circ' \psi(b)$, for all $a, b \in B_1$.

Given a left brace B and $a \in B$, let us denote by $\lambda_a \colon B \to B$ the map from B into itself defined by

$$\lambda_a(b) := -a + a \circ b,$$

for all $b \in B$. As shown in [**26**, Proposition 2] and [**11**, Lemma 1], these maps have special properties. We recall them in the following proposition.

Proposition 10. *Let B be a left brace. Then, the following are satisfied:*

(1) $\lambda_a \in \mathrm{Aut}(B, +)$, *for every* $a \in B$;

(2) *the map* $\lambda \colon B \to \mathrm{Aut}(B, +)$, $a \mapsto \lambda_a$, *is a group homomorphism from* $(B, \circ)$ *into* $\mathrm{Aut}(B, +)$.

For the following definition, we refer the reader to [**26**, p. 160] and [**11**, Definition 3].

Definition 11. Let B be a left brace. A subset I of B is said to be a *left ideal* if it is a subgroup of the multiplicative group and $\lambda_a(I) \subseteq I$, for every $a \in B$. Moreover, a left ideal is an *ideal* if it is a normal subgroup of the multiplicative group.

As one can expect, if I is an ideal of a left brace B, then the structure B/I is a left brace called the *quotient left brace* of B modulo I. Moreover, the ideal $\{0\}$ will be called the *trivial* ideal, and a non-zero left brace B which contains no ideals different from $\{0\}$ and B will be called a *simple* left brace.

A standard ideal of a left brace B, given in [**26**, corollary of Proposition 6] and indicated by B^2, is that given by the additive subgroup generated by the set $\{a * b \mid a, b \in B\}$, where $a * b := -a + a \circ b - b$ for all $a, b \in B$.

Other ideals can be obtained by left brace homomorphisms. Indeed, if B_1 and B_2 are left braces and ψ a homomorphism from B_1 to B_2, the kernel of ψ is an ideal of B_1, where the kernel, which we indicate by $\mathrm{Ker}(\psi)$, is the set given by $\mathrm{Ker}(\psi) := \{b \in B_1 \mid \psi(b) = 0\}$.

In [**26**], Rump also introduced another ideal that, in the terms of [**11**, Section 4], is the following.

Definition 12. Let B be a left brace. Then, the set

$$\mathrm{Soc}(B) := \{a \in A \mid \forall\, b \in B,\ a + b = a \circ b\}$$

is named the *socle* of B.

Clearly, $\mathrm{Soc}(B) = \{a \in B \mid \lambda_a = \mathrm{id}_B\}$. Moreover, we have that $\mathrm{Soc}(B)$ is an ideal of B. The two left braces given in Examples 9 are related by the socle. Indeed, given a cycle set X, one can show that the map $\theta\colon X \to \mathcal{G}(X)$ given by $x \mapsto \sigma_x^{-1}$ can be extended to a surjective left brace homomorphism $\bar{\theta}\colon \mathbb{Z}^X \to \mathcal{G}(X)$ such that $\mathrm{Ker}(\bar{\theta}) = \mathrm{Soc}(\mathbb{Z}^X)$.

Left brace homomorphisms are strongly related to cycle set homomorphisms. Indeed, every cycle set epimorphism $p\colon X \to Y$ induces a left brace epimorphism $p'\colon \mathbb{Z}^X \to \mathbb{Z}^Y$ by $x \mapsto p(x)$, for all $x \in X$ and $\bar{p}\colon \mathcal{G}(X) \to \mathcal{G}(Y)$ by $\sigma_x \mapsto \sigma_{p(x)}$, for all $x \in X$.

Definition 13. A cycle set epimorphism $p\colon X \to Y$ is said to be a *covering* if it induces a left brace isomorphism $\bar{p}\colon \mathcal{G}(X) \to \mathcal{G}(Y)$.

In the last part of the section, we recall the theory mainly developed in [**31, 28**], that allows us to detect more information on indecomposable cycle sets (and their epimorphic images) by left braces.

Proposition 14 ([**28**, Theorem 3])**.** *Let $(B, +, \circ)$ be a left brace, $Y \subset B$ a transitive cycle base, $a_1 \in Y$, and K a core-free subgroup of $(B, \circ)$, contained in the stabilizer B_{a_1} of a_1 (with respect to the action λ). Then, the pair $(X, \cdot)$ given by $X := B/K$ and $\sigma_{x\circ K}(y \circ K) := \lambda_x(a_1)^- \circ y \circ K$ gives rise to an indecomposable cycle set with $\mathcal{G}(X) \cong B$.*

Conversely, every indecomposable cycle set $(X, \cdot)$ with $\mathcal{G}(X) \cong B$ (as left braces) can be obtained in this way.

Convention. From now on, a cycle set obtained as in Proposition 14 will be indicated by C_{B,K,a_1}.

Proposition 15. *Let $p\colon X \to Y$ be a covering of indecomposable cycle sets, $\mathcal{G} := \mathcal{G}(X)$, and $x \in X$. Then, there exist two core-free subgroups K and H contained in $\mathcal{G}_{\sigma_x}$ with $K \leq H$ such that X can be identified with $C_{\mathcal{G},K,\sigma_x}$, Y can be identified with $C_{\mathcal{G},H,\sigma_x}$, and p can be identified with the epimorphism from $C_{\mathcal{G},K,\sigma_x}$ to $C_{\mathcal{G},H,\sigma_x}$ which sends an element $z \circ K$ to $z \circ H$.*

Conversely, up to isomorphisms, any covering of indecomposable cycle sets arises in this way.

Proof: It follows by [**28**, Theorem 3 and Corollary 2]. □

Proposition 16 ([**31**, Section 3])**.** *Let X, Y be indecomposable cycle sets and $p\colon X \to Y$ an epimorphism from X to Y. Then, the set $I := \{g \mid g \in \mathcal{G}(X),\, p(g(x)) = p(x)\ \forall\, x \in X\}$ is an ideal of $\mathcal{G}(X)$.*

Conversely, if X is an indecomposable cycle set and I an ideal of $\mathcal{G}(X)$, the I-orbits of X induce a cycle set structure, which we indicate by X/I, that gives rise to a canonical cycle set epimorphism $p\colon X \to X/I$.

From now on, if X, Y are indecomposable cycle sets and $p\colon X \to Y$ an epimorphism from X to Y, the ideal induced by p as in the previous result will be indicated by $I(p)$. If X is an indecomposable cycle set and I an ideal of $\mathcal{G}(X)$, the induced epimorphism from X to X/I will be indicated by p_I.

Proposition 17 ([**31**, Theorem 1])**.** *Let X, Y be indecomposable cycle sets and $p\colon X \to Y$ an epimorphism from X to Y. Up to isomorphism, there exists a unique factorization $p = qp_I$ for a suitable ideal I of $\mathcal{G}(X)$, where q is a covering of cycle sets. In particular, Y is an epimorphic image of X/I.*

2. Cycle sets of finite primitive level

After introducing the class of cycle sets of finite primitive level, in this section we give a characterization of cycle sets of finite primitive level by its associated permutation group.

We start with the definition of cycle sets of finite primitive level, given for the first time in [**13**] in terms of solutions.

Definition 18. A cycle set X is said to be *primitive* if $\mathcal{G}(X)$ acts primitively on X. Moreover, we say that a finite indecomposable cycle set X has *primitive level* k, and we will write $\mathrm{fpl}(X) = k$, if k is the biggest positive integer such that

(1) there exist cycle sets $X_1 = X, X_2, \dots, X_k$, with $|X_i| > |X_{i+1}| > 1$, for every $1 \le i \le k-1$;

(2) there exists an epimorphism of cycle sets $p_{i+1}\colon X_i \to X_{i+1}$, for every $1 \le i \le k-1$;

(3) X_k is primitive.

Clearly, every indecomposable cycle set having prime size is primitive, and by the main theorem of [**12**], there are no other finite primitive cycle sets such that $|X| > 1$.

Remark 19. As observed in [**9**, Corollary 5.5], if X is an indecomposable cycle set of finite primitive level and x an arbitrary element of X, then σ_x cannot have a fixed point y, i.e., an element $y \in X$ such that $x\cdot y = y$. However, this condition is not sufficient (see comment after [**9**, Corollary 5.7]).

The following lemma is a simple but useful result to state when an indecomposable cycle set has finite primitive level.

Lemma 20. *Let X be a finite indecomposable cycle set. Then X has finite primitive level if and only if there exist a trivial indecomposable cycle set Y with $|Y| > 1$ and an epimorphism $p\colon X \to Y$.*

Proof: Straightforward. □

Before giving the main result of this section, we give a preliminary lemma. This result is essentially a mixture between [**13**, Proposition 4.3] and the results contained in [**30**, Section 1].

Lemma 21. *Let X be an indecomposable cycle set. Then, the ideal $\mathcal{G}(X)^2$ of the left brace $\mathcal{G}(X)$ is equal to $\mathrm{Dis}(X)$. Moreover, the factor group $\mathcal{G}(X)/\mathrm{Dis}(X)$, regarded as a left brace, is a trivial left brace with a cyclic multiplicative (and hence additive) group.*

Now we are able to show the main result of the section.

Theorem 22. *Let X be a finite indecomposable cycle set. Then, X has finite primitive level if and only if $\mathrm{Dis}(X)$ does not act transitively on X.*

Proof: Suppose that X has finite primitive level. Then by Corollary 7 X is isomorphic to a dynamical extension $I \times_\alpha S$, where I is an indecomposable cycle set having prime size p. Moreover, by [**19**, Theorem 2.13], I is a trivial cycle set. These facts imply that $\mathrm{Dis}(X)$ fixes the subsets $\{i\} \times S$, for every $i \in I$, hence it cannot act transitively on X.

Conversely, suppose that $\mathrm{Dis}(X)$ does not act transitively on X, and let $\Delta := \{\Delta_1, \dots, \Delta_m\}$ the set of its orbits. Since by Lemma 21 $\mathrm{Dis}(X)$ is a normal subgroup of $\mathcal{G}(X)$, it follows that $\mathcal{G}(X)$ acts on Δ. Moreover, we have that $\sigma_x \mathrm{Dis}(X) =$

$\sigma_y \operatorname{Dis}(X)$ for every $x, y \in X$, therefore by Lemma 21 $\mathcal{G}(X)/\operatorname{Dis}(X)$ is a cyclic group generated by an element $\sigma_x \operatorname{Dis}(X)$. Now, since $\mathcal{G}(X)$ acts transitively on X, we have that every σ_x acts on Δ as a cycle δ of length m. Therefore, the map $r\colon X \to \Delta$, $x \mapsto \Delta_x$, where Δ_x is such that $x \in \Delta_x$, is an epimorphism from X to the trivial cycle set $(\Delta, \cdot)$ given by $\Delta_i \cdot \Delta_j := \delta(\Delta_j)$ for every $\Delta_i, \Delta_j \in \Delta$, hence the thesis follows by Lemma 20. □

As a corollary, we provide a necessary condition to test the simplicity of a cycle set. Recall that an indecomposable cycle set X is said to be *simple* if $|X| > 1$ and it has no epimorphic images different from itself and the singleton.

Corollary 23. *Let X be a finite indecomposable simple cycle set such that $|X|$ is not a prime number. Then,* $\operatorname{Dis}(X)$ *acts transitively on X.*

Proof: It follows directly by Theorem 22. □

In the particular case of cycle sets having prime-squared size, the condition of the previous result is also sufficient.

Corollary 24. *Let X be an indecomposable cycle set having size p^2, for a prime number p. Then, X is a simple cycle set if and only if* $\operatorname{Dis}(X)$ *acts transitively on X.*

Proof: Since $|X| = p^2$ for a prime number p, then by [**7**, Lemma 1] and [**19**, Theorem 2.13] X is simple if and only if it does not have finite primitive level, hence the thesis follows by Theorem 22. □

3. Examples and applications

In this section, we exhibit several examples and non-examples of cycle sets having finite primitive level, computing the exact primitive level in some cases. As an application of the main result of the previous section, we enumerate the indecomposable cycle sets of finite primitive level having small size. Moreover, we study some relations between these cycle sets and other classes recently considered in other papers.

Many examples of indecomposable cycle sets having finite primitive level appear in the literature. Below, we exhibit some of them.

Examples 25. (1) If X is an indecomposable cycle set with $\mathcal{G}(X)$ abelian and $|X| = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, where $p_1, \dots, p_n$ are distinct prime numbers, then $\operatorname{fpl}(X) = \alpha_1 + \dots + \alpha_n$ (see [**8**, Theorem 4.4] for more details). These cycle sets have been explicitly classified if $\operatorname{mpl}(X) = 2$ (see [**22**]) and if $\mathcal{G}(X)$ is cyclic (see [**23**]).

(2) Every finite indecomposable cycle set X having finite multipermutation level is a cycle set of finite primitive level (see [**8**, Corollary 4.5]) and $\operatorname{mpl}(X) \leq \operatorname{fpl}(X)$ (several concrete examples of indecomposable cycle sets belonging to the multipermutation ones are contained, for example, in [**21**] and [**15**]).

(3) Let k be an odd number and $(S, \cdot)$ be the trivial cycle set given by $S := \mathbb{Z}/k\mathbb{Z}$ and $x \cdot y := y + 1$ for all $x, y \in S$ and $(I, \star)$ the cycle set given by $I := \{1, 2, 3, 4\}$, $\sigma_1 := (1\ \ 4)$, $\sigma_2 := (1\ \ 3\ \ 4\ \ 2)$, $\sigma_3 := (2\ \ 3)$, and $\sigma_4 := (1\ \ 2\ \ 4\ \ 3)$. Then, the direct product $S \times I$ is a cycle set of finite primitive level, since the projection on the first component $S \times I \to S$, $(s, i) \mapsto s$, gives rise to a cycle set epimorphism. This family of cycle sets appears in [**7**, Example 9]. Note that these cycle sets are not of finite multipermutation level since $\operatorname{Ret}(S \times I) \cong I$.

(4) Let $G = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\cdot$ the binary operation on G given by

$$(i, j)\cdot (k, l) := (k - j, l + t_{k-i}),$$

where $t_x = 1$ if $x = 0$ and $t_x = 3$ otherwise. Then, $(G, \cdot)$ is the indecomposable cycle set constructed in [**13**, Remark 4.10]. By a standard calculation, one can show that $(G, \cdot)$ has the indecomposable trivial cycle set of size 2 as an epimorphic image.

As a generalization of (3) of Examples 25, one can easily show that the class of finite indecomposable cycle sets of finite primitive level is closed by indecomposable dynamical extensions.

Proposition 26. *Let I be an indecomposable cycle set of finite primitive level and let $I \times_\alpha S$ be a finite indecomposable dynamical extension. Then, $I \times_\alpha S$ is an indecomposable cycle set of finite primitive level.*

Proof: Straightforward. □

By the previous proposition, examples of indecomposable cycle sets of finite primitive level occur in abundance (see [**7**, Section 5] for several concrete examples).

In (1) of Examples 25, we exhibit a family of cycle sets for which the primitive level assumes the maximum possible value. In the next results, we show that this also happens for indecomposable cycle sets of square-free size and for those having a cyclic permutation left brace.

Theorem 27. *Let X be an indecomposable cycle set having size $p_1 \dots p_n$, where n is a positive integer and $p_1, \dots, p_n$ are distinct prime numbers. Then,* $\mathrm{fpl}(X) = n$.

Proof: We show the thesis by induction on n. If $n = 1$, the thesis directly follows by [**19**, Theorem 2.13]. Now, suppose that X is an indecomposable cycle set having size $p_1 \dots p_n$, where n is a positive integer and $p_1, \dots, p_n$ are distinct prime numbers. By [**15**, Theorem 4.1], we have that $\mathcal{G}(X) = P_1 \circ \cdots \circ P_n$, where P_i is the p_i-Sylow of $(\mathcal{G}(X), +)$, and without loss of generality we can suppose that $P_1 \circ \cdots \circ P_i$ is an ideal of $\mathcal{G}(X)$ for all $i \in \{1, \dots, n\}$. In particular, P_1 is an ideal of $\mathcal{G}(X)$ and hence a normal subgroup. Therefore, the orbits of P_1 form an imprimitive block system of X, hence necessarily every orbit of P_1 must have size p_1. By Proposition 16, P_1 induces a cycle set structure X/P_1 of size $p_2 \dots p_n$ and the natural map from X to X/P_1 is a cycle set epimorphism. By inductive hypothesis, it follows that $\mathrm{fpl}(X/P_1) = n - 1$, and since X/P_1 is an epimorphic image of X the thesis follows. □

Theorem 28. *Let X be an indecomposable cycle set having size $p_1^{\alpha_1} \dots p_n^{\alpha_n}$, where n is a positive integer and $p_1, \dots, p_n$ are distinct prime numbers. Moreover, suppose that $\mathcal{G}(X)$ is a permutation left brace with a cyclic additive group. Then,* $\mathrm{fpl}(X) = \alpha_1 + \cdots + \alpha_n$.

Proof: We show the thesis by induction on $\alpha_1+\cdots+\alpha_n$. If $\alpha_1+\cdots+\alpha_n = 1$, necessarily we have $n = 1$ and $\alpha_1 = 1$, therefore the thesis follows by [**19**, Theorem 2.13]. Now, suppose that $\alpha_1 + \cdots + \alpha_n > 1$. Then by [**27**, corollary of Proposition 14], since $\mathcal{G}(X)$ has a cyclic additive group, we have that $|\mathrm{Soc}(\mathcal{G}(X))| > 1$, therefore there exists a normal subgroup $(I, +)$ of $(\mathrm{Soc}(\mathcal{G}(X)), +)$ having prime size. Without loss of generality, we can suppose that $|I| = p_1$. Since I is a characteristic subgroup of $(\mathrm{Soc}(\mathcal{G}(X)), +)$, it follows that I is an ideal of $\mathcal{G}(X)$. Moreover, every orbit of X with respect to the action of I must have size p_1 and hence it induces a cycle set X/I of size $p_1^{\alpha_1-1} \dots p_n^{\alpha_n}$. By inductive hypothesis, we have that $\mathrm{fpl}(X/I) = \alpha_1 - 1 + \cdots + \alpha_n$ and hence $\mathrm{fpl}(X) \geq \mathrm{fpl}(X/I) + 1 = \alpha_1 + \cdots + \alpha_n$, therefore the thesis follows. □

In addition to Remark 19, in the following result we give a further information involving the cycle decomposition of the left multiplications. From now on, a k-cycle $(x_1 \dots x_k)$ will be called *trivial* if $k = 1$.

Proposition 29. *Let X be an indecomposable cycle set of finite primitive level, and let $\{\alpha_1, \dots, \alpha_n\}$ be the set of all the cycles (possibly trivial) belonging to at least one left multiplication σ_x. Then, there exists a prime divisor p of $|X|$ that divides the length of α_i, for all $i \in \{1, \dots, n\}$.*

Proof: Since X has finite primitive level, by Corollary 7 there exists a prime number p_1 such that X is isomorphic to a dynamical extension $I \times_\alpha S$ and I is an indecomposable cycle set of size p_1, which by [**19**, Theorem 2.13] can be identified with that given by $I := \mathbb{Z}/p_1\mathbb{Z}$ and $x \cdot y := y + 1$ for all $x, y \in I$. Now, let (i, s) and (j, t) be elements of $I \times S$. If (j, t) belongs to a z-cycle of $\sigma_{(i,s)}$ (by [**9**, Corollary 5.5] we must have $z > 1$), then $(j, t) = \sigma^z_{(i,s)}(j, t) = (j + z, t)$, therefore p_1 must divide z. Since (i, s) and (j, t) are arbitrary elements of X, p_1 is the desired prime number. □

In [**24**], Ramírez and Vendramin posed the following question.

Question 30. Let X be a cycle set. Is it true that if some σ_x contains a non-trivial cycle of length coprime with $|X|$, then X is decomposable?

As an application of Proposition 29, we give a positive answer when X has finite multipermutation level.

Corollary 31. *Let X be a finite multipermutation cycle set. Suppose that some σ_x contains a cycle of length coprime with $|X|$. Then, X is decomposable.*

Proof: If we suppose X to be indecomposable, then it has finite primitive level. Then, by Proposition 29 every cycle contained in an arbitrary σ_x does not have coprime length with $|X|$, a contradiction. □

Actually, we are not able to state if the hypothesis on the multipermutation level can be dropped in Corollary 31. In this context, note that a possible counterexample X to Question 30 would imply that $\mathcal{G}(X)$ is a *singular* left brace, where a left brace B is said to be singular if there exist an indecomposable cycle set X such that $B \cong \mathcal{G}(X)$ and a prime number p that divides the order of B but not the order of X (for this reason, these cycle sets also will be called singular). These left braces were recently characterized in [**31**]. Singular cycle sets seem to be very difficult to construct: indeed, only a counterexample of size 8, given in [**31**], is known in the literature. In the same paper, it was also shown that if X is a singular cycle set, then so is its retraction, therefore the research of these cycle sets can be reduced in some respects with the irretractable ones.

Below, we recall Rump's singular cycle set and we use Theorem 22 to show that it has finite primitive level. Moreover, we use this cycle set to construct, by dynamical extensions obtained in [**7**, Section 5], a family of irretractable singular cycle sets.

Example 32. Let $X := \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the indecomposable cycle set given by

$$\begin{aligned} \sigma_0 &= (07)(13)(25)(46), & \sigma_1 &= (0264)(1375), \\ \sigma_2 &= (01)(25)(34)(67), & \sigma_4 &= (0462)(1573), \\ \sigma_3 &= (02)(16)(34)(57), & \sigma_5 &= (0451)(2673), \\ \sigma_7 &= (07)(16)(23)(45), & \sigma_6 &= (0154)(2376). \end{aligned}$$

Then, the left brace $\mathcal{G}(X)$, which has size 24, is singular since 3 divides $|\mathcal{G}(X)|$ but not $|X|$. By a standard calculation, one can show that $\mathcal{G}(X)^2$ splits X into the orbits $\{0,3,5,6\}$ and $\{1,2,4,7\}$, hence by Theorem 22 X has finite primitive level. Now, let $S := \mathbb{Z}/k\mathbb{Z}$, with k an arbitrary number coprime with 3, A be the set given by $A := S \times S$, and α be the function from $X \times X \times A$ to $\mathrm{Sym}(A)$, $\alpha(x,y,(a,b)) \mapsto \alpha_{(x,y)}((a,b),-)$, given by

$$\alpha_{(x,y)}((a,b),(c,d)) := \begin{cases} (c,d+1) & \text{if } x=y \text{ and } a \neq c, \\ (c,d) & \text{if } x=y \text{ and } a=c, \\ (c-b-1,d) & \text{if } x \neq y \end{cases}$$

for all $(x,y,(a,b)) \in X \times X \times A$. By a standard calculation, one can show that the dynamical extension $X \times_\alpha A$ is an indecomposable cycle set and by [**7**, Proposition 10] is irretractable. Since $\mathcal{G}(X) \cong \mathcal{G}(X \times_\alpha A)/I$ for a suitable ideal I, we have that 3 divides $|\mathcal{G}(X \times_\alpha A)|$; on the other hand, 3 does not divide $|X \times A|$. Moreover, by Proposition 26, the cycle set $X \times_\alpha A$ is of finite primitive level.

In this context, an intriguing challenge is the construction of further singular cycle sets that are in some way different from the previous ones. For example, one could ask whether there exist singular cycle sets which do not have finite primitive level.

For a positive integer n, let $c(n)$ be the number of indecomposable cycle sets of size n, $m(n)$ be the number of indecomposable cycle sets of size n having finite multipermutation level, and $fp(n)$ be the number of indecomposable cycle sets of size n having finite primitive level. As an application of Theorem 22, by means of the GAP package [**33**], we computed, employing a small GAP code, the first values of $fp(n)$. We summarize our calculations in the following table.

n	$c(n)$	$m(n)$	$fp(n)$
2	1	1	1
3	1	1	1
4	5	3	3
5	1	1	1
6	10	10	10
7	1	1	1
8	100	39	70
9	16	13	13

Remark 33. For every $n \in \{2,\dots,9\}$, we have $m(n) \leq fp(n)$, and if n is a prime number, we obtain $m(n) = fp(n) = 1$: these facts agree with (2) of Examples 25 and [**19**, Theorem 2.13]. If $n = 6$, we have $c(n) = m(n) = fp(n)$: this is consistent with [**15**, Theorem 4.5] and Theorem 27.

In the last part of this section, we focus on some classes of cycle sets present in the literature that provide examples of indecomposable cycle sets which do not have finite primitive level.

Examples 34. (1) No non-trivial simple cycle set has a finite primitive level because the only epimorphic images are the whole cycle set and the cycle set of size 1 (see [**6, 13, 14**] for several concrete examples).

(2) Let $X := \{1,2,3,4,5,6,7,8\}$ and $\cdot$ the binary operation given by

$$\begin{aligned}\sigma_1 = \sigma_2 &:= (3\ \ 5\ \ 4\ \ 7),\\ \sigma_3 = \sigma_4 &:= (1\ \ 6\ \ 2\ \ 8),\\ \sigma_5 = \sigma_7 &:= (1\ \ 5\ \ 6\ \ 4\ \ 2\ \ 7\ \ 8\ \ 3),\\ \sigma_6 = \sigma_8 &:= (1\ \ 3\ \ 6\ \ 5\ \ 2\ \ 4\ \ 8\ \ 7).\end{aligned}$$

Then, 2 is a fixed point of σ_1; hence, by Remark 19, X cannot have finite primitive level. This cycle set was given in [**1**, Example 3.8]. Inspecting [**1**, Tables 3.2 and 3.3] and by means of Remark 19, one can find further examples of indecomposable cycle sets that are not of finite primitive level.

A large family of indecomposable cycle sets that are not of finite primitive level is given by the so-called *latin* cycle sets, where a cycle set is said to be latin if the right multiplication $\delta_x\colon X \to X$, $y \mapsto y\cdot x$, is bijective, for every $x \in X$ (see [**5**] for more details and concrete examples). Clearly, these cycle sets are always indecomposable.

Corollary 35. *Let X be a latin cycle set, with $|X| > 1$. Then, X does not have finite primitive level.*

Proof: If x, y, and z are elements of X, there exists $t \in X$ such that $y = t\cdot(\sigma_z^{-1}(x)) = \sigma_t(\sigma_z^{-1}(x))$, hence $\operatorname{Dis}(X)$ acts transitively on X. Therefore the thesis follows by Theorem 22. $\square$

In [**4**] the notion of a *soluble* (not necessarily involutive) solution was recently introduced. Here, we recall such a notion, restricting to an involutive setting and using the language of cycle sets, and we close the section showing that this class of cycle sets has empty intersection with that of the cycle sets of finite primitive level. First we recall that, if X, Y are cycle sets, every epimorphism f from X to Y induces a congruence $\sim_f$ on X, i.e., an equivalence relation in which $x \sim_f y$ and $x' \sim_f y'$ implies $x\cdot x' \sim_f y\cdot y'$, by $x \sim_f y :\iff f(x) = f(y)$, for all $x, y \in X$. The quotient of X by the equivalence relation $\sim_f$ will be indicated by X/Ker_f.

Definition 36. Let X be a cycle set. Assume that there exists a sequence of subsets $X_t \subseteq \cdots \subseteq X_1 \subseteq X_0 = X$ with $X_t = \{x_t\}$ such that, for every $1 \le i \le t$, there exist a cycle set Y_i and a cycle set epimorphism $f_i\colon X \to Y_i$ satisfying

(1) $X_i \in X/\operatorname{Ker}_{f_i}$ for all $1 \le i \le t$;
(2) $f_i(X_{i-1})$ is a trivial subcycle set of Y_i given by $x\cdot y = y$ for every $x, y \in f_i(X_{i-1})$, for all $1 \le i \le t$.

Then, X is said to be *soluble at x_t*.

Proposition 37. *Let X be an indecomposable cycle set having finite primitive level. Then, X is not a soluble cycle set.*

Proof: Suppose that X is soluble and let $X_0, \ldots, X_t$ and $f_1, \ldots, f_t$ be as in Definition 36. Then, we have that $X_t \in X/\operatorname{Ker}_{f_t}$, and by [**7**, Lemma 1] we have that f_t is bijective, therefore $X \cong Y_t$. By (2) of Definition 36, there exist $x, y \in Y_t$ such that $x\cdot y = y$, but this contradicts Remark 19. $\square$

4. Cycle sets of primitive level 2

In this section, we focus on cycle sets having primitive level 2. In particular, following [**13**, Question 3.2], we provide a description of all the indecomposable cycle sets having primitive level 2 by means of their permutation left braces.

We start with an easy case, considering cycle sets with a trivial permutation left brace.

Proposition 38. *Let X be an indecomposable cycle set with trivial permutation left brace $\mathcal{G}(X)$. Then,* $\mathrm{fpl}(X) = 2$ *if and only if X has size pq, where p and q are two prime numbers, not necessarily distinct.*

Proof: Since $\mathcal{G}(X)$ is a trivial left brace, it follows that X is a trivial cycle set and any epimorphic image of X is a trivial cycle set. Then, the thesis follows by the fact that if Y is a trivial indecomposable cycle set and the size of Y divides the size of X, then Y is an epimorphic image of X. □

By the previous proposition, we can focus on indecomposable cycle sets of primitive level 2 provided by non-trivial left braces. We start with some preliminary results.

Proposition 39. *Let X be a finite indecomposable cycle set and $p\colon X \to Y$ an epimorphism from X to a trivial indecomposable cycle set. Then, the size of Y divides $X/\mathcal{G}(X)^2$.*

Proof: By Corollary 7, X is isomorphic to a dynamical extension $Y \times_\alpha S$. Moreover, $\mathcal{G}(X)^2$ fixes every set $\{y\}\times S$, for all $y \in Y$, and by a standard calculation we have that x_1 and x_2 are in the same orbit with respect to $\mathcal{G}(X)^2$ if and only if $g(x_1)$ and $g(x_2)$ are in the same orbit with respect to $\mathcal{G}(X)^2$, for all $x_1, x_2 \in X$ and $g \in \mathcal{G}(X)$. Therefore, there exists a positive integer r such that $\mathcal{G}(X)^2$ splits every set $\{y\} \times S$ into r orbits. Hence, it follows that $|X/\mathcal{G}(X)^2| = r\cdot |Y|$. □

Corollary 40. *Let X be a non-trivial indecomposable cycle set having primitive level 2. Then, the action of $\mathcal{G}(X)^2$ on X splits X into p orbits, for a prime number p, and if $r\colon X \to Y$ is an epimorphism with $|Y|$ a prime number, then $|Y| = p$.*

Proof: Since X has primitive level 2, necessarily the action of $\mathcal{G}(X)^2$ on X splits X into p orbits, for a prime number p. Since $|Y|$ is a prime number, the thesis follows by the previous corollary. □

Now we are ready for the desired description.

Theorem 41. *Let $(B, +, \circ)$ be a non-trivial left brace, $Y \subset B$ a transitive cycle base, $a_1 \in Y$, and K a core-free subgroup contained in the stabilizer B_{a_1} of a_1 with respect to the action λ. Moreover, let $x \circ K$ be an arbitrary left coset of B/K and $B_{x\circ K}$ be the stabilizer of $x \circ K$ in B with respect to the left multiplication in $(B, \circ)$. Then, the cycle set C_{B,K,a_1} has primitive level 2 if and only if the following conditions hold:*

1. *the index of the subgroup $B^2 \circ B_{x\circ K}$ of B is a prime number p;*
2. *the action of B^2 on the left coset B/H by left multiplication is transitive, for every core-free subgroup H with $K < H \le B_{a_1}$;*
3. *if J is an ideal such that its action on the left coset B/K by left multiplication has o_J orbits, with $o_J > p$, then B^2 acts transitively (by the induced action) on the J-orbits of B/K.*

Moreover, every non-trivial indecomposable cycle set X having primitive level 2 can be constructed as C_{B,K,a_1} for suitable B, K, and a_1 satisfying the previous conditions.

Proof: Suppose that C_{B,K,a_1} has primitive level 2. Then, $C_{B,K,a_1}/B^2$ is a non-trivial quotient of C_{B,K,a_1} and the size of $C_{B,K,a_1}/B^2$ is a prime number p. Since the action of B^2 on the cycle set C_{B,K,a_1} is just the action by left multiplication on B/K, by [**18**, Exercise 9 on p. 117] we have that condition (1) follows. If (2) does not hold, there exist a core-free subgroup H, with $K < H$, that gives rise to a covering $p_1\colon C_{B,K,a_1} \to C_{B,H,a_1}$ and an epimorphism $p_1\colon C_{B,H,a_1} \to C_{B,H,a_1}/B^2$ with $|B/H| < |B/K|$ and $|C_{B,H,a_1}/B^2| > 1$, therefore by Theorem 22 C_{B,H,a_1} has finite primitive level and hence C_{B,K,a_1} has primitive level greater than 2, a contradiction. If (3) does not hold for a suitable ideal J, we obtain an epimorphism $p_1\colon C_{B,K,a_1} \to C_{B,K,a_1}/J$ and, if J' is such that $\mathcal{G}(C_{B,K,a_1}/J) \cong B/J'$, we have that $(B/J')^2$ does not act transitively on $C_{B,K,a_1}/J$. Therefore, by Theorem 22 $C_{B,K,a_1}/J$ has finite primitive level and hence C_{B,K,a_1} has primitive level greater than 2, a contradiction.

Conversely, suppose that (1), (2), and (3) hold. By condition (1) and [**18**, Exercise 9 on p. 117], the indecomposable cycle set $Z := C_{B,K,a_1}/B^2$ has prime size p. Moreover, there is a natural epimorphism r from C_{B,K,a_1} to Z. By Proposition 39 and [**19**, Theorem 2.13], p and Z are completely determined by condition (1), and there are no other trivial indecomposable cycle sets that are epimorphic images of C_{B,K,a_1}. Therefore, to demonstrate the thesis, it is sufficient to prove that there is no non-trivial indecomposable cycle set T, different from C_{B,K,a_1} and Z, such that T is an epimorphic image of C_{B,K,a_1} and Z is an epimorphic image of T. Suppose T is such a cycle set and $r_1\colon C_{B,H,a_1} \to T$ and $r_2\colon T \to Z$ are epimorphisms. If r_1 is a covering, by Proposition 15 $T = B/H$ for some subgroup H with $K < H \leq B_{a_1}$, and since T has finite primitive level, B^2 does not act transitively on the left cosets B/H, against condition (2). Then, by Proposition 17, without loss of generality we can suppose that T is an epimorphic image of the form $C_{B,H,a_1}/J$, for some non-trivial ideal J. Therefore $C_{B,H,a_1}/J$ is a non-trivial indecomposable cycle set of finite primitive level, with $|C_{B,H,a_1}/J| = o_j > p$, and this implies that B^2 does not act transitively on the J-orbits of B/K, but this contradicts (3).

Finally, by Proposition 14 every non-trivial indecomposable cycle set X having primitive level 2 can be constructed as C_{B,K,a_1} for suitable B, K, and a_1 satisfying conditions (1), (2), and (3). □

We conclude the section applying Theorem 41 to construct a family of indecomposable cycle sets having primitive level 2.

Example 42. Let B_1 be the left brace $B_{8,27}$ of [**33**] and B_2 the trivial left brace having p elements, for a prime number p different from 2, and set B the direct product of the left braces B_1 and B_2. Then, B has a transitive cycle base $Y = Y_1 \times \{y\}$, where Y_1 is a transitive cycle base of B_1, which has size 4, and y is a non-zero element of B_2. Moreover, every element a of Y_1 is stabilized by a core-free subgroup K'_a of $(B_1, \circ)$ having size 2. Therefore, if we set $a_1 \in Y$ and $K := K'_{a_1} \times \{0\}$, we obtain that C_{B,K,a_1} is an indecomposable cycle set having size $4p$. Now we show that it is of primitive level 2. If $x \circ K$ is a left coset of B/K, we obtain that $B^2 \circ B_{x\circ K}$ is equal to $B_1 \times \{0\}$, which is a subgroup of $(B, \circ)$ of index p, therefore condition (1) of Theorem 41 is satisfied. Since $(B_1, \circ)$ is the dihedral group of size 8 and $(B_2, \circ)$ is cyclic of prime order $p \neq 2$, condition (2) of Theorem 41 automatically follows. Finally, the ideals of B different from $\{0\}$ and B are: $B_1 \times \{0\}$, $\{0\} \times B_2$, $B_1^2 \times \{0\}$, $B_1^2 \times B_2$. We do not need to consider $B_1^2 \times B_2$, since it acts transitively on C_{B,K,a_1}. The ideals $B_1 \times \{0\}$ and $B_1^2 \times \{0\}$ split C_{B,K,a_1} in the same way into p orbits, hence the remaining case is the ideal $\{0\} \times B_2$. It splits C_{B,K,a_1} into four orbits, and $B^2 = B_1^2 \times \{0\}$ acts transitively on these orbits, therefore condition (3) of Theorem 41 also follows and hence C_{B,K,a_1} has primitive level 2.

Acknowledgements

The author thanks M. Bonatto for the discussion about latin cycle sets.

References

[1] E. Acri, R. Lutowski, and L. Vendramin, Retractability of solutions to the Yang–Baxter equation and p-nilpotency of skew braces, *Internat. J. Algebra Comput.* **30(1)** (2020), 91–115. DOI: 10.1142/S0218196719500656.

[2] D. Bachiller, F. Cedó, and E. Jespers, Solutions of the Yang–Baxter equation associated with a left brace, *J. Algebra* **463** (2016), 80–102. DOI: 10.1016/j.jalgebra.2016.05.024.

[3] D. Bachiller, F. Cedó, E. Jespers, and J. Okniński, A family of irretractable square-free solutions of the Yang–Baxter equation, *Forum Math.* **29(6)** (2017), 1291–1306. DOI: 10.1515/forum-2015-0240.

[4] A. Ballester-Bolinches, R. Esteban-Romero, P. Jiménez-Seral, and V. Pérez-Calabuig, Soluble skew left braces and soluble solutions of the Yang–Baxter equation, Preprint (2023). arXiv:2304.13475.

[5] M. Bonatto, M. Kinyon, D. Stanovský, and P. Vojtěchovský, Involutive Latin solutions of the Yang–Baxter equation, *J. Algebra* **565** (2021), 128–159. DOI: 10.1016/j.jalgebra.2020.09.001.

[6] M. Castelli, A characterization of finite simple set-theoretic solutions of the Yang–Baxter equation, *Proc. Amer. Math. Soc.* **151(12)** (2023), 5047–5057. DOI: 10.1090/proc/16329.

[7] M. Castelli, F. Catino, and G. Pinto, Indecomposable involutive set-theoretic solutions of the Yang–Baxter equation, *J. Pure Appl. Algebra* **223(10)** (2019), 4477–4493. DOI: 10.1016/j.jpaa.2019.01.017.

[8] M. Castelli, M. Mazzotta, and P. Stefanelli, Simplicity and finite primitive level of indecomposable set-theoretic solutions of the Yang–Baxter equation, Preprint (2021). arXiv:2107.11104.

[9] M. Castelli, M. Mazzotta, and P. Stefanelli, Simplicity of indecomposable set-theoretic solutions of the Yang–Baxter equation, *Forum Math.* **34(2)** (2022), 531–546. DOI: 10.1515/forum-2021-0189.

[10] M. Castelli, G. Pinto, and W. Rump, On the indecomposable involutive set-theoretic solutions of the Yang–Baxter equation of prime-power size, *Comm. Algebra* **48(5)** (2020), 1941–1955. DOI: 10.1080/00927872.2019.1710163.

[11] F. Cedó, E. Jespers, and J. Okniński, Braces and the Yang–Baxter equation, *Comm. Math. Phys.* **327(1)** (2014), 101–116. DOI: 10.1007/s00220-014-1935-y.

[12] F. Cedó, E. Jespers, and J. Okniński, Primitive set-theoretic solutions of the Yang–Baxter equation, *Commun. Contemp. Math.* **24(9)** (2022), Paper no. 2150105, 10 pp. DOI: 10.1142/S0219199721501054.

[13] F. Cedó and J. Okniński, Constructing finite simple solutions of the Yang–Baxter equation, *Adv. Math.* **391** (2021), Paper no. 107968, 40 pp. DOI: 10.1016/j.aim.2021.107968.

[14] F. Cedó and J. Okniński, New simple solutions of the Yang–Baxter equation and solutions associated to simple left braces, *J. Algebra* **600** (2022), 125–151. DOI: 10.1016/j.jalgebra.2022.02.011.

[15] F. Cedó and J. Okniński, Indecomposable solutions of the Yang–Baxter equation of square-free cardinality, *Adv. Math.* **430** (2023), Paper no. 109221, 26 pp. DOI: 10.1016/j.aim.2023.109221.

[16] F. Chouraqui, Garside groups and Yang–Baxter equation, *Comm. Algebra* **38(12)** (2010), 4441–4460. DOI: 10.1080/00927870903386502.

[17] V. G. Drinfeld, On some unsolved problems in quantum group theory, in: *Quantum Groups* (Leningrad, 1990), Lecture Notes in Math. **1510**, Springer-Verlag, Berlin, 1992, pp. 1–8. DOI: 10.1007/BFb0101175.

[18] D. S. Dummit and R. M. Foote, *Abstract Algebra*, Third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[19] P. Etingof, T. Schedler, and A. Soloviev, Set-theoretical solutions to the quantum Yang–Baxter equation, *Duke Math. J.* **100(2)** (1999), 169–209. DOI: 10.1215/S0012-7094-99-10007-X.

[20] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, *J. Algebra* **206(1)** (1998), 97–112. DOI: 10.1006/jabr.1997.7399.

[21] P. Jedlička and A. Pilitowska, Indecomposable involutive solutions of the Yang–Baxter equation of multipermutation level 2 with non-abelian permutation group, *J. Combin. Theory Ser. A* **197** (2023), Paper no. 105753, 35 pp. DOI: 10.1016/j.jcta.2023.105753.

[22] P. Jedlička, A. Pilitowska, and A. Zamojska-Dzienio, Indecomposable involutive solutions of the Yang–Baxter equation of multipermutational level 2 with abelian permutation group, *Forum Math.* **33(5)** (2021), 1083–1096. DOI: 10.1515/forum-2021-0130.

[23] P. Jedlička, A. Pilitowska, and A. Zamojska-Dzienio, Cocyclic braces and indecomposable cocyclic solutions of the Yang–Baxter equation, *Proc. Amer. Math. Soc.* **150(10)** (2022), 4223–4239. DOI: 10.1090/proc/15962.

[24] S. Ramírez and L. Vendramin, Decomposition theorems for involutive solutions to the Yang–Baxter equation, *Int. Math. Res. Not. IMRN* **2022(22)** (2022), 18078–18091. DOI: 10.1093/imrn/rnab232.

[25] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang–Baxter equation, *Adv. Math.* **193(1)** (2005), 40–55. DOI: 10.1016/j.aim.2004.03.019.

[26] W. Rump, Braces, radical rings, and the quantum Yang–Baxter equation, *J. Algebra* **307(1)** (2007), 153–170. DOI: 10.1016/j.jalgebra.2006.03.040.

[27] W. Rump, Classification of cyclic braces, II, *Trans. Amer. Math. Soc.* **372(1)** (2019), 305–328. DOI: 10.1090/tran/7569.

[28] W. Rump, Classification of indecomposable involutive set-theoretic solutions to the Yang–Baxter equation, *Forum Math.* **32(4)** (2020), 891–903. DOI: 10.1515/forum-2019-0274.

[29] W. Rump, One-generator braces and indecomposable set-theoretic solutions to the Yang–Baxter equation, *Proc. Edinb. Math. Soc. (2)* **63(3)** (2020), 676–696. DOI: 10.1017/s0013091520000073.

[30] W. Rump, Classification of non-degenerate involutive set-theoretic solutions to the Yang–Baxter equation with multipermutation level two, *Algebr. Represent. Theory* **25(5)** (2022), 1293–1307. DOI: 10.1007/s10468-021-10067-5.

[31] W. Rump, Primes in coverings of indecomposable involutive set-theoretic solutions to the Yang–Baxter equation, *Bull. Belg. Math. Soc. Simon Stevin* **30(2)** (2023), 260–280. DOI: 10.36045/j.bbms.230429.

[32] L. Vendramin, Extensions of set-theoretic solutions of the Yang–Baxter equation and a conjecture of Gateva-Ivanova, *J. Pure Appl. Algebra* **220(5)** (2016), 2064–2076. DOI: 10.1016/j.jpaa.2015.10.018.

[33] L. Vendramin and O. Konovalov, YangBaxter, Combinatorial Solutions for the Yang–Baxter equation, Version 0.10.6 (2024) (GAP package), https://gap-packages.github.io/YangBaxter.

[34] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.* **19** (1967), 1312–1315. DOI: 10.1103/PhysRevLett.19.1312.

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Received on December 19, 2023.

Accepted on May 10, 2024.