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ON GROUPS OF FINITE PRÜFER RANK

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Abstract: Let G be a group of finite rank and π any finite set of primes. We prove that G contains a characteristic subgroup H of finite index such that every finite π-image of H is nilpotent. Our conclusions are stronger if G is also soluble.

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A group has finite rank if there is an integer r such that each of its finitely generated subgroups can be generated by at most r elements, the least such r being its rank. In [**1**] Azarov and Romanovskii prove the following. Let G be a group of finite rank and π a finite set of primes. If G is either soluble or finitely generated, then G contains a subgroup H of finite index such that every finite π-image of H is nilpotent. They derive these two theorems from comparative results about profinite groups. Here, using quite different and more direct methods, we prove the following generalization that also covers both the above.

Theorem 1. *Let G be a group of finite rank and π any finite set of primes. Then G contains a characteristic subgroup H of finite index such that every finite π-image of H is nilpotent.*

A group G is said to have finite Hirsch number h if G has an ascending series running from $\langle 1\rangle$ to G with exactly h of the factors infinite cyclic, the remaining factors all being locally finite. Suppose G is a group with no non-trivial locally finite normal subgroups. If G has finite Hirsch number h, then G is soluble-by-finite with finite rank at most $7h/2+1$; see [**8**, Theorems 1 and 3]. Conversely if G is soluble-by-finite with finite rank r, then G has finite Hirsch number at most r.

Theorem 2. *Let G be a group with finite Hirsch number that also satisfies the minimal condition on p-subgroups for every prime p. If π is any finite set of primes, there exists a characteristic subgroup H of G of finite index such that if $X \geq Y$ are subgroups of H with $(H:Y)$ finite, Y normal in X, and X/Y a π-group, then X/Y is nilpotent. Moreover, if X is normal in H, then $[X, {}_k H] \leq Y$ for some integer k.*

Every finite extension of a soluble FAR group (see [**6**] especially page 86) satisfies all the hypotheses of Theorem 2 but not all those of Theorem 1. Clearly groups G in Theorem 2 need not be soluble-by-finite and need not have finite rank. Every soluble-by-finite group of finite rank satisfies the hypotheses of Theorem 2.

For elementary reasons the subgroup H in Theorem 1 does not necessarily satisfy the conclusions of Theorem 2, though of course other choices for H might. For example, let K be a finite perfect simple group, π the set of prime divisors of the order of K, C a cyclic group of prime order $p \notin \pi$, G the wreath product of K by C, X the base group of G, and $Y = \langle 1\rangle$. Then the only π-image of G is G/G, so $H = G$ satisfies the requirements of Theorem 1 but not those of Theorem 2, since X/Y is not nilpotent.

$\langle P', L\rangle(\mathbf{AF})$ denotes the smallest class of groups that contains all abelian groups and all finite groups and is closed under the ascending series operator P' and the local operator L. Any $\langle P', L\rangle(\mathbf{AF})$ group of finite rank is (locally finite)-by-soluble-by-finite by Theorem 1 of [**8**] and hence satisfies the requirements of Theorem 2. Consequently the following holds.

Corollary. *Let G be a $\langle P', L\rangle(\mathbf{AF})$ group of finite rank. If π is any finite set of primes, there exists a characteristic subgroup H of G of finite index such that if $X \geq Y$ are subgroups of H with $(H:Y)$ finite, Y normal in X, and X/Y a π-group, then X/Y is nilpotent. Moreover, if X is normal in H, then $[X, {}_kH] \leq Y$ for some integer k.*

Lemma 1. *Let N be a normal subgroup of the characteristic subgroup H of the group G, G being of finite rank at most r. Let M be the intersection of all $N\phi$ for ϕ ranging over* $\operatorname{Aut} G$. *If H/N is a finite π-group for some set π of primes, then H/M is also a finite π-group.*

Proof: Now H/M embeds into the Cartesian product of copies of H/N, so $\exp(H/M)=\exp(H/N)$ and H/M is a π-group. Let P be a Sylow p-subgroup of H/N of order p^n for some $p \in \pi$. Then any p-subgroup of H/M has a series of length n with elementary abelian factors and hence has order at most p^{nr}. It follows that H/M is finite. □

For any group G let $s(G)$ denote the subgroup of G generated by all the soluble normal subgroups of G. Then $s(G)$ is characteristic in G, is locally soluble, and is even soluble if G is finite or linear.

Lemma 2. *If r is a positive integer, there exists a positive integer n depending only on r such that if G is a finite group of rank at most r, then there exists a normal subgroup $J \geq s(G)$ of G such that G/J embeds into* $\operatorname{Sym}(r)$ *and $J's(G)/s(G)$ embeds into the direct product of r linear groups of degree n.*

Lemma 2 is effectively a very special case of Theorem 2 of [**9**]; in fact the first paragraph of the proof of that theorem suffices to prove Lemma 2 above.

For any group G and positive integer r, let $G(\operatorname{Sym}(r))$ denote the intersection of the kernels of all the homomorphisms of G into $\operatorname{Sym}(r)$.

Lemma 3. *Let G be a finite group of rank at most r with $G(\operatorname{Sym}(r)) = \langle 1\rangle$. Set $s = r^2$ and $e = \prod_p p^s$, where p ranges over all primes $p \leq r$. Then $|G|$ divides e.*

Proof: If P is a Sylow p-subgroup of G, then P embeds into the direct product of finitely many copies of $\operatorname{Sym}(r)$, so P has a series of length at most r with elementary abelian factors. Thus $|P| \leq (p^r)^r$ and the claim follows. □

Lemma 4. *Let π be a finite set of primes and suppose G is a finite linear π-group of degree n and rank at most r. Then there exists a positive integer m depending only on n, r, and π such that $|G/s(G)| \leq m$.*

Proof: If $\operatorname{char} G = 0$, this follows immediately from Jordan's theorem (e.g. [**3**, 5.7]). Let $\operatorname{char} G = q > 0$ and consider a Sylow q-subgroup Q of G. If $Q = \langle 1\rangle$, then again Lemma 4 follows from [**3**, 5.7]. If $Q \neq \langle 1\rangle$, then $q \in \pi$ and, being unipotent, Q has a series of length $n-1$ with elementary abelian factors. By the Brauer–Feit theorem, see [**2**], G has an abelian normal subgroup A with $(G:A)$ bounded in terms of n, r, and q only. The lemma follows. □

Lemma 5. *Let π be a finite set of primes and r some positive integer. If G is a finite soluble π-group of rank at most r, then $(G:\operatorname{Fitt}(G))$ divides $k = \prod_p GL(r,p)^2$, where p ranges over all of π.*

We make no attempt to obtain the best bound here or, for that matter, with various other bounds we require.

Proof: There exists a nilpotent normal subgroup B of G of class at most 2 with $Z = C_G(B) \le B$; see e.g. [**5**, 1.A.8]. Set

$$L = \bigcap_{p\in\pi}(C_G(Z/Z^p) \cap C_G(B/B^pZ).$$

Then L acts nilpotently on Z and B/Z and hence also on B. By stability theory $L/Z = L/C_G(B)$ is nilpotent. But L acts nilpotently on Z; consequently L is nilpotent. Clearly $(G : L)$ divides k. □

Proof of Theorem 1: Suppose no such H exists. By Lemma 1 there is a descending series $G = G_0 > G_1 > \cdots > G_i > \cdots$ of characteristic subgroups of G of finite index such that each G_i/G_{i+1} is a non-nilpotent π-group. Set $G_\omega = \bigcap_i G_i$. Suppose H/G_ω is a characteristic subgroup of G/G_ω of finite index all of whose finite π-images are nilpotent. Each G/G_i is a finite π-group. Thus HG_{i+1}/G_{i+1} is nilpotent, while G_i/G_{i+1} is not. But H covers all the G_i/G_{i+1} with at most $(G : H)$ exceptions. Since none of the G_i/G_{i+1} are nilpotent it follows that no such H exists. Therefore we may assume that $G_\omega = \langle 1\rangle$.

Set $S_i/G_i = s(G/G_i)$; clearly $S_{i+1}G_i \le S_i$ for all $i \ge 1$. With $r = \operatorname{rank} G$, $n = n(r)$ as in Lemma 2, and $K = G(\mathrm{Sym}(r))$, it follows that $K'S_i/S_i$ embeds into the direct product of r linear groups of degree n. Also by Lemma 3 we have $(G : K) \le e = e(r)$. By Lemma 4 there is an integer m depending only on r and π such that $(K'S_i/S_i : S_i) \le m$. Clearly then $(G : C_G(K'S_i/S_i) \le m!$.

Now $(KS_i \cap C_G(K'S_i/S_i)/S_i$ is soluble. Thus $(G : S_i) \le (m!)e$. Set $L_i/G_i = \mathrm{Fitt}(G/G_i)$, so $L_i \le S_i$ for all $i \ge 1$. By Lemma 5 we have $(G : L_i) \le (m!)ek$. Pick j so that $(G : L_j)$ is maximal. Clearly $L_{j+1} \le L_j$, so $L_{j+1} = L_j$. But then $G_j \le L_j \le L_{j+1}$, so $G_j/G_{j+1} \le L_{j+1}/G_{j+1}$ is nilpotent, which is false. This final contradiction completes the proof of the theorem. □

For any group G let $\tau(G)$ denote the unique maximal, locally finite, normal subgroup of G.

Lemma 6. *Let G be a group with finite Hirsch number h and with $\tau(G)$ finite. Then G has a characteristic series $\langle 1\rangle = N_0 \le N_1 \le \cdots \le N_d \le G$ of finite length such that G/N_d is finite and each N_i/N_{i-1} is torsion-free abelian of finite rank.*

This lemma is surely well known. It is immediate from Lemmas 4 and 6 of [**7**]; the soluble-by-finite case also follows from 5.2.4 and 5.2.5 of [**6**]. We can take $d \le h$ if we wish, since clearly h is equal to the sum of the ranks of the N_i/N_{i-1}.

Proof of Theorem 2: Suppose first that $T = \tau(G)$ is finite, so by Lemma 6 the group G has a characteristic series $\langle 1\rangle = N_0 \le N_1 \le \cdots \le N_d \le G$ of finite length such that G/N_d is finite and each N_i/N_{i-1} is torsion-free abelian of finite rank. We induct on d, the claims being obvious if $d \le 1$. Set $A = N_1$. Clearly A/A^m is finite for every positive integer m, so $K = \bigcap_{p\in\pi} C_G(A/A^p)$ is a characteristic subgroup of G of finite index containing A. Clearly $\tau(K/A)$ is finite.

By induction applied to K/A we may assume there is a characteristic subgroup H/A of K/A of finite index such that if $A \le Y \le X \le H$ with $(H : Y)$ finite, Y normal in X, and X/Y a π-group, then X/Y is nilpotent and further, if X is normal in H, then $[X, {}_eH] \le Y$ for some integer e. Now consider $Y \le X \le H$ with $(H : Y)$ finite, Y normal in X, and X/Y a π-group. Then $X/(X \cap A)Y \cong XA/YA$ is nilpotent.

By construction $[A, K] \leq A^p$ for every p in π. Clearly $(A : Y \cap A)$ is finite and $(X \cap A)/(Y \cap A) \cong (X \cap A)Y/Y$ is a finite π-group. If $m = (A : Y \cap A)$, then $A^m \leq Y \cap A$ and from the definition of K there is a positive integer f such that $[A/A^m, {}_f K]$ is a π'-group. It follows that $[X \cap A, {}_f X] \leq Y \cap A$. Hence we have $X/(X \cap A)Y$ nilpotent and $[(X \cap A)Y, {}_f X] \leq Y$. Consequently X/Y is nilpotent.

Now assume X is also normal in H. There is a positive π-integer m with $X^m \leq Y$. Then $X^m \leq Y_H = \bigcap_{x \in H} Y^x \leq Y$ and hence, using the series above, X/Y_H is a finite π-group. Thus we may assume Y as well as X is normal in H. By the induction hypothesis we have $[XA, {}_e H] \leq YA$ for some e, so $[X, {}_e H] \leq X \cap YA = (X \cap A)Y$. Also $[X \cap A, {}_f H] \leq Y \cap A$, so $[(X \cap A)Y, {}_f H] \leq Y$. Consequently $[X, {}_{e+f} H] \leq Y$. This completes the proof if T is finite.

We now consider the general case. By Belyaev's theorem (e.g. [**4**, 3.5.15] or see [**9**]) the group T has a locally soluble, characteristic subgroup of finite index, so by a theorem of Kargapolov (e.g. [**5**, 3.18]) there is a divisible abelian characteristic subgroup A of T such that T/A is residually finite with all its Sylow subgroups finite. Since π is finite it follows that $B/A = O_{\pi'}(T/A)$ is a characteristic π'-subgroup of T/A of finite index. Clearly B is characteristic in T and hence G.

Now $\tau(G/B) = T/B$ is finite. Thus by the special case above G/B has a characteristic subgroup H/B satisfying the conclusions of the theorem. Certainly H is characteristic and of finite index in G. Consider subgroups $X \geq Y$ of H with $(H : Y)$ finite, Y normal in X, and X/Y a π-group. Since $A \leq H$ and A is divisible, $A \leq Y$. Also $(X \cap B)/(Y \cap B) \cong (X \cap B)Y/Y$, which is a π-group and as a section of B/A it is also a π'-group. Thus $X \cap B = Y \cap B$. Further

$$XB/YB \cong X(X \cap YB) = X/(X \cap B)Y = X/Y.$$

But from the choice of H we have XB/YB nilpotent. Consequently X/Y is nilpotent.

Finally, suppose X is also normal in H. Then XB/B is normal in H/B, so there exists an integer k with $[XB, {}_k H] \leq YB$. Thus

$$[X, {}_k H] \leq X \cap YB = (X \cap B)Y = (Y \cap B)Y = Y.$$

The proof of Theorem 2 is now complete. □

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