Publ. Mat. **68** (2024), 545–557
DOI: 10.5565/PUBLMAT6822408

JORDAN PROPERTY FOR HOMEOMORPHISM GROUPS AND ALMOST FIXED POINT PROPERTY

Ignasi Mundet i Riera

Abstract: We study properties of continuous finite group actions on topological manifolds that hold true, for any finite group action, after possibly passing to a subgroup of index bounded above by a constant depending only on the manifold. These include the Jordan property, the almost fixed point property, as well as bounds on the discrete degree of symmetry. Most of our results apply to manifolds satisfying some restriction such as having nonzero Euler characteristic or having the integral homology of a sphere. For an arbitrary topological manifold X such that $H_*(X;\mathbb{Z})$ is finitely generated, we prove the existence of a constant C with the property that for any continuous action of a finite group G on X such that every $g \in G$ fixes at least one point of X, there is a subgroup $H \leq G$ satisfying $[G : H] \leq C$ and a point $x \in X$ which is fixed by all elements of H.

2020 Mathematics Subject Classification: 57S17, 55M35.

Key words: finite group actions, topological manifolds, symplectic manifolds.

1. Introduction

1.1. Main results. Our aim in this paper is to prove several results on continuous finite group actions on topological manifolds which, despite not being necessarily true for all group actions, are valid up to passing to subgroups of uniformly bounded index. This includes, for example, results on the Jordan property of homeomorphisms group, as we will explain below, but we will also consider a few other properties. Given the need to pass to subgroups of bounded index, our results are especially meaningful when considering continuous actions of large finite groups.

To materialize the previous idea we introduce some terminology. Let $\mathcal{P}$ and $\mathcal{P}'$ be two general properties of continuous finite group actions on topological manifolds. Both $\mathcal{P}$ and $\mathcal{P}'$ may refer to the algebraic structure of the finite group that acts (it may be abelian, nilpotent, a p-group for an arbitrary prime p, etc.), to the geometry of the action (being effective, free, with or without fixed points, or, if the manifold has the necessary additional structure, being smooth, symplectic, complex, etc.), or to both the group and the action.

Let X be a topological manifold. We say that *almost every* finite group action on X that enjoys some property $\mathcal{P}$ satisfies also property $\mathcal{P}'$ if there exists a constant C, depending only on X, such that the following is true:

> for every action of a finite group G on X which satisfies property $\mathcal{P}$ there exists a subgroup $G' \leq G$ such that $[G : G'] \leq C$ and such that the action of G' on X (defined by restricting the action of G) satisfies property $\mathcal{P}'$.

If property $\mathcal{P}'$ refers only to the group and not to the action, then we will say that almost every group acting on X with property $\mathcal{P}$ satisfies also property $\mathcal{P}'$.

This research was partially supported by the grant PID2019-104047GB-I00 from the Spanish Ministerio de Ciencia e Innovación. Partial support is also acknowledged from the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

Here is a simple example of this notion, which is [**17**, Lemma 2.6] with cohomology replaced by homology (the proof is identical).

Lemma 1.1. *Let X be a topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated, then almost every continuous finite group action on X induces the trivial action on the integral homology $H_*(X;\mathbb{Z})$.*

We next state the main results in this paper. From now on all finite group actions on topological manifolds will be implicitly assumed to be continuous. Topological manifolds will *not* be implicitly assumed to be compact in this paper, and they may have nonempty boundary. As usual, a closed manifold means a compact manifold without boundary.

Theorem 1.2. *If X is a connected and compact topological manifold of dimension at most 3, then almost every finite group acting effectively on X is abelian.*

For any real number a we denote by $[a]$ the biggest integer smaller than or equal to a. For any topological manifold X we denote as usual by $H_*(X;\mathbb{Z})=\bigoplus_{k\geq 0}H_k(X;\mathbb{Z})$ its integral homology. If $H_*(X;\mathbb{Z})$ is finitely generated, the Euler characteristic of X is defined to be

$$\chi(X) := \sum_{k\geq 0}(-1)^k \dim_{\mathbb{Q}} H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If X is a compact topological manifold, then $H_*(X;\mathbb{Z})$ is finitely generated, but this is not always true if X is not compact.

Theorem 1.3. *Let X be a connected n-dimensional topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated and $\chi(X)\neq 0$, then almost every finite group acting effectively on X is abelian and can be generated by $[n/2]$ or fewer elements.*

Theorem 1.4. *Let X be an n-dimensional topological manifold. If $H_*(X;\mathbb{Z}) \simeq H_*(S^n;\mathbb{Z})$, then almost every finite group acting effectively on X is abelian and can be generated by $[(n+1)/2]$ or fewer elements.*

As usual, when we say that the action of a group G on X has a fixed point we mean that there exists a point $x\in X$ fixed by all elements of G.

Theorem 1.5. *Let X be a connected and compact topological manifold. If $\chi(X)\neq 0$, then almost every finite group action on X has a fixed point.*

In Theorem 1.5, compactness is essential. This follows, for example, from the main result in [**11**] (see the end of [**21**, §1.1] for a detailed explanation).

Let us say that an action of a group G on a manifold X has the *weak fixed point property* if for every $g\in G$, there is some $x\in X$ such that $g\cdot x = x$. Here the point x may depend on g, so a priori the weak fixed point property does not imply the existence of a fixed point for the entire action. The next theorem implies that, nevertheless, for finite group actions this turns out to be the case, up to passing to a subgroup of controlled index.

Theorem 1.6. *Let X be a connected n-dimensional topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated, then, for almost every effective action of a finite group G on X with the weak fixed point property, the group G is abelian, it can be generated by $[n/2]$ or fewer elements, and its action on X has a fixed point.*

We next explain some applications of the previous theorem to symplectic geometry. Let (X,ω) be a symplectic manifold. The group of Hamiltonian diffeomorphisms of X, which we denote as usual by $\mathrm{Ham}(X,\omega)$, consists of those diffeomorphisms $\phi\in$

$\mathrm{Diff}(X)$ for which there exists a family of diffeomorphisms $\{\phi_t \in \mathrm{Diff}(X)\}_{t\in[0,1]}$, smoothly depending on t, satisfying $\phi_0 = \mathrm{Id}_X$, $\phi_1 = \phi$, and the following property:

> for every $t \in [0,1]$ there exists some $H_t \in \mathcal{C}^\infty(X)$ such that, for every vector field $\mathcal{F}$ on X, the equality $\omega(\partial_t\phi_t, \mathcal{F}) = dH_t(\mathcal{F})$ holds.

These properties imply in particular that ϕ is a symplectomorphism. Choosing $\{\phi_t\}$ and $\{H_t\}$ appropriately we may assume that $\{H_t\}$ extends to a $\mathbb{Z}$-periodic smooth function of $t \in \mathbb{R}$ (see e.g. [**1**, §4.1]). Suppose that X is compact. A standard argument implies that any $\phi \in \mathrm{Ham}(X,\omega)$ can be uniformly approximated by Hamiltonian diffeomorphisms all of whose fixed points are nondegenerate. We may thus apply Arnold's conjecture (which is now a theorem by [**10, 13**]) to conclude that each $\phi \in \mathrm{Ham}(X,\omega)$ has a fixed point in X. So Theorem 1.6 implies the following:

Corollary 1.7. *Let (X,ω) be a $2m$-dimensional closed symplectic manifold. For almost every action of a finite group G on X which is defined by a monomorphism $G \to \mathrm{Ham}(X,\omega)$ the group G is abelian, it can be generated by m or fewer elements, and its action on X has a fixed point.*

The statement that G is almost always abelian in the previous corollary (equivalently, the fact that $\mathrm{Ham}(X,\omega)$ is Jordan — see below) was proved in [**19**] using a different method. The existence of a fixed point seems to be a new result.

1.2. Jordan property. Following Popov [**25**], we say that a group $\mathcal{G}$ is Jordan if there exists some constant C such that any finite subgroup Γ of $\mathcal{G}$ has an abelian subgroup $A \leq \Gamma$ satisfying $[\Gamma : A] \leq C$. The most basic nontrivial examples are general linear groups, for which the Jordan property was proved by C. Jordan in 1878 [**12**], using, of course, a different terminology:

Theorem 1.8. *For any n, $\mathrm{GL}(n,\mathbb{C})$ is Jordan.*

É. Ghys asked around 1990 whether the diffeomorphism group of any closed smooth manifold is Jordan. A number of papers (see e.g. [**16, 20, 32**]) have appeared in the last few years giving partial positive answers to Ghys's question. In 2014 B. Csikós, L. Pyber, and E. Szabó ([**6**]) found the first counterexamples to Ghys's question, showing that the diffeomorphism group of $T^2 \times S^2$ is not Jordan (later, extending the ideas in [**6**], the author and D. R. Szabó found many other examples; see [**18, 27**]).

In contrast with diffeomorphism groups, the Jordan property for homeomorphism groups has received less attention so far, with the exception of a paper of S. Ye [**31**], which proves that closed flat manifolds such as tori have Jordan homeomorphism groups, and [**22**], which proves that rationally hypertoral manifolds have Jordan homeomorphism groups. The results stated in Subsection 1.1 imply the following.

- The homeomorphism group of any closed and connected topological manifold of dimension at most 3 is Jordan (Theorem 1.2).
- Let X be a connected topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated and $\chi(X) \neq 0$, then the homeomorphism group of X is Jordan (Theorem 1.3).
- If X is an n-dimensional manifold satisfying $H_*(X;\mathbb{Z}) \simeq H_*(S^n;\mathbb{Z})$, then the homeomorphism group of X is Jordan (Theorem 1.4).
- For any closed symplectic manifold (X,ω) the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(X,\omega)$ is Jordan (Corollary 1.7).

The analogue of the first statement for diffeomorphism groups was proved by Zimmermann in [**32**]. Analogues of the second and third statements for diffeomorphism groups have been proved in [**20**].

In view of [**6**], Ghys asked whether for any closed manifold X almost every finite group acting effectively on X is nilpotent. This has been recently proved by B. Csikós, L. Pyber, and E. Szabó [**7**] in full generality for continuous actions on topological manifolds.

1.3. Almost fixed point property. We say that a topological manifold X has the almost fixed point property if there exists a constant C such that for any action of a finite group G on X there exists a point $x \in X$ whose stabilizer $G_x = \{g \in G \mid g \cdot x = x\}$ satisfies $[G : G_x] \leq C$. In this terminology, we may rephrase Theorem 1.5 as follows.

- Let X be a connected, compact, topological manifold with nonzero Euler characteristic; then X has the almost fixed point property.

This generalizes the main result in [**21**] in two directions. First, it replaces smooth actions by continuous actions. Second, and more importantly, it replaces the hypothesis of not having odd cohomology (see [**21**] for a precise definition) by the much weaker condition of having nonzero Euler characteristic.

A simple but already interesting example is the case of a closed disk. Let n be a natural number and suppose that a finite group G acts on the closed n-disk D^n by homeomorphisms. By Brouwer's fixed point theorem, for every $g \in G$ there exists at least one point in D^n which is fixed by the action of g. A priori such a fixed point depends on g. If $n \leq 4$, then one can actually pick some $x \in D^n$ which is simultaneously fixed by all elements of G, but if $n \geq 6$, this is no longer true (see the introduction in [**21**] for references). However, Theorem 1.5 implies the existence, for each n, of a constant $\lambda \in (0, 1]$, depending only on n, with the property that for every action of a finite group G on D^n there exists some point in D^n which is fixed by at least $\lambda \cdot |G|$ elements of G.

1.4. Discrete degree of symmetry. A result of Mann and Su ([**14**]) states that for any closed manifold X there exists a constant M with the following property: for any prime p and any natural number m such that $(\mathbb{Z}/p)^m$ admits an effective action on X we have $m \leq M$. This result is also true for manifolds with boundary and whose integral homology is finitely generated (see Subsection 1.5). It follows that the set

$$\mu(X) = \{m \in \mathbb{N} \mid X \text{ supports effective actions of } (\mathbb{Z}/r)^m \text{ for arbitrarily large } r\}$$

is finite: more precisely, $\mu(X)$ is contained in $\{1, \ldots, M\}$ (of course, $\mu(X)$ may be empty). Following [**22**] we define the discrete degree of symmetry of X to be

$$\text{disc-sym}(X) = \max(\{0\} \cup \mu(X)).$$

By [**22**, Lemma 2.7], for any nonnegative integer k the inequality $\text{disc-sym}(X) \leq k$ is equivalent to the statement that almost every finite abelian group acting effectively on X can be generated by k or fewer elements (if $k = 0$, the latter means by convention that the abelian group is trivial). Consequently, Theorems 1.3 and 1.4 have the following implications, respectively:

- Let X be an n-dimensional connected topological manifold. If $H_*(X; \mathbb{Z})$ is finitely generated and $\chi(X) \neq 0$, then $\text{disc-sym}(X) \leq [n/2]$.
- Let X be an n-dimensional topological manifold such that $H_*(X; \mathbb{Z}) \simeq H_*(S^n; \mathbb{Z})$; then $\text{disc-sym}(X) \leq [(n + 1)/2]$.

It was asked in [**22**, Question 1.1] whether for any compact and connected n-dimensional manifold X we have $\text{disc-sym}(X) \leq n$. This was motivated by the well-known fact that if X supports a continuous and effective action of a torus $(S^1)^d$, then $d \leq n$, and the heuristic according to which the sequence of finite groups $(\mathbb{Z}/r)^m$, with m fixed and $r \to \infty$, can be thought of as increasingly good approximations of the torus T^m. But note that this heuristic has its limitations: as explained in [**22**, Theorem 1.11], not every compact and connected manifold X supports an effective and continuous action of $(S^1)^{\text{disc-sym}(X)}$. Some evidence for the conjectural inequality $\text{disc-sym}(X) \leq \dim X$ was given in [**22**], and the previous results give some additional evidence.

1.5. Some tools. As usual, by a finite p-group we mean a finite group whose cardinal is a power of an arbitrary prime number p.

We next state some results that will play a key role in the proofs of our theorems. While all the results that we mention in this section are originally proved only for manifolds without boundary, all of them extend to the case of manifolds with boundary thanks to the following result.

Lemma 1.9. *Let X be a topological manifold. There is a topological manifold X' without boundary, of the same dimension as X, and an embedding $X \hookrightarrow X'$ which is a homotopy equivalence and which has the property that any action of a group G on X extends to an action on X'.*

Proof: Let $U = \partial X \times [0, 1)$ and let $X' = (X \sqcup U)/\sim$, where $\sim$ identifies each $x \in \partial X$ with $(x, 0) \in U$. The natural inclusion $X \hookrightarrow X'$ is clearly a homotopy equivalence. Given an action of a group G on X we extend it to an action on X' by declaring the action of $g \in G$ on $(x, t) \in U$ to be $(g \cdot x, t)$. □

The first result, which is a particular case of [**5**, Theorem 1.8], is an extension of the theorem of Mann and Su [**14**] to not necessarily compact manifolds.

Theorem 1.10. *Let X be a topological manifold. If $H_*(X; \mathbb{Z})$ is finitely generated, then there exists a constant M with this property: for any prime p and any natural number m such that $(\mathbb{Z}/p)^m$ admits an effective action on X we have $m \leq M$.*

The next theorem is based on results from [**7**].

Theorem 1.11. *Let X be a topological manifold. Suppose that $H_*(X; \mathbb{Z})$ is finitely generated. For every positive number C there exists a positive number C' with the following property. Let G be a finite group acting effectively on X. Suppose that for every prime p and any Sylow p-subgroup G_p of G there is an abelian subgroup $G_p^a \leq G_p$ satisfying $[G_p : G_p^a] \leq C$. Then there is an abelian subgroup $G^a \leq G$ satisfying $[G : G^a] \leq C'$.*

Proof: Combine either Theorem 1.10, [**7**, Corollary 3.18], [**7**, Lemma 6.1], and [**7**, Lemma 5.3], or alternatively Theorem 1.10, [**7**, Lemma 6.1], [**7**, Lemma 5.3], and [**23**, Theorem 3.8]. □

It is crucial in the previous theorem that we only consider finite groups acting effectively on the manifold X. Indeed, for any $C' > 0$ there exists a finite group G all of whose Sylow subgroups are abelian and such that for any abelian subgroup $A \leq G$ we have $[G : A] > C'$ (see e.g. the comments after [**23**, Corollary 1.2]).

The following consequence of Theorem 1.11 is immediate.

Corollary 1.12. *Let X be a topological manifold such that $H_*(X; \mathbb{Z})$ is finitely generated. Suppose that almost every finite p-group that acts effectively on X is abelian. Then almost every finite group that acts effectively on X is abelian.*

If a group G acts on a manifold X, we denote the set of stabilizers of the action as

$$\text{Stab}(G, X) = \{G_x \mid x \in X\}.$$

The following is part of [**5**, Theorem 1.3].

Theorem 1.13. *Let X be a topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated, then there exists a constant C such that for almost every action of a finite p-group G on X we have $|\mathrm{Stab}\,(G,X)|<C$.*

1.6. Contents. Section 2 contains some results on almost fixed points, in Section 3 we prove how the existence of a fixed point for a finite p-group action on an n-manifold implies that the group can be embedded in $\mathrm{GL}(n,\mathbb{R})$, and finally in Section 4 we prove the theorems stated above.

1.7. Acknowledgements. The author is very pleased to thank L. Pyber for sending him a copy of the paper [**7**], on which most of the results in this paper are based. Many thanks also to E. Szabó for useful correspondence. Finally, the author would like to thank the referees for pointing out some corrections and for very useful suggestions to improve the text.

2. Almost fixed points

From now on, cohomology will implicitly refer to Alexander–Spanier cohomology. This is canonically isomorphic to singular cohomology for topological manifolds, but it is better behaved than singular homology when considering fixed point sets of finite p-group actions on topological manifolds. Similarly, Betti numbers are to be defined as dimensions of Alexander–Spanier cohomology groups, and the Euler characteristic is, when defined, the alternate sum of the dimensions of Alexander–Spanier cohomology groups.

Lemma 2.1. *Let X be a topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated and $\chi(X)\neq 0$, then almost every finite p-group action on X has a fixed point.*

Proof: Let p be a prime and let G be a finite p-group acting continuously and effectively on X. We are going to prove that there exists a subgroup $G'\leq G$ satisfying $[G:G']\leq|\chi(X)|$ and $X^{G'}\neq\emptyset$. Let the order of G be p^m. For any nonnegative integer i let X_i be the set of points in X whose stabilizer has order p^i. We may view $H^j(X_i;\mathbb{Z}/p)$ as a vector space over the field $\mathbb{Z}/p$, and we accordingly define $b_j(X_i;\mathbb{Z}/p)=\dim_{\mathbb{Z}/p}H^j(X_i;\mathbb{Z}/p)$. By [**30**, Theorem 2.5] we have $b_j(X_i;\mathbb{Z}/p)<\infty$ for each i, j; furthermore, $b_j(X_i;\mathbb{Z}/p)=0$ for big enough j, and $\chi(X_i)=\sum_{0\leq j}(-1)^j b_j(X_i;\mathbb{Z}/p)$ is divisible by p^{m-i} for every i; finally, $\chi(X)=\sum_{0\leq i}\chi(X_i)$. If we let p^s be the largest power of p that divides $\chi(X)$, which of course satisfies $p^s\leq|\chi(X)|$, we necessarily have $X_{m-k}\neq\emptyset$ for some $k\leq s$, because otherwise $\chi(X)$ would be equal to $\sum_{0\leq i<m-s}\chi(X_i)$, which is divisible by p^{s+1} because each summand is. Let x be a point belonging to X_{m-k} for some $k\leq s$, and let $G'\leq G$ be the stabilizer of x. The order of G' is $p^{m-k}\geq p^{m-s}$, so $[G:G']\leq p^s\leq|\chi(X)|$. □

The following is an analogue for continuous finite p-group actions on topological manifolds of [**21**, Lemma 9], which refers to smooth group actions on smooth manifolds. For the definition of a $\mathbb{Z}/p$-cohomology manifold, see [**4**, Chapter I].

Lemma 2.2. *Let X be a topological manifold without boundary. If $H_*(X;\mathbb{Z})$ is finitely generated, then there exists a constant D with the following property. Let p be any prime. Suppose given a chain of strict inclusions of $\mathbb{Z}/p$-cohomology submanifolds*

$$X_1\subsetneq X_2\subsetneq\cdots\subsetneq X_r\subseteq X,$$

where for each j there is a finite p-group G_j acting continuously on X in such a way that X_j is the union of some of the connected components of X^{G_j}. Then $r\leq D$.

Proof: For any action of a finite p-group G on a $\mathbb{Z}/p$-cohomology manifold X we have

$$|\pi_0(X^G)| = \dim_{\mathbb{F}_p} H^0(X^G;\mathbb{F}_p) \le \sum_j \dim_{\mathbb{F}_p} H^j(X^G;\mathbb{F}_p) \le \sum_j \dim_{\mathbb{F}_p} H^j(X;\mathbb{F}_p).$$

This follows from [**4**, Theorem III.4.3] and an easy induction on the cardinal of G (see e.g. [**19**, Lemma 5.1]). The lemma is a consequence of this estimate on $|\pi_0(X^G)|$ and the arguments in the proof of [**21**, Lemma 9], together with basic properties of the dimension function for cohomology submanifolds (namely, that dimension is well defined for connected $\mathbb{Z}/p$-cohomology manifolds and that it decreases when passing from a connected $\mathbb{Z}/p$-cohomology manifold to a proper connected $\mathbb{Z}/p$-cohomology submanifold). □

Given an action of a group G on a set X we denote for every $g \in G$

$$X^g = \{x \in X \mid g \cdot x = x\}$$

the set of points in X fixed by g.

Theorem 2.3. *Let X be a topological manifold. Suppose that $H_*(X;\mathbb{Z})$ is finitely generated. There exists a constant L with the following property. Let G be a finite p-group acting on X. There exists a subgroup $K \le G$ and an element $g \in K$ satisfying $[G : K] \le L$ and $X^K = X^g$.*

To avoid confusion, the reader should keep in mind that the previous theorem does not rule out the possibility that $X^K = \emptyset$.

Proof: By Lemma 1.9 it suffices to consider the case in which X has no boundary. So we assume in the remainder of the proof that this is the case. By Theorem 1.13 there exist numbers C, C', depending only on X, such that for any finite p-group G acting on X there exists a subgroup $G_s \le G$ satisfying $|\mathrm{Stab}\,(G_s\,, X)| < C$ and $[G : G_s] \le C'$. Let D be the number given by applying Lemma 2.2 to the manifold X. We claim that $L = (CC')^D C'$ satisfies the property given in the statement.

Let G be a finite p-group acting on X. We claim that there is a subgroup $G' \le G$ satisfying $[G : G'] \le (CC')^D$ and which has the property that any subgroup $H \le G'$ such that $X^{G'} \subsetneq X^H$ has index $[G' : H] > CC'$. Indeed, if no such subgroup G' existed, then we could construct a sequence of subgroups

$$G =: G_0 > G_1 > G_2 > \cdots > G_D$$

satisfying $X^{G_i} \subsetneq X^{G_{i+1}}$ and $[G_i : G_{i+1}] \le CC'$ for each $i \ge 0$. This would contradict Lemma 2.2, so the claim is proved.

By Theorem 1.13 there exists a subgroup $K \le G'$ satisfying $[G' : K] \le C'$ and $|\mathrm{Stab}\,(K, X)| < C$. The first property implies that

$$[G : K] = [G : G'] \cdot [G' : K] \le (CC')^D C' = L.$$

We next prove that there exists an element $g \in K$ such that $X^K = X^g$.

Let S be the collection of all $H \in \mathrm{Stab}(K, X)$ such that $H \ne K$. For any $H \in \mathrm{Stab}(K, X)$ the condition $H \ne K$ is equivalent to $X^K \subsetneq X^H$. By our choice of K the set S contains at most $|\mathrm{Stab}\,(K, X)| < C$ elements.

Let $H \in S$. There exists some $x \in X$ such that $H = K_x$ because $H \in \mathrm{Stab}(K, X)$. We have $K_x = G'_x \cap K$, so $G'_x \ne G'$ (for otherwise $H = K_x$ would be equal to K). Hence $G'_x \in \mathrm{Stab}(G', X)$ satisfies $X^{G'} \subsetneq X^{G'_x}$ and so, by the choice of G', we have $[G' : G'_x] > CC'$. The bound $[G' : K] \le C'$ implies that $|K| \ge |G'|/C'$, so

$$[K : H] = [K : K_x] = [K : G'_x \cap K] = \frac{|K|}{|G'_x \cap K|} \ge \frac{|G'|/C'}{|G'_x|} = \frac{[G' : G'_x]}{C'} > \frac{CC'}{C'} = C.$$

Hence for every $H \in S$ we have $|H| < |K|/C$.

We have

$$\left|\bigcup_{H\in S} H\right| \leq \sum_{H\in S}|H| < |S|\frac{|K|}{C} \leq |\mathrm{Stab}\,(K,X)|\frac{|K|}{C} < |K|.$$

So we may take some element $g \in K$ that does not belong to any $H \in S$. Certainly $X^K \subseteq X^g$. If the inclusion were strict, then we could take some $x \in X^g \setminus X^K$. Then K_x would belong to S and would contain g, which is a contradiction. Consequently, $X^K = X^g$. □

Theorem 2.4. *Let X be a topological manifold. If $H_*(X;\mathbb{Z})$ is finitely generated, then almost every finite nilpotent group action on X with the weak fixed point property has a fixed point.*

Proof: Let L be the number given by applying Theorem 2.3 to X. We are going to prove that for any action of a finite nilpotent group N on X with the weak fixed point property there is a subgroup $N' \leq N$ satisfying $[N : N'] \leq L^L$ and $X^{N'} \neq \emptyset$.

Let us fix a finite nilpotent group N and suppose that N acts on X with the weak fixed point property. Let $d = |N|$ and let $p_1, \dots, p_r$ be the primes dividing d. For each i let N_i denote the Sylow p_i-subgroup of N (recall that in a finite nilpotent group all Sylow subgroups are normal, and hence for each prime p it has a unique Sylow p-subgroup).

By Theorem 2.3, for each i there is a subgroup $N_i' \leq N_i$ and an element $a_i \in N_i'$ such that $X^{a_i} = X^{N_i'}$ and $[N_i : N_i'] \leq L$. The latter implies that $N_i' = N_i$ whenever $p_i > L$, since $[N_i : N_i']$ is divisible by p_i. Since the number of primes in $\{1, \dots, L\}$ is obviously at most L, it follows that

$$\prod_i |N_i'| \geq \prod_{p_i\leq L} \frac{|N_i|}{L} \prod_{p_i>L} |N_i| \geq d/L^L. \tag{1}$$

Let $a = a_1 a_2 \cdots a_r$. Since N is nilpotent, the map $N_1 \times \cdots \times N_r \to N$ sending $(g_1, \dots, g_r)$ to $g_1 g_2 \cdots g_r$ is an isomorphism of groups (this follows easily from the fact that each N_i is normal and the orders of the groups N_i are pairwise coprime). Consequently, for every integer e we have $a^e = a_1^e a_2^e \cdots a_r^e$. For each i let $d_i := |N_i|$ and let e_i denote an integer multiple of d/d_i such that $e_i - 1$ is divisible by d_i (e_i exists because d/d_i and d_i are coprime). Then $a^{e_i} = a_i$.

Since the action of N on X has the weak fixed point property, we have $X^a \neq \emptyset$. Choose some $x \in X^a$. For each i we have $x \in X^a \subseteq X^{a^{e_i}} = X^{a_i} = X^{N_i'}$, so the stabilizer N_x of x contains N_i' as a subgroup. Consequently, N_x contains $N_1' \times \cdots \times N_r'$. By (1), $N' := N_x$ satisfies $[N : N'] \leq L^L$, so the proof of the theorem is complete. □

3. Linearizing actions of finite *p*-groups at fixed points

It is well known that if a compact Lie group G acts smoothly and effectively on a connected smooth n-dimensional manifold X and $x \in X^G$ is a fixed point, then the linearization of the action provides an effective linear action of G on $T_x X$. This implies that G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

The last property is false for continuous actions. Indeed, Zimmermann has given in [**33**] an example, for each natural number $n > 5$, of a finite group which acts effectively on S^n and yet is not isomorphic to any subgroup of $\mathrm{GL}(n+1, \mathbb{R})$. Any such action induces an effective action on the cone $CS^n = (S^n \times [0,\infty))/(S^n \times \{0\}) \cong \mathbb{R}^{n+1}$ fixing the point that arises from collapsing $S^n \times \{0\}$ (i.e., the vertex of the cone).

In this section we prove two main results. The first one, Corollary 3.3, states that if we restrict to actions of finite p-groups, then the previous pathology cannot take place. The second one, Theorem 3.5, states that any finite p-group acting continuously on an n-dimensional manifold whose integral homology is isomorphic to $H_*(S^n;\mathbb{Z})$ is isomorphic to a subgroup of $\mathrm{GL}(n+1,\mathbb{R})$ (in contrast with the case of arbitrary finite group actions, again by Zimmermann [**33**]).

The proofs of these results follow in a straightforward way from the ideas in a paper of Dotzel and Hamrick [**8**] and from standard results on continuous actions of finite p-groups on topological manifolds mainly due to Smith and Borel.

The following lemma is a consequence of the arguments in [**8**, Section 1] (see also [**28**, Chapter III, Theorem 5.13]).

Lemma 3.1. *Let p be a prime and G be a finite p-group. Suppose given a function assigning a nonnegative integer $n(H)$ to each subgroup $H \le G$, satisfying these properties:*

(1) *For any two subgroups $K \lhd K' \le G$ such that K'/K is elementary abelian of rank 2 the following holds:*

$$n(K) - n(K') = \sum_H (n(H) - n(K')),$$

where the sum is over the subgroups $H \le K'$ satisfying $K \lhd H$ and $H/K \simeq \mathbb{Z}/p$.

(2) *For any two subgroups $K \le K' \le G$, we have $n(K') \le n(K)$, and if p is odd, then $n(K) - n(K')$ is even.*

(3) *For any subgroup $K \le G$ and any $g \in G$ we have $n(K) = n(gKg^{-1})$.*

(4) *For any three subgroups $K \lhd K' \lhd K'' \le G$ such that $K'/K \simeq \mathbb{Z}/2$ and K''/K is generalized quaternion (resp. cyclic of order 4), $n(K) - n(K')$ is divisible by 4 (resp. $n(K) - n(K')$ is even).*

If $n(K) < n := n(\{1\})$ for every subgroup $K \le G$ different from $\{1\}$, then G is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$.

Conditions (1) to (4) above are usually called Borel–Smith conditions. The result proved in [**8**, Section 1] and in [**28**, Chapter III, Theorem 5.13] states that given any function $G \ge H \mapsto n(H) \ge 0$ satisfying the Borel–Smith conditions there is a real representation $\rho\colon G \to \mathrm{GL}(V)$ with the property that $\dim V^H = n(H)$ for every $H \le G$. This implies in particular that $\dim V = \dim V^{\{1\}} = n := n(1)$ and that, if $n(K) < n$ for every subgroup $K \le G$ different from $\{1\}$, then the action of G on V is effective. Indeed, for any $g \in G$ we have $\mathrm{Ker}(\rho(g) - \mathrm{Id}) = V^{\langle g \rangle}$, where $\langle g \rangle \le G$ denotes the subgroup generated by g, and if $g \ne 1$, then $\dim V^{\langle g \rangle} < \dim V$, so $\rho(g) \ne 1$.

Theorem 3.2. *Let p be a prime, let G be a finite p-group, and suppose that G acts effectively on an n-dimensional connected $\mathbb{Z}/p$-cohomology manifold X without boundary. If $X^G \ne \emptyset$, then G is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{R})$.*

Proof: Take any $x \in X^G$. For any $H \le G$ the fixed point set X^H is a $\mathbb{Z}/p$-cohomology submanifold of X containing x (see [**4**, Chapter I] for the definition of a $\mathbb{Z}/p$-cohomology submanifold and [**4**, Chapter V, Theorem 2.2] for the proof of the statement on X^H). Let $n(H)$ be the dimension of the connected component of X^H containing x. We next explain why the function $G \ge H \mapsto n(H)$ satisfies the Borel–Smith conditions.

Condition (1) is proved in [**4**, Chapter XIII, Theorem 4.3]. The first part of condition (2) is a consequence of the fact, previously used in the proof of Lemma 2.2 above, that the dimension of cohomology manifolds is nonincreasing when passing to a submanifold (this follows from the general properties of dimension of cohomology manifolds; see [**4**, Chapter I]). The second part of condition (2) is proved in [**4**, Chapter V, Theorem 2.3, (a)]. Condition (3) follows from $X^{gKg^{-1}} = gX^K$ and the homeomorphism invariance of dimension. Finally, condition (4) follows from the same arguments as Propositions (4.31) and (4.32) in [**28**, Chapter III].

Since the action of G on X is effective and X is connected, we have $n(K) < n$ for every subgroup $K \leq G$ different from $\{1\}$. Consequently, the theorem follows from Lemma 3.1. □

Corollary 3.3. *Let G be a finite p-group, and suppose that G acts effectively on an n-dimensional connected topological manifold X. If $X^G \neq \emptyset$, then G is isomorphic to a subgroup of* $\mathrm{GL}(n, \mathbb{R})$.

Proof: If X has no boundary, then Theorem 3.2 applies to X for every prime p, so there is nothing to be proved. Suppose that $\partial X \neq \emptyset$. Let $f \colon X \hookrightarrow X'$ be the embedding given by applying Lemma 1.9 to X. Then X' is an n-dimensional topological manifold. Since f is a homotopy equivalence and X is connected, X' is also connected. In addition, for every prime p, X' is a $\mathbb{Z}/p$-cohomology manifold without boundary. Let G be a finite p-group acting effectively on X, and suppose that $X^G \neq \emptyset$. Consider the extension of this action to X' given by Lemma 1.9. Then $(X')^G \neq \emptyset$. Applying Theorem 3.2 to the action of G on X' we conclude that G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$. □

Corollary 3.4. *For any natural number n there exists a number C with the following property. Let N be a finite nilpotent group, and suppose that N acts effectively on an n-dimensional connected topological manifold X. For each prime p let $N_p \leq N$ denote the Sylow p-subgroup of N. If for each p we have $X^{N_p} \neq \emptyset$, then N has an abelian subgroup $A \leq N$ which can be generated by $[n/2]$ elements and which satisfies $[N : A] \leq C$.*

Proof: By Jordan's Theorem 1.8, there exists a constant C_n such that any finite subgroup of $\mathrm{GL}(n, \mathbb{R})$ has an abelian subgroup of index not bigger than C_n. In addition, if H is a finite abelian p-subgroup of $\mathrm{GL}(n, \mathbb{R})$, then there is a subgroup $H' \leq H$ which can be generated by $[n/2]$ or fewer elements and such that $[H : H']$ divides $2^{n-[n/2]}$. Now suppose that N is a finite nilpotent group acting on an n-dimensional connected topological manifold with the property given in the statement of the corollary, and denote as before by N_p the Sylow p-subgroup of N. The previous observations imply that for every prime p there is an abelian subgroup $A_p \leq N_p$ satisfying $[N_p : A_p] \leq \Lambda := C_n 2^{n-[n/2]}$ and such that A_p can be generated by $[n/2]$ or fewer elements. Since N is isomorphic to the direct product of its Sylow subgroups, N contains a subgroup A isomorphic to $\prod_p A_p$. The group A is then abelian and can be generated by $[n/2]$ or fewer elements. Finally, arguing as in the proof of Theorem 2.4 one proves that $[N : A] \leq C := \Lambda^\Lambda$. □

Theorem 3.5. *Let p be a prime, let G be a finite p-group, and suppose that G acts effectively on an n-dimensional $\mathbb{Z}/p$-cohomology sphere. Then G is isomorphic to a subgroup of* $\mathrm{GL}(n+1, \mathbb{R})$.

Proof: It follows from applying Theorem 3.2 to the induced action on the cone $CX = X \times [0, \infty)/(X \times \{0\})$, which is an $(n+1)$-dimensional $\mathbb{Z}/p$-cohomology manifold. Alternatively, one may use results for actions of finite p-groups on $\mathbb{Z}/p$-cohomology spheres analogous to the results on actions with fixed points that we used in the proof of Theorem 3.2 (see [**4, 28**]). □

4. Proofs of the theorems

4.1. Proof of Theorem 1.2: manifolds of dimension at most 3. If X is a compact manifold with nonempty boundary, we denote by $X^\sharp$ the double of X, resulting from taking the disjoint union of two copies of X and identifying the boundary of the first copy with the boundary of the second copy using the tautological identification between them. Then $X^\sharp$ is a closed manifold of the same dimension as X, and any group acting effectively on X also acts effectively on $X^\sharp$. Consequently, to prove Theorem 1.2 it suffices to consider closed manifolds.

Assume that X is a closed topological manifold of dimension 3. By Moise's theorem [**15**] (see also [**3**]), X has a unique smooth structure (actually both Moise and Bing refer to triangulations in their papers; for the existence of a unique smooth structure on triangulated manifolds of dimensions at most 3 see [**29**, Section 3.10]). By a recent result of Pardon [**24**] any finite group acting effectively and topologically on X admits effective smooth actions on X (although, of course, not any topological action is conjugate to a smooth action, as illustrated by the famous example due to Bing [**2**]). The previous facts also hold true in dimensions 1 and 2 with substantially simpler proofs. In dimension 1 they are an easy exercise. See [**26**] for the 2-dimensional version of Moise's theorem (Radó's theorem), and [**9**, pp. 340–341] and the references therein for the other statements.

Since the diffeomorphism group $\mathrm{Diff}(X)$ is known to be Jordan if X is closed and $\dim X \leq 3$ (see [**16**] for the case $\dim X \leq 2$ and Zimmermann [**32**] for the case $\dim X = 3$), it follows that $\mathrm{Homeo}(X)$ is Jordan as well, so Theorem 1.2 is proved. □

4.2. Proof of Theorem 1.3. Let X be a connected n-dimensional topological manifold with finitely generated integral homology and nonzero Euler characteristic. We first prove that almost every finite group acting effectively on X is abelian. By Corollary 1.12, it suffices to prove that almost every finite p-group acting effectively on X is abelian. By Lemma 2.1, almost every finite p-group action on X has a fixed point. Let G be a finite p-group acting effectively on X and suppose that $X^G \neq \emptyset$. By Corollary 3.3, G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$, and by Theorem 1.8, G has an abelian subgroup $G' \leq G$ with $[G : G']$ bounded above by a constant depending only on n. This concludes the proof that almost every finite group acting effectively on X is abelian. By Lemma 2.1 (and arguing as in the proof of Theorem 2.4), we conclude that for almost every effective action of a finite abelian group A on X each Sylow subgroup of A has a fixed point in X. Combining this fact with Corollary 3.4 we deduce that almost every finite abelian group acting effectively on X can be generated by $[n/2]$ or fewer elements. □

4.3. Proof of Theorem 1.4. The proof is almost identical to that of Theorem 1.3, except that to prove that almost every finite p-group acting on X is abelian we use Theorems 3.5 and 1.8. Regarding the use of Theorem 3.5, note that, since X is an n-dimensional topological manifold, the isomorphism $H_*(X; \mathbb{Z}) \simeq H_*(S^n; \mathbb{Z})$ implies that X is connected and orientable, so by Poincaré duality we have $H^*(X; \mathbb{Z}) \simeq H^*(S^n; \mathbb{Z})$, which implies that X is a $\mathbb{Z}/p$-cohomology sphere for every prime p. □

4.4. Proof of Theorem 1.5. Let X be a compact and connected n-dimensional topological manifold with nonzero Euler characteristic. Since X is compact, $H_*(X; \mathbb{Z})$ is finitely generated. By Lemma 1.1 almost every finite group action on X induces the trivial action on $H_*(X; \mathbb{Z})$. Since X is compact and has nonzero Euler characteristic, Lefschetz's formula ([**28**, Exercise 6.17.3]) implies that almost every finite group action on X has the almost fixed point property. Hence Theorem 1.5 follows from Theorem 1.6. □

4.5. Proof of Theorem 1.6. Let X be a connected topological manifold with finitely generated integral homology. By the solution to Ghys's conjecture in [**7**], for almost every effective action of a finite group G on X, the group N is nilpotent. Adding to this Theorem 2.4 and Corollary 3.4 we conclude that for almost every action of a finite group G on X with the weak fixed point property the group G is abelian, its action on X has a fixed point, and G can be generated by $[n/2]$ or fewer elements. □

References

[1] A. Abbondandolo and F. Schlenk, Floer homologies, with applications, *Jahresber. Dtsch. Math.-Ver.* **121(3)** (2019), 155–238. DOI: 10.1365/s13291-018-0193-x.

[2] R. H. Bing, A homeomorphism between the 3-sphere and the sum of two solid horned spheres, *Ann. of Math. (2)* **56(2)** (1952), 354–362. DOI: 10.2307/1969804.

[3] R. H. Bing, An alternative proof that 3-manifolds can be triangulated, *Ann. of Math. (2)* **69(1)** (1959), 37–65. DOI: 10.2307/1970092.

[4] A. Borel, *Seminar on Transformation Groups*, With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais, Ann. of Math. Stud. **46**, Princeton University Press, Princeton, NJ, 1960. DOI: 10.1515/9781400882670.

[5] B. Csikós, I. Mundet i Riera, L. Pyber, and E. Szabó, On the number of stabilizer subgroups in a finite group acting on a manifold, Preprint (2021). arXiv:2111.14450v1.

[6] B. Csikós, L. Pyber, and E. Szabó, Diffeomorphism groups of compact 4-manifolds are not always Jordan, Preprint (2014). arXiv:1411.7524v1.

[7] B. Csikós, L. Pyber, and E. Szabó, Finite subgroups of the homeomorphism group of a compact topological manifold are almost nilpotent, Preprint (2022). arXiv:2204.13375v1.

[8] R. M. Dotzel and G. C. Hamrick, p-group actions on homology spheres, *Invent. Math.* **62(3)** (1980/81), 437–442. DOI: 10.1007/BF01394253.

[9] A. L. Edmonds, Transformation groups and low-dimensional manifolds, in: *Group Actions on Manifolds* (Boulder, Colo., 1983), Contemp. Math. **36**, American Mathematical Society, Providence, RI, 1985, pp. 339–366. DOI: 10.1090/conm/036/780973.

[10] K. Fukaya and K. Ono, Arnold conjecture and Gromov–Witten invariant, *Topology* **38(5)** (1999), 933–1048. DOI: 10.1016/S0040-9383(98)00042-1.

[11] R. Haynes, S. Kwasik, J. Mast, and R. Schultz, Periodic maps on R^7 without fixed points, *Math. Proc. Cambridge Philos. Soc.* **132(1)** (2002), 131–136. DOI: 10.1017/S0305004101005345.

[12] M. C. Jordan, Mémoire sur les équations différentielles linéaires à intégrale algébrique, *J. Reine Angew. Math.* **84** (1878), 89–215. DOI: 10.1515/crelle-1878-18788408.

[13] G. Liu and G. Tian, Floer homology and Arnold conjecture, *J. Differential Geom.* **49(1)** (1998), 1–74. DOI: 10.4310/jdg/1214460936.

[14] L. N. Mann and J. C. Su, Actions of elementary p-groups on manifolds, *Trans. Amer. Math. Soc.* **106(1)** (1963), 115–126. DOI: 10.2307/1993717.

[15] E. E. Moise, Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung, *Ann. of Math. (2)* **56(1)** (1952), 96–114. DOI: 10.2307/1969769.

[16] I. Mundet i Riera, Jordan's theorem for the diffeomorphism group of some manifolds, *Proc. Amer. Math. Soc.* **138(6)** (2010), 2253–2262.

[17] I. Mundet i Riera, Finite group actions on 4-manifolds with nonzero Euler characteristic, *Math. Z.* **282(1-2)** (2016), 25–42. DOI: 10.1007/s00209-015-1530-8.

[18] I. Mundet i Riera, Non Jordan groups of diffeomorphisms and actions of compact Lie groups on manifolds, *Transform. Groups* **22(2)** (2017), 487–501. DOI: 10.1007/s00031-016-9374-9.

[19] I. Mundet i Riera, Finite subgroups of Ham and Symp, *Math. Ann.* **370(1-2)** (2018), 331–380. DOI: 10.1007/s00208-017-1566-7.

[20] I. Mundet i Riera, Finite group actions on homology spheres and manifolds with nonzero Euler characteristic, *J. Topol.* **12(3)** (2019), 744–758. DOI: 10.1112/topo.12100.

[21] I. Mundet i Riera, Almost fixed points of finite group actions on manifolds without odd cohomology, *Transform. Groups* **25(4)** (2020), 1269–1288. DOI: 10.1007/s00031-019-09534-7.

[22] I. Mundet i Riera, Discrete degree of symmetry of manifolds, Preprint (2021). arXiv:2112.05599v2.

[23] I. MUNDET I RIERA AND A. TURULL, Boosting an analogue of Jordan's theorem for finite groups, *Adv. Math.* **272** (2015), 820–836. DOI: 10.1016/j.aim.2014.12.021.
[24] J. PARDON, Smoothing finite group actions on three-manifolds, *Duke Math. J.* **170(6)** (2021), 1043–1084. DOI: 10.1215/00127094-2020-0052.
[25] V. L. POPOV, On the Makar–Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties, in: *Affine Algebraic Geometry*, CRM Proc. Lecture Notes **54**, American Mathematical Society, Providence, RI, 2011, pp. 289–311.
[26] T. RADÓ, Über den Begriff der Riemannschen Fläche, *Acta Sci. Math. (Szeged)* **2(2-2)** (1924–26), 101–121.
[27] D. R. SZABÓ, Special p-groups acting on compact manifolds, Preprint (2019). arXiv:1901.07319v2.
[28] T. TOM DIECK, *Transformation Groups*, De Gruyter Stud. Math. **8**, Walter de Gruyter & Co., Berlin, 1987. DOI: 10.1515/9783110858372.
[29] W. P. THURSTON, *Three-Dimensional Geometry and Topology*. Vol. 1, Edited by Silvio Levy, Princeton Math. Ser. **35**, Princeton University Press, Princeton, NJ, 1997.
[30] S. YE, Euler characteristics and actions of automorphism groups of free groups, *Algebr. Geom. Topol.* **18(2)** (2018), 1195–1204. DOI: 10.2140/agt.2018.18.1195.
[31] S. YE, Symmetries of flat manifolds, Jordan property and the general Zimmer program, *J. Lond. Math. Soc. (2)* **100(3)** (2019), 1065–1080. DOI: 10.1112/jlms.12260.
[32] B. P. ZIMMERMANN, On Jordan type bounds for finite groups acting on compact 3-manifolds, *Arch. Math. (Basel)* **103(2)** (2014), 195–200. DOI: 10.1007/s00013-014-0671-z.
[33] B. P. ZIMMERMANN, On topological actions of finite, non-standard groups on spheres, *Monatsh. Math.* **183(1)** (2017), 219–223. DOI: 10.1007/s00605-016-0959-0.

Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain, and Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, 08193, Barcelona, Spain
E-mail address: ignasi.mundet@ub.edu

Received on December 15, 2022.
Accepted on January 15, 2024.