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TOPOLOGICAL DIMENSION ZERO AND SOME RELATED PROPERTIES

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Abstract: In this paper, we introduce and study the C*-algebras with property (IC) and with other related properties. We prove that, surprisingly, residual (IC) is equivalent to topological dimension zero (and to another property), and that in the class of C*-algebras with topological dimension zero, pure infiniteness and strong pure infiniteness coincide, providing a partial positive answer to a question of Kirchberg and Rørdam in [**12**]. We also show that these last two properties are equivalent to weak pure infiniteness and to local pure infiniteness, in the residual (IS) case, giving a particular affirmative answer to an open question of Blanchard and Kirchberg in [**2**]. We prove, in particular, that in the class of purely infinite C*-algebras, the following properties are all equivalent: residual (IC), topological dimension zero, the ideal property, the weak ideal property, residual (IF), and residual (SP). We show that crossed products by finite solvable groups preserve the class of all separable C*-algebras with topological dimension zero (resp., the weak ideal property).

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Key words: C*-algebra, residual (IC), topological dimension zero, (strongly, weakly, locally) purely infinite, the (weak) ideal property, crossed product.

1. Introduction

One of the most important breakthroughs in the famous Elliott classification program was the classification of simple purely infinite UCT nuclear C*-algebras, by Kirchberg and Phillips. In order to extend this classification program beyond simple C*-algebras, it is natural and very important to generalize pure infiniteness to non-simple C*-algebras. This was done by Kirchberg and Rørdam: they defined strong pure infiniteness, pure infiniteness, and weak pure infiniteness, and they asked whether these notions are equivalent (see [**11**] and [**12**]). This was proved to be true by Kirchberg and Rørdam when the C*-algebra has real rank zero (see [**12**]), and by the first named author of this paper and Rørdam when the C*-algebra has the ideal property (see [**24**]). Also, Blanchard and Kirchberg introduced the notion of local pure infiniteness and they asked whether it is equivalent to weak pure infiniteness (see [**2**]). They proved that these two concepts are equivalent when the C*-algebra has real rank zero (see [**2**]). It is worth mentioning that these types of questions are very important not only for the classification of C*-algebras, but also for other areas of mathematics, e.g., the classification of extensions.

In this paper we introduce several closely related properties, the most important being residual (IC), and also residual (IF) and residual (IS). Residual (IC) can be seen as a natural generalization of the ideal property (all the ideals are generated by their projections). Indeed, let A be a C*-algebra. If A has the ideal property, then any non-zero ideal-quotient of A (a quotient of different ideals) has a non-zero projection p, and hence (by a general argument), the ideal generated by p in the ideal-quotient

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has compact primitive spectrum. If we request that any non-zero ideal-quotient of a C*-algebra A contains a non-zero C*-subalgebra with compact (resp., finite) primitive spectrum, we say that A has residual (IC) (resp., residual (IF)), and if we request that any non-zero ideal-quotient of a C*-algebra A contains a non-zero simple ideal, we say that A has residual (IS) (see Definitions 3.1 and 3.2, and Remark 3.9) (note that, obviously, in the simple case, the primitive spectrum is trivial, and hence finite; also, observe that the C*-algebras with finite primitive spectrum have residual (IS)). We show that the class of separable C*-algebras with the weak ideal property, the class of separable C*-algebras with topological dimension zero, and the class of separable stable C*-algebras with residual (IF) are left invariant by crossed products by finite solvable groups (see Theorem 4.10 and Remark 4.13).

We now describe the main results of this paper. We prove that under the hypothesis of residual (IC), the notions of pure infiniteness and strong pure infiniteness are equivalent; and under the hypothesis of residual (IS), the four notions of strong pure infiniteness, pure infiniteness, weak pure infiniteness, and local pure infiniteness are all equivalent (see Theorem 3.10). These results give particular affirmative answers to the two above important questions raised by Kirchberg and Rørdam, and by Blanchard and Kirchberg. We show that for separable nuclear C*-algebras with topological dimension zero (the primitive spectrum of the C*-algebra has a basis of compact open sets), the notion of pure infiniteness is equivalent to being O_∞-stable (see Corollary 3.14). This result extends Kirchberg's O_∞-absorption theorem for the case of simple C*-algebras (see [**25**, Theorem 7.2.6(ii)]). We characterize residual (IC) and we prove that it is equivalent to topological dimension zero, and that under the assumption of pure infiniteness, it is also equivalent to residual (IF) and also to some known and important properties, namely: the ideal property, the weak ideal property, and residual (SP) (see Theorem 3.13).

The paper is organized as follows. In Section 2, we introduce some concepts and results needed in the next sections. In Section 3, we first define the notions of residual (IC), residual (IF), and residual (IS) (see Definitions 3.1 and 3.2), and then we prove the main results of this paper, described above (see Theorems 3.10 and 3.13, and Corollary 3.14). In Section 4, we prove that the weak ideal property, topological dimension zero, and residual (IF) have a good behavior with respect to crossed products by finite solvable groups (see Theorem 4.10 and Remark 4.13), and we also characterize the weak ideal property and topological dimension zero for crossed products of unital C*-algebras by finite groups (see Theorem 4.16 and Corollary 4.17).

2. Preliminaries

In this paper, by an ideal, we always mean a closed two-sided ideal, unless otherwise specified. For an ideal I (resp., C*-subalgebra B) in a C*-algebra A, we write $I \trianglelefteq A$ (resp., $B \leq A$). Also, for a hereditary (resp., and full) C*-subalgebra B in a C*-algebra A, we write $B \leq_h A$ (resp., $B \leq_{h,f} A$). For a subset S of a C*-algebra A, $\overline{ASA}$ denotes the ideal generated by S, where we simply write $\overline{AaA}$, when $S = \{a\}$. An element $a \in A$ is called *full* if $A = \overline{AaA}$ (see [**1**, p. 91]). Let $\mathrm{Prim}(A)$ be the set of primitive ideals in a C*-algebra A. Then $\mathrm{Prim}(A)$ is a topological space with the Jacobson (hull-kernel) topology [**1**].

Throughout this paper, the symbol $\otimes$ will mean the minimal tensor product of C*-algebras, and the C*-algebra of all compact linear bounded operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{K}(\mathcal{H})$. When $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, we denote $\mathcal{K} := \mathcal{K}(\mathcal{H})$.

Given $a, b \in A_+$, we say that a is Cuntz subequivalent to b (and write $a \precsim b$), if there is a sequence $\{x_k\}_{k=1}^{\infty} \subseteq A$ such that $x_k^* b x_k \to a$, in norm. We say that a and b are *Cuntz equivalent* (and write $a \sim_{cu} b$), if $a \precsim b$ and $b \precsim a$ (see [**11**, Definition 2.1]). A positive element a in a C*-algebra A is called *infinite* if there exists a non-zero positive element b in A such that $a \oplus b \precsim a \oplus 0$ in $M_2(A)$. If a is not infinite, then we say that a is *finite*. If a is non-zero and if $a \oplus a \precsim a \oplus 0$ in $M_2(A)$, then a is said to be *properly infinite* (see [**11**, Definition 3.2]). A C*-algebra A is said to be *purely infinite* if there are no characters on A, and for every pair of positive elements a, b in A, if $a \in \overline{AbA}$, then $a \precsim b$ (see [**11**, Definition 4.1]). A C*-algebra A will be said to have property *pi-n* if the n-fold direct sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite in $M_n(A)$ for every non-zero positive element a in A. If A is pi-n for some n, then we shall call A *weakly purely infinite* (see [**12**, Definition 4.3]). Also, A is said to be *strongly purely infinite* if for every

$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)_+,$$

and every $\varepsilon > 0$, there are $d_1, d_2 \in A$ such that

$$\left\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| \leq \varepsilon$$

(see [**12**, Definition 5.1]). A C*-algebra A is said to be *locally purely infinite* if, for every primitive ideal J of A and every element $b \in A_+$ with $\|b + J\| > 0$, there is a non-zero stable C*-subalgebra D of the hereditary C*-subalgebra generated by b such that D is not included in J (see [**2**, Definition 1.3]).

A C*-algebra A is of *real rank zero* if every self-adjoint element in A is the norm limit of self-adjoint elements of finite spectrum (see [**25**, Definition 1.1.7]). A C*-algebra A is said to have the *ideal property* if each ideal in A is generated (as an ideal) by its projections (see [**25**, Definition 1.5.2]). We say that a C*-algebra A has the *weak ideal property* if whenever $I \subseteq J \subseteq K \otimes A$ are ideals in $K \otimes A$ such that $I \neq J$, then J/I contains a non-zero projection (see [**21**, Definition 8.1]). Also, a C*-algebra A has the *projection property* if every ideal in A has an increasing approximate unit consisting of projections [**15**].

A C*-algebra A has *property (SP)* if every non-zero hereditary C*-subalgebra of A contains a non-zero projection, and A has *residual (SP)* if every quotient of A has property (SP) (see [**21**, p. 959 and Definition 7.1]).

An ideal J in a C*-algebra A is called a *compact ideal* if whenever $(J_\lambda)_{\lambda \in \Lambda}$ is an increasing net of ideals in A such that $J = \overline{\bigcup_{\lambda \in \Lambda} J_\lambda}$, then $J = J_\lambda$, for some λ (see [**24**, Remark 2.2]). Equivalently, an ideal I in a C*-algebra A is compact if and only if $\mathrm{Prim}(I)$ is a compact open (but not necessarily closed) subset of $\mathrm{Prim}(A)$ (see [**22**, p. 1389]).

An element $a \in A_+$ is *strictly full* if $(a - \varepsilon)_+$ is full for some (and so for all sufficiently small) $\varepsilon > 0$ [**13**, p. 46].

In the next section, we need [**26**, Lemma 3.12] repeatedly; and so we here state it as a lemma.

Lemma 2.1 ([**26**, Lemma 3.12])**.** *Let A be a C*-algebra. Then* $\mathrm{Prim}(A)$ *is compact if and only if A has a strictly full element.*

By the way, here is a short sketch of parts of the argument in [**26**, Lemma 3.12]: it is shown there that for a C*-algebra A, if $\mathrm{Prim}(A)$ is compact, then A has a full positive element h (by considering an approximate unit for A), and the lower semicontinuous function $\check{h}\colon \operatorname{Prim}(A) \to \mathbb{R}_+$, given by $\check{h}(J) = \|h/J\|$ (see [**1**, Proposition II.6.5.6(i)]), attains its minimum value, and it is concluded that h is a strictly full element in A. Conversely, if A has a strictly full element, then by applying the definition of a compact ideal and the Pedersen ideal, it is shown that $\operatorname{Prim}(A)$ is compact.

A C*-algebra A has *topological dimension zero* if $\operatorname{Prim}(A)$ has a basis consisting of compact open sets [**4**]; equivalently, every ideal in A is the closure of the union of an increasing net of compact ideals in A (see [**24**, Remark 2.2]). It is known that the real rank zero property implies the ideal property, the ideal property implies the weak ideal property, and the weak ideal property implies topological dimension zero (see [**22**, Theorem 2.8]).

3. C*-algebras with property (IC)

In this section, we define and study C*-algebras with property (IC) and with other related properties.

Definition 3.1. Let A be a C*-algebra.

(i) A is said to have *property (IC)* or to be ideal-compact (resp., *property (IF)* or ideal-finite) if every non-zero ideal in A has a non-zero C*-subalgebra with compact (resp., finite) primitive spectrum.

(ii) A is said to have *residual (IC)* (resp., *residual (IF)*) if every quotient of A has property (IC) (resp., property (IF)).

Definition 3.2. Let A be a C*-algebra.

(i) A is said to have *property (IS)* if every non-zero ideal in A has a non-zero simple hereditary C*-subalgebra.

(ii) A is said to have *residual (IS)* if every quotient of A has property (IS).

Note that the zero C*-algebra has all the properties mentioned in the two definitions above.

Remark 3.3. We clearly have the following:

(i) property (IS) $\Rightarrow$ property (IF) $\Rightarrow$ property (IC).

(ii) residual (IS) $\Rightarrow$ residual (IF) $\Rightarrow$ residual (IC).

The following lemma is probably well known. Indeed, it is a standard well-known fact that every C*-algebra with finitely many ideals has a finite composition series. Thus we have the following statement:

Lemma 3.4. *Let A be a non-zero C*-algebra such that* $\operatorname{Prim}(A)$ *is finite. Then A has a non-zero simple ideal.*

Using Lemma 3.4 one can easily prove the following result:

Proposition 3.5. *Let A be a C*-algebra. The following are equivalent:*

(i) *A has property (IF) (resp., residual (IF)).*

(ii) *Every non-zero ideal (resp., of every quotient) of A has a non-zero simple C*-subalgebra.*

Proposition 3.6. *Let A and B be separable C*-algebras such that A or B is exact, and B is simple. Assume that A has residual (IF). Then $A \otimes B$ has residual (IF).*

Proof: Let I and J be different ideals of $A \otimes B$ with $J \subseteq I$. Then, a theorem of Kirchberg implies that there are different ideals I_0 and J_0 of A with $J_0 \subseteq I_0$, such that $I = I_0 \otimes B$ and $J = J_0 \otimes B$ (see [**10**, Proposition 2.13]; see also [**23**, Theorem 1.3] and [**2**, Proposition 2.16(ii) and Proposition 2.17(2)]). Then, since A or B is exact, we have that

$$I/J \cong (I_0/J_0) \otimes B$$

(see [**2**, Proposition 2.16(iv) and Proposition 2.17(2)]). Since A has residual (IF), it follows that there is a non-zero C*-subalgebra C of I_0/J_0 with $\mathrm{Prim}(C)$ finite. Then, $C \otimes B$ is a non-zero C*-subalgebra of $(I_0/J_0) \otimes B$, and

$$\mathrm{Prim}(C \otimes B)(\cong \mathrm{Prim}(C) \times \mathrm{Prim}(B) \cong \mathrm{Prim}(C))$$

is finite (we used the hypothesis, [**2**, Proposition 2.16(iii)], and a theorem of Dixmier in [**7**] saying that if D is a separable C*-algebra, then $\mathrm{prime}(D) = \mathrm{Prim}(D)$). Hence, $A \otimes B$ has residual (IF). □

We denote by $\mathrm{Lat}(A)$ the set of all ideals in a C*-algebra A. Of course, it is well known that the set of all ideals in a C*-algebra is a complete lattice under inclusion.

Lemma 3.7. *Let A be a C*-algebra, $0 \neq D \leq_h A$, and $L = \overline{ADA}$. Then there exists a lattice isomorphism* $\mathrm{Lat}(D) \cong \mathrm{Lat}(L)$.

Proof: We denote by $\mathrm{Lat}_L(A)$ the set of all ideals of A contained in L. Then $\mathrm{Lat}_L(A) = \mathrm{Lat}(L)$. On the other hand, by [**1**, Section II.5.3.5] there is a bijection

$$\Phi\colon\ \mathrm{Lat}(D) \to \mathrm{Lat}_L(A);\ K \mapsto \overline{AKA},$$

whose inverse is

$$\Phi^{-1}\colon\ \mathrm{Lat}_L(A) \to \mathrm{Lat}(D);\ M \mapsto M \cap D.$$

Note that both Φ and Φ^{-1} preserve inclusions, and hence for every two ideals K_1 and K_2 of D we have that

$$K_1 \subseteq K_2 \Leftrightarrow \Phi(K_1) \subseteq \Phi(K_2).$$

It is clear that Φ^{-1} is a lattice isomorphism. Thus $\mathrm{Lat}(D) \cong \mathrm{Lat}(L)$. □

Proposition 3.8. *Let A be a C*-algebra. The following are equivalent:*

(i) *A has property (IS).*

(ii) *Every non-zero ideal in A contains a non-zero simple ideal.*

(iii) *Every non-zero hereditary C*-subalgebra of A contains a non-zero simple hereditary C*-subalgebra.*

The proof is easy and uses Lemma 3.7.

Remark 3.9. Using Proposition 3.8, it can be proved that a C*-algebra A has residual (IS) if and only if every non-zero ideal in every quotient of A contains a non-zero simple ideal, if and only if every non-zero hereditary C*-subalgebra of every quotient of A contains a non-zero simple hereditary C*-subalgebra.

Theorem 3.10. *Let A be a C*-algebra.*

(i) *If A has residual (IC), then pure infiniteness and strong pure infiniteness are equivalent for A.*

(ii) *If A has residual (IS), then local pure infiniteness, weak pure infiniteness, pure infiniteness, and strong pure infiniteness are all equivalent for A.*

Proof: (i) Let A be a purely infinite C*-algebra with residual (IC). According to [**24**, Proposition 2.11 ((iv) $\Rightarrow$ (ii))] (where separability is not necessary), we show that every non-zero hereditary C*-subalgebra in any quotient of A contains an infinite projection. In this case, A has the ideal property, and hence the first assertion holds, by [**24**, Proposition 2.14].

Let Q be a quotient of A, $0 \neq D \leq_h Q$, and $J_D := \overline{QDQ}$. Since A has residual (IC), J_D has a non-zero C*-subalgebra B_D with compact primitive spectrum. By Lemma 2.1, B_D has a strictly full element $b \in B_D^+$, and so there is $\varepsilon > 0$ such that $(b-\varepsilon)_+$ is full in B_D.

According to [**11**, Propositions 4.3 and 4.17], pure infiniteness passes to hereditary C*-subalgebras and quotients. Thus since J_D is purely infinite, [**11**, Theorem 4.16] implies that every non-zero positive element in J_D is properly infinite. In particular, b and $(b-\varepsilon)_+$ are properly infinite in J_D. Since the ideal $L_1 = \overline{B_D b B_D}$ is equal to $L_2 = \overline{B_D(b-\varepsilon)_+ B_D}$, we have that

$$\overline{J_D L_1 J_D} = \overline{J_D L_2 J_D},$$

and so

$$I := \overline{J_D b J_D} = \overline{J_D (b-\varepsilon)_+ J_D}.$$

This implies that $b \sim_{cu} (b-\varepsilon)_+$ in J_D, by [**11**, Proposition 3.5(ii)]. Now, according to the last two paragraphs of the proof of Proposition 2.7 ((i) $\Rightarrow$ (ii)) in [**24**], there is a full projection p in $I(\trianglelefteq J_D)$, and hence $p \neq 0$ (we also use [**11**, Proposition 3.3]). But $D \leq_{h,f} J_D$, and J_D is purely infinite. Thus every non-zero projection in J_D is equivalent to a properly infinite projection in D, by [**24**, Lemma 2.9]. Thus there is a properly infinite projection $q \in D$ such that $p \sim q$. In particular, q is infinite, and so D contains an infinite projection.

(ii) Let A be a locally purely infinite C*-algebra with residual (IS). If we show that every non-zero hereditary C*-subalgebra in any quotient of A contains an infinite projection, then A is purely infinite and has the ideal property, by [**24**, Proposition 2.11 ((iv) $\Rightarrow$ (ii))] (where separability is not necessary). In this case, since A has the ideal property, all three properties (weak pure infiniteness, pure infiniteness, and strong pure infiniteness) coincide, by [**24**, Proposition 2.14]. Moreover, every weakly purely infinite C*-algebra is locally purely infinite (see [**2**, Proposition 4.11]). Thus the assertion holds.

Let Q be a quotient of A and $0 \neq D \leq_h Q$. According to Remark 3.9, if A is a C*-algebra with the residual (IS), then every non-zero hereditary C*-subalgebra of every quotient of A contains a non-zero simple hereditary C*-subalgebra. Thus D has a non-zero simple hereditary C*-subalgebra, say H_D. Since local pure infiniteness passes to hereditary C*-subalgebras and quotients (see [**2**, Proposition 4.1(iii)]), H_D is locally purely infinite. Now, according to [**2**, Proposition 3.1], H_D is purely infinite (and simple). This implies that H_D (and so D) has an infinite projection. □

In particular, every locally purely infinite C*-algebra with residual (IS) has the ideal property.

Theorem 3.11. *Let A be a C*-algebra. The following are equivalent:*

(i) *A has property (IC).*
(ii) *Every non-zero ideal in A contains a non-zero compact ideal.*
(iii) *Every non-zero hereditary C*-subalgebra of A contains a non-zero compact ideal.*

If A is purely infinite, then the conditions (i)–(iii) above are equivalent to the following statements:

(iv) *A has property (SP).*
(v) *Every non-zero ideal in A contains a non-zero projection.*
(vi) *A has property (IF).*

Proof: (i) $\Rightarrow$ (ii) Assume that A has property (IC). Let I be a non-zero ideal of A. Then, there exists a non-zero C*-subalgebra E of I such that $\mathrm{Prim}(E)$ is compact.

Using Lemma 2.1, it follows that E has a strictly full element b, that is, b is a positive element of E and there is $\varepsilon > 0$ such that

$$E = \overline{E(b-\varepsilon)_+E}. \tag{1}$$

Let J be the ideal of I generated by E:

$$J := \overline{IEI}. \tag{2}$$

From (1) and (2) we get that

$$J = \overline{I(b-\varepsilon)_+I} \tag{3}$$

or

$$J = \overline{J(b-\varepsilon)_+J}. \tag{4}$$

Using (3), (4), and Lemma 2.1, we deduce that J is a non-zero ideal of I and $\mathrm{Prim}(J)$ is compact.

(ii) $\Rightarrow$ (iii) Let H be a non-zero hereditary C*-subalgebra of A, and $I := \overline{AHA}$. The hypothesis (ii) implies that I has a non-zero compact ideal I_0, and we have that $I_0 \in \mathrm{Lat}(I) \cong \mathrm{Lat}(H)$, by Lemma 3.7. This implies that H contains a non-zero compact ideal.

(iii) $\Rightarrow$ (i) This implication is obvious.

(ii) $\Rightarrow$ (iv) Let $0 \neq H \leq_h A$, and $I := \overline{AHA}$. Then I has a compact ideal I_0, by (ii). Since $\mathrm{Prim}(I_0)$ is compact, I_0 has a strictly full element h, by Lemma 2.1. Thus there is $\varepsilon > 0$ such that

$$I_0 = \overline{I_0(h-\varepsilon)_+I_0}$$

and $h \sim_{cu} (h-\varepsilon)_+$. Now since I_0 is purely infinite, h and $(h-\varepsilon)_+$ are properly infinite, and hence the last two paragraphs of the proof of Proposition 2.7 ((i) $\Rightarrow$ (ii)) in [**24**] imply that (I_0 and so) I has an infinite projection p. But we have that $H \leq_{h,f} I$. Thus [**24**, Lemma 2.9] implies that there is a projection q in H such that $p \sim q$.

(iv) $\Rightarrow$ (ii) If $0 \neq I \trianglelefteq A$, then the hypothesis (iv) implies that I has a non-zero projection p. Thus $J := \overline{IpI}$ is a non-zero compact ideal in I, by Lemma 2.1, because, for every $0 < \varepsilon < 1$, we have that

$$p \sim_{cu} (p-\varepsilon)_+,$$

and hence p is a strictly full element in J.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i) These implications are clear. Note that for the implication (v) $\Rightarrow$ (vi), if I is an ideal of A, and p is a non-zero projection of I, then $\mathbb{C}p \cong \mathbb{C}$ is a non-zero C*-subalgebra with finite primitive spectrum of I. □

Lemma 3.12. *Let A be a C*-algebra. If A has residual (SP), then A has the weak ideal property, and hence A has topological dimension zero.*

Proof: Assume that A has residual (SP). Let I and J be ideals of A, with $I \subsetneq J$. Since J/I is a non-zero ideal of A/I, and hence a non-zero hereditary C*-subalgebra of A/I, and since A has residual (SP), it follows that J/I contains a non-zero projection p. Let q be a non-zero projection of $\mathcal{K}$. Then $p \otimes q$ is a non-zero projection of $(J/I) \otimes \mathcal{K}$. It follows that A has the weak ideal property. Since also the weak ideal property implies topological dimension zero (by Theorem 2.8 of [**22**]), the proof is over. □

Theorem 3.13. *Let A be a C*-algebra. The following are equivalent:*

(i) *A has residual (IC).*

(ii) *For every two ideals I and J of A, with $I \subsetneq J$, there exists an ideal K of A, with $I \subsetneq K \subseteq J$, such that $\mathrm{Prim}(K/I)$ is compact.*

(iii) *A has topological dimension zero.*

If A is purely infinite, then the conditions (i)–(iii) are equivalent to the following statements:

(iv) *A has the ideal property.*

(v) *A has the weak ideal property.*

(vi) *A has residual (SP).*

(vii) *A has residual (IF).*

If A is separable, purely infinite, and stable, then the conditions (i)–(vii) above are equivalent to the following statement:

(viii) *A has the projection property.*

Proof: (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) We have that (i) $\Leftrightarrow$ (ii) follows from Theorem 3.11 ((i) $\Leftrightarrow$ (ii)), and (ii) $\Rightarrow$ (iii) follows from Theorem 1.11 ((2) $\Rightarrow$ (1)) of [**17**].

(iii) $\Rightarrow$ (i) This follows from the fact that topological dimension zero passes to ideals and quotients (by [**4**, Proposition 2.6]), and that a C*-algebra B has topological dimension zero if and only if every ideal J in B is equal to $\overline{\bigcup_\alpha J_\alpha}$ for some increasing net $\{J_\alpha\}_\alpha$ of compact ideals (see [**24**, p. 53] and [**22**, Definition 2.1]).

(i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (vi) Let A be a purely infinite C*-algebra with residual (IC). The proof of Theorem 3.10(i) shows that every non-zero hereditary C*-subalgebra in any quotient of a purely infinite C*-algebra with residual (IC) (in particular, A) contains an infinite projection. This shows that A has residual (SP), and also [**24**, Proposition 2.11 ((iv) $\Rightarrow$ (ii))] (where separability is not necessary) implies that A has the ideal property.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) It is obvious that the ideal property implies the weak ideal property. Moreover, in [**22**, Theorem 2.8] it was shown that every C*-algebra with the weak ideal property has topological dimension zero.

(vi) $\Rightarrow$ (i) This follows from Lemma 3.12 and the equivalence (i) $\Leftrightarrow$ (iii).

(vi) $\Rightarrow$ (vii) $\Rightarrow$ (i) These implications are obvious.

(iv) $\Rightarrow$ (viii) This implication holds, by [**24**, Proposition 2.13].

(viii) $\Rightarrow$ (iv) This implication is obvious. □

Corollary 3.14. *Let A be a nuclear and separable C*-algebra with topological dimension zero. Then A is purely infinite if and only if it is $\mathcal{O}_\infty$-stable. In this case, A is also $\mathcal{Z}$-stable and it has nuclear dimension one.*

Proof: Since A is nuclear, separable, and with topological dimension zero (or residual (IC), by Theorem 3.13), then A is purely infinite if and only if A is strongly purely infinite if and only if A is $\mathcal{O}_\infty$-stable, where the first equivalence follows from Theorem 3.10(i), and the second equivalence follows from the fact that if B is a separable nuclear C*-algebra, then B is strongly purely infinite if and only if B is $\mathcal{O}_\infty$-stable (see [**27**, Corollary 3.2] and [**12**, Proposition 5.11(iii) and Theorem 8.6]).

The second assertion also holds, because $\mathcal{O}_\infty$ is $\mathcal{Z}$-stable (see [**29**]), and every separable, nuclear, and $\mathcal{O}_\infty$-stable C*-algebra has nuclear dimension one (see [**3**, Theorem A]). ☐

Corollary 3.14 is an extension of Kirchberg's $\mathcal{O}_\infty$-absorption theorem for simple separable nuclear C*-algebras (see [**25**, Theorem 7.2.6(ii)]).

Example 3.15. (i) Every simple projectionless C*-algebra has residual (IS) but not the ideal property (note that every hereditary C*-subalgebra of a simple C*-algebra is simple).

(ii) In [**14**, Theorem 5.1], a C*-algebra A is constructed which is the extension of two simple C*-algebras (with real rank zero) but which does not have the ideal property. Therefore, $\mathrm{Prim}(A)$ is finite (it has in fact only two elements, as A is an extension of two simple C*-algebras), and therefore A has residual (IS). However, A does not have the ideal property.

(iii) The C*-algebra $A = C([0,1])$ does not have residual (IC), because a commutative C*-algebra A has topological dimension zero if and only if $\mathrm{Prim}(A)$ is totally disconnected. Indeed, it follows from Corollary 2.4 of [**16**] (taking there $B = \mathbb{C}$) that a commutative C*-algebra A has the ideal property if and only if $\mathrm{Prim}(A)$ is totally disconnected. On the other hand, it follows from Proposition 4 of [**18**] that for a type I C*-algebra (in particular, a commutative C*-algebra), the ideal property and topological dimension zero are equivalent.

Also, the Toeplitz algebra $\mathcal{T}$ does not have residual (IC). Indeed, since the Toeplitz algebra $\mathcal{T}$ has a quotient isomorphic to $C(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus (i.e., the unit circle) and $\mathbb{T}$ is not totally disconnected (being connected), it follows that $C(\mathbb{T})$ does not have topological dimension zero, and hence the same is true about the quotient, and therefore about $\mathcal{T}$ (the topological dimension zero passes to quotients; see [**4**, Proposition 2.6]).

(iv) Every separable connective C*-algebra A does not have property (IC), since $\mathrm{Prim}(A)$ has no non-empty compact open subsets (see [**6**, Proposition 2.7(i)] and Theorem 3.11 ((1) $\Leftrightarrow$ (2))).

4. Crossed products

In this section, which could be seen as a natural continuation of [**19**], we prove the invariance of some classes of C*-algebras with residual (IF), the weak ideal property or topological dimension zero under crossed products by finite solvable groups.

The question of the invariance of the weak ideal property and of topological dimension zero with respect to crossed products by finite groups has been investigated in [**19**], [**20**], and [**21**]. The concept of residual (IF) is close to the above two notions, and sometimes all three are equivalent (e.g., under the assumption of pure infiniteness; see Theorem 3.13). Hence, it is natural to ask the following question:

Question 4.1. Is it true that crossed products by finite groups preserve the class of all separable C*-algebras which have residual (IF)?

Theorem 4.2. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. If the fixed point algebra A^α has residual (IF), then A has residual (IF).*

Proof: The proof follows immediately from [**21**, Lemma 8.8], since if C is the image of a C*-algebra B with $\mathrm{Prim}(B)$ finite by an injective homomorphism, then B and C are isomorphic, and hence $\mathrm{Prim}(B)$ and $\mathrm{Prim}(C)$ are homeomorphic, and therefore $\mathrm{Prim}(C)$ is finite. ☐

Corollary 4.3. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite abelian group G on a C*-algebra A. If A has residual (IF), then the crossed product $C^*(G, A, \alpha)$ has residual (IF).*

Proof: Apply Theorem 4.2 with $C^*(G, A, \alpha)$ in place of A, and the dual action $\hat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(C^*(G, A, \alpha))$ in place of α (see [**28**, Section 7.1] for dual actions on crossed products by locally compact abelian groups). □

Proposition 4.4. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable C*-algebra A, and let H be a normal subgroup of G of index n. Assume that the answer to Question 4.1 is "yes" for G/H. If $C^*(H, A, \alpha)$ has residual (IF), then $C^*(G, A, \alpha) \otimes M_n$ has residual (IF).*

Proof: The proof is similar to the proof of "$\Leftarrow$" of Theorem 1.25(1) of [**19**], and it replaces "the weak ideal property" with "residual (IF)", uses once Proposition 3.6 in place of Theorem 8.5(6) of [**21**], and uses that the answer to Question 4.1 is "yes" for G/H in place of Corollary 8.10 of [**21**]. □

Corollary 4.5. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable C*-algebra A, and let H be a normal subgroup of G of index n. Assume that the answer to Question 4.1 is "yes" for both H and G/H. If A has residual (IF), then $C^*(G, A, \alpha) \otimes M_n$ has residual (IF). In particular, $C^*(G, A, \alpha)$ is stably isomorphic to a separable C*-algebra which has residual (IF).*

Proof: Use Proposition 4.4. □

Since the answer to Question 4.1 seems to be elusive, it is natural to ask the following related question (see Remark 4.7):

Question 4.6. Is it true that crossed products by finite groups preserve the class of all separable stable C*-algebras which have residual (IF)?

Remark 4.7. If the answer to Question 4.1 is "yes", then the answer to Question 4.6 is also "yes". This follows by using that crossed products by discrete groups of σ-unital stable C*-algebras are stable (by [**9**, Corollary 4.5]).

Proposition 4.8. *The answer to Question 4.6 is "yes" for all finite abelian groups.*

Proof: The proof follows from Corollary 4.3 and [**9**, Corollary 4.5]. □

Theorem 4.9. *Let G be a finite group and let H be a normal subgroup of G. Assume that the answer to Question 4.6 is "yes" for both H and G/H. Then the answer to Question 4.6 is "yes" for G.*

Proof: Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of the finite group G on a C*-algebra A. Assume that A is separable, stable, and has residual (IF). We shall prove that $C^*(G, A, \alpha)$ is separable, stable, and has residual (IF). Working essentially as in the proof of Theorem 1.25(1) of [**19**], using also Proposition 3.6, and denoting by n the index of H in G, we deduce that

(5) $C^*(G, A, \alpha) \otimes M_n$ is separable, stable, and has residual (IF).

On the other hand, crossed products by discrete groups of σ-unital stable C*-algebras are stable (by [**9**, Corollary 4.5]); hence, since A is stable and σ-unital (being separable), it follows that

(6) $C^*(G, A, \alpha)$ is stable.

Combining (5) and (6), we get that $C^*(G, A, \alpha) \otimes M_n \cong C^*(G, A, \alpha)$, and hence, by (5), we get that $C^*(G, A, \alpha)$ is separable, stable, and has residual (IF). In conclusion, the answer to Question 4.6 is “yes” for G. $\square$

Recall that a finite group is called *solvable* if there is a finite chain of subgroups $G_0 = \{1\} \subset G_1 \subset \cdots \subset G_n = G$ such that G_i is normal in G_{i+1}, and G_{i+1}/G_i is simple and abelian, for $i = 0, 1, \dots, n-1$.

Theorem 4.10. *Crossed products by finite solvable groups G preserve the class of all separable stable C*-algebras which have residual (IF). In particular, this happens if G is a finite group whose order is either odd (by the Feit–Thompson theorem,* [**8**]*), or of the form $p^\alpha q^\beta$, where p and q are distinct primes and α and β are non-negative integers (by the Burnside $p^\alpha q^\beta$ theorem,* [**5**]*), or less than* 60.

Proof: We have to prove that the answer to Question 4.6 is “yes” for any finite solvable group G.

Fix now an arbitrary finite solvable group G. Then, there exists a finite chain of subgroups $G_0 = \{1\} \subset G_1 \subset \cdots \subset G_n = G$ such that G_i is normal in G_{i+1}, and G_{i+1}/G_i is simple and abelian, for $i = 0, 1, \dots, , n-1$. First note that since $G_1 = G_1/G_0$ is a finite abelian group, Proposition 4.8 implies that the answer to Question 4.6 is “yes” for G_1. Assume now that for some $0 \leq i \leq n-1$ the answer to Question 4.6 is “yes” for G_i. Since G_{i+1}/G_i is a finite abelian group, Proposition 4.8 implies that the answer to Question 4.6 is “yes” for G_{i+1}/G_i. Then, these facts and Theorem 4.9 imply that the answer to Question 4.6 is “yes” for G_{i+1}. Using now a standard induction argument on n, we obtain that the answer to Question 4.6 is “yes” for any finite solvable group G. $\square$

Question 4.11. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable C*-algebra A. Assume that A has the weak ideal property. Does $C^*(G, A, \alpha)$ have the weak ideal property?

Question 4.12. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable C*-algebra A. Assume that A has topological dimension zero. Does $C^*(G, A, \alpha)$ have topological dimension zero?

Remark 4.13. Working essentially as above and using also results from [**20**] and [**21**], we can prove that the answer to Questions 4.11 and 4.12 is “yes” for any finite solvable group G.

Observation 4.14. Let G be a fixed finite group and let H be a fixed normal subgroup of G. Assume that the answer to Question 4.11 (resp., Question 4.12) is “yes” for both H and G/H. Then, the answer to Question 4.11 (resp., Question 4.12) is “yes” for G.

Proof: The proof is similar to the proof of Theorem 4.9. $\square$

Observation 4.15. The following are equivalent:

(i) The answer to Question 4.11 (resp., Question 4.12) is “yes”.
(ii) The answer to Question 4.11 (resp., Question 4.12) is “yes” for any simple non-commutative finite group G.

Proof: The proof of (i) $\Rightarrow$ (ii) is obvious. The proof of (ii) $\Rightarrow$ (i) is easy and uses mathematical induction on card(G), what was before Observation 4.14, Corollary 8.10 of [**21**], and Theorem 3.17 of [**20**]. $\square$

Theorem 4.16. *Let A be a unital C^*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A, and let I be a maximal α-invariant ideal of A. Then, the following are equivalent:*

(i) *$C^*(G, A, \alpha)$ has the weak ideal property (resp., topological dimension zero).*
(ii) *$C^*(G, I, \alpha)$ has the weak ideal property (resp., topological dimension zero).*
(iii) *Whenever J is a proper α-invariant ideal of A, then $C^*(G, J, \alpha)$ has the weak ideal property (resp., topological dimension zero).*

Proof: We first give the proof in the case of the weak ideal property.

The proof of (i) $\Rightarrow$ (iii) in the case of the weak ideal property follows from the fact that for any proper α-invariant ideal J of A, $C^*(G, J, \alpha)$ is an ideal of $C^*(G, A, \alpha)$, and the weak ideal property passes to ideals by Theorem 8.5(5) of [**21**].

The proof of (iii) $\Rightarrow$ (ii) is obvious in the case of the weak ideal property.

We now prove that (ii) $\Rightarrow$ (i) in the case of the weak ideal property.

Assume (ii) in the case of the weak ideal property. We have the following exact sequence of C*-algebras:

$$0 \to C^*(G, I, \alpha) \to C^*(G, A, \alpha) \to C^*(G, A/I, \alpha) \to 0. \tag{7}$$

Note that since A/I is an α-simple C*-algebra which has a full projection (its unit) (see Definition 1.5 of [**19**]), Lemma 1.12 of [**19**] implies that $C^*(G, A/I, \alpha)$ has the ideal property, and and hence it has the weak ideal property. Since also $C^*(G, I, \alpha)$ has the weak ideal property, by (ii) in the case of the weak ideal property, and since the weak ideal property passes to extensions by Theorem 8.5(5) of [**21**], (7) implies that $C^*(G, A, \alpha)$ has the weak ideal property. The proof of (ii) $\Rightarrow$ (i) in the case of the weak ideal property is over.

The proof of the theorem in the case of topological dimension zero is similar to the above proof and uses Proposition 2.6 of [**4**] instead of Theorem 8.5(5) of [**21**], and also uses the fact that every C*-algebra with the weak ideal property has topological dimension zero (see Theorem 2.8 of [**22**]). □

Corollary 4.17. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Assume that* $\mathrm{Prim}(A)$ *has two elements. Then, the following are equivalent:*

(i) *$C^*(G, A, \alpha)$ has the weak ideal property (resp., topological dimension zero).*
(ii) *There exists an α-invariant ideal I of A such that $I \neq \{0\}$, $I \neq A$, and $C^*(G, I, \alpha)$ has the weak ideal property (resp., topological dimension zero).*

Proof: Since $\mathrm{card}(\mathrm{Prim}(A)) = 2$, it follows that A has a unique ideal I such that $I \neq \{0\}$ and $I \neq A$. Therefore, I is the unique maximal α-invariant ideal of A. Now the proof follows from Theorem 4.16. □

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