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A NON-TRIVIAL VARIANT OF HILBERT'S INEQUALITY, AND AN APPLICATION TO THE NORM OF THE HILBERT MATRIX ON THE HARDY–LITTLEWOOD SPACES

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Abstract: Hilbert's inequality for non-negative sequences states that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m{}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n{}^q\right)^{\frac{1}{q}},$$

where $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. This implies that the norm of the Hilbert matrix as an operator on the sequence space ℓ^p equals $\frac{\pi}{\sin\frac{\pi}{p}}$.

In this article we prove the non-trivial variant

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}$$

of Hilbert's inequality, and we use it to prove that the norm of the Hilbert matrix as an operator on the Hardy–Littlewood space K^p equals $\frac{\pi}{\sin\frac{\pi}{p}}$, where K^p consists of all functions $f(z)=\sum_{m=0}^{\infty} a_m z^m$ analytic in the unit disk with $\|f\|_{K^p}^p=\sum_{m=0}^{\infty}(m+1)^{p-2}|a_m|^p<\infty$. We also see that $\frac{\pi}{\sin\frac{\pi}{p}}$ is the norm of the Hilbert matrix on the space ℓ_{p-2}^p of sequences (a_m) with $\|(a_m)\|_{\ell_{p-2}^p}^p=\sum_{m=1}^{\infty} m^{p-2}|a_m|^p<\infty$.

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1. Preliminaries

The Hilbert matrix is the infinite matrix whose entries are

$$\frac{1}{m+n-1}, \quad n,m=1,2\dots$$

The well known Hilbert's inequality ([**8**, Theorem 323]; see also [**8**, Theorem 315] for a weaker inequality) states that if (a_m), (b_n) are sequences of non-negative terms such that $(a_m)\in\ell^p$, $(b_n)\in\ell^q$, then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m{}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n{}^q\right)^{\frac{1}{q}},$$

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where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the smallest possible for this inequality. This implies that the Hilbert matrix induces a bounded operator $\mathscr{H}$,

$$\mathscr{H}\colon (a_m) \longmapsto \mathscr{H}(a_m) = \sum_{m=1}^{\infty} \frac{a_m}{m+n-1}$$

on the spaces ℓ^p, $1 < p < \infty$, with norm

$$\|\mathscr{H}\|_{\ell^p\to\ell^p} = \frac{\pi}{\sin\frac{\pi}{p}}.$$

The operator $\mathscr{H}$ can also be considered as an operator on spaces of analytic functions by its action on the sequence of Taylor coefficients of any such function.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$.

The Hardy space H^p, $0 < p < \infty$, consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0\le r<1} M_p(r, f) < \infty,$$

where $M_p^p(r, f)$ are the integral means

$$M_p^p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta.$$

If $p \ge 1$, then H^p is a Banach space under the norm $\|\cdot\|_{H^p}$. If $0 < p < 1$, then H^p is a complete metric space.

For $f(z) = \sum\limits_{m=0}^{\infty} a_m z^m \in H^1$, Hardy's inequality ([**6**, p. 48])

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} \le \pi\|f\|_{H^1}$$

implies that the power series

$$\mathscr{H}(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right) z^n$$

has bounded coefficients. Therefore $\mathscr{H}(f)$ is an analytic function of the unit disk for any $f \in H^1$ and hence for any $f \in H^p$, $p \ge 1$.

The Bergman space A^p, $0 < p < \infty$, consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p\, dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$. If $p \ge 1$, then A^p is a Banach space under the norm $\|\cdot\|_{A^p}$.

If $f(z) = \sum\limits_{m=0}^{\infty} a_m z^m \in A^p$ and $p > 2$, then by [**10**, Lemma 4.1] we have

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} < \infty.$$

Thus $\mathscr{H}(f)$ is an analytic function in $\mathbb{D}$ for each function $f \in A^p$, $p > 2$.

E. Diamantopoulos and A. G. Siskakis initiated the study of the Hilbert matrix as an operator on Hardy and Bergman spaces in [**3, 4**] and showed that $\mathscr{H}(f)$ has the following integral representation:

$$\mathscr{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz}\,dt, \quad z \in \mathbb{D}.$$

Then, considering $\mathscr{H}$ as an average of weighted composition operators, they showed that it is a bounded operator on H^p, $p>1$, and on A^p, $p>2$, and they estimated its norm. Their study was further extended by M. Dostanić, M. Jevtić, and D. Vukotić in [**5**] and by V. Božin and B. Karapetrović in [**1**] (see also [**9**]). Summarizing their results, we now know that

$$\|\mathscr{H}\|_{H^p\to H^p} = \|\mathscr{H}\|_{A^{2p}\to A^{2p}} = \frac{\pi}{\sin\frac{\pi}{p}}, \quad 1<p<\infty.$$

The Hardy–Littlewood space K^p, $0<p<\infty$, is defined as the space of all $f(z) = \sum_{m=0}^{\infty} a_m z^m \in H(\mathbb{D})$ such that

$$\|f\|_{K^p}^p = \sum_{m=0}^{\infty}(m+1)^{p-2}|a_m|^p < \infty.$$

If $p \geq 1$, then K^p is a Banach space under the norm $\|\cdot\|_{K^p}$.

According to the classical Hardy–Littlewood inequalities, [**7**, Theorems 5 and 6], [**6**, Theorems 6.2 and 6.3], if $f(z) = \sum_{m=0}^{\infty} a_m z^m \in H^p$, $0<p\leq 2$, then

$$\sum_{m=0}^{\infty}(m+1)^{p-2}|a_m|^p \leq c_p\|f\|_{H^p}^p$$

and hence $f\in K^p$. Also, if $2\leq p<\infty$ and $f(z) = \sum_{m=0}^{\infty} a_m z^m \in K^p$, then

$$\|f\|_{H^p}^p \leq c_p \sum_{m=0}^{\infty}(m+1)^{p-2}|a_m|^p$$

and hence $f\in H^p$. In both cases c_p is a constant independent of f.

If $p\geq 1$, and in the special case where the sequence (a_m) is real and decreasing to zero, then for $f(z) = \sum_{m=0}^{\infty} a_m z^m$ we have that $f\in H^p$ if and only if $f \in K^p$ [**11**, Theorems A and 1.1].

Now it is clear that the proper domain of definition of the operator $\mathscr{H}$ acting on analytic functions in the unit disk is the space K^1. Indeed, if $f(z) = \sum_{m=0}^{\infty} a_m z^m \in K^1$, then

$$\sum_{m=0}^{\infty}\frac{|a_m|}{m+1} < \infty$$

and hence $\mathscr{H}(f)\in H(\mathbb{D})$.

Moreover, when $1 < p < \infty$ and $f \in K^p$, we consider q so that $\frac{1}{p} + \frac{1}{q} = 1$ and we apply Hölder's inequality to find

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} = \sum_{m=0}^{\infty} (m+1)^{\frac{2}{p}-2}(m+1)^{1-\frac{2}{p}}|a_m|$$

$$\leq \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^2}\right)^{\frac{1}{q}} \left(\sum_{m=0}^{\infty} (m+1)^{p-2}|a_m|^p\right)^{\frac{1}{p}} < \infty.$$

Hence $K^p \subseteq K^1$ and so, if $f \in K^p$, then $\mathscr{H}(f)$ defines an analytic function in $\mathbb{D}$.

Recently, in [**12**, Theorem 1] (see also [**2**]), the boundedness of the generalized Volterra operators

$$T_g(f)(z) = \int_0^z f(w)g'(w)\,dw, \quad z \in \mathbb{D},$$

induced by symbols $g \in H(\mathbb{D})$ with non-negative Taylor coefficients and acting from a space X to H^∞, was associated to the K^q-norm of the function $\mathscr{H}(g')$. In this result X can be H^p or K^p or the Dirichlet-type space D^p_{p-1}.

2. A variant of Hilbert's inequality

Our first result is a non-trivial variant of the classical Hilbert's inequality.

Before we state our first main result we shall mention two more variants of Hilbert's inequality. The first, in [**13**], is

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}$$

and the second, in [**14**], is

$$\sum_{m,n=1}^{\infty} \left(\frac{n-\frac{1}{2}}{m-\frac{1}{2}}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}.$$

In fact Yang proves a whole family of such inequalities depending on a parameter. In all these variants, as well as in the original Hilbert's inequality, the kernel involved in the double sum is of the form

$$\left(\frac{k(n)}{k(m)}\right)^{c_p} \frac{1}{(Ak(m)+Bk(n))^\lambda}$$

which is homogeneous of degree $-\lambda$. As a consequence, in order to prove these variants one needs to apply the standard arguments used in the proof of the original Hilbert's inequality. The kernel

$$\left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{1}{m+n-1}$$

in our variant of Hilbert's inequality, which appears in the following Theorem 1, lacks any homegeneity and the standard arguments do not apply.

Theorem 1. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $(a_m) \in \ell^p$, $(b_n) \in \ell^q$ are sequences of non-negative terms, then*

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}.$$

The constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the smallest possible for this inequality.

Proof: It is sufficient to consider the case $1 < q \le 2 \le p < \infty$.

We assume

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1, \quad \alpha \ge 0,\ \beta \ge 0,$$

where α, β will be chosen appropriately later; the choice of α, β will depend on p.

By Hölder's inequality,

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1}$$

$$= \sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{(\frac{1}{pq}-\frac{1}{p})+(\frac{1}{q}-\frac{1}{pq})} \frac{a_m b_n}{(m+n)^{\frac{1}{p}}(m+n)^{\frac{1}{q}}} \left(\frac{m+n}{m+n-1}\right)^{\frac{\alpha}{p}} \left(\frac{m+n}{m+n-1}\right)^{\frac{\beta}{q}}$$

$$\le \left(\sum_{m=1}^{\infty} a_m^p \left(\sum_{n=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{1}{p}} \frac{1}{(m+n)^{1-\alpha}(m+n-1)^{\alpha}}\right)\right)^{\frac{1}{p}}$$

$$\times \left(\sum_{n=1}^{\infty} b_n^q \left(\sum_{m=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}} \frac{1}{(m+n)^{1-\beta}(m+n-1)^{\beta}}\right)\right)^{\frac{1}{q}}.$$

Hence it is enough to prove

$$\sum_{n=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{1}{p}} \frac{1}{(m+n)^{1-\alpha}(m+n-1)^{\alpha}} \le \frac{\pi}{\sin\frac{\pi}{p}}, \quad m \ge 1, \tag{2.1}$$

and

$$\sum_{m=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}} \frac{1}{(m+n)^{1-\beta}(m+n-1)^{\beta}} \le \frac{\pi}{\sin\frac{\pi}{q}}, \quad n \ge 1, \tag{2.2}$$

where, of course, $\sin\frac{\pi}{p} = \sin\frac{\pi}{q}$.

Now we observe that, for all $\alpha \ge 0$, $p > 0$, $m \ge 1$, the positive function

$$f(t) = t^{-\frac{1}{p}}(m+t)^{\alpha-1}(m+t-1)^{-\alpha}, \quad t > 0,$$

is convex. Indeed, taking the second derivative of the logarithm of $f(t)$, we get

$$\frac{f(t)f''(t) - f'(t)^2}{f(t)^2} = \frac{t^{-2}}{p} + (m+t)^{-2} + \alpha((m+t-1)^{-2} - (m+t)^{-2}) > 0,$$

which proves that $f''(t) > 0$. In fact, this calculation proves more: that f is logarithmically convex.

The convexity of f implies

$$f(n) \le \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)\,dt, \quad n \ge 1.$$

Adding these inequalities we get for the left-hand side of (2.1) that

$$\sum_{n=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{1}{p}} \frac{1}{(m+n)^{1-\alpha}(m+n-1)^{\alpha}} \le \int_{\frac{1}{2}}^{\infty} \left(\frac{m}{t}\right)^{\frac{1}{p}} \frac{1}{(m+t)^{1-\alpha}(m+t-1)^{\alpha}}\,dt$$

$$= \int_{\frac{1}{2m}}^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)^{1-\alpha}(t+1-\frac{1}{m})^{\alpha}}\,dt$$

by the change of variables $t \mapsto mt$.

Therefore, in order to prove (2.1) it is enough to prove

$$\int_{\frac{1}{2m}}^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)^{1-\alpha}(t+1-\frac{1}{m})^{\alpha}}\,dt \le \frac{\pi}{\sin\frac{\pi}{p}}, \quad m \ge 1. \tag{2.3}$$

We now consider the function

$$\begin{aligned} F(y) &= \int_{y}^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)^{1-\alpha}(t+1-2y)^{\alpha}}\,dt \\ &= \int_{0}^{\infty} \frac{1}{(t+y)^{\frac{1}{p}}(t+1+y)^{1-\alpha}(t+1-y)^{\alpha}}\,dt, \quad 0 \le y \le \frac{1}{2}. \end{aligned}$$

Hence in order to prove (2.3) it is enough to prove

$$F(y) \le \frac{\pi}{\sin\frac{\pi}{p}}, \quad 0 \le y \le \frac{1}{2}. \tag{2.4}$$

Now, exactly as before, we observe that, for all $\alpha \ge 0$, $p > 0$, $t > 0$, the positive function

$$g_t(y) = (t+y)^{-\frac{1}{p}}(t+1+y)^{\alpha-1}(t+1-y)^{-\alpha}, \quad 0 \le y \le \frac{1}{2},$$

is convex. Indeed, we take the second derivative of the logarithm of $g_t(y)$ and we get

$$\frac{g_t(y)g_t''(y) - g_t'(y)^2}{g_t(y)^2} = \frac{(t+y)^{-2}}{p} + (t+1+y)^{-2} + \alpha((t+1-y)^{-2} - (t+1+y)^{-2}) > 0,$$

which proves that $g_t''(y) > 0$.

Thus $F(y) = \int_0^\infty g_t(y)\,dt$ is also convex and, as such, it satisfies

$$F(y) \le \max\left\{F(0), F\left(\frac{1}{2}\right)\right\}.$$

Since

$$F(0) = \int_0^\infty \frac{1}{t^{\frac{1}{p}}(t+1)}\,dt = \frac{\pi}{\sin\frac{\pi}{p}},$$

in order to prove (2.4) it is enough to prove

$$F\left(\frac{1}{2}\right) \le \frac{\pi}{\sin\frac{\pi}{p}}.$$

Since

$$F\left(\frac{1}{2}\right) = \int_{1/2}^\infty \frac{(t+1)^{\alpha-1}}{t^{\frac{1}{p}+\alpha}}\,dt = \int_0^2 \frac{(t+1)^{\alpha}}{t^{1-\frac{1}{p}}(t+1)}\,dt$$

after the change of variables $t \mapsto \frac{1}{t}$, we conclude that in order to prove (2.1) it is enough to prove

$$\int_0^2 \frac{(t+1)^{\alpha}}{t^{1-\frac{1}{p}}(t+1)}\,dt \le \frac{\pi}{\sin\frac{\pi}{p}}.$$

In exactly the same manner, we see that in order to prove (2.2) it is enough to prove

$$\int_0^2 \frac{(t+1)^{\beta}}{t^{1-\frac{1}{q}}(t+1)}\,dt \le \frac{\pi}{\sin\frac{\pi}{q}}.$$

We make the change of notation

$$x = \frac{1}{p}, \quad 1 - x = \frac{1}{q},$$

and, after $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, we write

$$\beta = \frac{1 - \alpha x}{1 - x},$$

where $0 \le \alpha x \le 1$. Then our last two inequalities become

$$\int_0^2 \frac{(t+1)^\alpha}{t^{1-x}(t+1)}\, dt \le \frac{\pi}{\sin \pi x} = \int_0^\infty \frac{1}{t^{1-x}(t+1)}\, dt \tag{2.5}$$

and

$$\int_0^2 \frac{(t+1)^{\frac{1-\alpha x}{1-x}}}{t^x(t+1)}\, dt \le \frac{\pi}{\sin \pi x} = \int_0^\infty \frac{1}{t^x(t+1)}\, dt. \tag{2.6}$$

Now, inequality (2.5) is equivalent to

$$\int_0^2 \frac{(t+1)^\alpha - 1}{t^{1-x}(t+1)}\, dt \le \int_2^\infty \frac{1}{t^{1-x}(t+1)}\, dt$$

or, after the change of variables $t \mapsto 2t$, to

$$\int_0^1 \frac{(2t+1)^\alpha - 1}{t^{1-x}(2t+1)}\, dt \le \int_1^\infty \frac{1}{t^{1-x}(2t+1)}\, dt,$$

or finally, substituting $t \mapsto \frac{1}{t}$ in the left-hand integral, to the inequality

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^\alpha - 1}{t^x(t+2)}\, dt \le \int_1^\infty \frac{1}{t^{1-x}(2t+1)}\, dt, \quad 0 < x \le \frac{1}{2}. \tag{2.7}$$

Similarly, inequality (2.6) is equivalent to

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^{\frac{1-\alpha x}{1-x}} - 1}{t^{1-x}(t+2)}\, dt \le \int_1^\infty \frac{1}{t^x(2t+1)}\, dt, \quad 0 < x \le \frac{1}{2}. \tag{2.8}$$

So we have come to the point where, for every x with $0 < x \le \frac{1}{2}$, we have to prove inequalities (2.7) and (2.8) for a proper choice of α with $0 \le \alpha \le \frac{1}{x}$.

A very useful observation for what follows is that for fixed α with $0 \le \alpha \le 1$, if (2.7) holds for some x, then it holds for all larger x. The reason is that the left-hand side in (2.7) is a decreasing function of x and the right-hand side in (2.7) is an increasing function of x. Similarly, if (2.8) holds for some x, then it holds for all smaller x. It helps to see that for fixed α with $0 \le \alpha \le 1$ the function $\frac{1-\alpha x}{1-x}$ is increasing.

Now we split the interval $0 < x \le \frac{1}{2}$ into three subintervals in each of which we make the corresponding choices $\alpha = 0$, $\alpha = 1$, and $\alpha = \frac{1}{2}$.

The case $0 < x \le \frac{1}{3}$.

Let $\alpha = 0$. First of all, it is obvious that (2.7) is true for all $0 < x \le \frac{1}{2}$. We claim that (2.8) is valid for all $0 < x \le \frac{1}{3}$ and, as we have observed, it is enough to prove it for $x = \frac{1}{3}$.

Observe now that $0 < x \le \frac{1}{2}$ implies $0 < \frac{x}{1-x} \le 1$, so by Bernoulli's inequality we get

$$\left(1+\frac{2}{t}\right)^{\frac{1}{1-x}} = \left(1+\frac{2}{t}\right)\left(1+\frac{2}{t}\right)^{\frac{x}{1-x}} \le \left(1+\frac{2}{t}\right)\left(1+\frac{x}{1-x}\frac{2}{t}\right)$$
$$= 1+\frac{2}{t}+\frac{x}{1-x}\frac{2(t+2)}{t^2}.$$

Hence

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^{\frac{1}{1-x}}-1}{t^{1-x}(t+2)}\,dt \le \int_1^\infty \frac{2}{t^{2-x}(t+2)}\,dt + \frac{2x}{1-x}\int_1^\infty \frac{1}{t^{3-x}}\,dt.$$

Using

$$\frac{2}{t(t+2)} = \frac{1}{t} - \frac{1}{t+2} \tag{2.9}$$

the last inequality becomes

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^{\frac{1}{1-x}}-1}{t^{1-x}(t+2)}\,dt \le \int_1^\infty \frac{1}{t^{2-x}}\,dt - \int_1^\infty \frac{1}{t^{1-x}(t+2)}\,dt + \frac{2x}{(1-x)(2-x)}$$
$$= \frac{2+x}{(1-x)(2-x)} - \int_1^\infty \frac{1}{t^{1-x}(t+2)}\,dt.$$

Hence in order to prove (2.8) it is enough to have

$$\frac{2+x}{(1-x)(2-x)} \le \int_1^\infty \frac{1}{t^{1-x}(t+2)}\,dt + \int_1^\infty \frac{1}{t^x(2t+1)}\,dt$$
$$= \int_0^1 \frac{1}{t^x(2t+1)}\,dt + \int_1^\infty \frac{1}{t^x(2t+1)}\,dt = \int_0^\infty \frac{1}{t^x(2t+1)}\,dt$$
$$= 2^{x-1}\int_0^\infty \frac{1}{t^x(t+1)}\,dt = 2^{x-1}\frac{\pi}{\sin \pi x}.$$

For $x = \frac{1}{3}$ this becomes $\frac{21}{10} \le \frac{2^{\frac{1}{3}}\pi}{\sqrt{3}}$, which is true and proves our claim. We proved that when $\alpha = 0$ both (2.7) and (2.8) hold for $0 < x \le \frac{1}{3}$.

The case $\frac{2}{5} \le x \le \frac{1}{2}$.

Let $\alpha = 1$. In this case (2.7) becomes

$$\int_1^\infty \frac{2}{t^{1+x}(t+2)}\,dt \le \int_1^\infty \frac{1}{t^{1-x}(2t+1)}\,dt. \tag{2.10}$$

We claim that this inequality is true for $\frac{2}{5} \le x \le \frac{1}{2}$ and it suffices to prove it for $x = \frac{2}{5}$.

Using (2.9), the left-hand side of (2.10) becomes

$$\int_1^\infty \frac{2}{t^{1+x}(t+2)}\,dt = \int_1^\infty \frac{1}{t^{1+x}}\,dt - \int_1^\infty \frac{1}{t^x(t+2)}\,dt$$
$$= \frac{1}{x} - \int_1^\infty \frac{1}{t^x(t+2)}\,dt.$$

Therefore, (2.10) amounts to showing the inequality

$$\frac{1}{x} \leq \int_1^\infty \frac{1}{t^x(t+2)}\,dt + \int_1^\infty \frac{1}{t^{1-x}(2t+1)}\,dt = \int_0^\infty \frac{1}{t^x(t+2)}\,dt$$
$$= 2^{-x}\int_0^\infty \frac{1}{t^x(t+1)}\,dt = 2^{-x}\frac{\pi}{\sin(\pi x)}$$

for $x = \frac{2}{5}$. But the inequality $\frac{\sin \pi x}{\pi x} \leq 2^{-x}$ can be easily proved for $x = \frac{2}{5}$ using the first three non-zero terms of the Taylor expansion of $\frac{\sin \pi x}{\pi x}$ and doing a few straightforward calculations. Thus, (2.7) is valid for $\frac{2}{5} \leq x \leq \frac{1}{2}$.

We now turn to (2.8), and we claim that it holds for $0 < x \leq \frac{1}{2}$ and it suffices to prove it for $x = \frac{1}{2}$. When $\alpha = 1$, (2.8) becomes

$$\int_1^\infty \frac{2}{t^{2-x}(t+2)}\,dt \leq \int_1^\infty \frac{1}{t^x(2t+1)}\,dt$$

or, by the use of (2.9),

$$\int_1^\infty \frac{1}{t^{2-x}}\,dt - \int_1^\infty \frac{1}{t^{1-x}(t+2)}\,dt \leq \int_1^\infty \frac{1}{t^x(2t+1)}\,dt.$$

This is equivalent to

$$\frac{1}{1-x} \leq \int_1^\infty \frac{1}{t^{1-x}(t+2)}\,dt + \int_1^\infty \frac{1}{t^x(2t+1)}\,dt = \int_0^\infty \frac{1}{t^{1-x}(t+2)}\,dt = 2^{x-1}\frac{\pi}{\sin \pi x}.$$

When $x = \frac{1}{2}$ this becomes $2\sqrt{2} \leq \pi$ and it is clearly true. We proved that when $\alpha = 1$ both (2.7) and (2.8) hold for $\frac{2}{5} \leq x \leq \frac{1}{2}$.

The case $\frac{1}{3} \leq x \leq \frac{2}{5}$.

Let $\alpha = \frac{1}{2}$. We first deal with inequality (2.7), which we shall prove for $\frac{1}{3} \leq x \leq \frac{2}{5}$. As we know, it is enough to prove it for $x = \frac{1}{3}$. When $\alpha = \frac{1}{2}$, (2.7) becomes

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^{\frac{1}{2}} - 1}{t^x(t+2)}\,dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)}\,dt.$$

Bernoulli's inequality gives

$$\left(1+\frac{2}{t}\right)^{\frac{1}{2}} \leq 1 + \frac{1}{2}\frac{2}{t} = 1 + \frac{1}{t}$$

and in view of (2.7) it suffices to show that

$$\int_1^\infty \frac{1}{t^{1+x}(t+2)}\,dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)}\,dt$$

for $x = \frac{1}{3}$. This is indeed true, since

$$t^{\frac{2}{3}}(2t+1) \leq t^{\frac{4}{3}}(t+2), \quad t \geq 1,$$

as we easily see by raising to the third power.

We now turn to (2.8), which for $\alpha = \frac{1}{2}$ becomes

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^{\frac{1}{2}\frac{2-x}{1-x}} - 1}{t^{1-x}(t+2)}\,dt \le \int_1^\infty \frac{1}{t^x(2t+1)}\,dt,$$

and we claim it holds for $\frac{1}{3} \le x \le \frac{2}{5}$. Again it suffices to prove this inequality for $x = \frac{2}{5}$. Namely, it suffices to show

$$\int_1^\infty \frac{\left(1+\frac{2}{t}\right)^{\frac{4}{3}} - 1}{t^{\frac{3}{5}}(t+2)}\,dt \le \int_1^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)}\,dt. \tag{2.11}$$

Taking into account Bernoulli's inequality, we have

$$\left(1+\frac{2}{t}\right)^{\frac{4}{3}} = \left(1+\frac{2}{t}\right)\left(1+\frac{2}{t}\right)^{\frac{1}{3}} \le \left(1+\frac{2}{t}\right)\left(1+\frac{1}{3}\frac{2}{t}\right) = 1 + \frac{4}{3t^2}(2t+1),$$

so instead of (2.11), it suffices to prove

$$\frac{4}{3}\int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}(t+2)}\,dt \le \int_1^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)}\,dt. \tag{2.12}$$

Observe that the left-hand side of (2.12), in view of (2.9), is equal to

$$\begin{aligned}
\frac{4}{3}\int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}(t+2)}\,dt &= \frac{2}{3}\int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}}\,dt - \frac{2}{3}\int_1^\infty \frac{2t+1}{t^{1+\frac{3}{5}}(t+2)}\,dt \\
&= \frac{4}{3}\int_1^\infty \frac{1}{t^{1+\frac{3}{5}}}\,dt + \frac{2}{3}\int_1^\infty \frac{1}{t^{2+\frac{3}{5}}}\,dt \\
&\quad - \frac{4}{3}\int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)}\,dt - \frac{2}{3}\int_1^\infty \frac{1}{t^{1+\frac{3}{5}}(t+2)}\,dt \\
&= \frac{20}{9} + \frac{5}{12} - \frac{4}{3}\int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)}\,dt \\
&\quad - \frac{1}{3}\int_1^\infty \frac{1}{t^{1+\frac{3}{5}}}\,dt + \frac{1}{3}\int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)}\,dt,
\end{aligned}$$

where we have used (2.9) for the last equality. Thus, altogether we have

$$\frac{4}{3}\int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}(t+2)}\,dt = \frac{25}{12} - \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)}\,dt.$$

Therefore, (2.12) is equivalent to the inequality

$$\frac{25}{12} \le \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)}\,dt + \int_1^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)}\,dt = \int_0^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)}\,dt = \frac{2^{-\frac{3}{5}}\pi}{\sin\frac{3\pi}{5}}.$$

This inequality is an easy consequence of the inequality $\frac{\sin\frac{2\pi}{5}}{\frac{2\pi}{5}} < 2^{-\frac{2}{5}}$, which we proved when we considered the case $\alpha = 1$, and of the equality $\sin\frac{3\pi}{5} = \sin\frac{2\pi}{5}$.

We proved that when $\alpha = \frac{1}{2}$ both (2.7) and (2.8) hold for $\frac{1}{3} \le x \le \frac{2}{5}$.

Therefore, we have proved the inequality of our theorem and now we shall show that the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible in this inequality. The proof follows the lines of Hardy's corresponding proof for the original Hilbert's inequality [**8**, proof of Theorem 317, p. 232], adapted to our weighted setting. For the sake of completeness, we provide the details.

We consider any $\epsilon > 0$ and the sequences $(a_m(\epsilon))$ and $(b_n(\epsilon))$ defined by

$$a_m(\epsilon) = m^{-\frac{1+\epsilon}{p}}, \quad b_n(\epsilon) = n^{-\frac{1+\epsilon}{q}}.$$

We then have

$$\|(a_m(\epsilon))\|_{\ell^p}^p = \sum_{m=1}^{\infty} \tfrac{1}{m^{1+\epsilon}}.$$

Now, since $\frac{1}{x^{1+\epsilon}}$ is decreasing for $x \geq 1$, we have

$$\frac{1}{\epsilon} = \int_1^\infty \frac{1}{x^{1+\epsilon}}\,dx \leq \sum_{m=1}^{\infty} \frac{1}{m^{1+\epsilon}} \leq 1 + \int_1^\infty \frac{1}{x^{1+\epsilon}}\,dx = 1 + \frac{1}{\epsilon}.$$

Setting $\phi(\epsilon) = \sum\limits_{m=1}^{\infty} \frac{1}{m^{1+\epsilon}} - \frac{1}{\epsilon}$, we get

$$\|(a_m(\epsilon))\|_{\ell^p}^p = \frac{1}{\epsilon} + \phi(\epsilon), \quad 0 \leq \phi(\epsilon) \leq 1. \tag{2.13}$$

Respectively, setting $\psi(\epsilon) = \sum\limits_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} - \frac{1}{\epsilon}$, we have

$$\|(b_n(\epsilon)\|_{\ell^q}^q = \frac{1}{\epsilon} + \psi(\epsilon), \quad 0 \leq \psi(\epsilon) \leq 1. \tag{2.14}$$

In addition, we have that

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n-1} \geq \sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n}. \tag{2.15}$$

Now for (x, y) in the square $[m, m+1) \times [n, n+1)$, $m \geq 1$, $n \geq 1$, we have

$$\begin{aligned}\left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n} &= \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{m^{-\frac{1+\epsilon}{p}} n^{-\frac{1+\epsilon}{q}}}{m+n} = \frac{m^{-\frac{1}{q}-\frac{\epsilon}{p}} n^{-\frac{1}{p}-\frac{\epsilon}{q}}}{m+n} \\ &\geq \frac{x^{-\frac{1}{q}-\frac{\epsilon}{p}} y^{-\frac{1}{p}-\frac{\epsilon}{q}}}{x+y} = \left(\frac{y}{x}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{x^{-\frac{1+\epsilon}{p}} y^{-\frac{1+\epsilon}{q}}}{x+y}.\end{aligned}$$

Therefore

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n} \geq I(\epsilon), \tag{2.16}$$

where $I(\epsilon)$ is defined by

$$I(\epsilon) = \int_1^\infty \int_1^\infty \left(\frac{y}{x}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{x^{-\frac{1+\epsilon}{p}} y^{-\frac{1+\epsilon}{q}}}{x+y}\,dx\,dy = \int_1^\infty \int_1^\infty \frac{x^{-\frac{1}{q}-\frac{\epsilon}{p}} y^{-\frac{1}{p}-\frac{\epsilon}{q}}}{x+y}\,dx\,dy.$$

Applying the change of variables $y \mapsto xy$, we get

$$I(\epsilon) = \int_1^\infty \frac{1}{x^{1+\epsilon}} \int_{\frac{1}{x}}^\infty \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)}\,dy\,dx.$$

Another change of variables $x \mapsto \frac{1}{x}$ gives

$$\begin{aligned}
I(\epsilon) &= \int_0^1 x^{\epsilon-1} \int_x^\infty \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)}\,dy\,dx \\
&= \int_0^1 \frac{1}{\epsilon}(x^\epsilon)' \int_x^\infty \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)}\,dy\,dx \\
&= \frac{1}{\epsilon}\left(\int_1^\infty \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)}\,dy + \int_0^1 \frac{1}{x^{\frac{1}{p}-\frac{\epsilon}{p}}(1+x)}\,dx\right)
\end{aligned}$$

by integration by parts. From this we notice that

$$\epsilon I(\epsilon) \to \int_0^\infty \frac{1}{t^{\frac{1}{p}}(1+t)}\,dt = \frac{\pi}{\sin\frac{\pi}{p}}$$

when $\epsilon \to 0^+$. This together with (2.13), (2.14), (2.15), and (2.16) implies

$$\frac{\sum_{m,n=1}^\infty \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n-1}}{\|(a_m(\epsilon))\|_{\ell^p}\|(b_n(\epsilon))\|_{\ell^q}} \geq \frac{\epsilon I(\epsilon)}{(1+\epsilon\,\phi(\epsilon))^{\frac{1}{p}}(1+\epsilon\,\psi(\epsilon))^{\frac{1}{q}}} \to \frac{\pi}{\sin\frac{\pi}{p}}$$

when $\epsilon \to 0^+$. □

3. The norm of the Hilbert matrix on the Hardy–Littlewood spaces and on weighted sequence spaces

Our second result is the determination of the exact value of the norm $\|\mathscr{H}\|_{K^p\to K^p}$ for $1 < p < \infty$. To that effect we shall use the variant of Hilbert's inequality in Theorem 1.

Theorem 2. *If $1 < p < \infty$, then the Hilbert matrix operator is bounded on the Hardy–Littlewood space K^p with norm*

$$\|\mathscr{H}\|_{K^p\to K^p} = \frac{\pi}{\sin\frac{\pi}{p}}.$$

Proof: Let $f(z) = \sum\limits_{m=0}^\infty a_m z^m \in K^p$. Then

$$\mathscr{H}(f)(z) = \sum_{n=0}^\infty \left(\sum_{m=0}^\infty \frac{a_m}{m+n+1}\right) z^n,$$

and

$$\begin{aligned}
\|\mathscr{H}(f)\|_{K^p} &\leq \left(\sum_{n=0}^\infty (n+1)^{p-2}\left(\sum_{m=0}^\infty \frac{|a_m|}{m+n+1}\right)^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{n=0}^\infty \left(\sum_{m=0}^\infty (n+1)^{\frac{p-2}{p}} \frac{|a_m|}{m+n+1}\right)^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Due to the duality of ℓ^p spaces,

$$\|\mathscr{H}(f)\|_{K^p} \leq \sup_{\|(b_n)\|_{\ell^q}=1,\, b_n\geq 0} \sum_{m,n=0}^\infty (n+1)^{\frac{p-2}{p}} \frac{|a_m|b_n}{m+n+1},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Setting $A_m = |a_m|(m+1)^{\frac{p-2}{p}}$, we have that $\|(A_m)\|_{\ell^p} = \|f\|_{K^p}$ and

$$\sup_{\|f\|_{K^p}=1} \|\mathscr{H}(f)\|_{K^p} \le \sup_{\substack{\|(A_m)\|_{\ell^p}=1,\\ \|(b_n)\|_{\ell^q}=1,\, b_n\ge 0}} \sum_{m,n=0}^{\infty} \left(\frac{n+1}{m+1}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{A_m b_n}{m+n+1} = \frac{\pi}{\sin\frac{\pi}{p}},$$

because of Theorem 1. This proves that $\|\mathscr{H}\|_{K^p\to K^p} \le \frac{\pi}{\sin\frac{\pi}{p}}$.

The equality $\|\mathscr{H}\|_{K^p\to K^p} = \frac{\pi}{\sin\frac{\pi}{p}}$ follows when we consider $f(z) = \sum_{m=0}^{\infty} a_m z^m \in K^p$ with non-negative coefficients a_m, since then all previous inequalities become equalities. □

One final remark is that the proof of Theorem 2 applies unchanged and in an obvious way to show that the Hilbert matrix $\mathscr{H}$ induces a bounded operator on the weighted space l^p_{p-2} of sequences (a_m) with norm defined by

$$\|(a_m)\|^p_{\ell^p_{p-2}} = \sum_{m=1}^{\infty} m^{p-2}|a_m|^p,$$

and that the norm $\|\mathscr{H}\|_{l^p_{p-2}\to l^p_{p-2}}$ of this operator is again equal to $\frac{\pi}{\sin\frac{\pi}{p}}$.

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