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A COMBINATORIAL CHARACTERISATION OF d-KOSZUL AND (D, A)-STACKED MONOMIAL ALGEBRAS THAT SATISFY (Fg)

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Abstract: Condition **(Fg)** was introduced in [**6**] to ensure that the theory of support varieties of a finite-dimensional algebra, established by Snashall and Solberg, has some similar properties to that of a group algebra. In this paper we give some easy-to-check combinatorial conditions that are equivalent to **(Fg)** for monomial d-Koszul algebras. We then extend this to monomial (D, A)-stacked algebras. We also extend the description of the Yoneda algebra of a d-Koszul algebra in [**10**] to (D, A)-stacked monomial algebras.

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Introduction

Let Λ be an indecomposable finite-dimensional algebra over a field K. Support varieties for modules over Λ were introduced by Snashall and Solberg in [**23**], as a geometric tool to study the representation theory of Λ, using the Hochschild cohomology $\mathrm{HH}^*(\Lambda)$. It was then proved in [**6**] that many of the properties of support varieties for group algebras have analogues in this more general case, provided some finiteness conditions hold. These are now known as **(Fg)** and can be expressed in the following way. Let $\mathfrak{r}$ be the Jacobson radical of Λ and let $E(\Lambda) = \mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ be its Yoneda algebra. Then Condition **(Fg)** states that:

(Fg) there is a commutative Noetherian graded subalgebra H of $\mathrm{HH}^*(\Lambda)$ with $H^0 = \mathrm{HH}^0(\Lambda)$ such that $E(\Lambda)$ is a finitely generated H-module.

In particular, it was shown in [**6**] that if Λ satisfies Condition **(Fg)**, then Λ is necessarily Gorenstein, that the variety of a module is trivial if and only if the module has finite projective dimension, and that periodic modules can be characterised up to projective summands as those whose support variety is a line. Moreover, the converse of the first result mentioned was proved for monomial algebras in [**4**], that is, a Gorenstein monomial algebra satisfies **(Fg)**.

Support varieties for group algebras have been very effective in the study of the representations of these algebras. Therefore Condition **(Fg)** has been much studied as it ensures a similarly useful theory of support varieties for finite-dimensional algebras. For instance, Condition **(Fg)** is invariant under various constructions, such as derived equivalence or singular equivalence of Morita type; see [**16, 22, 19**]. Condition **(Fg)** has been studied or shown to hold for large families of algebras in [**24, 26, 25, 5**] among others, and support varieties have been studied for algebras that satisfy **(Fg)**; see for instance [**8, 20**].

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Since Hochschild cohomology is generally very difficult to compute, Condition **(Fg)** can be difficult to establish for a given algebra. It is therefore useful to have necessary, sufficient or equivalent conditions for **(Fg)** to hold for a given algebra. One such result was proved by Erdmann and Solberg in [**7**], where they showed that if **(Fg)** holds for Λ, then the graded centre $Z_{\mathrm{gr}}(E(\Lambda))$ of the Yoneda algebra is a Noetherian algebra and $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module; moreover, they proved that this is an equivalence when the algebra Λ is Koszul. For monomial algebras, **(Fg)** was proved in [**4**] to be equivalent to the related condition that the A_∞-centre $Z_\infty(E(\Lambda))$ is a Noetherian algebra and $E(\Lambda)$ is a finitely generated $Z_\infty(E(\Lambda))$-module. We note that if Λ has finite global dimension, then both $E(\Lambda)$ and $\mathrm{HH}^*(\Lambda)$ are finite-dimensional as vector spaces, and Λ has **(Fg)**. Thus we are particularly interested in algebras of infinite global dimension.

The aim of this paper is to prove that a number of conditions are equivalent to **(Fg)** for a large category of algebras, namely finite-dimensional d-Koszul, and more generally (D, A)-stacked, monomial algebras. This is motivated in particular by a result of the first author in her PhD thesis [**15**], a result that provides a sufficient and not-difficult-to-check condition for d-Koszul monomial algebras to satisfy **(Fg)**; this result is Theorem 2.7 in this paper.

Berger introduced d-Koszul algebras in [**3**] as a natural generalisation of Koszul algebras (which occur as 2-Koszul algebras). They are the algebras such that the n-th projective module in a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a Λ-module is generated in a specific degree denoted by $\delta(n)$ (with $\delta(n) = n$ if $d = 2$). Moreover, they were characterised in [**10**] as the algebras Λ that are d-homogeneous (that is, their ideal of relations can be generated by a set of homogeneous elements of degree d) and such that $E(\Lambda)$ is generated in degrees 0, 1, and 2. The (D, A)-stacked monomial algebras, where $D \geqslant 2$ and $A \geqslant 1$ are integers, were introduced by Green and Snashall in [**13**], and those of infinite global dimension were characterised by the same authors in [**12**] as the monomial algebras such that the n-th projective module in a minimal projective resolution of $\Lambda/\mathfrak{r}$ as a Λ-module is generated in precisely one degree and such that $E(\Lambda)$ is finitely generated (in which case $E(\Lambda)$ is generated in degrees 0, 1, 2, and 3). In particular, when $A = 1$, a $(D, 1)$-stacked monomial algebra is D-Koszul. Thus (D, A)-stacked monomial algebras are natural generalisations of d-Koszul and indeed Koszul monomial algebras.

In this paper, we consider Condition **(Fg)** for d-Koszul monomial algebras and more generally for (D, A)-stacked monomial algebras. We introduce some combinatorial conditions in 2.4 (in the d-Koszul case) and in 3.6 (in the (D, A)-stacked case) that are easy to check in terms of a minimal set of relations for the algebra Λ, and we prove that they are equivalent to **(Fg)** when K is algebraically closed. This gives a very practical way of checking whether a monomial d-Koszul or (D, A)-stacked algebra satisfies **(Fg)**, because it is easy to check that a monomial algebra is d-Koszul using [**10**, Theorem 10.2] (recalled in Property 2.1) or (D, A)-stacked using [**13**, Section 3] (recalled in Property 3.2).

To summarise, if Λ is a finite-dimensional monomial algebra over an algebraically closed field K, which is d-Koszul with $d \geqslant 2$ or (D, A)-stacked with $D \neq 2A$ whenever $A > 1$, then the following conditions are equivalent:

(C1) Λ satisfies **(Fg)**.

(C2) Λ satisfies some combinatorial conditions defined in Condition 2.4 when Λ is d-Koszul monomial and in Condition 3.6 when Λ is (D, A)-stacked monomial (this follows from Theorems 2.8 and 3.10).

(C3) $Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module (by [**7**] and Theorems 2.8 and 3.10).

(C4) $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module (again by Theorems 2.8 and 3.10).

(C5) $Z_{\infty}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\infty}(E(\Lambda))$-module (by [**4**]).

(C6) Λ is Gorenstein (by [**4**]).

The assumption that K is algebraically closed is needed for the implication (C1) $\Rightarrow$ (C3) of [**7**], which we use in our proof of (C1) $\Rightarrow$ (C2); however, (C2) implies (C1) without this assumption.

The paper is organised as follows. In Section 1 we give some background on monomial algebras and the notion of overlaps, as well as on the Yoneda algebra and the Hochschild cohomology of a monomial algebra. Section 2 is devoted to the proof of the implications (C2) $\Rightarrow$ (C1) and (C4) $\Rightarrow$ (C2) for d-Koszul monomial algebras, which completes the equivalence of all the conditions above. The first implication relies on a presentation of the Hochschild cohomology for (D, A)-stacked monomial algebras from [**13**] and the second one uses a description of the Yoneda algebra $E(\Lambda)$ of a d-Koszul algebra Λ as a graded subspace of the Koszul dual algebra ${}^{!}\Lambda$ from [**10**]. In Section 3, we extend these results to (D, A)-stacked monomial algebras, where $D \neq 2A$ whenever $A > 1$. Here again, we use a description of $E(\Lambda)$ as a subspace of an analogue ${}^{\natural}\Lambda$ of the Koszul dual of Λ; this description is detailed and proved in the appendix, and is a generalisation of the corresponding result of [**10**] to (D, A)-stacked monomial algebras.

General assumptions. Throughout the paper, Λ is an indecomposable finite-dimensional algebra over a field K with $\mathrm{char}(K) \neq 2$ that is not necessarily algebraically closed. Moreover, we assume that $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is a finite quiver (it has a finite number of vertices and arrows) and I is an admissible ideal in $K\mathcal{Q}$. If $\Lambda = K\mathcal{Q}/I$ is also a monomial algebra, then I is generated by a minimal set ρ of paths (monomials) and Λ is graded by the length of the paths; we denote by $\ell(p)$ the length of a path p. Note that paths in any algebra given by quiver and relations are written from left to right. For any $j \geqslant 0$, we shall denote by $\mathcal{Q}_j$ the set of paths of length j in $\mathcal{Q}$.

In order to use the results in [**13**], we shall need to assume that $\mathrm{gldim}\,\Lambda \geqslant 4$. However, if Λ is a monomial algebra with finite global dimension, all the conditions (C1)–(C6) hold for Λ (we note that Condition (C2) is necessarily empty in this case). Therefore we do not lose any generality in making this assumption.

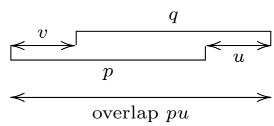
1. Some background on monomial algebras and their cohomology

1.1. Overlaps. Keeping the above assumptions, let $\Lambda = K\mathcal{Q}/I$ be a monomial algebra so that $\Lambda = \bigoplus_{i\geqslant 0} \Lambda_i$ is a graded algebra with the length grading. We denote by $\mathfrak{r} = \bigoplus_{i\geqslant 1} \Lambda_i$ the radical of Λ. An arrow α starts at the vertex $\mathfrak{o}(\alpha)$ and ends at the vertex $\mathfrak{t}(\alpha)$. If $p = \alpha_1\alpha_2\cdots\alpha_n$ is a path with $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathcal{Q}_1$, then $\mathfrak{o}(p) = \mathfrak{o}(\alpha_1)$ and $\mathfrak{t}(p) = \mathfrak{t}(\alpha_n)$.

A path p is a *prefix* of a path q if there is some path p' such that $q = pp'$; if an arrow α is a prefix of q, then we say that q *begins* with α. A path p is a *suffix* of a path q if there is some path p' such that $q = p'p$; if an arrow α is a suffix of q, then we say that q *ends* with α.

We use the concept of overlaps of [**9**] and [**14**] to describe the minimal projective resolution of $\Lambda_0 \cong \Lambda/\mathfrak{r}$ over Λ, and to describe the minimal projective resolution of Λ over Λ^e, where Λ^e is the enveloping algebra $\Lambda^{\mathrm{op}} \bigotimes_K \Lambda$ of Λ. We recall the relevant definitions here using the notation of [**13**].

Definition 1.1. (1) A path q overlaps a path p with overlap pu if there are paths u and v such that $pu = vq$ and $1 \leqslant \ell(u) < \ell(q)$. We illustrate the definition with the following diagram.



Note that we allow $\ell(v) = 0$ here.

(2) A path q properly overlaps a path p with overlap pu if q overlaps p and $\ell(v) \geqslant 1$.

(3) A path p has no overlaps with a path q if p does not properly overlap q and q does not properly overlap p.

We now define sets $\mathcal{R}^n$ recursively. Let

$$\mathcal{R}^0 = \mathcal{Q}_0, \text{ the set of vertices of } \mathcal{Q};$$
$$\mathcal{R}^1 = \mathcal{Q}_1, \text{ the set of arrows of } \mathcal{Q};$$
$$\mathcal{R}^2 = \rho, \text{ the minimal generating set for } I.$$

For $n \geqslant 3$, the construction is as follows.

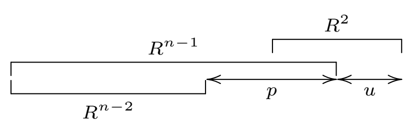
Definition 1.2. (1) For $n \geqslant 3$, we say that $R^2 \in \mathcal{R}^2$ maximally overlaps $R^{n-1} \in \mathcal{R}^{n-1}$ with overlap $R^n = R^{n-1}u$ if

(a) $R^{n-1} = R^{n-2}p$ for some path p;

(b) R^2 overlaps p with overlap pu;

(c) there is no element of $\mathcal{R}^2$ which overlaps p with overlap being a proper prefix of pu.

We may also say that R^n is a maximal overlap of $R^2 \in \mathcal{R}^2$ with $R^{n-1} \in \mathcal{R}^{n-1}$. The construction of R^n is illustrated in the following diagram.



(2) For $n \geqslant 3$, the set $\mathcal{R}^n$ is defined to be the set of all overlaps R^n formed in this way.

We also recall from [**14**] that if $R^n_1 p = R^n_2 q$, for $R^n_1, R^n_2 \in \mathcal{R}^n$ and paths p, q, then $R^n_1 = R^n_2$ and $p = q$. Any element R^n in $\mathcal{R}^n$ may be expressed uniquely as $R^{n-1}_j a_j$ and as $b_k R^{n-1}_k$ for some R^{n-1}_j, R^{n-1}_k in $\mathcal{R}^{n-1}$ and paths a_j, b_k. We say that the elements R^{n-1}_j and R^{n-1}_k *occur in* R^n.

1.2. The Ext algebra $E(\Lambda)$. The Ext algebra $E(\Lambda)$ is given by $E(\Lambda) = \operatorname{Ext}^*_\Lambda(\Lambda_0, \Lambda_0) = \bigoplus_{n\geqslant 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$ with the Yoneda product. In the terminology of overlaps, the n-th projective module in a minimal projective Λ-resolution of Λ_0 is $\bigoplus_{R^n \in \mathcal{R}^n} \mathfrak{t}(R^n)\Lambda$. Then $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$ has a basis indexed by $\mathcal{R}^n$ and $E(\Lambda)$ has a basis indexed by $\bigcup_{n\geqslant 0} \mathcal{R}^n$ (see [**14, 9**]). We identify $R^n_i \in \mathcal{R}^n$ with the corresponding element of $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$, that is, with the map $\bigoplus_{R^n \in \mathcal{R}^n} \mathfrak{t}(R^n)\Lambda \to \Lambda_0$ given by

$$\mathfrak{t}(R^n)\lambda \longmapsto \begin{cases} \mathfrak{t}(R^n_i)\lambda + \mathfrak{r} & \text{if } R^n = R^n_i; \\ 0 & \text{otherwise.} \end{cases}$$

1.3. The Hochschild cohomology ring $\mathbf{HH^*(\Lambda)}$. Let $(\mathcal{P}^*, \partial^*)$ be the minimal projective Λ^e-resolution of Λ from [**1**]. We write $\otimes$ for $\bigotimes_K$ throughout. Then

$$\mathcal{P}^n = \bigoplus_{R^n \in \mathcal{R}^n} \Lambda\mathfrak{o}(R^n) \otimes \mathfrak{t}(R^n)\Lambda.$$

The maps are given as follows. In odd degrees, if $R^{2n+1} = R_j^{2n} a_j = b_k R_k^{2n} \in \mathcal{R}^{2n+1}$, then $\partial^{2n+1} \colon \mathcal{P}^{2n+1} \to \mathcal{P}^{2n}$ is given by

$$\mathfrak{o}(R^{2n+1}) \otimes \mathfrak{t}(R^{2n+1}) \longmapsto \mathfrak{o}(R_j^{2n}) \otimes a_j - b_k \otimes \mathfrak{t}(R_k^{2n}),$$

where the first tensor lies in the summand corresponding to R_j^{2n} and the second tensor lies in the summand corresponding to R_k^{2n}.

For even degrees, any element R^{2n} in $\mathcal{R}^{2n}$ may be expressed in the form $p_j R_j^{2n-1} q_j$ for some $R_j^{2n-1} \in \mathcal{R}^{2n-1}$ and paths p_j, q_j with $n \geqslant 1$. Let $R^{2n} = p_1 R_1^{2n-1} q_1 = \cdots = p_r R_r^{2n-1} q_r$ be all expressions of R^{2n} which contain some element of $\mathcal{R}^{2n-1}$ as a subpath. Then, for $R^{2n} \in \mathcal{R}^{2n}$, the map $\partial^{2n} \colon \mathcal{P}^{2n} \to \mathcal{P}^{2n-1}$ is given by

$$\mathfrak{o}(R^{2n}) \otimes \mathfrak{t}(R^{2n}) \longmapsto \sum_{j=1}^{r} p_j \otimes q_j,$$

where the tensor $p_j \otimes q_j$ lies in the summand of $\mathcal{P}^{2n-1}$ corresponding to R_j^{2n-1}.

If not specified, then it will always be clear from the context in which summand of a projective module our tensors lie.

The Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ is given by

$$\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda) = \bigoplus_{n \geqslant 0} \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$$

with the Yoneda product.

2. Characterisations of d-Koszul monomial algebras that satisfy (Fg)

2.1. Notation and properties of d-Koszul monomial algebras. Let $\Lambda = K\mathcal{Q}/I$ be a monomial algebra, where $\mathcal{Q}$ is a finite quiver and I is an admissible ideal in $K\mathcal{Q}$ generated by a minimal set ρ of paths. Recall that the algebra $\Lambda = \bigoplus_{i \geqslant 0} \Lambda_i$ is graded by the length of the paths. We can express Λ as a quotient $\Lambda = \mathbb{T}_{\Lambda_0}(\Lambda_1)/I$ of the tensor algebra, where $\Lambda/\mathfrak{r} \cong \Lambda_0 = K\mathcal{Q}_0$ and $\Lambda_1 = K\mathcal{Q}_1$ and I is an ideal generated by a minimal set ρ of monomials. The algebra $\Lambda_0 \cong K^{|\mathcal{Q}_0|}$ is isomorphic to a finite product of copies of the base field K; it is therefore a semisimple and commutative K-algebra. We denote by e_i the idempotent in Λ_0 corresponding to the vertex i.

Let $d \geqslant 2$ be an integer. We assume that Λ is a d-Koszul algebra, that is, for any minimal projective right Λ-module resolution of Λ_0, the n-th projective module is generated in degree $\delta(n)$, where

$$\delta(n) = \begin{cases} \dfrac{n}{2} d & \text{if } n \text{ is even;} \\[2ex] \dfrac{n-1}{2} d + 1 & \text{if } n \text{ is odd.} \end{cases}$$

It follows that Λ is d-homogeneous (that is, ρ consists of paths of length d).

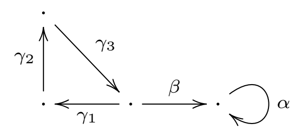
The monomial d-Koszul algebras can be characterised as follows.

Property 2.1 ([**10**, Theorem 10.2])**.** *A finite-dimensional d-homogeneous monomial algebra $\Lambda = K\mathcal{Q}/I$ is d-Koszul if, and only if, ρ is d-*covering*, that is, for any paths p, q, and r in $\mathcal{Q}$,*

$$(pq \in \rho, qr \in \rho, \ell(q) \geqslant 1) \Longrightarrow (\textit{all subpaths of } pqr \textit{ of length } d \textit{ are in } \rho).$$

Note that this condition is always satisfied if $d = 2$; it is indeed well known that all finite-dimensional quadratic monomial algebras are Koszul; see [**14**] and [**18**, Corollary 2.4.3].

Example 1. Let $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the quiver



and the ideal I has minimal generating set $\rho = \{\alpha^3, \gamma_1\gamma_2\gamma_3, \gamma_2\gamma_3\gamma_1, \gamma_3\gamma_1\gamma_2\}$. Then Λ is a 3-Koszul monomial algebra.

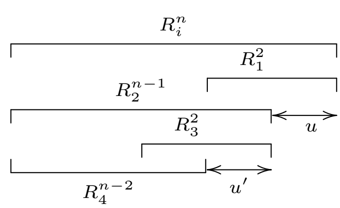
From now on, we assume that $\Lambda = K\mathcal{Q}/I$ is a finite-dimensional d-Koszul monomial algebra with $d \geqslant 2$.

We have the following consequences of Property 2.1.

Consequence 2.2 ([**15**, Proposition 7.13])**.** *Let R_i^n be an element in $\mathcal{R}^n$. Then all subpaths of R_i^n of length d are in ρ.*

Proof: The result is proved by induction. It is clear when $n = 2$. Moreover, if $n = 3$, since $R_i^3 \in \mathcal{R}^3$ is a maximal overlap of two elements in $\mathcal{R}^2$, it follows from Property 2.1.

Now let $n \geqslant 4$ be an integer and take $R_i^n \in \mathcal{R}^n$. Then R_i^n is a maximal overlap of $R_1^2 \in \mathcal{R}^2$ with $R_2^{n-1} \in \mathcal{R}^{n-1}$ so that $R_i^n = R_2^{n-1}u$ for some path u, and R_2^{n-1} is a maximal overlap of $R_3^2 \in \mathcal{R}^2$ with $R_4^{n-2} \in \mathcal{R}^{n-2}$ so that $R_2^{n-1} = R_4^{n-2}u'$ for some path u'. This can be illustrated as follows:



Moreover, $\ell(u'u) = \ell(R_i^n) - \ell(R_4^{n-2}) = \delta(n) - \delta(n-2) = d$ so $u'u = R_1^2$. By induction, every subpath of R_2^{n-1} of length d is in ρ. Any other subpath of length d of R_i^n is either $u'u = R_1^2 \in \mathcal{R}^2$ or a proper subpath of $R_3^2 u$; therefore it is in ρ by Property 2.1. We have proved the induction step. $\square$

A *trail* in $\mathcal{Q}$ is a path $T = \alpha_1 \cdots \alpha_n$ with $n \geqslant 1$ such that the arrows α_i are all distinct. We say that the trail is *closed* when $\mathfrak{t}(\alpha_n) = \mathfrak{o}(\alpha_1)$. A path q is said to *lie on* the closed trail T if q is a subpath of T^m for some $m \geqslant 1$. We say that two trails are *distinct* if neither lies on the other.

We now have a second consequence of Property 2.1.

Consequence 2.3 ([**15**, Proposition 7.14])**.** *Suppose that $T = \alpha_1 \cdots \alpha_n$ is a closed trail in $\mathcal{Q}$ and that $d \geqslant n + 1$. Then all paths of length d that lie on the closed trail T are in ρ.*

Proof: Since Λ is finite-dimensional, there is a path $R_2 \in \rho$ that lies on T. Now, $\ell(R_2) = d$ and $d \geqslant n+1$ so, without loss of generality, we may suppose that $R_2 = (\alpha_1\alpha_2\cdots\alpha_n)^m\alpha_1\alpha_2\cdots\alpha_s$ for some $1 \leqslant s \leqslant n$ with $d = nm+s$ and $m \geqslant 1$. Let $p = (\alpha_1\alpha_2\cdots\alpha_n)^m$, $q = \alpha_1\alpha_2\cdots\alpha_s$, and $r = (\alpha_{s+1}\cdots\alpha_n\alpha_1\cdots\alpha_s)^m$. Then $pq = R_2 = qr$ and we can apply Property 2.1 so that all subpaths of pqr of length d are in ρ. Now, any path of length d that lies on the closed trail T is a subpath of pqr and hence is in ρ. $\square$

We now introduce Condition 2.4. Jawad showed in her PhD thesis [**15**] that this condition is sufficient for Λ to satisfy **(Fg)**; we give a proof in Theorem 2.7 below.

Condition 2.4 ([**15**, Theorems 7.11 and 7.15])**.** We say that a d-Koszul monomial algebra Λ satisfies Condition 2.4, or (C2), when the following properties (1) and (2) both hold:

(1) Let α be a loop in $\mathcal{Q}_1$. Then $\alpha^d \in \rho$ but there is no path in ρ of the form $\alpha^{d-1}\beta$ or $\beta\alpha^{d-1}$, where β is an arrow that is distinct from α.

(2) Let $T = \alpha_1\cdots\alpha_n$ be a closed trail in $\mathcal{Q}$ with $n > 1$ and $\alpha_i \in \mathcal{Q}_1$ for all i and such that $\rho_T := \{\alpha_1\cdots\alpha_d, \alpha_2\cdots\alpha_d\alpha_{d+1}, \ldots, \alpha_n\alpha_1\cdots\alpha_{d-1}\} \subseteq \rho$. Then there are no elements in $\rho \setminus \rho_T$ which begin or end with the arrow α_i, for all i.

Remark 2.5. If $T = \alpha_1\cdots\alpha_n$ is a closed trail, then the subscript i of α_i is taken modulo n within the range $1 \leqslant i \leqslant n$. Thus ρ_T is the set of all paths of length d that lie on the closed trail T.

Remark 2.6. Suppose that Condition 2.4 is non-empty, that is, there is a loop or a closed trail with the given properties. Then the description of the projective modules in Subsection 1.2 using overlaps shows that Λ_0 has infinite projective dimension as a Λ-module, and hence Λ has infinite global dimension.

2.2. Condition 2.4 is sufficient for Λ to satisfy (Fg). The proof of Theorem 2.7 uses the description of the Hochschild cohomology ring modulo nilpotence of a (D, A)-stacked monomial algebra from [**13**, Theorem 3.4]. We recall the definition of a (D, A)-stacked monomial algebra in Subsection 3.1. The Hochschild cohomology ring modulo nilpotence is the quotient $\mathrm{HH}^*(\Lambda)/\mathcal{N}$, where $\mathcal{N}$ is the ideal of $\mathrm{HH}^*(\Lambda)$ that is generated by the homogeneous nilpotent elements. It is well known that $\mathrm{HH}^*(\Lambda)$ is a graded commutative ring, so, since $\mathrm{char}(K) \neq 2$, every homogeneous element of odd degree squares to zero. Moreover, $\mathcal{N}$ is the set of all nilpotent elements of $\mathrm{HH}^*(\Lambda)$. Our calculations involving $\mathrm{HH}^*(\Lambda)$ use the minimal projective Λ^e-resolution $(\mathcal{P}^*, \partial^*)$ of Λ from [**1**]; see Subsection 1.3.

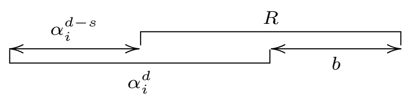
Noting that a d-Koszul monomial algebra is a $(d, 1)$-stacked monomial algebra (see [**13**]), we apply [**13**, Theorem 3.4] in the special case where $D = d$ and $A = 1$, and this simplifies the hypotheses. Specifically, if there is a closed path C in $\mathcal{Q}$ with $C^{D/A} \in \rho$, then $C^d \in \rho$ and it is immediate that C has length 1 and is necessarily a loop.

Theorem 2.7 ([**15**, Theorems 7.11 and 7.15])**.** *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional d-Koszul monomial algebra with $d \geqslant 2$. Assume that Λ satisfies Condition 2.4. Then Λ satisfies* **(Fg)**.

Proof: We keep the notation of Condition 2.4.

Let $\alpha_1, \ldots, \alpha_u$ be the loops in the quiver $\mathcal{Q}$, and suppose that α_i is a loop at the vertex v_i. Since Λ is a finite-dimensional d-Koszul monomial algebra, α_i^d is necessarily in the minimal generating set ρ. By Condition 2.4(1), for each $i = 1, \ldots, u$, there are no elements in ρ of the form $\alpha_i^{d-1}\beta$ or $\beta\alpha_i^{d-1}$, where β is an arrow that is distinct from α_i.

We need to show that there are no overlaps of α_i^d with any element of $\rho\setminus\{\alpha_i^d\}$. This is immediate if $d=2$, so suppose that $d\geqslant 3$. If $R\in\rho\setminus\{\alpha_i^d\}$ and R overlaps α_i^d, then either $R=\alpha_i^s b$ or $R=b\alpha_i^s$, where $1\leqslant s\leqslant d-1$ and b is a path of length $d-s$ which does not begin (respectively, end) with the arrow α_i. Suppose first that $R=\alpha_i^s b$. Then R overlaps α_i^d with overlap of length $2d-s$ as follows:



This is a maximal overlap since α_i is not the first arrow of b and thus gives an element $R_1^3\in\mathcal{R}^3$. However, $\ell(R_1^3)=d+1$ since Λ is d-Koszul. Thus $2d-s=d+1$ and so $s=d-1$. But then $R=\alpha_i^{d-1}b$ and b is an arrow distinct from α_i, which contradicts our hypothesis. The case where $R=b\alpha_i^s$ is similar. So there are no overlaps of α_i^d with any element of $\rho\setminus\{\alpha_i^d\}$. Moreover, as Λ is a finite-dimensional monomial algebra, it follows that the vertices $v_1,\dots,v_u$ are distinct.

Let $T_{u+1},\dots,T_r$ be the distinct closed trails in $\mathcal{Q}$ such that all paths of length d that lie on these closed trails are contained in ρ. For each $i=u+1,\dots,r$, we write $T_i=\alpha_{i,1}\cdots\alpha_{i,m_i}$, where the $\alpha_{i,j}$ are arrows, and set

$$\rho_{T_i}=\{\alpha_{i,1}\cdots\alpha_{i,d},\alpha_{i,2}\cdots\alpha_{i,d+1},\dots,\alpha_{i,m_i}\alpha_{i,1}\cdots\alpha_{i,d-1}\}.$$

Then ρ_{T_i} is contained in ρ. By Condition 2.4(2), for each closed trail T_i ($i=u+1,\dots,r$), there are no elements in $\rho\setminus\rho_{T_i}$ which begin or end with the arrow $\alpha_{i,j}$, for all $j=1,\dots,m_i$. So no arrow $\alpha_{i,j}$ has overlaps with any element in $\rho\setminus\rho_{T_i}$.

For $i=u+1,\dots,r$, let $T_{i,1},\dots,T_{i,m_i}$ be defined by

$$\begin{aligned}T_{i,1}&=T_i=\alpha_{i,1}\alpha_{i,2}\cdots\alpha_{i,m_i},\\ T_{i,2}&=\alpha_{i,2}\alpha_{i,3}\cdots\alpha_{i,m_i}\alpha_{i,1},\\ &\ \ \vdots\\ T_{i,m_i}&=\alpha_{i,m_i}\alpha_{i,1}\cdots\alpha_{i,m_i-1}.\end{aligned}$$

Then the paths $T_{i,1},\dots,T_{i,m_i}$ are all of length m_i and lie on the closed path T_i.

We now describe a commutative Noetherian graded subalgebra H of $\mathrm{HH}^*(\Lambda)$ with $H^0=\mathrm{HH}^0(\Lambda)$. As noted above, Λ is a $(d,1)$-stacked monomial algebra. Moreover, Condition **(Fg)** is always satisfied if the global dimension of Λ is finite; therefore we may assume that $\operatorname{gldim}\Lambda\geqslant 4$. Hence we can apply [**13**, Theorem 3.4], which gives $\mathrm{HH}^*(\Lambda)/\mathcal{N}\cong K[x_1,\dots,x_r]/\langle x_ax_b \text{ for } a\neq b\rangle$, where

- for $i=1,\dots,u$, the vertices $v_1,\dots,v_u$ are distinct and the element x_i corresponding to the loop α_i is in degree 2 and is represented by the map $\mathcal{P}^2\to\Lambda$, where for $R^2\in\mathcal{R}^2$,

$$\mathfrak{o}(R^2)\otimes\mathfrak{t}(R^2)\longmapsto\begin{cases}v_i & \text{if } R^2=\alpha_i^d;\\ 0 & \text{otherwise}\end{cases}$$

- and for $i=u+1,\dots,r$, the element x_i corresponding to the closed trail $T_i=\alpha_{i,1}\cdots\alpha_{i,m_i}$ is in degree $2\mu_i$ such that $\mu_i=m_i/\gcd(d,m_i)$ and is represented by the map $\mathcal{P}^{2\mu_i}\to\Lambda$, where for $R^{2\mu_i}\in\mathcal{R}^{2\mu_i}$,

$$\mathfrak{o}(R^{2\mu_i})\otimes\mathfrak{t}(R^{2\mu_i})\longmapsto\begin{cases}\mathfrak{o}(T_{i,k}) & \text{if } R^{2\mu_i}=T_{i,k}^{d/\gcd(d,m_i)} \text{ for all } k=1,\dots,m_i;\\ 0 & \text{otherwise.}\end{cases}$$

Let H be the subring of $\mathrm{HH}^*(\Lambda)$ generated by $Z(\Lambda)$ and $\{x_1, \dots, x_r\}$. Since $Z(\Lambda) = \mathrm{HH}^0(\Lambda)$ and $\mathrm{HH}^*(\Lambda)$ is graded commutative, it follows that

$$H = Z(\Lambda)[x_1, \dots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$$

and so H is a commutative ring. Moreover, $Z(\Lambda)$ is finite-dimensional so is a commutative Noetherian ring. Thus H is a Noetherian ring (see [**21**, Corollary 8.11]).

The rest of this proof shows that Λ satisfies **(Fg)**, with the algebra H that we have just described. Following the discussion in Subsection 1.2, we identify $\bigcup_{n\geqslant 0} \mathcal{R}^n$ with a basis of $E(\Lambda)$. The action of a homogeneous element $x \in \mathrm{HH}^n(\Lambda)$ on $E(\Lambda)$ is then given by left multiplication by $\sum_j R^n_j$, where the sum is over all j such that $x(\mathfrak{o}(R^n_j)\otimes\mathfrak{t}(R^n_j)) \neq 0$. Thus if $x_i \in \mathrm{HH}^2(\Lambda)$ corresponds to the loop α_i, then the action of x_i on $E(\Lambda)$ is given by left multiplication by α_i^d. And if x_i in degree $2\mu_i$ corresponds to the closed trail T_i, then the action of x_i on $E(\Lambda)$ is given by left multiplication by $\sum_{k=1}^{m_i} T_{i,k}^{d/\gcd(d,m_i)}$.

Set $N = \max\{3, |x_1|, \dots, |x_r|, |\mathcal{Q}_1|\}$. We show that $\bigcup_{n=0}^{N} \mathcal{R}^n$ is a generating set for $E(\Lambda)$ as a left H-module and thus $E(\Lambda)$ is finitely generated as a left H-module.

Let $R \in \mathcal{R}^n$ with $n > N$. Then $\ell(R) = \delta(n) \geqslant 2d$ and we can write $R = a_1 a_2 \cdots a_{\delta(n)}$, where the a_i are in $\mathcal{Q}_1$. From Consequence 2.2, all subpaths of R of length d are in ρ, so we may illustrate R with the following diagram:

$$\underbrace{a_1\, a_2 \cdots a_d}\!\!\overbrace{\, a_{d+1}} \cdots \underbrace{a_{\delta(n)-d+1} \cdots a_{\delta(n)}}$$

Now, $n > N \geqslant |\mathcal{Q}_1|$ so there is some repeated arrow. Choose j, k with k minimal and $k \geqslant 1$ such that a_j is a repeated arrow, $a_j, \dots, a_{j+k-1}$ are all distinct arrows and $a_{j+k} = a_j$. Write

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_{\delta(n)}).$$

There are two cases to consider.

Case (1): $k = 1$. Then $a_j = a_{j+1}$ and so a_j is a loop. It follows that

$$R = (a_1 \cdots a_{j-1})(a_j a_j)(a_{j+2} \cdots a_{\delta(n)}).$$

Suppose first that $j \leqslant d-1$. Then $j + d - 1 \leqslant \delta(n)$, so from Consequence 2.2, $a_j^2 a_{j+2} \cdots a_{j+d-1}$ is in ρ. But $a_j^d \in \rho$ and we have already shown that there are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$. Thus $a_j = a_{j+2} = \cdots = a_{j+d-1}$. Inductively we see that $R = (a_1 \cdots a_{j-1})a_j^{\delta(n)-j+1}$. Similarly, $a_1 \cdots a_{j-1}a_j^{d-j+1}$ is in ρ and $d - j + 1 \geqslant 2$. Again, there are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$ so $a_j = a_1 = \cdots = a_{j-1}$. Thus $R = a_j^{\delta(n)}$.

Now suppose that $j \geqslant d$. Then $j-d+1 \geqslant 1$, so by Consequence 2.2, $a_{j-d+1} \cdots a_{j-1}a_j$ is in ρ. As there are no overlaps of a_j^d with any element of $\rho \setminus \{a_j^d\}$, it follows that $a_{j-d+1} = \cdots = a_{j-1} = a_j$, and inductively $R = a_j^{j+1}(a_{j+2} \cdots a_{\delta(n)})$. Using Consequence 2.2 again, $a_j^{d-1}a_{j+2}$ is in ρ so $a_j = a_{j+2}$. Inductively, we have $R = a_j^{\delta(n)}$.

Hence, for all j,

$$R = a_j^{\delta(n)} = \begin{cases} (a_j^d)^{(n/2)} & \text{if } n \text{ is even;} \\ (a_j^d)^{((n-1)/2)} a_j & \text{if } n \text{ is odd.} \end{cases}$$

Let x_i be the generator in H corresponding to the loop a_j, so $1 \leqslant i \leqslant u$ and $|x_i| = 2$. Then x_i acts on $E(\Lambda)$ as left multiplication by a_j^d. Hence

$$R = \begin{cases} (x_i)^{(n/2)}\mathfrak{o}(a_j) & \text{if } n \text{ is even;} \\ (x_i)^{((n-1)/2)}a_j & \text{if } n \text{ is odd} \end{cases}$$

with $x_i \in H$, $\mathfrak{o}(a_j) \in \mathcal{R}^0$, and $a_j \in \mathcal{R}^1$, so that $\mathfrak{o}(a_j)$ and a_j are in $\bigcup_{n=0}^N \mathcal{R}^n$.

Case (2): $k > 1$. We note by our choice of j, k that $a_j \cdots a_{j+k-1}$ is a closed trail of length k, which we denote by T. Let ρ_T be the set of all paths of length d which lie on T.

The first step is to show that ρ_T is contained in ρ. If $d \geqslant k+1$, then this follows from Consequence 2.3. So, suppose that $d \leqslant k$. Recall that

$$R = (a_1 \cdots a_{j-1})(a_j \cdots a_{j+k-1})(a_j a_{j+k+1} \cdots a_{\delta(n)}).$$

Then:

$$\begin{aligned} &a_j a_{j+1} \cdots a_{j+d-1}, \\ &a_{j+1} a_{j+2} \cdots a_{j+d}, \\ &\qquad\vdots \\ &a_{j+k-d} a_{j+k-d+1} \cdots a_{j+k-1}, \\ &a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j \end{aligned}$$

are all paths of length d which are subpaths of R, and so, by Consequence 2.2, are in ρ.

Now $a_j a_{j+1} \cdots a_{j+d-1}$ overlaps $a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j$. So there is an element $R_1^2 \in \rho$ such that R_1^2 maximally overlaps $a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j$ with maximal overlap of length $d+1$. Then we have that

$$R_1^2 = a_{j+k-d+2} a_{j+k-d+3} \cdots a_{j+k-1} a_j a_{j+1}$$

and this maximal overlap is $(a_{j+k-d+1} a_{j+k-d+2} \cdots a_{j+k-1} a_j) a_{j+1} = a_{j+k-d+1} R_1^2$. Continuing in this way, $a_{j+1} a_{j+2} \cdots a_{j+d}$ overlaps R_1^2. So there is an element $R_2^2 \in \rho$ such that R_2^2 maximally overlaps R_1^2 with maximal overlap of length $d+1$. So

$$R_2^2 = a_{j+k-d+3} a_{j+k-d+4} \cdots a_{j+k-1} a_j a_{j+1} a_{j+2}$$

and this maximal overlap is $R_1^2 a_{j+2} = a_{j+k-d+2} R_2^2$. Inductively, we see that every path of length d on the closed trail T is in ρ. Hence ρ_T is contained in ρ.

It follows from Condition 2.4(2), that there are no paths in $\rho \setminus \rho_T$ which begin or end with any of the arrows $a_j, a_{j+1}, \ldots, a_{j+k-1}$.

Next we show that R can be written in the form $R = p_1 T^q p_2$, where p_1 is a suffix of T and p_2 is a prefix of T. If $d = 2$, then $a_j a_{j+k+1}$ is a subpath of R of length 2 and hence is in ρ. By Condition 2.4(2), $a_j a_{j+k+1}$ must be in ρ_T and so $a_{j+k+1} = a_{j+1}$. Then $a_{j+k+1} a_{j+k+2} = a_{j+1} a_{j+k+2}$ and is a subpath of R of length 2, so we must have that $a_{j+k+2} = a_{j+2}$. Inductively, we see that R lies on the closed trail T. So $R = p_1 T^q p_2$, where p_1 is a suffix of T and p_2 is a prefix of T.

So let $d \geqslant 3$, and suppose first that $d \leqslant k$. Then $a_{j+k-d+2} \cdots a_{j+k-1} a_j a_{j+k+1}$ is a subpath of R of length d which begins with the arrow $a_{j+k-d+2} \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So, by Consequence 2.2 and Condition 2.4(2), this path is in ρ_T and hence $a_{j+k+1} = a_{j+1}$. Inductively, we have $a_{j+k+2} = a_{j+2}, a_{j+k+3} = a_{j+3}, \ldots$ Similarly, $a_{j-1} a_j \cdots a_{j+d-2}$ is a subpath of R of length d which ends with the arrow $a_{j+d-2} \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So this path is in ρ_T and hence $a_{j-1} = a_{j+k-1}$. Inductively, we have $a_{j-2} = a_{j+k-2}$, $a_{j-3} = a_{j+k-3}, \ldots$ So we may write $R = p_1 T^q p_2$, where p_1 is a suffix of T and p_2 is a prefix of T.

Now suppose that $d \geqslant k+1$ (with $d \geqslant 3$). We consider $j \leqslant d-1$ and $j \geqslant d$ separately. Let $j \leqslant d-1$. Then $j+d < \delta(n)$, so $a_{j+1}a_{j+2}\cdots a_{j+k-1}a_j a_{j+k+1}\cdots a_{j+d}$ is a subpath of R of length d and starts with the arrow $a_{j+1} \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So by Consequence 2.2 and Condition 2.4(2), this path is in ρ_T and hence $a_{j+k+1} = a_{j+1}$, $a_{j+k+2} = a_{j+2}, \ldots$ So, inductively, we may write $R = (a_1 \cdots a_{j-1})T^q p_2$, where p_2 is a prefix of T. Now $a_1 a_2 \cdots a_{j-1} \cdots a_d$ is a subpath of R of length d and ends with the arrow $a_d \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So by Condition 2.4(2), this path is in ρ_T and hence $a_{j-1} = a_{j+k-1}$, $a_{j-2} = a_{j+k-2}, \ldots$ Thus $R = p_1 T^q p_2$, where p_1 is a suffix of T and p_2 is a prefix of T. Finally, suppose $j \geqslant d$. Then, we know that $a_{j+k-d}\cdots a_{j-1}a_j \cdots a_{j+k-1}$ is a subpath of R of length d and ends with the arrow $a_{j+k-1} \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So by Consequence 2.2 and Condition 2.4(2), this path is in ρ_T and hence $a_{j-1} = a_{j+k-1}, a_{j-2} = a_{j+k-2}, \ldots$ Also $a_{j+k-d+2}\cdots a_{j-1}a_j \cdots a_{j+k+1}$ is a subpath of R of length d and starts with the arrow $a_{j+k-d+2}$. But we have just shown that $a_{j+k-d+2} \in \{a_j, a_{j+1}, \ldots, a_{j+k-1}\}$. So again, this path is in ρ_T and hence $a_{j+k+1} = a_{j+1}$. Inductively, $a_{j+k+2} = a_{j+2}, \ldots$ Thus $R = p_1 T^q p_2$, where p_1 is a suffix of T and p_2 is a prefix of T.

So, in all cases, $R = p_1 T^q p_2$, where $T = a_j \cdots a_{j+k-1}$, p_1 is a suffix of T and p_2 is a prefix of T.

Without loss of generality, relabel the trail T and write $R = T^q p$, where $T = a_1 \cdots a_k$, p is a prefix of T, $\delta(n) = kq + \ell(p)$, and we choose $\ell(p)$ in the range $1 \leqslant \ell(p) \leqslant k$. Note that R has a repeated arrow so $q \geqslant 1$, and if $q = 1$, then $\ell(p) \geqslant 1$; moreover, if $\ell(p) = k$, then $p = T$ and $R = T^{q+1}$.

Let x_i be the generator in H corresponding to this closed trail T, so $u+1 \leqslant i \leqslant r$. Let

$$\begin{aligned} T_{i,1} &= T = a_1 a_2 \cdots a_k, \\ T_{i,2} &= a_2 a_3 \cdots a_k a_1, \\ &\vdots \\ T_{i,k} &= a_k a_1 \cdots a_{k-1}. \end{aligned}$$

The action of x_i on $E(\Lambda)$ is left multiplication by

$$T_{i,1}^{d/\gcd(d,k)} + T_{i,2}^{d/\gcd(d,k)} + \cdots + T_{i,k}^{d/\gcd(d,k)}$$

and $|x_i| = 2k/\gcd(d,k)$. Consequently, $N \geqslant 2k/\gcd(d,k)$. Now $R = T^q p$ with $1 \leqslant \ell(p) \leqslant k$. Write $q = \frac{d}{\gcd(d,k)}c + w$ with $0 \leqslant w \leqslant \frac{d}{\gcd(d,k)} - 1$. Then

$$R = (T^{d/\gcd(d,k)})^c (T^w p).$$

Moreover, from the construction of R as a maximal overlap, we see that $T^w p$ is also constructed as a maximal overlap and so corresponds to a basis element of $E(\Lambda)$. We have $\ell(T^w p) = kw + \ell(p) \leqslant k\big(\frac{d}{\gcd(d,k)} - 1\big) + k = kd/\gcd(d,k) = \delta(2k/\gcd(d,k))$. So $T^w p$ corresponds to a basis element of $E(\Lambda)$ of degree at most $2k/\gcd(d,k)$, that is, $T^w p$ is in $\mathcal{R}^m$ for some $m \leqslant N$.

Let $2 \leqslant l \leqslant k$; we show that $T_{i,l}^{d/\gcd(d,k)}(T^w p) = 0$ in $E(\Lambda)$. We have $T_{i,l} = a_l a_{l+1} \cdots a_k a_1 \cdots a_{l-1}$, $T = a_1 a_2 \cdots a_k$, and $p = a_1 a_2 \cdots a_{\ell(p)}$ with $1 \leqslant \ell(p) \leqslant k$. If $T_{i,l}^{d/\gcd(d,k)}(T^w p)$ represents a non-zero element in $E(\Lambda)$, then $\mathfrak{t}(a_{l-1}) = \mathfrak{o}(a_1)$ so that $a_1 \cdots a_{l-1}$ is a closed trail. But $l - 1 < k$, so this contradicts the minimality of k. Hence $T_{i,l}^{d/\gcd(d,k)}(T^w p) = 0$ in $E(\Lambda)$ for $2 \leqslant l \leqslant k$.

A similar argument also shows that

$$\left(\sum_{l=1}^{k} T_{i,l}^{d/\gcd(d,k)}\right)^{c} = \sum_{l=1}^{k} (T_{i,l}^{d/\gcd(d,k)})^{c}.$$

Thus

$$R = (T^{d/\gcd(d,k)})^{c}(T^{w}p) = \sum_{l=1}^{k} (T_{i,l}^{d/\gcd(d,k)})^{c}(T^{w}p) = \left(\sum_{l=1}^{k} T_{i,l}^{d/\gcd(d,k)}\right)^{c}(T^{w}p).$$

Hence $R = x_i^c(T^w p)$ with x_i in H, and $T^w p \in \mathcal{R}^m$ for some $m \leqslant N$.

Hence each $R \in \mathcal{R}^n$ with $n > N$ can be written in the form hr for some $h \in H$ and $r \in \bigcup_{n=0}^{N} \mathcal{R}^n$. It follows that $\bigcup_{n=0}^{N} \mathcal{R}^n$ is a generating set for $E(\Lambda)$ as a left H-module. Thus we conclude that Λ has **(Fg)**. $\square$

Example 2. We return to Example 1. Condition 2.4 is satisfied: the only closed trails that are not loops are the cycles of length 3 (whose arrows are the γ_i); for all of these closed trails T, we have $\rho_T = \rho \setminus \{\alpha^3\}$. Hence by Theorem 2.7 the algebra $\Lambda = K\mathcal{Q}/I$ satisfies **(Fg)**.

2.3. Conditions equivalent to (Fg) for a d-Koszul monomial algebra. Our aim is now to prove the converse, and more precisely, the following theorem.

Theorem 2.8. *Let Λ be an indecomposable finite-dimensional d-Koszul monomial K-algebra with $d \geqslant 2$. Consider the following statements:*

(C1) *Λ satisfies* **(Fg)**.
(C2) *Condition 2.4 holds for Λ.*
(C3) *$Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module.*
(C4) *$E(\Lambda)$ is finitely generated as a module over $Z_{\mathrm{gr}}(E(\Lambda))$.*

Then (C4) *implies* (C2)*, which in turn implies* (C1).

Moreover, if the field K is algebraically closed, then the four statements are equivalent.

We shall need the description of the Ext algebra $E(\Lambda)$ from [**10**], which we recall here.

Let $\mathcal{Q}^{\mathrm{op}}$ be the opposite quiver of $\mathcal{Q}$, so that $\mathcal{Q}_0^{\mathrm{op}} = \mathcal{Q}_0$ and $\mathcal{Q}_1^{\mathrm{op}} = \{\overline{\alpha}\colon j \to i \mid$ there is $\alpha\colon i \to j$ in $\mathcal{Q}_1\}$.

Now consider ${}^!\Lambda = K\mathcal{Q}^{\mathrm{op}}/J$ with $J = ({}^{\perp}\rho)$, where the orthogonal is taken with respect to the natural bilinear form $K\mathcal{Q}_d^{\mathrm{op}} \times K\mathcal{Q}_d \to K$, that is, $\langle \overline{\beta}_d \cdots \overline{\beta}_1, \alpha_1 \cdots \alpha_d \rangle$ is equal to 1 if $\alpha_1 \cdots \alpha_d = \beta_1 \cdots \beta_d$ and is equal to 0 otherwise. (Recall that, for any $n \geqslant 0$, $\mathcal{Q}_n$ denotes the set of paths of length n in $\mathcal{Q}$.)

If $d = 2$, set $B = {}^!\Lambda$, and if $d \geqslant 3$, let $B = \bigoplus_{n \geqslant 0} B_n$ be the algebra defined as follows:

- $B_n = {}^!\Lambda\delta(n)$;
- for $x \in B_n$ and $y \in B_m$, define $x \cdot y \in B_{n+m}$ by

$$x \cdot y = \begin{cases} 0 & \text{if } n \text{ and } m \text{ are odd;} \\ xy & (\text{in } {}^!\Lambda) \text{ if } n \text{ or } m \text{ is even.} \end{cases}$$

Note that if n or m is even, $\delta(n) + \delta(m) = \delta(n+m)$, so that this defines a graded algebra B.

Then by [**11, 2, 10**], the algebras $E(\Lambda)$ and B are isomorphic (for $d \geqslant 2$).

Since Λ is monomial it is easy to see that the algebra ${}^!\Lambda$ is d-homogeneous monomial and that the set σ of paths $\overline{\alpha}_d\cdots\overline{\alpha}_1 \in \mathcal{Q}_d^{\mathrm{op}}$ such that $\alpha_1\cdots\alpha_d \in \mathcal{Q}_d$ is not in ρ is a minimal generating set for J consisting of paths of length d. There is a basis $\mathcal{B}_{{}^!\Lambda}$ of ${}^!\Lambda$ consisting of all paths $\overline{p}$ in $\mathcal{Q}^{\mathrm{op}}$ such that no subpath of $\overline{p}$ is in σ. It follows from Consequence 2.2 that no subpath of length d of $\overline{R}_i^n$ is in σ. Therefore $\overline{R}_i^n \in \mathcal{B}_{{}^!\Lambda}$.

As we mentioned in Subsection 1.2, there is a basis of $E(\Lambda)$ indexed by $\bigcup_{n\geqslant 0}\mathcal{R}^n$ that corresponds, via the isomorphism with the algebra B, to the set of paths $\overline{R}_i^n$ for $n \geqslant 0$ and $R_i^n \in \mathcal{R}^n$. We then have an embedding of the basis $\bigcup_{n\geqslant 0}\overline{\mathcal{R}}^n$ of B into $\mathcal{B}_{{}^!\Lambda}$, where $\overline{\mathcal{R}}^n = \{\overline{R}_i^n \mid R_i^n \in \mathcal{R}^n\}$; denote by $\mathcal{B}_B$ its image, which is a basis of B.

We now define several gradings on Λ, ${}^!\Lambda$, and B.

There are natural gradings on Λ and on ${}^!\Lambda$ given by the lengths of the paths; denote the length by ℓ for both algebras. The degree of a homogeneous element x in B will be denoted by $|x|$, so $x \in {}^!\Lambda_{\delta(|x|)}$ or, in other terms, $|x| = k$ if, and only if, $\ell(x) = \delta(k)$.

The algebra ${}^!\Lambda$ is also multi-graded by $\mathbb{N}^{\mathcal{Q}_1}$: for each path $\overline{p}$ in $\mathcal{Q}^{\mathrm{op}}$, we define an element $\mathfrak{d}(\overline{p}) = (\mathfrak{d}_\alpha(\overline{p}))_{\alpha\in\mathcal{Q}_1} \in \mathbb{N}^{\mathcal{Q}_1}$ as follows:

- if $\ell(\overline{p}) = 0$, then $\mathfrak{d}(\overline{p}) = (0)_{\alpha\in\mathcal{Q}_1}$;
- if $\ell(\overline{p}) > 0$, then $\mathfrak{d}_\alpha(\overline{p})$ is the number of occurrences of α in p (it is 0 if α does not occur in p).

Since ${}^!\Lambda$ is monomial, the ideal J is homogeneous with respect to this multi-degree and therefore ${}^!\Lambda$ is multi-graded. In B, if x and y are homogeneous and $|x|$ or $|y|$ is even, then $\mathfrak{d}_\alpha(xy) = \mathfrak{d}_\alpha(x) + \mathfrak{d}_\alpha(y)$, but if both degrees are odd, then $\mathfrak{d}_\alpha(xy) = 0$.

Let $Z := Z_{\mathrm{gr}}(B)$ be the graded centre of B. It is generated as a subring of B by the homogeneous elements $z \in B$ such that, for all homogeneous $y \in B$, $zy = (-1)^{|y||z|}yz$. Note that $Z \subset \bigoplus_{e\in\mathcal{Q}_0} eBe$; therefore Z is generated by elements that are linear combinations of (non-zero) cycles in $\mathcal{Q}^{\mathrm{op}}$.

Moreover, it can be checked easily that the graded centre Z of B is generated by elements z that are homogeneous with respect to the grading $|\cdot|$ and the multi-degree $\mathfrak{d}$ and such that, for any element $y \in B$ that is homogeneous with respect to the grading $|\cdot|$, we have $zy = (-1)^{|y||z|}yz$.

Remark 2.9. If $d = 2$, then $B \cong E(\Lambda)$ is generated in degrees 0 and 1, so in order to check that an element of B is in Z, it is sufficient to check that it is a linear combination of cycles and that it commutes or anti-commutes with all arrows in $\mathcal{Q}^{\mathrm{op}}$.

If $d \geqslant 3$, then $B \cong E(\Lambda)$ is generated in degrees 0, 1, and 2. Therefore, when checking that an element is in Z, we need to check that it is a linear combination of cycles and that it commutes or anti-commutes with paths of degrees 1 and 2, that is, arrows and (non-zero) paths of length d in $\mathcal{Q}^{\mathrm{op}}$.

The proof of Theorem 2.8 relies on some preliminary results. These are Lemma 2.10, Proposition 2.11, and Lemma 2.13. For the first of these, we start with the following observation about loops. Suppose α is a loop in $\mathcal{Q}_1$. Since Λ is finite-dimensional, there is some integer N such that $\alpha^N \in I$; therefore α^N has a subpath of length d that is in ρ and so $\alpha^d \in \rho$. Therefore $\overline{\alpha}^d \notin \sigma$ and it follows that $\overline{\alpha}^j \neq 0$ in ${}^!\Lambda$ for all $j \geqslant 0$ and that $\overline{\alpha}^{\delta(j)} \neq 0$ in B for all $j \geqslant 0$.

Lemma 2.10. *Let α be a loop in $\mathcal{Q}_1$ and let $n \geqslant 2$. Then*

(1) *if $d = 2$, $\overline{\alpha}^n \in Z$ if, and only if, n is even and α satisfies Condition 2.4(1);*
(2) *if $d \geqslant 3$, $\overline{\alpha}^{\delta(n)} \in Z$ if, and only if, α satisfies Condition 2.4(1).*

Proof: First note that if n is odd, then

- if $d = 2$, $\overline{\alpha}^n\overline{\alpha} = \overline{\alpha}^{n+1} \neq (-1)^n\overline{\alpha}\,\overline{\alpha}^n$, therefore $\overline{\alpha}^{\delta(n)} = \overline{\alpha}^n \notin Z$;
- if $d \geqslant 3$, $\overline{\alpha}^{\delta(n)}$ anti-commutes with all arrows since the products are 0 in B.

Therefore we may assume that $n \geqslant 2$ is an even integer and that $\overline{\alpha}^{\delta(n)} \in Z$. Set $e = \mathfrak{o}(\alpha)$.

Let β be an arrow ending at e with $\beta \neq \alpha$. Then, in B,

$$\overline{\alpha}^{\delta(n)+d-1}\overline{\beta} = \overline{\alpha}^{\delta(n)} \cdot \overline{\alpha}^{d-1}\overline{\beta} = \overline{\alpha}^{d-1}\overline{\beta} \cdot \overline{\alpha}^{\delta(n)} = \overline{\alpha}^{d-1}\overline{\beta}\overline{\alpha}^{\delta(n)}$$

(the element $\overline{\alpha}^{d-1}\overline{\beta}$ is in degree 2). Therefore we have an equality $\overline{\alpha}^{\delta(n)+d-1}\overline{\beta} = \overline{\alpha}^{d-1}\overline{\beta}\overline{\alpha}^{\delta(n)}$ between two paths in the monomial algebra ${}^!\Lambda$, so that both paths are zero. In particular, $\overline{\alpha}^{\delta(n)+d-1}\overline{\beta}$ contains a subpath in σ, and since $\overline{\alpha}^d \notin \sigma$, we must have $\overline{\alpha}^{d-1}\overline{\beta} \in \sigma$. It follows that $\beta\alpha^{d-1} \notin \rho$.

Similarly, if β is an arrow that starts at e with $\beta \neq \alpha$, then $\alpha^{d-1}\beta \notin \rho$.

Therefore α satisfies Condition 2.4(1).

Conversely, assume that α satisfies Condition 2.4(1).

Let $\beta \neq \alpha$ be an arrow and let $n \geqslant 2$. By assumption, $\alpha^{d-1}\beta \notin \rho$ and $\beta\alpha^{d-1} \notin \rho$. It follows that $\overline{\beta}\overline{\alpha}^{d-1} = 0 = \overline{\alpha}^{d-1}\overline{\beta}$ in ${}^!\Lambda$ (either in σ or not composable) and therefore that $\overline{\alpha}^{\delta(n)}\overline{\beta} = 0 = \overline{\beta}\overline{\alpha}^{\delta(n)}$ since $\delta(n) \geqslant \delta(2) \geqslant d-1$. The path $\overline{\alpha}^{\delta(n)}$ anti-commutes with all elements of degree 1.

In particular, if $d = 2$ and n is even, then $\overline{\alpha}^n \in Z$.

Now assume that $d \geqslant 3$ and consider commutation with elements in B_2. As a vector space, B_2 is generated by the paths $\overline{p}$ of length d such that $p \in \rho$. Let $p = \beta_1 \cdots \beta_d$ be a path in ρ. Since $\overline{p}$ has degree 2 in B, products of elements in B with $\overline{p}$ in B and in ${}^!\Lambda$ are equal.

If $p = \alpha^d$, then clearly $\overline{\alpha}^{\delta(n)}\overline{p} = \overline{p}\,\overline{\alpha}^{\delta(n)}$.

If $p \neq \alpha^d$, set $j = \min\{i \mid 1 \leqslant i \leqslant d,\ \beta_i \neq \alpha\}$ and $k = \max\{i \mid 1 \leqslant i \leqslant d,\ \beta_i \neq \alpha\}$. By assumption, $\alpha^{d-1}\beta_j \notin \rho$ and $\beta_k\alpha^{d-1} \notin \rho$, therefore $\overline{\beta}_j\overline{\alpha}^{d-1} = 0$ and $\overline{\alpha}^{d-1}\overline{\beta}_k = 0$ in ${}^!\Lambda$. We have assumed that $n \geqslant 2$, so $\delta(n) \geqslant d$ and therefore $\overline{\alpha}^{\delta(n)}\overline{p} = 0 = \overline{p}\,\overline{\alpha}^{\delta(n)}$ in ${}^!\Lambda$ and in B, and $\overline{\alpha}^{\delta(n)}$ anti-commutes with elements of degree 2.

Finally, we have proved that $\overline{\alpha}^{\delta(n)}$ is in Z whenever $d = 2$ and $n \geqslant 2$ is even or $d \geqslant 3$ and $n \geqslant 2$. □

For the next result, we need some more terminology for closed trails. Let $n \geqslant 2$ and let $T = \alpha_1 \cdots \alpha_n$ be a closed trail in $\mathcal{Q}$ with $\rho_T = \{\alpha_i \cdots \alpha_{i+d-1} \mid 1 \leqslant i \leqslant n\} \subseteq \rho$. A *subcycle* of T is a cycle of the form $q = \alpha_i \cdots \alpha_j$ with $1 \leqslant i \leqslant j \leqslant n$ and $\ell(q) < \ell(T)$. We say that T has a *repeated vertex* if $T = \alpha_1 \cdots \alpha_{i-1} v \alpha_i \cdots \alpha_{i+k-1} v \alpha_{i+k} \cdots \alpha_n$ for some i, k, and vertex v such that the paths $\alpha_i \cdots \alpha_{i+k-1}$ and $\alpha_{i+k} \cdots \alpha_n\alpha_1 \cdots \alpha_{i-1}$ are non-trivial paths in $K\mathcal{Q}$.

We make the following assumptions that we use in Proposition 2.11 and Lemma 2.13. The reason for these specific assumptions becomes clear in the proof of Theorem 2.8.

(i) None of the α_i are loops.
(ii) No subcycle of T satisfies the same assumptions as T (that is, there is no subcycle q of T of length at least 2 with $\rho_q \subseteq \rho$).

Proposition 2.11. *Let $T = \alpha_1 \cdots \alpha_n$ be a closed trail with $n \geqslant 2$, $\rho_T \subseteq \rho$, and such that assumptions (i) and (ii) hold. Let p be a path of length d such that $\mathfrak{d}_\beta(\overline{p}) = 0$ if $\beta \notin \{\alpha_1, \ldots, \alpha_n\}$ and which does not lie on T. Then $p \notin \rho$.*

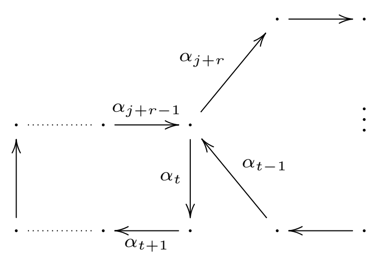
Proof: Let $p = \gamma_1 \cdots \gamma_d$ with $\gamma_i \in \{\alpha_1, \ldots, \alpha_n\}$ for $i = 1, \ldots, d$. The path p is a non-zero path in $K\mathcal{Q}$ so $\mathfrak{t}(\gamma_i) = \mathfrak{o}(\gamma_{i+1})$ for all i. Suppose that $\gamma_1 = \alpha_j$ so $p = \alpha_j\gamma_2 \cdots \gamma_d$ and $\gamma_2 \in \{\alpha_1, \ldots, \alpha_n\}$ with $\mathfrak{t}(\alpha_j) = \mathfrak{o}(\alpha_{j+1}) = \mathfrak{o}(\gamma_2)$. If T does not have a repeated vertex, then necessarily $\alpha_{j+1} = \gamma_2$. Inductively, $p = \alpha_j\alpha_{j+1} \cdots \alpha_{j+d-1}$ and hence p lies on the trail T. This contradicts our hypothesis. Hence T has a repeated vertex.

Suppose that v is a repeated vertex so that T has a proper subpath q of length k with $q = vqv$ for some k; thus $2 \leqslant k \leqslant n-1$ since T does not have any loops and q is a closed trail. We claim that $k \geqslant d$. Indeed, if we had $k < d$, then by Consequence 2.3 every path of length d that lies on q would be in ρ, that is, $\rho_q \subseteq \rho$, with $\ell(q) < \ell(T)$. But this contradicts assumption (ii). Hence $k \geqslant d$.

Now suppose for contradiction that $p \in \rho$. As above, let α_j be the first arrow in p. We know that p does not lie on T and that T has a repeated vertex, so we may write

$$p = \alpha_j \cdots \alpha_{j+r-1}\gamma_{r+1} \cdots \gamma_d,$$

where $r \geqslant 1$, $\gamma_{r+1}, \ldots, \gamma_d \in \{\alpha_1, \ldots, \alpha_n\}$, $\mathfrak{t}(\alpha_{j+r-1}) = \mathfrak{o}(\alpha_{j+r}) = \mathfrak{o}(\gamma_{r+1})$, and $\gamma_{r+1} \neq \alpha_{j+r}$. Then there is some t such that $\gamma_{r+1} = \alpha_t$ and $t \not\equiv j + r \pmod{n}$. Moreover, $\mathfrak{t}(\alpha_{t-1}) = \mathfrak{o}(\alpha_t) = \mathfrak{o}(\alpha_{j+r})$. We may illustrate this as follows:



(We make no assumption as to whether γ_{r+2} is or is not equal to α_{t+1}.) We note that the path $\alpha_{j+r} \cdots \alpha_{t-1} \cdot \alpha_t \cdots \alpha_{j+r-1}$ is a cyclic permutation of T and has length n. Moreover, from the previous part of this proof, both $\alpha_{j+r} \cdots \alpha_{t-1}$ and $\alpha_t \cdots \alpha_{j+r-1}$ are paths of length at least d.

Let $S = \alpha_t\alpha_{t+1} \cdots \alpha_{j+r-1}$. Then S is a closed path in $K\mathcal{Q}$ of length at least 2 and is a subcycle of T. There is an overlap $\alpha_{j+r-d} \cdots \alpha_j \cdots \alpha_{j+r-1}$ with p so the subpath $\alpha_{j+r-d+1} \cdots \alpha_{j+r-1}\alpha_t$ must also be in ρ. Then we have an overlap of $\alpha_{j+r-d+1} \cdots \alpha_{j+r-1}\alpha_t$ with $\alpha_t\alpha_{t+1} \cdots \alpha_{t+d-1}$; by Property 2.1, all subpaths of length d of the path

$$\alpha_{j+r-d+1} \cdots \alpha_{j+r-1}\alpha_t\alpha_{t+1} \cdots \alpha_{t+d-1}$$

must also be in ρ. Thus every path of length d that lies on S is in ρ. Hence $\rho_S \subseteq \rho$.

So S is a subcycle of T that satisfies the same assumptions as T, and this contradicts assumption (ii). Hence $p \notin \rho$. $\square$

Remark 2.12. We keep the assumptions and notation of Proposition 2.11. Then $\overline{T}$ and all the paths lying on $\overline{T}$ are in the basis $\mathcal{B}_{{}^!\Lambda}$ since none of their subpaths of length d are in σ; those of length $\delta(k)$ for some $k \geqslant 0$ are in the basis $\mathcal{B}_B$.

Set $T_i = \alpha_i \cdots \alpha_n\alpha_1 \cdots \alpha_{i-1}$ so that $\overline{T}_i = \overline{\alpha}_{i-1} \cdots \overline{\alpha}_1\overline{\alpha}_n \cdots \overline{\alpha}_i \in {}^!\Lambda$ for all i. Then for any $j \geqslant 1$ we have

$$\overline{T}_i^j\overline{\alpha}_k \neq 0 \iff k = i - 1,$$

$$\overline{\alpha}_k\overline{T}_i^j \neq 0 \iff k = i.$$

Lemma 2.13. *Let $T = \alpha_1 \cdots \alpha_n$ be a closed trail with $n \geqslant 2$ and $\rho_T \subseteq \rho$ that satisfies assumptions (i) and (ii) and set $z_j = \sum_{i=1}^n \overline{T}_i^j$ with $nj = \delta(u)$ for some integer $u \geqslant 2$ and with $nj = u$ even if $d = 2$. Then $z_j \in Z$ if, and only if, T satisfies Condition 2.4(2).*

Moreover, if T does not satisfy Condition 2.4(2), then no element in B that is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$ (when viewed in ${}^!\Lambda$) and that is a linear combination of non-trivial cycles that lie on $\overline{T}$ is in Z.

Proof: First assume that $z_j \in Z$. Fix an integer i and let $e = \mathfrak{t}(\alpha_i)$. Suppose for contradiction that there is a path p of length $d-1$ starting at e such that $\alpha_i p \in \rho$ and $p \neq \alpha_{i+1} \cdots \alpha_{i+d-1}$. By Remark 2.12, at least one arrow in p is not in $\{\alpha_1, \ldots, \alpha_n\}$, therefore $\overline{p}\,\overline{\alpha}_i$ is not a subpath of any of the paths that occur in z_j (note that $\ell(z_j) \geqslant \delta(2) = d \geqslant \ell(\overline{p})$). The relation $\overline{\alpha_i p} z_j = z_j \overline{\alpha_i p}$ in B becomes, in ${}^!\Lambda$,

$$\overline{p}\,\overline{\alpha}_i \overline{T}_i^j = z_j \overline{p}\,\overline{\alpha}_i.$$

Since ${}^!\Lambda$ is monomial, it follows that $\overline{p}\,\overline{\alpha}_i \overline{T}_i^j$ contains a subpath in σ, that is, there is a subpath of length d of $T_i^j \alpha_i p$ that is not in ρ. It cannot be a subpath of $T_i^j \alpha_i$ because we have assumed that $\rho_T \subseteq \rho$. Moreover, we have assumed that $\alpha_i p \in \rho$. We also have $\alpha_{i-d+1} \cdots \alpha_{i-1}\alpha_i \in \rho_T \subseteq \rho$ and ρ is d-covering (Property 2.1); therefore every subpath of length d of $\alpha_{i-d+1} \cdots \alpha_{i-1}\alpha_i p$ is also in ρ and so is every subpath of length d of $T_i^j \alpha_i p$, and we have obtained a contradiction. Therefore T satisfies Condition 2.4(2).

Conversely, assume that T satisfies Condition 2.4(2). We prove that, for all $j \geqslant 1$ such that $nj = \delta(u)$ for some integer $u \geqslant 2$, and such that $nj = u$ is even if $d = 2$, we have $z_j \in Z$.

- First note that if $d \geqslant 3$ and $|z_j|$ is odd, then z_j anti-commutes with all arrows (the products are 0 in B). Therefore assume that $|z_j|$ is even and that $d \geqslant 2$, and let β be an arrow.

  If $\beta = \alpha_k$ for some k, then $\overline{\alpha}_k z_j = \overline{\alpha}_k \overline{T}_k^j$ and $z_j \overline{\alpha}_k = \overline{T}_{k+1}^j \alpha_k$ using Remark 2.12, and these paths are indeed equal, so that $\overline{\beta} z_j = z_j \overline{\beta} = (-1)^{|z_j||\overline{\beta}|} z_j \overline{\beta}$.

  If $\beta \notin \{\alpha_1, \ldots, \alpha_n\}$, then $\overline{\beta}\,\overline{T}_i^j = 0 = \overline{T}_i^j \overline{\beta}$ for all i by assumption; therefore $\overline{\beta} z_j = 0 = (-1)^{|z_j||\overline{\beta}|} z_j \overline{\beta}$.
- Now assume that $d \geqslant 3$ (and $|z_j|$ is still even) and consider commutation with elements in B_2. As a vector space, B_2 is generated by the paths $\overline{p}$ of length d such that $p \in \rho$. Let $p = \beta_1 \cdots \beta_d$ be a path in ρ with $\beta_i \in \mathcal{Q}_1$ for all i. Since $\overline{p}$ has degree 2 in B, products of elements in B with $\overline{p}$ in B and in ${}^!\Lambda$ are equal.

  If p lies on T, then $p = \alpha_k \cdots \alpha_{k+d-1}$ for some k, and it is easy to check that $\overline{p} z_j = z_j \overline{p} = (-1)^{|z_j||\overline{p}|} z_j \overline{p}$ (as for commutation with an arrow).

  If p does not lie on T, then by Condition 2.4(2), the first and last arrows in p are not in $\{\alpha_1, \ldots, \alpha_n\}$. Moreover, for all i, we have $\alpha_i \beta_1 \cdots \beta_{d-1} \notin \rho$; it follows that $\overline{p} z_j = 0$. Similarly, for all i, we have $\beta_2 \cdots \beta_d \alpha_i \notin \rho$ (since it ends with α_i and is not in ρ_T), therefore $z_j \overline{p} = 0$. Finally, $\overline{p} z_j = z_j \overline{p} = (-1)^{|z_j||\overline{p}|} z_j \overline{p}$.

We have proved that $z_j \in Z$.

Now let z be an element in Z that is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$ and that is a linear combination of cycles that lie on $\overline{T}$. Assume that $\ell(z) > 0$; by (i) z is not a linear combination of arrows so $|z| \geqslant 2$. Put $z = \sum_{i=1}^m \lambda_i c_i$ with $\lambda_i \in K$ and the c_i cycles in $\mathcal{Q}^{\mathrm{op}}$ that lie on $\overline{T}$. Since z is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$ and the c_i lie on $\overline{T}$, the c_i are cyclic permutations of c_1. Up to relabelling, we may write $c_i = \overline{T}_i^{j-1} \overline{\alpha}_{i-1} \cdots \overline{\alpha}_{i-s}$ for some fixed s with $1 \leqslant s \leqslant n$ (and $m = n$).

We first consider the case where $|z|$ is even. Then we must have $\overline{\alpha}_k z = z\overline{\alpha}_k$ for all k, that is, $\sum_{i=1}^n \lambda_i \overline{\alpha}_k \overline{T}_i^{j-1} \overline{\alpha}_{i-1}\cdots\overline{\alpha}_{i-s} = \sum_{i=1}^n \lambda_i \overline{T}_i^{j-1}\overline{\alpha}_{i-1}\cdots\overline{\alpha}_{i-s}\overline{\alpha}_k$. Using Remark 2.12, this is equivalent to

$$\lambda_k \overline{\alpha}_k \overline{T}_k^{j-1}\overline{\alpha}_{k-1}\cdots\overline{\alpha}_{k-s} = \lambda_{k+s+1}\overline{T}_{k+s+1}^{j-1}\overline{\alpha}_{k+s}\cdots\overline{\alpha}_{k+1}\overline{\alpha}_k.$$

Therefore $\lambda_k = 0 = \lambda_{k+s+1}$ or $k+s \equiv k \pmod n$ (that is, $s=n$), and $\lambda_k = \lambda_{k+1}$.

This is true for all k, so if $z \neq 0$, then $z = \lambda_1 z_j$ with $nj = \ell(z) = \delta(|z|)$.

We now consider the case where $|z|$ is odd.

If $d=2$, then the same reasoning as in the even case shows that $\lambda_{k+1} = (-1)^{|z|}\lambda_k$ for all k with $1 \leqslant k \leqslant n-1$ and $\lambda_1 = (-1)^{|z|}\lambda_n$; therefore $\lambda_1 = (-1)^{n|z|}\lambda_1 = -\lambda_1$ (because $nj = \ell(z) = |z|$ is odd, hence n is odd) so that $\lambda_k = 0$ for all k and finally $z=0$.

Now assume that $d \geqslant 3$. Since $z \in Z$, we have $\overline{\alpha}_{k+d-1}\cdots\overline{\alpha}_k z = z\overline{\alpha}_{k+d-1}\cdots\overline{\alpha}_k$ for all k, that is,

$$\sum_{i=1}^n \lambda_i \overline{\alpha}_{k+d-1}\cdots\overline{\alpha}_k \overline{T}_i^{j-1}\overline{\alpha}_{i-1}\cdots\overline{\alpha}_{i-s} = \sum_{i=1}^n \lambda_i \overline{T}_i^{j-1}\overline{\alpha}_{i-1}\cdots\overline{\alpha}_{i-s}\overline{\alpha}_{k+d-1}\cdots\overline{\alpha}_k.$$

Using Remark 2.12, this is equivalent to

$$\lambda_k\overline{\alpha}_{k+d-1}\cdots\overline{\alpha}_k\overline{T}_k^{j-1}\overline{\alpha}_{k-1}\cdots\overline{\alpha}_{k-s} = \lambda_{k+s+d}\overline{T}_{k+s+d}^{j-1}\overline{\alpha}_{k+s+d-1}\cdots\overline{\alpha}_{k+d}\overline{\alpha}_{k+d-1}\cdots\overline{\alpha}_k.$$

Therefore $\lambda_k = 0 = \lambda_{k+s+d}$ or $k+s+d-1 \equiv k+d-1 \pmod n$ (that is, $s=n$), and $\lambda_k = \lambda_{k+d}$.

When $s=n$, we have $nj = \ell(z) = \delta(|z|) = \frac{|z|-1}{2}d+1$, therefore n and d are coprime. It follows that if $z \neq 0$, then all the λ_i are equal so that $z = \lambda_1 z_j$.

We have proved that if z is a non-zero element in Z that is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$ and which is a linear combination of non-trivial cycles that lie on $\overline{T}$, then if $d=2$, we must have $|z|$ even, and for all $d \geqslant 2$, z is then a non-zero scalar multiple of z_j. Therefore z_j is in Z and by the first part of the proof, T satisfies Condition 2.4(2). $\square$

We now have all the tools we need for the proof of Theorem 2.8.

Proof of Theorem 2.8: The fact that (C2) implies (C1) is Theorem 2.7. The implication (C3) $\Rightarrow$ (C4) is clear, and if in addition K is algebraically closed, then the implication (C1) $\Rightarrow$ (C3) follows from [**7**].

We now prove that (C4) implies (C2). Suppose that (C2) does not hold, that is, Condition 2.4 does not hold. Assume for contradiction that B is a finitely generated Z-module, generated by elements $\overline{g}_1,\ldots,\overline{g}_t$ that are homogeneous with respect to both the grading $|\cdot|$ and the multi-grading $\mathfrak{d}$.

We first assume that Condition 2.4(1) does not hold, so that there is a loop α that does not satisfy this condition. Then for all $j \geqslant 2$, $\overline{\alpha}^{\delta(j)} \notin Z$ by Lemma 2.10.

Now consider $\overline{\alpha}^{\delta(k)} \in B$ for some even integer $k \geqslant 2$. Then

$$\mathfrak{d}_\beta(\overline{\alpha}^{\delta(k)}) = \begin{cases} \delta(k) & \text{if } \beta = \alpha; \\ 0 & \text{if } \beta \neq \alpha \end{cases}$$

and $|\overline{\alpha}^{\delta(k)}| = k$. By assumption, and using the fact that $\overline{\alpha}^{\delta(k)}, \overline{g}_1,\ldots,\overline{g}_t$ are homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, there exist elements $\overline{u}_i^{(k)}$ in Z, $1 \leqslant i \leqslant t$, that are homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, such that $\overline{\alpha}^{\delta(k)} = \sum_{i=1}^t \overline{u}_i^{(k)}\overline{g}_i$.

Since $\overline{\alpha}^{\delta(k)}$ is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, we can assume that for all i we have $|\overline{u}_i^{(k)}\overline{g}_i| = k$ and $\mathfrak{d}(\overline{u}_i^{(k)}) + \mathfrak{d}(\overline{g}_i) = \mathfrak{d}(\overline{u}_i^{(k)}\overline{g}_i) = \mathfrak{d}(\overline{\alpha}^{\delta(k)})$. If $\beta \neq \alpha$, then $\mathfrak{d}_\beta(\overline{u}_i^{(k)}) + \mathfrak{d}_\beta(\overline{g}_i) = 0$ so $\mathfrak{d}_\beta(\overline{u}_i^{(k)}) = 0$ and $\mathfrak{d}_\beta(\overline{g}_i) = 0$. It follows that $\overline{u}_i^{(k)}$ and $\overline{g}_i$ are powers of $\overline{\alpha}$; since $\overline{u}_i^{(k)} \in Z$, we must have $|u_i^{(k)}| = 0$ or 1 by assumption. If $|\overline{u}_i^{(k)}| = 1$, then $|\overline{g}_i| = k-1$ is odd and hence, in B, we have $\overline{u}_i^{(k)}\overline{g}_i = 0$. Therefore we may assume that $\overline{u}_i^{(k)} \in Z^0 = K$. It follows that the sum contains only one term and that $\overline{g}_i$ is a (non-zero) scalar multiple of $\overline{\alpha}^{\delta(k)}$ so that $\overline{\alpha}^{\delta(k)} \in \operatorname{span}_K\{\overline{g}_1, \dots, \overline{g}_t\}$.

We have shown that $\{\overline{\alpha}^{\delta(k)} \mid k \geqslant 1,\ k \text{ even}\} \subseteq \operatorname{span}_K\{\overline{g}_1, \dots, \overline{g}_t\}$. However, using the grading $|\cdot|$, we see that the $\overline{\alpha}^{\delta(k)}$, $k \geqslant 1$, are linearly independent over K: we have reached a contradiction.

Therefore the Yoneda algebra $E(\Lambda) = B$ is not finitely generated as a Z-module when Condition 2.4(1) does not hold.

We now assume that Condition 2.4(2) does not hold, so that there is a closed trail $T = \alpha_1 \cdots \alpha_n$ with $n \geqslant 2$ and $\rho_T \subseteq \rho$ that does not satisfy this condition. We can make the following assumptions (and therefore use Proposition 2.11):

(i) None of the α_i are loops. Indeed, if α_i is a loop, then the paths $\alpha_i\alpha_{i+1}\cdots\alpha_{i+d-1}$ and α_i^d are in ρ and properly overlap, hence $\alpha_i^{d-1}\alpha_{i+1}$ is in ρ because ρ is d-covering, therefore α does not satisfy Condition 2.4(1), and in this case we already know that $E(\Lambda)$ is not a finitely generated Z-module.

(ii) No subcycle of T satisfies the same hypotheses as T (otherwise replace T with the shortest such subcycle).

We have seen in Lemma 2.13 that no linear combination of non-trivial cycles that lie on $\overline{T}$, that is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, is in Z.

Now consider $z_{\delta(k)} = \sum_{i=1}^n \overline{T}_i^{\delta(k)} \in B$ for some even integer $k \geqslant 2$. Then $|z_{\delta(k)}| = nk$ and

$$\mathfrak{d}_\beta(z_{\delta(k)}) = \begin{cases} \delta(k) & \text{if } \beta \in \{\alpha_1, \dots, \alpha_n\}; \\ 0 & \text{if } \beta \notin \{\alpha_1, \dots, \alpha_n\}. \end{cases}$$

Since $z_{\delta(k)}$ and the $\overline{g}_i$ are homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, by assumption there exist elements $\overline{u}_i^{(k)}$ in Z, $1 \leqslant i \leqslant t$, that are homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, such that $z_{\delta(k)} = \sum_{i=1}^t \overline{u}_i^{(k)}\overline{g}_i$. Note that $Z \subseteq \bigoplus_{e \in \mathcal{Q}_0} e^! \Lambda e$, therefore the $\overline{u}_i^{(k)}$ are linear combinations of cycles.

Fix an integer j with $1 \leqslant j \leqslant n$. Then $\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j\overline{T}_j^{\delta(k)} = \overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j\overline{z}_{\delta(k)} = \sum_{i=1}^t \overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j\overline{u}_i^{(k)}\overline{g}_i$.

Since $\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j\overline{T}_j^{\delta(k)}$ is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$, we can assume that for all i we have $2 + |\overline{u}_i^{(k)}| + |\overline{g}_i| = |\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j\overline{u}_i^{(k)}\overline{g}_i| = |\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j z_{\delta(j)}| = nk+2$ and $\mathfrak{d}(\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j) + \mathfrak{d}(\overline{u}_i^{(k)}) + \mathfrak{d}(\overline{g}_i) = \mathfrak{d}(\overline{\alpha}_{j+d-1}\cdots\alpha_j\overline{u}_i^{(k)}\overline{g}_i) = \mathfrak{d}(\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j\overline{T}_i^{\delta(k)}) = \mathfrak{d}(\overline{\alpha}_{j+d-1}\cdots\overline{\alpha}_j) + \mathfrak{d}(\overline{T}_i^{\delta(k)})$, and therefore the only arrows that occur in $\overline{u}_i^{(k)}\overline{g}_i$ are the $\overline{\alpha}_j$, $1 \leqslant j \leqslant n$. Moreover, Proposition 2.11 and the fact that the $\overline{u}_i^{(k)}$ are linear combinations of cycles show that the $\overline{u}_i^{(k)}$ must be linear combinations of cycles lying on $\overline{T}$ (otherwise, one at least of these cycles has a subpath of length d that does not lie on $\overline{T}$, hence that is in σ, and this cycle vanishes in B and does not occur in $u_i^{(k)}$). Our assumption shows that these cycles must be trivial (of length 0), and since $Z^0 = K$ we see that $\overline{u}_i^{(k)} \in K$. Therefore $\overline{\alpha}_{j+d-1} \cdots \overline{\alpha}_j \overline{T}_j^{\delta(k)}$ is a linear combination of the $\overline{\alpha}_{j+d-1} \cdots \overline{\alpha}_j \overline{g}_i$, hence $\{\overline{\alpha}_{j+d-1} \cdots \overline{\alpha}_j \overline{T}_j^{\delta(k)} \mid 1 \leqslant j \leqslant n,\, k \geqslant 1,\, k \text{ even}\} \subseteq \operatorname{span}_K\{\overline{\alpha}_{j+d-1} \cdots \overline{\alpha}_j \overline{g}_i \mid 1 \leqslant j \leqslant n,\, 1 \leqslant i \leqslant t\}$.

The set $\{\overline{\alpha}_{j+d-1} \cdots \overline{\alpha}_J \overline{T}_J^{\delta(k)} \mid 1 \leqslant j \leqslant n,\, k \geqslant 2,\, k \text{ even}\}$ is linearly independent over K (using the grading $|\cdot|$), therefore we have reached a contradiction.

Therefore the Yoneda algebra $E(\Lambda) = B$ is not finitely generated as a Z-module when Condition 2.4(2) does not hold. Hence (C4) $\Rightarrow$ (C2). ☐

Remark 2.14. In the case where K is algebraically closed, we have extended the equivalence between (C1) and (C3), already known for Koszul algebras from [**7**], to d-Koszul monomial algebras with $d \geqslant 3$. In particular, we have the following corollary.

Corollary 2.15. *Let Λ be a d-Koszul monomial algebra over an algebraically closed field with $d \geqslant 2$. Assume that $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module. Then the algebra $Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian.*

3. Extension to (D, A)-stacked monomial algebras

3.1. Notation and properties of (D, A)-stacked monomial algebras. Let $\Lambda = K\mathcal{Q}/I$ be a monomial algebra with the length grading as before. Let D and A be integers with $D > A \geqslant 1$. From [**13**, Definition 3.1], Λ is then a (D, A)-stacked monomial algebra if, for any minimal projective right Λ-module resolution of Λ_0, the n-th projective module is generated in degree $\delta_A(n)$, where

$$\delta_A(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1; \\ \dfrac{n}{2}D & \text{if } n \geqslant 2 \text{ is even;} \\ \dfrac{n-1}{2}D + A & \text{if } n \geqslant 3 \text{ is odd.} \end{cases}$$

When $A = 1$, we retrieve the definition of a D-Koszul algebra, so that a $(D, 1)$-stacked monomial algebra is a D-Koszul monomial algebra.

It was shown in [**13**, Proposition 3.3] that if $\operatorname{gldim} \Lambda \geqslant 4$, then A divides D; in particular, $D \geqslant 2A$. If the global dimension of Λ is finite, then Condition **(Fg)**, and in fact all the conditions (C1)–(C6) stated in the introduction, are satisfied by Λ. Therefore we shall assume throughout this section that Λ is a (D, A)-stacked monomial algebra with $\operatorname{gldim} \Lambda \geqslant 4$ and set $d = \frac{D}{A}$. We define

$$\delta(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1; \\ \dfrac{n}{2}d = \dfrac{\delta_A(n)}{A} & \text{if } n \geqslant 2 \text{ is even;} \\ \dfrac{n-1}{2}d + 1 = \dfrac{\delta_A(n)}{A} & \text{if } n \geqslant 3 \text{ is odd.} \end{cases}$$

Definition 3.1. For $A \geqslant 1$, we define an *A-path* as a non-zero path $p = \alpha_1 \cdots \alpha_n$, where all the α_i are paths of length A (that is, $\alpha_i \in \mathcal{Q}_A$ for all i). An *A-trail* is an A-path in which all the α_i are distinct. An *A-cycle* is a closed A-path and finally an *A-loop* is an A-cycle of length A.

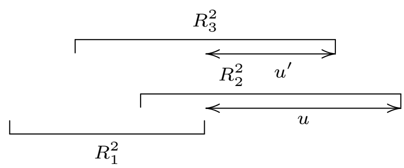
Given an A-path p as above, an *A-subpath* of p is an A-path of the form $\alpha_i \cdots \alpha_j$ with $1 \leqslant i \leqslant j \leqslant n$ (note that not every A-path that is a subpath of p is an A-subpath of p). An *A-subcycle* of p is a closed A-subpath of one of the non-zero A-paths $\alpha_i \cdots \alpha_n \alpha_1 \cdots \alpha_{i-1}$ with $1 \leqslant i \leqslant n$.

We also define the *A-length* $\ell_A(p)$ of an A-path $p = \alpha_1 \cdots \alpha_n$, where the α_i are paths of length A, as $\ell_A(p) = n$, that is, $\ell(p) = A\ell_A(p)$.

We will need the following result from [**13**].

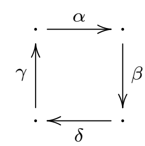
Property 3.2 ([**13**, Section 3])**.** *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional monomial algebra. Then Λ is (D, A)-stacked if, and only if, $\rho = \mathcal{R}^2$ has the following properties:*

(1) *every path in ρ is of length D;*
(2) *if $R_2^2 \in \mathcal{R}^2$ properly overlaps $R_1^2 \in \mathcal{R}^2$ with overlap $R_1^2 u$, then $\ell(u) \geqslant A$ and there exists $R_3^2 \in \mathcal{R}^2$ which properly overlaps R_1^2 with overlap $R_1^2 u'$, $\ell(u') = A$, and u' is a prefix of u.*



Therefore ρ consists of paths of length D, and if Λ is (D, A)-stacked with $\operatorname{gldim} \Lambda \geqslant 4$, we view ρ as a set of A-paths of A-length d.

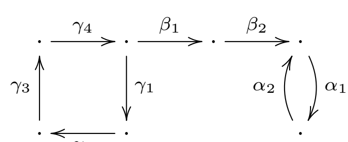
Example 3. We include first an example from [**8**, Example 3.2]. Let $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the quiver



and the ideal I has minimal generating set $\rho = \{\alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta\}$. Then Λ is a $(6, 2)$-stacked monomial algebra.

The closed 2-trails are all the paths of length 4.

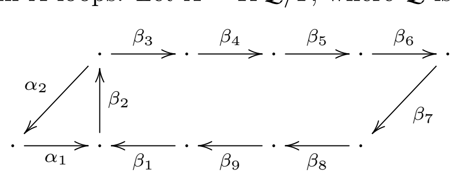
Example 4. Now we give an example where, as well as closed A-trails, there are A-loops. Let $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the quiver



and the ideal I has minimal generating set $\rho = \{(\alpha_1\alpha_2)^2, (\gamma_1\gamma_2)(\gamma_3\gamma_4), (\gamma_3\gamma_4)(\gamma_1\gamma_2)\}$. Then Λ is a $(4, 2)$-stacked monomial algebra.

The closed 2-trails are the paths of length 4 whose arrows are the γ_i and the 2-loops are $\alpha_1\alpha_2$ and $\alpha_2\alpha_1$.

Example 5. Finally, we give an example in which an arrow, namely β_2, occurs both in closed A-trails and in A-loops. Let $\Lambda = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the quiver



and the ideal I has minimal generating set

$$\rho = \{(\alpha_1\beta_2\alpha_2)^2, (\beta_1\beta_2\beta_3)(\beta_4\beta_5\beta_6), (\beta_4\beta_5\beta_6)(\beta_7\beta_8\beta_9), (\beta_7\beta_8\beta_9)(\beta_1\beta_2\beta_3)\}.$$

Then Λ is a $(6, 3)$-stacked monomial algebra.

The closed 3-trails are all the cycles of length 9 whose arrows are the β_i and the 3-loops are $\alpha_1\beta_2\alpha_2$, $\alpha_2\alpha_1\beta_2$, and $\beta_2\alpha_2\alpha_1$.

We have the following consequences of Property 3.2.

Consequence 3.3. *We keep the notation of Property 3.2, with $D = dA$. Then the length of u must be a multiple of A, so that R_1^2u is an A-path, and every A-subpath of A-length d of R_1^2u is in ρ. Moreover, no other subpath of length D of R_1^2u is in ρ.*

Proof: Write $\ell(u) = qA + r$ with $q \geqslant 1$ and $0 \leqslant r < A$. We prove the result by induction on q.

If $q = 1$, then the path $R_2^2 \in \rho$ overlaps $R_3^2 \in \rho$ with overlap $R_3^2u_3$ for some path u_3. If this overlap is a proper overlap (that is, $R_3^2 \neq R_2^2$), then $\ell(u) = \ell(u') + \ell(u_3) = A + \ell(u_3)$ so that $\ell(u_3) = (q-1)A + r = r$ and $0 < r < A$. Therefore by Property 3.2 we have a contradiction. It follows that $R_3^2 = R_2^2$ and $u = u'$ has length A and that R_1^2 and R_3^2 are the only A-subpaths of A-length d of R_1^2u and they are in ρ. Moreover, any other subpath of length D of R_1^2u is a proper overlap of R_1^2 of length strictly smaller than $D + A$, which is impossible by Property 3.2.

Let $q > 1$ be such that $\ell(u) = qA + r$ with $0 \leqslant r < A$ and assume that the result is true for any proper overlap of a path in ρ of length $D + q'A + r'$ with $q' < q$ and $0 \leqslant r' < A$. The path $R_2^2 \in \rho$ properly overlaps $R_3^2 \in \rho$ with overlap $R_3^2u_3$ for some path u_3 with $\ell(u) = \ell(u') + \ell(u_3) = A + \ell(u_3)$ so that $\ell(u_3) = (q-1)A + r$ and the overlap $R_3^2u_3$ has length $D + (q-1)A + r$. By induction, $\ell(u_3)$ is a multiple of A, therefore $r = 0$ and $\ell(u)$ is a multiple of A. Any A-subpath of A-length d of R_1^2u is either R_1^2 or an A-subpath of A-length d of $R_3^2u_3$. Again by induction, they are all in ρ. Finally, a subpath of length D of R_1^2u which is not an A-subpath either is a subpath of length D of $R_3^2u_3$ that is not an A-subpath, therefore not in ρ by induction, or properly overlaps R_1^2 with overlap R_1^2u''' with $0 < \ell(u''') < A$, which is impossible by Property 3.2. $\square$

Consequence 3.4. *Suppose that $D = dA$. Let $n \geqslant 2$ and let R_i^n be an element of $\mathcal{R}^n$. Write $R_i^n = \alpha_1 \cdots \alpha_{\delta(n)}$, where each α_i is a path of length A. Then for all i with $1 \leqslant i \leqslant \delta(n) - d + 1$, the path $\alpha_i \cdots \alpha_{i+d-1}$ is in ρ, that is, all the A-subpaths of A-length d of R_i^n are in ρ. Moreover, no other subpath of R_i^n of length D is in ρ.*

Proof: The result is proved by induction. It is clear when $n = 2$. Moreover, if $n = 3$, since $R_i^3 \in \mathcal{R}^3$ is a maximal overlap of two elements in $\mathcal{R}^2$, it follows from Property 3.2 and using the notation therein that $R_i^3 = R_1^2u' = v'R_3^2$, where v' is the prefix of R_1^2 of length A. By Consequence 3.3, the only subpaths of length D of R_i^3 that are in $\rho = \mathcal{R}^2$ are R_1^2 and R_3^2.

Now let $n \geqslant 4$ and take $R_i^n \in \mathcal{R}^n$. Then R_i^n is a maximal overlap of $R_1^2 \in \mathcal{R}^2$ with $R_2^{n-1} \in \mathcal{R}^{n-1}$ so that $R_i^n = R_2^{n-1}u$ for some path u. Write $R_2^{n-1} = \alpha_1 \cdots \alpha_{\delta(n-1)}$ with $\ell(\alpha_i) = A$ for all i. By the induction assumption, we have $\alpha_i \cdots \alpha_{i+d-1} \in \mathcal{R}^2$ for all i with $i + d - 1 \leqslant \delta(n-1)$. In particular, $R_3^2 := \alpha_{\delta(n-1)-d+1} \cdots \alpha_{\delta(n-1)}$ is in $\mathcal{R}^2$. Since R_1^2 overlaps R_3^2 with overlap R_3^2u, by Property 3.2 we have $\ell(u) = A$ and $\alpha_{\delta(n-1)-d+2} \cdots \alpha_{\delta(n-1)}u = R_1^2 \in \mathcal{R}^2$. Since $R_i^n = R_2^{n-1}u$, we have proved the first part of the result for R_i^n.

Now let p be another subpath of R_i^n of length D. We already know by induction that if p is a subpath of R_2^{n-1}, then p is not in $\mathcal{R}^2$. Therefore p is a subpath of R_3^2u which is neither R_3^2 nor R_1^2. By Consequence 3.3, p is not in ρ. We have proved that $p \notin \mathcal{R}^2$ and the induction step is complete. $\square$

Consequence 3.5. *Suppose that $D = dA$. Let $T = \alpha_1 \cdots \alpha_n$ be a closed A-trail in $\mathcal{Q}$ with $\alpha_i \in \mathcal{Q}_A$ for all i and suppose that $d \geqslant n+1$. Assume also that T is the prefix of an A-path in ρ and the suffix of an A-path in ρ. Then all A-subpaths of A-length d of powers of the closed trail T are in ρ.*

Proof: By assumption, there exist A-paths T' and T'' such that $T'T \in \rho$ and $TT'' \in \rho$. Since Λ is finite-dimensional, there is a path $R_2 \in \rho$ that lies on T, and $\ell(R_2) = D = dA > \ell(T) = nA$. Therefore R_2 is a subpath of length D of $T^N = (\alpha_1 \cdots \alpha_n)^N$ for some $N \geqslant 2$. If $R_2 = T^m$ is a power of T with $m \geqslant 2$ (and $d = nm$), then R_2 overlaps itself with overlap T^{2m-1} and the result follows using Consequence 3.3 (every A-subpath of A-length d of a power of T is an A-subpath of T^{2m-1}). Otherwise, TT'' overlaps R_2 or R_2 overlaps $T'T$ and we can use Consequence 3.3 again to prove that R_2 is an A-subpath of T^N and then that every A-subpath of A-length d of the overlap is in ρ; since every A-subpath of A-length d of a power of T is one of these, we obtain the result. □

3.2. Characterisations of (D, A)-stacked monomial algebras that satisfy (Fg). We now give our combinatorial condition for (D, A)-stacked monomial algebras Λ.

Condition 3.6. We say that a (D, A)-stacked monomial algebra Λ satisfies Condition 3.6, or (C2), when the following properties (1) and (2) both hold:

(1) Let c be an A-loop in $\mathcal{Q}_A$. Write $c = a_1 \cdots a_A$ with $a_i \in \mathcal{Q}_1$ for all i and $c_j = a_j \cdots a_A a_1 \cdots a_{j-1}$ for $j \in \{1, \ldots, A\}$. Then there exists j such that $c_j^d \in \rho$ but there is no path in ρ of the form $c_j^{d-1}\beta$ or βc_j^{d-1}, where β is a path of length A that is distinct from c_j.

(2) Let $T = \alpha_1 \cdots \alpha_n$ be a closed A-trail in $\mathcal{Q}$ with $n \geqslant 2$ and $\alpha_i \in \mathcal{Q}_A$ for all i and such that $\rho_T := \{\alpha_1 \cdots \alpha_d, \alpha_2 \cdots \alpha_d \alpha_{d+1}, \ldots, \alpha_n \alpha_1 \cdots \alpha_{d-1}\} \subseteq \rho$. Then there are no elements in $\rho \setminus \rho_T$ which begin or end with the path α_i, for all i.

Remark 3.7. In part (1) of the condition, there is exactly one j such that $c_j^d \in \rho$. Indeed, if c_j^d and c_k^d were in ρ, they would overlap with an overlap of length at most $D + A - 1$, hence by Property 3.2 we must have $c_j^d = c_k^d$ and therefore $j = k$.

Remark 3.8. If $A = 1$, then Condition 3.6 is equivalent to Condition 2.4.

We first prove that this condition is sufficient for Λ to satisfy **(Fg)**.

Theorem 3.9. *Let $\Lambda = K\mathcal{Q}/I$ be a finite-dimensional (D, A)-stacked monomial algebra. Assume that Λ satisfies Condition 3.6. Then Λ satisfies* **(Fg)**.

Proof: The case $D \geqslant 2$ and $A = 1$ corresponds to d-Koszul monomial algebras (with $D = d$) and is proved in Theorem 2.7. Therefore we assume that $A > 1$ so that necessarily $D > 2$. If $\operatorname{gldim} \Lambda$ is finite, then Λ satisfies **(Fg)** (and Condition 3.6 is empty), so we also assume that $\operatorname{gldim} \Lambda \geqslant 4$ so that $D = dA$.

The structure of this proof follows that of Theorem 2.7 by replacing each arrow in $\mathcal{Q}_1$ with a path of length A in $\mathcal{Q}_A$. We do not give all the details here, but indicate those places where we need to provide additional arguments.

The first part of the proof is to show that the hypotheses of [**13**, Theorem 3.4] hold.

Let $c_1, \ldots, c_u$ be the A-loops in $\mathcal{Q}$ such that $c_i^d \in \rho$ for $i = 1, \ldots, u$. (We remark that, in the terminology of [**13**], these are precisely the closed paths in $\mathcal{Q}$ such that for each c_i we have $c_i \neq p_i^{r_i}$ for any path p_i with $r_i \geqslant 2$ and $c_i^d \in \rho$. Firstly, $c_i^d \in \rho$

implies that $\ell(c_i) = A$. Then, if $c_i = p_i^{r_i}$ for some path p_i with $r_i \geqslant 2$, we have $1 \leqslant \ell(p_i) < A$. Now $p_i^{dr_i}$ is in ρ and $p_i^{dr_i}$ overlaps itself with overlap $p_i^{dr_i+1}$, so there is a maximal overlap in $\mathcal{R}^3$ of length $\leqslant D + \ell(p_i) < D + A$. But this is a contradiction since Λ is a (D, A)-stacked monomial algebra. So $c_i \neq p_i^{r_i}$.) By Condition 3.6(1), for each $i = 1, \ldots, u$, there are no elements in ρ of the form $c_i^{d-1}\beta$ or βc_i^{d-1}, where β is a path of length A that is distinct from c_i.

We need to show that there are no overlaps of c_i^d with any element of $\rho \setminus \{c_i^d\}$. If $R \in \rho \setminus \{c_i^d\}$ and R overlaps c_i^d, then, by Consequence 3.3, either $R = c_i^s b$ or $R = bc_i^s$, where $1 \leqslant s \leqslant d-1$ and b is an A-path with $\ell_A(b) = d - s$ and that does not begin (respectively, end) with the path c_i. Suppose that $R = c_i^s b$. Then R overlaps c_i^d with overlap of length $A(2d - s)$. By Consequence 3.3, this is a maximal overlap since c_i is not a prefix of b and thus gives an element $R_1^3 \in \mathcal{R}^3$. However, $\ell(R_1^3) = D + A = (d+1)A$. Thus $2d - s = d + 1$ and so $s = d - 1$. But then $R = c_i^{d-1}b$ and b is a path of length A distinct from c_i, which is a contradiction. The case $R = bc_i^s$ is similar. So there are no overlaps of c_i^d with any element of $\rho \setminus \{c_i^d\}$.

Let $T_{u+1}, \ldots, T_r$ be the distinct closed A-trails in $\mathcal{Q}$ with $\ell_A(T_i) > 1$ such that the sets ρ_{T_i} of Condition 3.6(2) are contained in ρ. For each $i = u+1, \ldots, r$, we write $T_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$, where the $\alpha_{i,j}$ are in $\mathcal{Q}_A$ so that $\ell_A(T_i) = m_i > 1$ and

$$\rho_{T_i} = \{\alpha_{i,1} \cdots \alpha_{i,d}, \alpha_{i,2} \cdots \alpha_{i,d+1}, \ldots, \alpha_{i,m_i}\alpha_{i,1} \cdots \alpha_{i,d-1}\} \subseteq \rho.$$

By Condition 3.6(2), for each closed A-trail T_i $(i = u+1, \ldots, r)$, there are no elements in $\rho \setminus \rho_{T_i}$ which begin or end with the path $\alpha_{i,j}$, for all $j = 1, \ldots, m_i$. So no path $\alpha_{i,j}$ of length A has overlaps with any element in $\rho \setminus \rho_{T_i}$.

The next step is to describe a commutative Noetherian graded subalgebra H of $\mathrm{HH}^*(\Lambda)$ with $H^0 = \mathrm{HH}^0(\Lambda)$. Applying [**13**, Theorem 3.4] gives $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \ldots, x_r]/\langle x_a x_b \text{ for } a \neq b\rangle$, where

- for $i = 1, \ldots, u$, the vertices $v_1, \ldots, v_u$ are distinct and the element x_i corresponding to the A-loop c_i is in degree 2 and is represented by the map $\mathcal{P}^2 \to \Lambda$, where for $R^2 \in \mathcal{R}^2$,

$$\mathfrak{o}(R^2) \otimes \mathfrak{t}(R^2) \longmapsto \begin{cases} v_i & \text{if } R^2 = c_i^d; \\ 0 & \text{otherwise} \end{cases}$$

- and for $i = u+1, \ldots, r$, the element x_i corresponding to the closed A-trail $T_i = \alpha_{i,1} \cdots \alpha_{i,m_i}$ is in degree $2\mu_i$ such that $\mu_i = m_i/\gcd(d, m_i)$ and is represented by the map $\mathcal{P}^{2\mu_i} \to \Lambda$, where for $R^{2\mu_i} \in \mathcal{R}^{2\mu_i}$,

$$\mathfrak{o}(R^{2\mu_i}) \otimes \mathfrak{t}(R^{2\mu_i}) \longmapsto \begin{cases} \mathfrak{o}(T_{i,k}) & \text{if } R^{2\mu_i} = T_{i,k}^{d/\gcd(d,m_i)} \text{ for all } k = 1, \ldots, m_i; \\ 0 & \text{otherwise.} \end{cases}$$

Let H be the subring of $\mathrm{HH}^*(\Lambda)$ generated by $Z(\Lambda)$ and $\{x_1, \ldots, x_r\}$. As in Theorem 2.7, H is a commutative Noetherian ring.

Now we show that Λ satisfies **(Fg)** with this algebra H. Again, we identify $\bigcup_{n\geqslant 0} \mathcal{R}^n$ with a basis of $E(\Lambda)$. Set $N = \max\{3, |x_1|, \ldots, |x_r|, |\mathcal{Q}_A|\}$. We show that $\bigcup_{n=0}^{N} \mathcal{R}^n$ is a generating set for $E(\Lambda)$ as a left H-module and thus $E(\Lambda)$ is finitely generated as a left H-module.

Let $R \in \mathcal{R}^n$ with $n > N$. Then $\ell_A(R) = \delta(n) \geqslant 2d$ and we can write $R = a_1 a_2 \cdots a_{\delta(n)}$, where the a_i are in $\mathcal{Q}_A$. The proof now follows that of Theorem 2.7 by replacing each arrow with a path of length A, and with extensive use of Consequences 3.3, 3.4, and 3.5, and Condition 3.6. Thus we conclude that Λ has **(Fg)**. $\square$

Example 6. We return to Examples 3, 4, and 5. In all these examples, Condition 3.6 is satisfied and therefore **(Fg)** holds for Λ.

For instance, in Example 3, the only closed 2-trails T such that $\rho_T \subseteq \rho$ are $\alpha\beta\gamma\delta$ and $\gamma\delta\alpha\beta$ and, in both cases, $\rho_T = \rho$. In Example 5, the closed 3-trails T such that $\rho_T \subseteq \rho$ are those that start with β_1, β_4, and β_7, in all cases we have $\rho_T = \rho \setminus \{(\alpha_1\beta_2\alpha_2)^2\}$ and $(\alpha_1\beta_2\alpha_2)^2$ does not start or end with a β_i.

By [**6**, Theorem 2.5], it follows that Λ is Gorenstein in each case. Moreover, it was proved in [**8**] that the algebra in Example 3 has injective dimension 2.

Our aim is now to prove the following theorem, and in particular the converse of Theorem 3.9.

Theorem 3.10. *Let Λ be an indecomposable finite-dimensional (D, A)-stacked monomial algebra. Suppose that $D \neq 2A$ whenever $A > 1$. Consider the following statements:*

(C1) *Λ satisfies* **(Fg)**.

(C2) *Condition 3.6 holds for Λ.*

(C3) *$Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian and $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module.*

(C4) *$E(\Lambda)$ is finitely generated as a module over $Z_{\mathrm{gr}}(E(\Lambda))$.*

Then (C4) *implies* (C2), *which in turn implies* (C1).

Moreover, if the field K is algebraically closed, then the four statements are equivalent.

We shall need, as in the d-Koszul case, a description of the Ext algebra of Λ. We give the details of this in the appendix, and we briefly describe it here. Since we have already proved Theorem 3.10 when Λ is d-Koszul, we assume here that $D > A > 1$ and, in addition, that $D \neq 2A$.

Let Γ be the quiver with the same vertices as $\mathcal{Q}$ and whose set of arrows corresponds to the set of paths of length A in $\mathcal{Q}$, that is, $\Gamma_1 = \{\overline{\alpha}\colon i \to j \mid \text{there exists } \alpha \in \mathcal{Q}_A,\ \alpha\colon j \to i\}$. Let ${}^\perp\rho$ be the orthogonal of ρ for the bilinear form $K\Gamma_d \times K(\mathcal{Q}_A)_d \to K$ defined on paths of length d in Γ and A-paths of A-length d in $\mathcal{Q}$ by $\langle \overline{\alpha}_d \cdots \overline{\alpha}_1, \beta_1 \cdots \beta_d\rangle = 1$ if $\alpha_1 \cdots \alpha_d = \beta_1 \cdots \beta_d$ and 0 otherwise, where the α_i and β_i are in $\mathcal{Q}_A$. Set ${}^\natural\Lambda = K\Gamma/J$, where $J = ({}^\perp\rho)$; it is a monomial algebra and the ideal J has a minimal generating set σ given by all the paths $\overline{\alpha}_d \cdots \overline{\alpha}_1$ such that the A-path $\alpha_1 \cdots \alpha_d$ is not in ρ.

Let $B = \bigoplus_{n \geqslant 0} B_n$ be the algebra defined as follows:

- $B_n = {}^\natural\Lambda_{\delta(n)}$;
- for $x \in B_n$ and $y \in B_m$, define $x \cdot y \in B_{m+n}$ by

$$x \cdot y = \begin{cases} 0 & \text{if } n \text{ and } m \text{ are odd;} \\ 0 & \text{if } n \text{ or } m \text{ is equal to 1 and } n \geqslant 1,\ m \geqslant 1; \\ xy & \text{in } {}^\natural\Lambda \text{ otherwise.} \end{cases}$$

Observe that if n or m is even and both are larger than 1, $\delta(n) + \delta(m) = \delta(n+m)$, so that the algebra B is a graded K-algebra, generated in degrees 0, 1, 2, and 3. Note that this is also true of $E(\Lambda)$ by [**12**]. Moreover, we prove in the appendix that the algebras $E(\Lambda)$ and B are isomorphic, generalising the description given in [**10**] when Λ is a d-Koszul algebra. This isomorphism uses the assumption that $D \neq 2A$.

There is a basis $\mathcal{B}_{{}^\natural\Lambda}$ of ${}^\natural\Lambda$ consisting of all paths p in Γ such that no path in σ is a subpath of p, and basis $\mathcal{B}_B$ of B contained in $\mathcal{B}_{{}^\natural\Lambda}$ consisting of all $\overline{R}_i^m$ for all $m \geqslant 0$ and all $R_i^m \in \mathcal{R}^m$.

We now define several gradings, on ${}^\natural\Lambda$ and on B.

There is a natural grading on ${}^\natural\Lambda$ given by the length ℓ of the paths. Note that if p is an A-path in $\mathcal{Q}$, then $\ell(\overline{p}) = \ell_A(p)$. The degree of a homogeneous element x in B will be denoted by $|x|$, so $x \in {}^\natural\Lambda_{\delta(|x|)}$ or, in other terms, $|x| = k$ if, and only if, $\ell(x) = \delta(k)$.

The algebra ${}^\natural\Lambda$ is also multi-graded by $\mathbb{N}^{\mathcal{Q}_1}$: for each path $\overline{p}$ in Γ, we define an element $\mathfrak{d}(\overline{p}) = (\mathfrak{d}_\alpha(\overline{p}))_{\alpha\in\mathcal{Q}_A} \in \mathbb{N}^{\mathcal{Q}_1}$ as follows: write the A-path p in $\mathcal{Q}$ as $p = \alpha_1 \cdots \alpha_n$, where each α_i is in $\mathcal{Q}_A$;

- if $\ell(\overline{p}) = 0$, then $\mathfrak{d}(\overline{p}) = (0)_{\alpha\in\mathcal{Q}_1}$;
- if $\ell(\overline{p}) > 0$, then $\mathfrak{d}_\alpha(\overline{p})$ is the number of α_i that are equal to α (it is 0 if none of the α_i are equal to α).

Note that even if α is a subpath of p, we can have $\mathfrak{d}_\alpha(\overline{p}) = 0$ (if α is not one of the α_i, that is, $p = q\alpha r$, where q and r are paths in $\mathcal{Q}$ whose lengths are not multiples of A).

Since ${}^\natural\Lambda$ is monomial, the ideal J is homogeneous with respect to this multi-degree and therefore ${}^\natural\Lambda$ is multi-graded. In B, if x and y are homogeneous and $|x|$ or $|y|$ is even with both degrees at least 2, then $\mathfrak{d}_\alpha(xy) = \mathfrak{d}_\alpha(x) + \mathfrak{d}_\alpha(y)$, but $\mathfrak{d}_\alpha(xy) = 0$ otherwise.

Let $Z := Z_{\mathrm{gr}}(B)$ be the graded centre of B. As in the d-Koszul case, it is generated by elements z that are homogeneous with respect to the grading $|\cdot|$ and the multi-degree and such that, for any element $y \in B$ that is homogeneous with respect to the grading $|\cdot|$, we have $zy = (-1)^{|y||z|}yz$.

Remark 3.11. Recall that $B \cong E(\Lambda)$ is generated in degrees 0, 1, 2, and 3 and that the product of an element of degree 1 with any other element vanishes. Therefore when checking that an element is in Z, we need to check that it is a linear combination of cycles and that it commutes or anti-commutes with paths of degrees 2 and 3, that is, (non-zero) paths of length d and of length $d+1$ in Γ.

The proof of Theorem 3.10 relies on some preliminary results, namely Lemma 3.12, Proposition 3.13, and Lemma 3.15. We start with some comments on A-loops in $\mathcal{Q}_A$. Let c be an A-loop in $\mathcal{Q}_A$. Since Λ is finite-dimensional, there exists an integer N such that $c^N = 0$ in Λ and therefore there is some j such that $c_j^d \in \rho$. To simplify notation and without loss of generality, write $c = c_j$. Then $c^d \in \rho$, therefore $\overline{c}^d \notin \sigma$ and it follows that $\overline{c}^k \neq 0$ in ${}^\natural\Lambda$ for all $k \geqslant 0$ and that $\overline{c}^{\delta(k)} \neq 0$ in B for all $k \geqslant 0$.

Lemma 3.12. *Let c be an A-loop in $\mathcal{Q}_A$ and let $n \geqslant 2$ be an integer. Then $\overline{c}^{\delta(n)} \in Z$ if, and only if, c satisfies Condition 3.6(1).*

Proof: The proof is very similar to that of Lemma 2.10, using A-paths and Remark 3.11. □

We shall now consider part (2) of Condition 3.6.

Let $T = \alpha_1 \cdots \alpha_n$ be a closed A-trail in $\mathcal{Q}$ with $\alpha_i \in \mathcal{Q}_A$ for all i. Assume that $n \geqslant 2$ and that $\rho_T = \{\alpha_i \cdots \alpha_{i+d-1} \mid 1 \leqslant i \leqslant n\} \subseteq \rho$. Then $\overline{T}$ and all the paths lying on $\overline{T}$ are in $\mathcal{B}_{{}^\natural\Lambda}$ (none of their subpaths of length d are in σ); those of length $\delta(k)$ for some $k \geqslant 0$ are in $\mathcal{B}_B$.

In a similar way to Subsection 2.3, we make the following assumptions.

(i) None of the α_i are A-loops.
(ii) No A-subcycle of T satisfies the same assumptions as T (that is, there is no A-subcycle q of T of A-length at least 2, and $\rho_q \subseteq \rho$).

Proposition 3.13. *Let $T = \alpha_1 \cdots \alpha_n$ be a closed A-trail with $n \geqslant 2$, $\rho_T \subseteq \rho$, and such that assumptions (i) and (ii) hold. Let p be an A-path of A-length d such that $\mathfrak{d}_\beta(\overline{p}) = 0$ if $\beta \in \mathcal{Q}_A \setminus \{\alpha_1, \dots, \alpha_n\}$ and which is not an A-subpath of a power of T. Then $p \notin \rho$.*

Proof: The proof is very similar to that of Proposition 2.11, replacing paths with A-paths and using Consequence 3.5 in the proof that $d > k$. □

Remark 3.14. We keep the assumptions and notation of Proposition 3.13. Set $T_i = \alpha_i \cdots \alpha_n \alpha_1 \cdots \alpha_{i-1}$. Then for any $j \geqslant 1$, we have

$$\overline{T}_i^j \overline{\alpha}_k \neq 0 \iff k = i - 1,$$
$$\overline{\alpha}_k \overline{T}_i^j \neq 0 \iff k = i.$$

Lemma 3.15. *Let $T = \alpha_1 \cdots \alpha_n$ be a closed A-trail that satisfies assumptions (i) and (ii) and set $z_j = \sum_{i=1}^n \overline{T}_i^j$ with $nj = \delta(u)$ for some $u \geqslant 1$. Then $z_j \in Z$ if, and only if, T satisfies Condition 3.6(2).*

Moreover, if T does not satisfy Condition 3.6(2), then no element in B that is homogeneous with respect to $|\cdot|$ and $\mathfrak{d}$ (when viewed in ${}^\natural\Lambda$) and that is a linear combination of non-trivial cycles lying on $\overline{T}$ is in Z.

Proof: The proof is very similar to that of Lemma 2.13, replacing paths with A-paths, again using Remark 3.11, replacing the d-covering property with Consequence 3.3 and Proposition 2.11 with Proposition 3.13. Note also that, for the proof of the last part, testing commutation with paths in B_2 gives $s = n$ and $\lambda_k = \lambda_{k+d}$ for all k and hence the result if $|z|$ is odd; and if $|z|$ is even, we must use the fact that z commutes with elements in B_3 in a similar way to obtain, in addition, that $\lambda_k = \lambda_{k+d+1}$ for all k and hence that $\lambda_k = \lambda_{k+1}$ for all k. □

Proof of Theorem 3.10: We note first that if $\operatorname{gldim} \Lambda$ is finite, then Λ satisfies **(Fg)** and Condition 3.6 is empty. The implication (C2) $\Rightarrow$ (C1) is Theorem 3.9. Again, the implication (C3) $\Rightarrow$ (C4) is clear, and if in addition K is algebraically closed, then the implication (C1) $\Rightarrow$ (C3) follows from [**7**]. It remains to prove that (C4) implies (C2). The proof is similar to that of Theorem 2.8, again replacing paths with A-paths (we need not assume that the integers k are even). □

Remark 3.16. Suppose that K is algebraically closed. We have now extended the equivalence between (C1) and (C3), already known for Koszul algebras from [**7**], as well as d-Koszul monomial algebras by Theorem 2.8, to (D, A)-stacked monomial algebras with $D \neq 2A$ whenever $A > 1$.

In particular, we can extend Corollary 2.15 to (D, A)-stacked monomial algebras.

Corollary 3.17. *Let Λ be a (D, A)-stacked monomial algebra over an algebraically closed field with $D \neq 2A$ whenever $A > 1$. Assume that $E(\Lambda)$ is a finitely generated $Z_{\mathrm{gr}}(E(\Lambda))$-module. Then the algebra $Z_{\mathrm{gr}}(E(\Lambda))$ is Noetherian.*

Appendix A. The Ext algebra of a (D, A)-stacked monomial algebra

Leader and Snashall gave in [**17**] a presentation of the Yoneda algebra $E(\Lambda)$ of a (D, A)-stacked monomial algebra by quiver and relations. However, in our proof of Theorem 2.8 that (C4) implies (C2) for d-Koszul monomial algebras, we used the description from [**10**, Sections 8 and 9] of $E(\Lambda)$ as an algebra contained, as a graded vector space, in the Koszul dual ${}^!\Lambda$. In this appendix, we generalise this description to (D, A)-stacked monomial algebras.

Throughout this section, $\Lambda = KQ/I$ is a (D, A)-stacked monomial algebra with $D = dA$ and $d \geqslant 2$, where I is an ideal generated by a set ρ of A-paths of A-length $d = \frac{D}{A}$. We view Λ as a quotient of the tensor algebra: $\Lambda = \mathbb{T}_{\Lambda_0}(\Lambda_1)/I$.

All tensor products are taken over Λ_0 and we write $\overline{\otimes}$ for $\bigotimes_{\Lambda_0}$. The subspace $\mathcal{S} = I \cap (\Lambda_1^{\overline{\otimes} D}) = \operatorname{span}(\rho)$ of $\mathbb{T} = \mathbb{T}_{\Lambda_0}(\Lambda_1)$ is a Λ_0-Λ_0-submodule of $\Lambda_1^{\overline{\otimes} D}$; it is finite-dimensional over K. For an element $x \in \mathbb{T}$, write $\overline{x}$ for its image in Λ. Note that for $0 \leqslant i < D$ we have $\Lambda_i = \Lambda_1^{\overline{\otimes} i}$.

A.1. Generalised Koszul complex of $\mathcal{S}$. Define spaces $H_{\delta(n)} \subseteq \mathbb{T}^n_{\Lambda_0}(\Lambda_1)$ as follows:

$$H_0 = \Lambda_0,\ H_1 = \Lambda_1 \text{ and, for } n \geqslant 2,\ H_{\delta(n)} = \bigcap_{i+j=\delta(n)-d} (\Lambda_A^{\overline{\otimes} i}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}).$$

For $n \geqslant 0$, let $\mathbf{P}^n$ be the right Λ-module defined by $\mathbf{P}^n = H_{\delta(n)} \overline{\otimes} \Lambda$; it is projective.

We have $H_{\delta(1)} = \Lambda_1 = H_{\delta(0)} \overline{\otimes} \Lambda_1$, $H_{\delta(2)} = \mathcal{S} \subseteq \Lambda_1^{\overline{\otimes} D} = H_{\delta(1)} \overline{\otimes} \Lambda_1^{\overline{\otimes} D-1}$, and, for any $n \geqslant 3$, $H_{\delta(n)} \subseteq H_{\delta(n-1)} \overline{\otimes} \Lambda_A^{\overline{\otimes}(\delta(n)-\delta(n-1))}$. Indeed, for any $k \geqslant 1$,

$$\begin{aligned} H_{\delta(2k+2)} = H_{(k+1)d} &= \bigcap_{j=0}^{kd} (\Lambda_A^{\overline{\otimes}(kd-j)}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}) \subseteq \bigcap_{j=d-1}^{kd} (\Lambda_A^{\overline{\otimes}(kd-j)}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}) \\ &= \bigcap_{j=0}^{(k-1)d+1} (\Lambda_A^{\overline{\otimes}((k-1)d+1-j)}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}) \overline{\otimes} (\Lambda_A^{\overline{\otimes}(d-1)}) = H_{\delta(2k+1)} \overline{\otimes} (\Lambda_A^{\overline{\otimes}(d-1)}); \end{aligned}$$

$$\begin{aligned} H_{\delta(2k+1)} = H_{kd+1} &= \bigcap_{j=0}^{kd+1} (\Lambda_A^{\overline{\otimes}((k-1)d+1-j)}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}) \subseteq \bigcap_{j=A}^{kd+1} (\Lambda_A^{\overline{\otimes}((k-1)d+1-j)}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}) \\ &= \bigcap_{j=0}^{kd} (\Lambda_A^{\overline{\otimes}((k-1)d-j)}) \overline{\otimes} \mathcal{S} \overline{\otimes} (\Lambda_A^{\overline{\otimes} j}) \overline{\otimes} \Lambda_A = H_{\delta(2k)} \overline{\otimes} \Lambda_A. \end{aligned}$$

It follows that the maps $F^1 \colon \Lambda_1 \overline{\otimes} \Lambda \to \Lambda_0 \overline{\otimes} \Lambda \cong \Lambda$, $F^2 \colon \Lambda_A^{\overline{\otimes} d} \overline{\otimes} \Lambda \to \Lambda_1 \overline{\otimes} \Lambda$, and, for $n \geqslant 3$, $F^n \colon \Lambda_A^{\overline{\otimes}\delta(n)} \overline{\otimes} \Lambda \to \Lambda_A^{\overline{\otimes}\delta(n-1)} \overline{\otimes} \Lambda$ defined by

$$\begin{aligned} F^1(x_1 \overline{\otimes} \lambda) &= x_1 \lambda, \\ F^2(x_1 \overline{\otimes} \cdots \overline{\otimes} x_D \overline{\otimes} \lambda) &= x_1 \overline{\otimes} x_2 \cdots x_D \lambda, \\ F^n(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(n)} \overline{\otimes} \lambda) &= y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(n-1)} \overline{\otimes} y_{\delta(n-1)+1} \cdots y_{\delta(n)} \lambda, \end{aligned}$$

where $x_i \in \Lambda_1$ and $y_i \in \Lambda_A$ for all i, induce maps $b^n \colon \mathbf{P}^n \to \mathbf{P}^{n-1}$. More specifically, for all $k \geqslant 1$,

$$F^{2k+1}(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{kd+1} \overline{\otimes} \lambda) = y_1 \overline{\otimes} \cdots \overline{\otimes} y_{kd} \overline{\otimes} y_{kd+1} \lambda \text{ if } n = 2k+1 \text{ is odd};$$

$$F^{2k+2}(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{(k+1)d} \overline{\otimes} \lambda) = y_1 \overline{\otimes} \cdots \overline{\otimes} y_{kd+1} \overline{\otimes} y_{kd+2} \cdots y_{(k+1)d} \lambda \text{ if } n = 2k+2 \text{ is even}.$$

Define also $b^0 \colon \mathbf{P}^0 = \Lambda_0 \overline{\otimes} \Lambda \cong \Lambda \to \Lambda_0$, which identifies with the natural projection. Moreover, for $n \geqslant 3$ we have

$$\begin{aligned} H_{\delta(n+1)} &\subseteq H_{\delta(n)} \overline{\otimes} \Lambda_A^{\overline{\otimes}(\delta(n+1)-\delta(n))} \\ &\subseteq H_{\delta(n-1)} \overline{\otimes} \Lambda_A^{\overline{\otimes}(\delta(n)-\delta(n-1))} \overline{\otimes} \Lambda_A^{\overline{\otimes}(\delta(n+1)-\delta(n))} = H_{\delta(n-1)} \overline{\otimes} \Lambda_A^{\overline{\otimes} d} \end{aligned}$$

and $H_{\delta(n+1)} \subseteq \Lambda_A^{\overline{\otimes}\delta(n-1)}\overline{\otimes}\mathcal{S}$, we have $H_{\delta(n+1)} \subseteq (\Lambda_A^{\overline{\otimes}\delta(n-1)}\overline{\otimes}\mathcal{S}) \cap (H_{\delta(n-1)}\overline{\otimes}\Lambda_A^{\overline{\otimes}d}) = H_{\delta(n-1)}\overline{\otimes}\mathcal{S}$ (all the spaces involved are finitely generated and projective over Λ_0), hence $b^n \circ b^{n+1} = 0$. It is easy to check that $b^n \circ b^{n+1} = 0$ when $n = 1$, 2, or 3.

Therefore we have a complex $(\mathbf{P}^n, b^n)$ of projective right Λ-modules.

Theorem A.1. *Let $\Lambda = K\mathcal{Q}/I$ be a monomial algebra with I generated in degree $D = dA$ with $d \geqslant 2$. Then Λ is (D, A)-stacked monomial if, and only if, $(\mathbf{P}^\bullet, b^\bullet)$ is a minimal projective right Λ-module resolution of Λ_0.*

Proof: By construction, $\mathbf{P}^n$ is generated in degree $\delta(n)$. Therefore, if $(\mathbf{P}^\bullet, b^\bullet)$ is a minimal projective right Λ-module resolution of Λ_0, then Λ is a (D, A)-stacked monomial algebra.

Conversely, assume that Λ is a (D, A)-stacked monomial algebra. We already have a complex $(\mathbf{P}^\bullet, b^\bullet)$ of right Λ-modules such that $\mathbf{P}^n$ is generated in degree $\delta(n)$. The beginning $\mathbf{P}^1 \xrightarrow{b^1} \mathbf{P}^0 \xrightarrow{b^0} \Lambda_0 \to 0$ is exact.

We prove exactness at $\mathbf{P}^{2n+1}$ for $n \geqslant 1$ (the proof of exactness at $\mathbf{P}^{2n}$ and at $\mathbf{P}^1$ is similar, without the need for Consequence 3.3).

First note that $b^{2n+2}(\mathbf{P}^{2n+2})$ is generated in degree $\delta(2n+2) = (n+1)D$ in $\mathbf{P}^{2n+1}$.

Let $z = \sum_i x_{nd+1,i}\overline{\otimes}\cdots\overline{\otimes}x_{1,i}\overline{\otimes}\lambda_i$ be an element in $\operatorname{Ker} b^{2n+1}$ with $x_{j,i} \in \Lambda_A$ and $\lambda_i \in \Lambda$ for all i, j. Then $\sum_i x_{nd+1,i}\overline{\otimes}\cdots\overline{\otimes}x_{2,i}\overline{\otimes}x_{1,i}\lambda_i$ is in $\mathbb{T}\overline{\otimes}I$. It follows that $\lambda_i \in \bigoplus_{k\geqslant D-A}\Lambda_k$ and that $\operatorname{Ker} b^{2n+1}$ is generated in degrees at least $(n+1)D$. Therefore z can be rewritten as $z = \sum_i x_{nd+1,i}\overline{\otimes}\cdots\overline{\otimes}x_{1,i}\overline{\otimes}y_{d-1,i}\cdots y_{1,i}\lambda'_i$ with $y_{j,i} \in \Lambda_A$ and $\lambda'_i \in \Lambda$ for all i, j, with the $y_{j,i}$ right uniform and $\mathfrak{t}(y_{1,i}) \neq \mathfrak{t}(y_{1,k})$ when $i \neq k$. Write $\lambda'_i = \sum_{l\geqslant 0}\lambda'_{i,l}$ with $\lambda'_{i,l} \in \Lambda_l$ for all i, l, and $\lambda'_{i,0} = \mathfrak{t}(y_{1,i})$. Then each of the $\sum_i x_{nd+1,i}\overline{\otimes}\cdots\overline{\otimes}x_{1,i}\overline{\otimes}y_{d-1,i}\cdots y_{1,i}\lambda'_{i,l}$ is in $\operatorname{Ker} b^{2n+1}$ so in particular $z' := \sum_i x_{nd+1,i}\overline{\otimes}\cdots\overline{\otimes}x_{1,i}\overline{\otimes}y_{d-1,i}\cdots y_{1,i} \in \operatorname{Ker} b^{2n+1}$.

Consider $z'' := \sum_i x_{nd+1,i}\overline{\otimes}\cdots\overline{\otimes}x_{1,i}\overline{\otimes}y_{d-1,i}\overline{\otimes}\cdots\overline{\otimes}y_{1,i}\overline{\otimes}\mathfrak{t}(y_{1,i}) \in \Lambda_A^{\overline{\otimes}((n+1)d)}\overline{\otimes}\Lambda$. We show that $z'' \in \mathbf{P}^{2n+2}$; this will imply that $z' = b^{2n+2}(z'') \in \operatorname{Im} b^{2n+2}$. Since Λ is monomial and $b^{2n+1}(z') = 0$, we may assume that, for all i, $x_{1,i}y_{d-1,i}\cdots y_{1,i}$ is a path; since it is in I and has degree D, it is in $\rho = \mathcal{R}^2$. By definition of $\mathbf{P}^{2n+1}$, we may assume that z is written so that, for each i, $x_{d,i}\cdots x_{1,i}$ is a path in $\rho = \mathcal{R}^2$. The path $x_{1,i}y_{d-1,i}\cdots y_{1,i}$ properly overlaps $x_{d,i}\cdots x_{1,i}$, therefore using Consequence 3.3 it follows that for all k with $1 \leqslant k \leqslant d-1$ we have $x_{k,i}\cdots x_{1,i}y_{d-1,i}\cdots y_{k,i} \in \rho$ and hence $z'' \in \bigcap_{k=1}^{d-1}(\Lambda_A^{\overline{\otimes}(nd-k+1)})\overline{\otimes}\mathcal{S}\overline{\otimes}(\Lambda_A^{\overline{\otimes}(k-1)})\overline{\otimes}\Lambda$. Finally, using the fact that $z \in \mathbf{P}^{2n+1} = H_{nd+1}\overline{\otimes}\Lambda$, we get $z'' \in H_{(n+1)d}\overline{\otimes}\Lambda = \mathbf{P}^{2n+2}$. We have proved that $(\operatorname{Ker} b^{2n+1})_{(n+1)d} \subseteq \operatorname{Im} b^{2n+2}$.

Since $\operatorname{Im} b^{2n+2}$ is generated in degree $(n+1)d$ and $\operatorname{Ker} b^{2n+1}$ is generated in degrees at least $(n+1)d$, it follows that $\operatorname{Ker} b^{2n+1} \subseteq \operatorname{Im} b^{2n+2}$ and lastly that $\operatorname{Ker} b^{2n+1} = \operatorname{Im} b^{2n+2}$.

Finally, since $\operatorname{Im} b^{n+1}$ is generated in degree $\delta(n+1)$ and $\mathbf{P}^n$ is generated in degree $\delta(n) < \delta(n+1)$, $\operatorname{Im} b^{n+1} \subseteq \mathfrak{r}\mathbf{P}^n$ for all n and therefore the resolution is minimal. □

A.2. The Ext algebra.

A.2.1. Some duality results. We recall without proof some results stated in [**2**]. All modules are finitely generated Λ_0-Λ_0-bimodules. All claims are easily checked for bimodules that are free and finitely generated as left or as right Λ_0-modules, and follow for arbitrary finitely generated modules (since Λ_0 is semisimple, all the Λ_0-modules (left or right) are projective).

For any bimodules V and W, define $V^*=\mathrm{Hom}_{\Lambda_0-}(V,\Lambda_0)$ and ${}^*W=\mathrm{Hom}_{-\Lambda_0}(W,\Lambda_0)$; they are Λ_0-Λ_0-bimodules, for the actions given for all e, e' in Λ_0, $\alpha \in V^*$, $\beta \in {}^*W$, and $v \in V$, $w \in W$, by:

$$(e\alpha e')(v) = \alpha(ve)e' \text{ and } (e\beta e')(w) = e\beta(e'w).$$

There are natural isomorphisms of Λ_0-Λ_0-bimodules $\Lambda_0^* \cong \Lambda_0 \cong {}^*\Lambda_0$ which we view as identifications.

There are also natural bimodule isomorphisms $V \cong {}^*(V^*)$ and $W \cong ({}^*W)^*$ of bimodules.

If $V_1 \subseteq V$ (respectively $W_1 \subseteq W$), define $V_1^\perp = \{\alpha \in V^* \mid \alpha(V_1) = 0\}$ (respectively ${}^\perp W_1 = \{\beta \in {}^*W \mid \beta(W_1) = 0\}$). If V_1 (respectively W_1) is a sub-bimodule, then they are sub-bimodules of V^* and ${}^*W$ respectively.

Fix a Λ_0-Λ_0-bimodule V. Then:

- (i) if W is another Λ_0-Λ_0-bimodule, then there is an isomorphism of Λ_0-Λ_0-bimodules $\varphi_{V,W}\colon {}^*V\overline{\otimes}{}^*W \to {}^*(W\overline{\otimes}V)$ given for all $\alpha \in {}^*V$, $\beta \in {}^*W$, $v \in V$, and $w \in W$ by $\varphi_{V,W}(\alpha\overline{\otimes}\beta)(w\overline{\otimes}v) = \alpha(\beta(w)v)$. There is a similar isomorphism with right duals, which sends $\alpha\overline{\otimes}\beta$ to the map $w\overline{\otimes}v \mapsto \beta(w\alpha(v))$;
- (ii) if U and W are sub-bimodules of V, then ${}^\perp(U+W) = {}^\perp U \cap {}^\perp W$ and $(U+W)^\perp = U^\perp \cap W^\perp$;
- (iii) if U is a sub-bimodule of V, then $(V/U)^* \cong U^\perp$ and ${}^*(V/U) \cong {}^\perp U$;
- (iv) if W is a sub-bimodule of V, then for any idempotents e_i, e_j with $(i,j) \in \mathcal{Q}_0^2$ we have $e_i{}^*We_j = {}^*(e_jWe_i)$;
- (v) if U is a sub-bimodule of V, then for all i, j in $\mathcal{Q}_0$ we have $\dim e_j{}^\perp Ue_i = \dim e_iVe_j - \dim e_iUe_j = \dim e_jU^\perp e_i$;
- (vi) if U is a sub-bimodule of V, then under the identification of V with ${}^*(V^*)$ we have ${}^\perp(U^\perp)$ and under the identification of V with $({}^*V)^*$ we have $({}^\perp U)^\perp$;
- (vii) if U is a sub-bimodule of V and W and Z are Λ_0-Λ_0-bimodules, there are bimodule isomorphisms ${}^\perp(W\overline{\otimes}U\overline{\otimes}Z) \cong {}^*Z\overline{\otimes}{}^\perp U\overline{\otimes}{}^*W$ and $(W\overline{\otimes}U\overline{\otimes}Z)^\perp \cong Z^*\overline{\otimes}U^\perp\overline{\otimes}W^*$.

A.2.2. Description of the Ext algebra. From the above, there is a natural isomorphism ${}^*(\Lambda_A^{\overline{\otimes}i}) \cong ({}^*\Lambda_A)^{\overline{\otimes}i}$, which we view as an identification. We also view $\mathcal{S}$ as a subspace of $\Lambda_A^{\overline{\otimes}d}$ (rather than $\Lambda_1^{\overline{\otimes}D}$). We may then consider ${}^\perp\mathcal{S} = \{f \in ({}^*\Lambda_A)^{\overline{\otimes}d} \mid f(x) = 0 \text{ for all } x \in \mathcal{S}\}$. The dual algebra of Λ is then ${}^\natural\Lambda = \mathbb{T}_{\Lambda_0}({}^*\Lambda_A)/({}^\perp\mathcal{S})$. It is a graded d-homogeneous algebra since ${}^\perp\mathcal{S}$ is contained in $({}^*\Lambda_A)^{\overline{\otimes}d}$, therefore ${}^\natural\Lambda = \bigoplus_{n\geqslant 0} {}^\natural\Lambda_n$.

In terms of quivers, we have $\Lambda_0 = K\mathcal{Q}_0$ and $\Lambda_A = K\mathcal{Q}_A$, the vector space whose basis is the set $\mathcal{Q}_A$ of paths of length A in $\mathcal{Q}$; moreover, ${}^*\Lambda_A \cong K\mathcal{Q}_A^{\mathrm{op}}$ using (iv).

Then ${}^\natural\Lambda$ is isomorphic to $K\Gamma/({}^\perp\rho)$, where Γ is the quiver with the same vertices as $\mathcal{Q}$ and whose set of arrows is $\Gamma_1 = \{\overline{\alpha}\colon i \to j \mid \text{there exists } \alpha \in \mathcal{Q}_A,\ \alpha\colon j \to i\}$ and where ${}^\perp\rho$ is the left orthogonal of the set ρ viewed as a set of A-paths, for the bilinear form $K\Gamma_d \times K(\mathcal{Q}_A)_d \to K$ defined on paths of length d in Γ and A-paths of A-length d in $\mathcal{Q}$ by $\langle \overline{\alpha}_d \cdots \overline{\alpha}_1, \beta_1 \cdots \beta_d\rangle = 1$ if $\alpha_1 \cdots \alpha_d = \beta_1 \cdots \beta_d$ and 0 otherwise, where the α_i and β_i are paths of length A.

The algebra $K\Gamma/({}^\perp\rho)$ is monomial, and the ideal $({}^\perp\rho)$ has a minimal generating set σ given by all the paths $\overline{\alpha}_d \cdots \overline{\alpha}_1$ such that the A-path $\alpha_1 \cdots \alpha_d$ is not in ρ. In particular, if $\overline{\gamma} = \overline{\alpha}_r \ldots \overline{\alpha}_1$ is a path in Γ, with $\alpha_i \in \mathcal{Q}_A$ for all i, then $\overline{\gamma} \notin (\sigma)$ if, and only if, for each i with $1 \leqslant i \leqslant r-d+1$, the path $\alpha_i \cdots \alpha_{i+d-1}$ is in ρ (we use the fact that Λ is monomial here).

Lemma A.2. *There is an isomorphism* $({}^\natural\Lambda_{\delta(n)})^* \cong H_{\delta(n)}$.

Proof: By definition, ${}^\natural\Lambda_{\delta(n)} = \frac{({}^*\Lambda_A)^{\overline{\otimes}\delta(n)}}{\sum_{i=0}^{\delta(n)-d}({}^*\Lambda_A)^{\overline{\otimes}(\delta(n)-d-i)}\overline{\otimes}{}^\perp\mathcal{S}\overline{\otimes}({}^*\Lambda_A)^{\overline{\otimes}i}}$. Therefore we have

$$\begin{aligned}
({}^\natural\Lambda_{\delta(n)})^* &\cong \left(\sum_{i=0}^{\delta(n)-d}({}^*\Lambda_A)^{\overline{\otimes}(\delta(n)-d-i)}\overline{\otimes}{}^\perp\mathcal{S}\overline{\otimes}({}^*\Lambda_A)^{\overline{\otimes}i}\right)^\perp \\
&= \bigcap_{i=0}^{\delta(n)-d}\left(({}^*\Lambda_A)^{\overline{\otimes}(\delta(n)-d-i)}\overline{\otimes}{}^\perp\mathcal{S}\overline{\otimes}({}^*\Lambda_A)^{\overline{\otimes}i}\right)^\perp \\
&= \bigcap_{i=0}^{\delta(n)-d}\left((({}^*\Lambda_A)^{\overline{\otimes}i})^*\overline{\otimes}({}^\perp\mathcal{S})^\perp\overline{\otimes}(({}^*\Lambda_A)^{\overline{\otimes}(\delta(n)-d-i)})^*\right) \\
&\cong \bigcap_{i=0}^{\delta(n)-d}\left(\Lambda_A^{\overline{\otimes}i}\overline{\otimes}\mathcal{S}\overline{\otimes}\Lambda_A^{\overline{\otimes}(\delta(n)-d-i)}\right) \\
&= H_{\delta(n)}.
\end{aligned}$$

This isomorphism takes $x = \sum x_1\overline{\otimes}\cdots\overline{\otimes}x_{\delta(n)} \in H_{\delta(n)}$ to the map $g_x\colon {}^\natural\Lambda_{\delta(n)} \to \Lambda_0$ defined by

$$g_x(\overline{\gamma_{\delta(n)}\overline{\otimes}\cdots\overline{\otimes}\gamma_1}) = \sum \gamma_{\delta(n)}(\gamma_{\delta(n)-1}(\dots(\gamma_2(\gamma_1(x_1)x_2)\dots)x_{\delta(n)-1})x_{\delta(n)}),$$

where the x_i are in Λ_A and the γ_i are in ${}^*\Lambda_A$. □

Lemma A.3. *There is an isomorphism $\psi\colon {}^\natural\Lambda_{\delta(n)} \to \mathrm{Hom}_\Lambda(\mathbf{P}^n,\Lambda_0)$ given by*

$$\psi(\overline{f})(x_1\overline{\otimes}\cdots\overline{\otimes}x_{\delta(n)}\overline{\otimes}\lambda) = f_{\delta(n)}(f_{\delta(n-1)}(\dots(f_1(x_1))\dots x_{\delta(n)-1})x_{\delta(n)})\overline{\lambda},$$

where $\overline{f} = \overline{f_{\delta(n)}\overline{\otimes}\cdots\overline{\otimes}f_1} \in {}^\natural\Lambda_{\delta(n)}$ with $f_i \in {}^*\Lambda_1$ for all i, $x_i \in \Lambda_1$ for all i and $\lambda \in \Lambda$.

Proof: The isomorphism ψ is the composition of the following isomorphisms:

- ${}^\natural\Lambda_{\delta(n)} \to \mathrm{Hom}_{-\Lambda_0}(({}^\natural\Lambda_{\delta(n)})^*,\Lambda_0)$, which sends $\overline{f}$ to the map $[g \mapsto g(\overline{f})]$;
- $\mathrm{Hom}_{-\Lambda_0}(({}^\natural\Lambda_{\delta(n)})^*,\Lambda_0) \to \mathrm{Hom}_{-\Lambda_0}(H_{\delta(n)},\Lambda_0)$, which sends a map h to the map $[x \mapsto h(g_x)]$, where g_x is as in the proof of Lemma A.2;
- $\mathrm{Hom}_{-\Lambda_0}(H_{\delta(n)},\Lambda_0) \to \mathrm{Hom}_{-\Lambda}(H_{\delta(n)}\overline{\otimes}\Lambda,\Lambda_0)$, which sends a map k to the map $[x\overline{\otimes}\lambda \mapsto k(x)\overline{\lambda}]$.

Applying these isomorphisms to $\overline{f}$ gives the expression in the statement. □

Let B be the vector space $B = \bigoplus_{n\geqslant 0} B_n$, where $B_n = {}^\natural\Lambda_{\delta(n)}$. Define a multiplication on B as follows: for $x \in B_n$ and $y \in B_m$, set

$$x.y = \begin{cases} 0 & \text{if } n \text{ and } m \text{ are odd;} \\ 0 & \text{if } n \text{ or } m \text{ is equal to 1 and } n \geqslant 1,\ m \geqslant 1; \\ xy & \text{in } {}^\natural\Lambda \text{ otherwise.} \end{cases}$$

The algebra B is a graded K-algebra generated in degrees 0, 1, 2, and 3.

We want to prove that $E(\Lambda) \cong B$ when $A > 1$ and $D \neq 2A$. We first need a description of the Yoneda product.

Proposition A.4. *Let Λ be a (D, A)-stacked monomial algebra with $A > 1$, $D = dA$, and $d \geqslant 3$. The Yoneda product of $f_n \in \operatorname{Hom}_\Lambda(\mathbf{P}^n, \Lambda_0)$ and $f_m \in \operatorname{Hom}_\Lambda(\mathbf{P}^m, \Lambda_0)$ is given by*

$$f_n f_m = \begin{cases} 0 & \text{if } n \text{ and } m \text{ are odd;} \\ 0 & \text{if } n \geqslant 1,\ m \geqslant 1, \text{ and } n = 1 \text{ or } m = 1; \\ \sum x_1 \overline{\otimes} \cdots \overline{\otimes} x_{\delta(n)+\delta(m)} \overline{\otimes} \lambda \longmapsto f_n\Big(\sum f_m(x_1 \overline{\otimes} \cdots \overline{\otimes} x_{\delta(m)} \overline{\otimes} 1) x_{\delta(m)+1} & \\ \qquad\qquad \overline{\otimes} x_{\delta(m)+2} \overline{\otimes} \cdots \overline{\otimes} x_{\delta(m)+\delta(n)} \overline{\otimes} \lambda\Big) & \text{otherwise,} \end{cases}$$

where the x_i are all in Λ_A.

Proof: If m and n are odd or if $m \geqslant 1$ or $n \geqslant 1$ is equal to 1, then, under the assumption that $A > 1$ and $D \neq 2A$, the Yoneda products vanish by [**17**, Theorem 3.4]. We now assume that m or n is even and that $m \neq 1$ and $n \neq 1$.

Let $\sigma\colon \Lambda_0 \to \Lambda$ be the natural inclusion.

Consider $f_m\colon \mathbf{P}^m \to \Lambda_0$; it lifts to $f_m^0 = \sigma \circ f_m\colon \mathbf{P}^m \to \Lambda$. We now define further liftings $f_m^i\colon \mathbf{P}^{m+i} \to \mathbf{P}^i$ for $i \geqslant 1$ as follows:

$$f_m^1(x_1 \overline{\otimes} \cdots \overline{\otimes} x_{\delta(m+1)} \overline{\otimes} \lambda) = f_m^0(x_1 \overline{\otimes} \cdots \overline{\otimes} x_{\delta(m)} \overline{\otimes} 1) x_{\delta(m)+1} \overline{\otimes} x_{\delta(m)+2} \cdots x_{\delta(m+1)} \lambda;$$

$$f_m^i(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m+i)} \overline{\otimes} \lambda) = f_m^0(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)} \overline{\otimes} 1) y_{\delta(m)+1} \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m+i)} \overline{\otimes} \lambda \quad \text{if } m \text{ or } i \geqslant 2 \text{ is even;}$$

$$f_m^i(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m+i)} \overline{\otimes} \lambda) = f_m^0(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)} \overline{\otimes} 1) y_{\delta(m)+1} \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m+i-1)+1} \quad \text{if } m \text{ and } i \geqslant 2 \text{ are odd,}$$

where the x_i are in Λ_1 and the y_i are in Λ_A. The proof that $(f_m^i)_{i \geqslant 0}$ is a family of liftings of f_m, that is, $f^{i-1} \circ b^{i+m} = b^i \circ f_m^i$ for all $i \geqslant 1$, is tedious but straightforward.

Finally, if n or m is even and $n \geqslant 2$, $m \geqslant 2$, then

$$\begin{aligned} f_n f_m(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)+\delta(n)} \overline{\otimes} \lambda) &= f_n \circ f_m^n(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m+n)} \overline{\otimes} \lambda) \\ &= f_n(f_m^0(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)} \overline{\otimes} 1) y_{\delta(m)+1} \overline{\otimes} y_{\delta(m)+2} \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)+\delta(n)} \overline{\otimes} \lambda) \\ &= f_n(f_m(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)} \overline{\otimes} 1) y_{\delta(m)+1} \overline{\otimes} y_{\delta(m)+2} \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)+\delta(n)} \overline{\otimes} \lambda). \qquad \square \end{aligned}$$

Theorem A.5. *If Λ is a (D, A)-stacked monomial algebra, with $A > 1$, $D = dA$, and $d \geqslant 3$, then $E(\Lambda)$ and B are isomorphic as graded algebras. In particular, $\operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)$ is isomorphic to ${}^\natural\Lambda_{\delta(n)}$.*

Proof: We use the isomorphisms in Lemma A.3 and the cup product described in Proposition A.4. If $\overline{f} = \overline{f_{\delta(n)} \overline{\otimes} \cdots \overline{\otimes} f_1} \in B_n$ and $\overline{g} = \overline{g_{\delta(m)} \overline{\otimes} \cdots \overline{\otimes} g_1} \in B_m$, where m or n is even and both are at least 2, then

$$\begin{aligned} &\psi(\overline{f})\psi(\overline{g})(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)+\delta(n)} \overline{\otimes} \lambda) \\ &\quad = \psi(\overline{f})(\psi(\overline{g})(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)} \overline{\otimes} 1) y_{\delta(m)+1} \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m+n)} \overline{\otimes} \lambda) \\ &\quad = f_{\delta(n)}(f_{\delta(n)-1}(\ldots(f_2(f_1(\psi(\overline{g})(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)} \overline{\otimes} 1) y_{\delta(m)+1}) y_{\delta(m)+2}) \ldots) \\ &\qquad\qquad \cdot y_{\delta'(m+n)-1}) y_{\delta(m+n)}) \overline{\lambda} \\ &\quad = f_{\delta(n)}(f_{\delta(n)-1}(\ldots(f_2(f_1(g_{\delta(m)}(\ldots(g_1(y_1)) \ldots y_{\delta(m)}) y_{\delta(m)+1}) y_{\delta(m)+2}) \ldots) \\ &\qquad\qquad \cdot y_{\delta(m+n)-1}) y_{\delta(m+n)}) \overline{\lambda} \\ &\quad = \psi(\overline{fg})(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)+\delta(n)} \overline{\otimes} \lambda) \\ &\quad = \psi(\overline{f} \cdot \overline{g})(y_1 \overline{\otimes} \cdots \overline{\otimes} y_{\delta(m)+\delta(n)} \overline{\otimes} \lambda), \end{aligned}$$

therefore $\varphi(\overline{f}\cdot\overline{g}) = \psi(\overline{f})\psi(\overline{g})$ and we have proved that ψ is an isomorphism of graded algebras. $\square$

Remark A.6. Recall from Subsection 1.2 that the m-th projective in a minimal projective right Λ-module resolution of Λ_0 is $L^m = \bigoplus_{R_i^m\in\mathcal{R}^m} \mathfrak{t}(R_i^m)\Lambda$. By Consequence 3.4, $R_i^m \in \mathbf{P}^m$. We then have an isomorphism $\mathbf{P}^m \to L^m$ which is determined by $R_i^m \mapsto \mathfrak{t}(R_i^m)$ for all i.

As we mentioned in Subsection 1.2, the authors of [**14**] also gave a basis of $E(\Lambda)$, namely the set $\{g_i^m \in \mathrm{Hom}_\Lambda(L^m,\Lambda_0) \mid R_i^m \in \mathcal{R}^m\}$, where $g_i^m(\mathfrak{t}(R_j^m)) = \mathfrak{t}(R_i^m)$ if $j = i$ and 0 otherwise. The element g_i^m corresponds to a map in $\mathrm{Hom}_\Lambda(\mathbf{P}^m,\Lambda_0)$ which we denote again by g_i^m and that is defined by $g_i^m(R_j^m) = \mathfrak{t}(R_i^m)$ if $j = i$ and 0 otherwise.

We have isomorphisms $K\mathcal{Q}_A \cong K\Gamma_1$ and $K\mathcal{Q}_A \cong {}^*\Lambda_A = {}^*(K\mathcal{Q}_A)$. Combining them, we obtain an isomorphism which associates to $\overline{\alpha} \in \Gamma_1$ the linear form f_α on $K\mathcal{Q}_A$ that sends $\beta \in \mathcal{Q}_A$ to $\mathfrak{t}(\alpha)$ if $\beta = \alpha$ and to 0 otherwise. This extends to an isomorphism between the algebras $K\Gamma/(\sigma)$ and ${}^\natural\Lambda$ that sends a path $\overline{p}$ of length n to the class of the linear map $f_p \in ({}^*\Lambda_A)^{\overline{\otimes} n}$ defined on A-paths by $f_p(q) = \mathfrak{t}(p)$ if $q = p$ and 0 otherwise.

Now consider $R_i^m = \alpha_1\cdots\alpha_{\delta(m)}$ with $\alpha \in \mathcal{Q}_A$ for all i and $\overline{R}_i^m = \overline{\alpha}_{\delta(m)}\cdots\overline{\alpha}_1$ in $K\Gamma/(\sigma)$. It is easy to check that $g_i^m = \psi(\overline{f_{\alpha_{\delta(m)}}\overline{\otimes}\cdots\overline{\otimes}f_{\alpha_1}})$ so that it corresponds, via the isomorphism above, to $\overline{R}_i^m$.

Therefore we have a basis $\mathcal{B}_B = \{\overline{R}_i^m \mid R_i^m \in \mathcal{R}^m\}$ of B.

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