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COMPLEXITY OF PUISEUX SOLUTIONS OF DIFFERENTIAL AND q-DIFFERENCE EQUATIONS OF ORDER AND DEGREE ONE

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Abstract: We relate the complexity of both differential and q-difference equations of order one and degree one and their solutions. Our point of view is to show that if the solutions are complicated, the initial equation is complicated too. In this spirit, we bound from below an invariant of the differential or q-difference equation, the height of its Newton polygon, in terms of the characteristic factors of a solution. The differential and the q-difference cases are treated in a unified way.

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1. Introduction

The "Poincaré problem", which consists in finding an upper bound for the algebraic degree of an invariant curve of a polynomial differential equation in the complex plane [**22**], has greatly influenced the study of singular holomorphic foliations. See [**4, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19, 21, 23, 24, 25**] for just a possibly biased collection of relevant citations. A related problem, which might be called the *local Poincaré problem*, consists in trying to find upper bounds for the *multiplicity* at a point of an invariant analytic curve of a holomorphic foliation defined in a germ of complex surface. As a matter of fact, the solution to this problem in the non-dicritical case is an essential part of the proof of the main result in [**9**]: $\deg(\Gamma) \leq \deg(\mathcal{F}) + 2$, where Γ is an invariant algebraic curve of a holomorphic foliation $\mathcal{F}$ in $\mathbb{CP}^2$ with no dicritical singularities and deg stands for the degree. In this work, we focus on this local problem, and solve it *at the same time* for differential and q-difference equations, as we shall show.

As the original Poincaré problem is stated for differential equations, the usual techniques for solving it are geometric in nature and derived from the general theory of singularities of curves and of plane holomorphic foliations. There is, however, a less known and powerful tool called the *Newton polygon* or *diagram* [**20**], introduced by the renowned physicist and mathematician as a tool for computing solutions of algebraic equations, and later applied by Cramer [**13**] for computing power series $y = \sum_{\iota>0} a_\iota x^\iota$ with rational exponents that are solutions of analytic equations $f(x, y) = 0$. This tool, which is purely algorithmic and makes no reference to the geometric nature of the problem, can be applied to any two-variable problem involving power series for which one seeks a solution in terms of well-ordered power series in one of the variables. In [**5**],

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this technique is applied to differential equations in two variables, whereas in [**2**], it is used in the context of q-algebraic equations. The modifications required for these applications are minimal, and one can unify the arguments and state general results regardless of the context.

Specifically, let σ denote either the differential operator $y(x) \mapsto dy(x)/dx$ or the q-difference one $y(x) \mapsto y(qx)$, and let $P \equiv A(x,y) + B(x,y)\sigma(y)$ be a "first order and first degree" polynomial in the operator σ: $A(x,y)$ and $B(x,y)$ are power series over $\mathbb{C}$, and a solution of $P = 0$ is a power series with rational exponents $s(x) \in \bigcup_{m\in\mathbb{N}} \mathbb{C}[[x^{\frac{1}{m}}]]$ such that $P(x, s(x)) \equiv 0$. Given such an analytic differential or q-difference equation in two variables, our main objective is to compute an upper bound for the complexity of a power series solution $s(x)$ in terms of some property of the original equation. Obviously it is critical to consider the right notion of complexity. Notice that if γ is the local, possibly formal, plane branch given by the parametrization $(x, s(x))$, then there is no way to bound the multiplicity of γ in terms of algebraic invariants of the equation and, specifically, its multiplicity: the differential equation $nx\, dy - my\, dx = 0$, with $m, n \in \mathbb{N}$ and $\gcd(m,n) = 1$, has multiplicity 1 but its power series solutions are $s(x) = cx^{\frac{m}{n}}$ for any c. The curve $(x, cx^{m/n})$ has multiplicity $\min(m,n)$ if $c \in \mathbb{C}^*$, that can be arbitrarily large. For q-difference equations, the same issue occurs with the equations $y - q^{\frac{n}{m}}\sigma(y) = 0$ of multiplicity 0, where σ is the q-difference operator: the solutions are $s(x) = cx^{\frac{m}{n}}$ for $c \in \mathbb{C}$. This problem cannot be overcome, which leads to seeking a different criterion for the complexity of a Puiseux power series solution.

In this paper, we consider the characteristic exponents of $s(x)$ as the measure of such complexity, following the point of view of [**8**]. Notice that the characteristic exponents of $s(x)$ are intimately related to the Puiseux characteristic of the curve Γ defined by $(x, s(x))$ [**26**] but, when Γ is tangent to $x = 0$, one has to use the well-known *inversion formula* [**27, 1**] in order to compute one set of exponents from the others. The characteristic exponents are significant invariants for germs of plane curves: for instance, their number, which has come to be called the *genus*, is deeply related to the topology of the curve γ, as it measures the levels of interlacing of the associated knot [**26**]. Given $s(x) \in \bigcup_{m\in\mathbb{N}} \mathbb{C}[[x^{\frac{1}{m}}]]$ with $s(0) = 0$, let $g \geq 0$ be the genus of $s(x)$ and n its multiplicity. One can derive, from the characteristic exponents, positive integers $r_1, \ldots, r_g$, which we later call the *characteristic factors* in Definition 2, that are greater than 1 and such that if n is the least common denominator of the exponents of $s(x)$, then $r_1 \cdots r_g = n$. Our results hinge on these factors and the notion of the *dicritical exponent.* Assume $s(x) = \sum a_i x^{i/m}$ is a solution of $P = 0$. Roughly speaking, an exponent k/m of $s(x)$ is *dicritical* if, for all but finitely many $c \in \mathbb{C}$, there exists another solution $s_c(x)$ of $P = 0$ such that $s_c(x) - \sum_{i<k} a_i x^{i/m} = x^{k/m} u(x)$ where $u(0) = c$. We include a brief excursus in Subsection 2.6 relating our definition to the classical definition of the dicritical divisor of a singular holomorphic foliation.

In what follows, $P \equiv A(x,y) + B(x,y)\sigma(y)$ is an operator with $A(x,y), B(x,y) \in \mathbb{C}[[x,y]]$ such that $A(0,0) = B(0,0) = 0$, and $s(x) \in \bigcup_{m\in\mathbb{N}} \mathbb{C}[[x^{\frac{1}{m}}]]$, with $s(0) = 0$ is a solution of $P = 0$ with $r_1, \ldots, r_g$ its characteristic factors. After constructing the Newton diagram $\mathcal{N}(P)$, we shall attach to P and $s(x)$ several invariants: $H(P)$, the *height of* P, which is the topmost vertex of $\mathcal{N}(P)$; the *multiplicity of* P, $\nu_0(P)$, which is the minimum multiplicity of $A(x,y)$ and $B(x,y)$; and a number $H(P, s(x))$, a kind of *relative height*, which is, roughly speaking, the topmost vertex of the part of $\mathcal{N}(P)$ corresponding to the order of $s(x)$, $\mathrm{ord}(s(x))$. By definition, we have

$$H(P) \geq H(P, s(x)), \quad \text{for any } s(x),$$

and also,

$$\nu_0(P) + 1 \geq H(P, s(x)), \quad \text{if } \mathrm{ord}(s(x)) \geq 1.$$

Our main results provide bounds of $H(P)$ and of $\nu_0(P)$ from below in terms of the characteristic factors $r_1, \dots, r_g$.

Theorem A. *If $1 \leq i_1 < \cdots < i_d \leq g$ is the sequence of indices of dicritical characteristic exponents of $s(x)$, then*

$$H(P) \geq H(P, s(x)) \geq \prod_{j=1}^{g} r_j - \sum_{k=1}^{d} \left(\prod_{j=1}^{i_k} r_j - \prod_{j=1}^{i_k-1} r_j \right).$$

If, moreover, $\operatorname{ord}(s(x)) \geq 1$*, then $\nu_0(P) + 1$ is greater than or equal to the right hand side of the inequality.*

By convention and unless expressly stated otherwise, we consider that an empty sum is equal to 0 and that an empty product is equal to 1. In particular, the right hand side is equal to 1 if $g = 0$. Note that if no characteristic exponent corresponds to a dicritical element, then Theorem A reads

$$H(P) \geq H(P, s(x)) \geq r_1 \cdots r_g.$$

The inequality can be improved with every instance of consecutive dicritical characteristic exponents (Lemma 13). In this way, one obtains a simplified form in which no assumptions need to be made about the dicritical exponents.

Corollary A. *Let $r_1, \dots, r_g$ be the characteristic factors of $s(x)$. Then*

$$H(P) \geq H(P, s(x)) > \prod_{j=1}^{g-1} r_j - \prod_{j=1}^{g-2} r_j.$$

If, moreover, $\operatorname{ord}(s(x)) \geq 1$*, then $\nu_0(P)$ is greater than or equal to the right hand side of the inequality.*

We can improve this result in the differential (see [**8**]), in the *generic* q-difference, and in the contracting, i.e. $|q| < 1$, q-difference cases. To this end, we introduce the concept of *reasonable equations* (see Definition 17), that encompasses the previous cases.

Theorem B. *If $s(x)$ is a Puiseux solution of genus g of the reasonable equation $P = 0$, then*

$$H(P) \geq H(P, s(x)) \geq r_1 \cdots r_{g-1}.$$

If, moreover, $\operatorname{ord}(s(x)) \geq 1$*, then one also has $\nu_0(P) + 1 \geq r_1 \cdots r_{g-1}$.*

As a consequence we get also a bound for the genus g of a solution $s(x)$, namely $g \leq 1 + \log_2(H(P))$, and if $\operatorname{ord}(s(x)) \geq 1$, then $g \leq 1 + \log(\nu_0(P) + 1)$.

We end our paper showing how the bound for the multiplicity of a differential equation found in [**8**] can be obtained exclusively by means of the Newton polygon using our technique. Let $\nu_0(\mathcal{F})$ be the multiplicity at $0 \in \mathbb{C}^2$ of the singular foliation defined by the differential equation $A(x, y)\, dx + B(x, y)\, dy = 0$, assuming $A(x, y)$ and $B(x, y)$ have no common factors.

Corollary B. *Let $\mathcal{F}$ be a germ of singular holomorphic foliation in a neighborhood of the origin in $\mathbb{C}^2$ that has a formal irreducible invariant curve Γ whose characteristic factors are $r_1, \dots, r_g$. Then, we obtain*

$$\nu_0(\mathcal{F}) \geq r_1 \cdots r_{g-1},$$

where an empty product is 1*.*

To summarize, we apply the Newton polygon technique *simultaneously* to both differential and q-difference equations in order to obtain lower bounds for the height of the Newton polygon in terms of the characteristic factors of a solution $s(x)$ that parametrizes an irreducible curve. Similar results were first proved in [**8**] in the differential case using geometric techniques related to the desingularization of the curve defined by $s(x)$. Those bounds are valid in the differential and the generic q-difference case, which includes the contracting ($|q|<1$) case. In the case of a non-generic non-contracting q-difference equation P, those lower bounds for $H(P)$ are just somewhat worse. A final section is devoted to improving the bound in the case of differential equations, and obtaining the same bound as in [**8**], just with the Newton polygon technique.

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2. Notation and preliminary results

From now on, a complex number $q \in \mathbb{C}^*$ is chosen with $|q| \neq 1$, and also a specific determination of the complex logarithm, which we shall denote by $\log(z)$ for $z \in \mathbb{C}$ whenever required. There is no indetermination, as the reader will notice.

Let σ be one of the following operators on the set of Puiseux series over x with non-negative exponents:

$$\sigma\left(\sum_{i\geq 0} a_i x^{i/n}\right) = \begin{cases} \displaystyle\sum_{i\geq 0} \frac{i}{n} a_i x^{(i-n)/n}, \\ \displaystyle\sum_{i\geq 0} q^{i/n} a_i x^{i/n}. \end{cases} \tag{1}$$

The first one will be called the *differential operator* and the second one the *q-difference operator*. The operator σ is extended to a variable y giving $\sigma(y) = y_1$ (the variable "operated"). This way, we can write any differential equation of order and degree one, or any q-difference equation in which the q-difference operation only appears to degree and order one, as

$$A(x,y) + B(x,y)y_1 = 0. \tag{2}$$

Before defining the concept of solutions we gather all the equations we are going to study under a single concept:

Definition 1. An *m-covered equation* is an equation (2) where:

(i) Both $A(x,y)$ and $B(x,y)$ are formal power series in $\mathbb{C}[[x^{\frac{1}{m}}, y]]$ with $A(0,0) = B(0,0) = 0$;

(ii) y_1 stands for $\sigma(y)$, where σ is any of the operators in equation (1).

We say that the equation is *covered* if it is m-covered for some $m \in \mathbb{N}$. A *solution* of such an equation is a Puiseux series $s(x)$ in $\bigcup_{m\in\mathbb{N}} \mathbb{C}[[x^{\frac{1}{m}}]]$ such that (as a Puiseux series)

$$A(x, s(x)) + B(x, s(x))\sigma(s(x)) = 0$$

holds, where σ is the appropriate operator. Finally, the *order* o_σ of σ is 0 for the q-difference operator and 1 for the differential operator.

Our aim is to use the Newton–Puiseux polygon (from now on just *Newton polygon*) to relate the complexity of the solutions of a 1-covered equation (2) to some specific invariant. Along the way, we carry out some auxiliary operations that transform equation (2) into m-covered equations for m possibly higher than 1: this explains why Definition 1 is relevant. From now on, we fix a Puiseux power series

$$s(x) = \sum_{i\geq 1} a_i x^{i/n} \in \bigcup_{m\in\mathbb{N}} \mathbb{C}[[x^{\frac{1}{m}}]], \tag{3}$$

where n is the minimal $m \in \mathbb{N}$ such that $s(x) \in \mathbb{C}[[x^{\frac{1}{m}}]]$. Indeed, if $s \not\equiv 0$, n is the least common denominator of the exponents having a non-zero coefficient: in technical terms, $s(x)$ is a *reduced power series* and the series $s(x)$ is a formal power series if and only if $n = 1$. In the case of differential equations, we shall consider, in the last section, the analytic branch Γ associated with $s(x)$, and relate its multiplicity to the notion of multiplicity of the associated foliation. This requires us to perform a change of coordinates, that will be introduced in the last section, which has no equivalent in the q-difference case.

There are two cases: either $n = 1$ or there exists a first index e_1 such that $a_{e_1} \neq 0$ and $e_1/n \notin \mathbb{Z}$. In the former case we define $g = 0$ whereas in the latter case we write

$$\frac{e_1}{n} = \frac{p_1}{r_1}$$

with p_1, r_1 mutually prime with $r_1 \geq 2$. Assuming e_i, p_i, r_i are defined, either $r_1 \cdots r_i = n$ and we define $g = i$ or there exists a first index e_{i+1} such that $a_{e_{i+1}} \neq 0$ and $e_{i+1}/n \notin \frac{1}{r_1 \cdots r_i}\mathbb{Z}$, and write

$$r_1 \cdots r_i \frac{e_{i+1}}{n} = \frac{p_{i+1}}{r_{i+1}} \tag{4}$$

with $\gcd(p_{i+1}, r_{i+1}) = 1$ and $r_{i+1} \in \mathbb{N}_{\geq 2}$. This construction ends at some $g \geq 0$ when $r_1 \cdots r_g = n$.

As we shall work with power series and only in the case of foliations we shall consider the associated germ of analytic curve, we use the following definition, associated with $s(x)$ and not with the germ of curve Γ defined by it. However, the *genus* is a measure both of the complexity of $s(x)$ and of the topological complexity of Γ [**26**]:

Definition 2. The numbers $e_1, \ldots, e_g$ are called the *characteristic exponents* of $s(x)$, and the factors $r_1, \ldots, r_g$ will be called the *characteristic factors*. The number g of characteristic exponents is the so-called *genus* of $s(x)$. If $g = 0$, then $n = 1$ and the Puiseux series $s(x)$ is said to be *non-singular*.

2.1. The Newton polygon. Given a covered equation such as (2), the *Newton polygon or diagram* is a graphical help for computing its solutions. Its construction follows.

Fix a covered equation $P = P(x, y, y_1) \equiv A(x, y) + B(x, y)y_1 = 0$ and write

$$A(x,y) = \sum a_{\iota j} x^\iota y^j = \sum A_{\iota j} x^\iota y^j, \quad B(x,y) = \sum b_{\iota j} x^\iota y^j = \sum B_{\iota - o_\sigma\, j+1} x^\iota y^j, \tag{5}$$

where $\iota \in \mathbb{Q}_{\geq 0}$ and $j \in \mathbb{N}$, where we use ι instead of i to emphasize that it may not be an integer. The *supports* of $A(x, y)$ and $B(x, y)$ are the sets

$$\operatorname{supp}(A) = \{(\iota, j) : A_{\iota j} \neq 0\} \quad \text{and} \quad \operatorname{supp}(B) = \{(\iota, j) : B_{\iota j} \neq 0\}$$

respectively. The support of B is obtained from $\{(\iota, j) : b_{\iota j} \neq 0\}$ by pushing one step up, because of the factor y_1, and one step left in the case of differential equations, because σ decreases the order of each monomial x^μ by one.

Definition 3. The *cloud of points* of P is the set $\mathcal{C}(P) = \operatorname{supp}(A) \cup \operatorname{supp}(B)$.

Consider the following subset of $\mathbb{R}_{\geq -1} \times \mathbb{R}_{\geq 0}$:

$$\mathcal{Q}(P) = \bigcup_{(\iota,j) \in \mathcal{C}(P)} (\iota, j) + (\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}),$$

where we place a positive quadrant at each point of the cloud.

Definition 4. The *Newton polygon* $\mathcal{N}(P)$ of P is the convex envelope of $\mathcal{Q}(P)$.

Example 1. Consider the equation

$$P \equiv y^4 + x^3y^3 + xy^2 - x^3y + x^5 + (xy^3 - x^2y)y_1. \tag{6}$$

Its Newton polygon is shown in Figure 1. The points $(1,4)$ and $(2,2)$, the unfilled circles, correspond to $(xy^3 - x^2y)y_1$ in the q-difference case. The Newton polygon, however, is the same, in this case, for both equations.

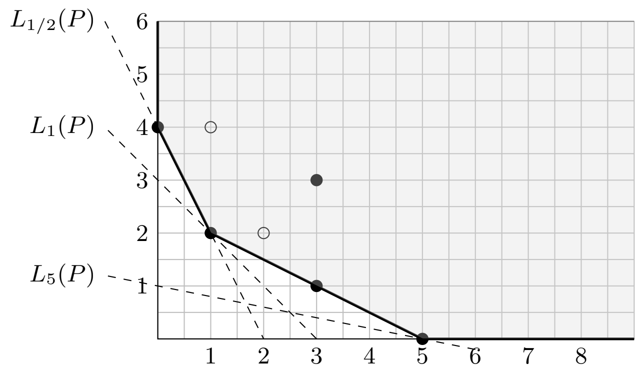

FIGURE 1. Cloud of points, Newton polygon, and some supporting lines (see Definition 7) of the equation $P = 0$ in (6). The two unfilled points correspond only to the q-difference case, whereas the filled ones correspond to both cases.

2.2. Newton polygon and solutions. The interest of this construction will become apparent (hopefully) at the end of this section. Take $s(x)$ as in (3). From now on, we fix a covered equation

$$P \equiv A(x,y) + B(x,y)y_1 = 0,$$

and denote by σ the corresponding operator.

Definition 5. The *k-th truncation* of $s(x)$ is the Puiseux series

$$s_k(x) = \sum_{0<i\leq k} a_i x^{i/n}.$$

Note that the truncation *includes* the term $a_k x^{k/n}$ of $s(x)$. By convention, $s_0(x) = 0$.

Definition 6. The *k-th substitution* (of $s(x)$, but this will always be implicit) in P is the equation

$$P_k \equiv A(x, y + s_k(x)) + B(x, y + s_k(x))(y_1 + \sigma(s_k(x))) = A^k(x,y) + B^k(x,y)y_1,$$

where $s_k(x)$ is the k-th truncation of $s(x)$ (thus, $P_0 = P$). The *total substitution* of s in P is the equation

$$P_\infty \equiv A(x, y + s(x)) + B(x, y + s(x))(y_1 + \sigma(s(x))).$$

As $P_k = A^k(x,y) + B^k(x,y)y_1$, the expressions $A^k_{\iota j}$, $B^k_{\iota j}$ will denote the corresponding coefficients of $A^k(x,y)$ and $B^k(x,y)$, following (5).

Notice that if $R(x,y,y_1) = P_k$, then

$$R(x, y + a_{k+1}x^{(k+1)/n}, y_1 + \sigma(a_{k+1}x^{(k+1)/n})) = P_{k+1},$$

and the definition of P_k can be made iterative, substitution by substitution. This is Newton and Cramer's construction, which allowed the latter to find approximate solutions of algebraic equations. Some geometric concepts are required for the proper application of the Newton polygon to solving covered equations.

Definition 7. Given $\mu \in \mathbb{R}_{>0}$, let $L_\mu(P)$ denote the line

$$L_\mu(P) \equiv \left\{(i,j) \in \mathbb{R}^2 : j = \frac{-1}{\mu} i + \alpha\right\}$$

with α maximum satisfying the following property: if $L^+_\mu(P) \equiv \{(i,j) \mid j \geq -i/\mu+\alpha\}$, then $\mathcal{N}(P) \subset L^+_\mu(P)$. This line will be called the *supporting line of* $\mathcal{N}(P)$ *of co-slope* μ.

Notice that $L_\mu(P) \cap \mathcal{N}(P)$ is either a vertex of $\mathcal{N}(P)$ or a side:

Definition 8. The *element of co-slope* μ of $\mathcal{N}(P)$ is

$$E_{P,\mu} = L_\mu(P) \cap \mathcal{N}(P),$$

and it will be called either the *vertex of co-slope* μ or the *side of co-slope* μ if $E_{P,\mu}$ is a single point or otherwise. We shall denote $E_{k,\mu} = E_{P_k,\mu}$ and, because $E_{k,k/n}$ will be our main concern, $E_k = E_{k,k/n}$.

We stress the fact that E_k is the element of co-slope k/n *after applying the k-th substitution*, whereas $E_{k-1,k/n}$ is the element of the same co-slope *just before* that substitution has been applied. We refer the reader to the later example of Subsection 2.3 for this important distinction.

Example 2. In Figure 1, the elements $E_{P,\mu}$ for $\mu \in [1/2, 2]$ are the following. To begin with, $E_{P,1/2}$ is the segment joining $(0,4)$ and $(1,2)$ that corresponds to the dashed line $L_{1/2}(P)$. Then, for $1/2 < \mu < 2$, $E_{P,\mu}$ is just the vertex $(1,2)$. Finally, $E_{P,2}$ is the segment from $(1,2)$ to $(5,0)$, which contains the point $(3,1)$.

Let μ be a co-slope and $E_{P,\mu}$ the corresponding element of P. We can unify the notation for differential and q-difference equations using the δ coefficient:

Definition 9. The δ coefficient corresponding to the co-slope μ is the number:

$$\delta_\mu = \begin{cases} \mu & \text{if } P \text{ is a differential equation,} \\ q^\mu & \text{if } P \text{ is a } q\text{-difference equation.} \end{cases}$$

This allows us to unify the first key concept:

Definition 10. The *initial polynomial* of P of co-slope μ is $\Phi_{P,\mu}(C)$, given by:

$$\Phi_{P,\mu}(C) = \sum_{(\iota,j) \in E_{P,\mu}} (A_{\iota j} + \delta_\mu B_{\iota j}) C^j.$$

When working with P_k, we shall normally use the notation $\Phi_{k,\mu}(C)$ instead of $\Phi_{P_k,\mu}(C)$.

Definition 11. If the initial polynomial is identically zero, i.e. $\Phi_{k-1,k/n}(C) \equiv 0$, then the exponent k/n of $s(x)$, the element $E_{k-1,k/n}$, and the co-slope k/n are called *dicritical*.

In Subsection 2.6 we relate this notion to the geometric concept of *dicritical foliations*.

The following results are classical for differential equations [**5, 6**] and trivially extended to q-difference equations (see [**2, 7**] for instance). Fix P and a solution $s(x)$ as above.

Lemma 1. *For any* $k > 0$, *we have*

$$\Phi_{k-1,k/n}(a_k) = 0.$$

A cornerstone of the method is that the operation with the term a_k does not modify the Newton polygon "up to the part corresponding to a_{k-1}":

Lemma 2. *If (ι, t) is the topmost point of $E_{k-1,k/n}$, then, for any $l \geq k$,*

$$\mathcal{N}(P_{k-1}) \cap (\mathbb{R}_{\geq -1} \times \mathbb{R}_{\geq t}) = \mathcal{N}(P_l) \cap (\mathbb{R}_{\geq -1} \times \mathbb{R}_{\geq t}),$$

that is, both polygons are equal at (ι, t) and above. Even more, if

$$P_{k-1} = A^{k-1}(x, y) + B^{k-1}(x, y)y_1,$$

then $A^{k-1}_{\iota t} = A^l_{\iota t}$ and $B^{k-1}_{\iota t} = B^l_{\iota t}$ for any $l \geq k-1$. As a consequence, for all $k \geq 1$ and $l \geq k$, we have

$$E_k = E_{l,k/n} = E_{P_\infty,k/n}.$$

Finally, solutions are characterized by their "flattening" of the Newton polygon from below:

Theorem (see [**5, 2**]). *Let $P \equiv A(x, y) + B(x, y)y_1 = 0$ be a covered equation and $s(x)$ a Puiseux series with $a_0 = 0$ (no independent term). The following statements are equivalent:*

(i) *The power series $s(x)$ is a solution of P.*
(ii) *The power series 0 is a solution of P_∞.*
(iii) *The Newton polygon of P_∞ has a horizontal side at height greater than 0.*

Therefore, if $s(x) = \sum a_i x^{i/n}$ is a solution of P, then a_i is a root of the corresponding initial polynomial $\Phi_{i-1,i/n}(C)$. If this holds for each i, then $s(x)$ is indeed a solution of P. Due to Lemma 2, the polygon construction is, thus, an iterative process in which each coefficient a_i is a zero of the initial polynomial of the unique element of P_{i-1} of co-slope i/n. Furthermore, also by Lemma 2, this latter element is *to the right and not above* the element of P_{i-1} of co-slope $(i-1)/n$.

The following definition covers all the main invariants associated to P and $s(x)$.

Definition 12. Let $P \equiv A(x, y) + B(x, y)y_1 = 0$ be a covered equation and $s(x)$ be a Puiseux series with order $\operatorname{ord}(s(x)) > 0$. The *height of* P, denoted by $H(P)$, is the ordinate of the leftmost vertex of $\mathcal{N}(P)$. Consider a co-slope μ and the corresponding element $E_{P,\mu}$ of P of co-slope μ. The *top (or height)* of $E_{P,\mu}$ is the highest ordinate of the points of $E_{P,\mu}$, and the *bottom* of $E_{P,\mu}$ is the lowest. They will be denoted as $\operatorname{Top}(E_{P,\mu})$ and $\operatorname{Bot}(E_{P,\mu})$, respectively. We denote $H(P, s(x)) = \operatorname{Top}(E_{P,\mu})$ for $\mu = \operatorname{ord}(s(x))$. Finally, the *multiplicity* of P at the origin is

$$\nu_0(P) = \min\{\operatorname{ord}_{(x,y)}(A(x, y)), \operatorname{ord}_{(x,y)}(B(x, y))\}.$$

Remark 3. As $\mathcal{N}(P)$ has a finite number of sides, we have

(i) The map $\mu \to \operatorname{Top}(E_{P,\mu})$ is a decreasing function from $\mathbb{R}^+$ to $\mathbb{Z}$.
(ii) For any $\mu \in \mathbb{R}^+$, we have $H(P) \geq \operatorname{Top}(E_{P,\mu})$; in particular for any $s(x)$,

$$H(P) \geq \operatorname{Top}(E_{P,\mu}) \geq H(P, s(x)), \quad 0 < \mu \leq \operatorname{ord}(s(x)).$$

If, moreover, $\operatorname{ord}(s(x)) \geq 1$, then

$$\nu_0(P) + 1 \geq \operatorname{Top}(E_{P,1}) \geq H(P, s(x)).$$

(iii) There is $\mu_0 > 0$ such that $H(P) = \operatorname{Top}(E_{P,\mu})$ for any $0 < \mu < \mu_0$.

We are interested, for a 1-covered equation P, in bounding $H(P)$ and $H(P, s(x))$ from below in terms of the characteristic factors of a solution. In lay terms, we wish to prove that an equation with a complicated solution must already be "complicated" where the complexity is measured by $H(P)$ or $H(P, s(x))$. In the last section, devoted to the case of differential equations, we shall see how, up to a linear change of coordinates, the multiplicity of the associated foliation at $0 \in \mathbb{C}^2$ is greater than or equal to $H(P, s(x)) - 1$. We shall use our general results to bound this multiplicity from below.

Lemma 2 implies the following property which we shall use freely:

Lemma 4. *For any $k \geq 0$,*

$$\mathrm{Top}(E_{k-1,k/n}) = \mathrm{Top}(E_k) \geq \mathrm{Bot}(E_k) \geq \mathrm{Top}(E_{k,(k+1)/n}) = \mathrm{Top}(E_{k+1}).$$

Remark 5. Let $(\iota_0, 0)$ be the point of intersection of the x-axis and $L_{k/n}(P_k)$. Later on we shall see that $A^k_{\iota_0 0} = \Phi_{k-1,k/n}(a_k)$ in equation (9). Thus, Lemma 1 gives $\mathrm{Top}(E_k) \geq \mathrm{Bot}(E_k) \geq 1$ for any $k \geq 1$: after each substitution, the element E_k does not meet the x-axis. On the other hand, Lemma 4 implies $\mathrm{Bot}(E_k) \geq \mathrm{Top}(E_{l-1,l/n})$ for $k < l$. Moreover, if for some j with $k < j < l$ the element E_j contains more than one vertex, then $\mathrm{Bot}(E_k) > \mathrm{Top}(E_{l-1,l/n})$ because, in this case,

$$\mathrm{Bot}(E_k) \geq \mathrm{Top}(E_j) > \mathrm{Bot}(E_j) \geq \mathrm{Top}(E_{l-1,l/n}).$$

This last remark is quite relevant because it will provide a *descent* argument for our bounds. We shall see in Lemma 6 that all characteristic exponents k/n, except possibly the last one, give rise to sides in the Newton polygon, i.e. $\mathrm{Top}(E_k) > \mathrm{Bot}(E_k)$. Moreover, one of our main results (Proposition 11) provides a qualitative estimate of the gap between $\mathrm{Bot}(E_k)$ and $\mathrm{Top}(E_k)$.

2.3. An example. For the benefit of the reader, we include an exhaustive example in this section, in order to clarify the technique, the notation, and some of the results.

Consider the differential equation associated with the following polynomial:

(7) $$P_0 = y^4+4y^3x+5y^2x^2+2yx^3+yx^4+4x^5+x^7+(-y^3x-4y^2x^2-5yx^3-2x^4+3x^5)y_1.$$

We know in advance – this is the initial assumption in this work – that P_0 admits a solution with the following Puiseux expansion:

$$s(x) = -x - \sqrt{11}x^{3/2} - \frac{121}{30}x^2 + \cdots,$$

where the exponents of the remaining terms belong to $\frac{1}{2}\mathbb{Z}$ and are greater than 2. Thus, $e_1 = 3$ is the single characteristic exponent of $s(x)$. So, setting $n = 2$, we have $a_2 = -1$, $a_3 = -\sqrt{11}$, $a_4 = -\frac{121}{30}$. The clouds of points, Newton polygons corresponding to each substitution, and their respective elements are depicted in Figure 2. These substitutions are computed in the following paragraphs. Recall that $\delta_{k/n} = k/n = k/2$ because P_0 is differential and $n = 2$.

(i) The first exponent is $1 = 2/2$, so that $k = 2$. Thus, the relevant element of $\mathcal{N}(P_1) = \mathcal{N}(P_0)$ is $E_{1,2/2}$. The initial polynomial is

$$\Phi_{1,2/2}(C) = (1-1)C^4 + (4-4)C^3 + (5-5)C^2 + (2-2)C \equiv 0,$$

that is, $E_{1,2/2}$ is a dicritical element.

(ii) Once the substitution $y = y - x$ is performed, we obtain

$$P_2 \equiv y^4 + xy^3 + x^4y + x^7 + (-xy^3 - x^2y^2 + 3x^5)y_1,$$

whose element $E_2 := E_{2,2/2}$ of co-slope 1 is, in this case, a shorter subsegment of $E_{1,2/2}$. Notice that it might have been a single point or, if $E_{1,2/2}$ were shorter, it might have been longer but by Remark 5, $\mathrm{Bot}(E_2) \geq 1$, in any case.

Following $s(x)$, the next relevant element is $E_{2,3/2}$, which corresponds to the single characteristic exponent $e_1 = 3$, which gives $r_1 = 2$. The initial polynomial $P_{2,3/2}(C)$ is

$$\Phi_{2,3/2}(C) = -\frac{1}{2}C^3 + \frac{11}{2}C,$$

whose roots are $C = 0$ and $C = \pm\sqrt{11}$, that include, certainly, $a_3 = -\sqrt{11}$.

(iii) Performing the substitution $y = y - \sqrt{11}x^{3/2}$ corresponding to $a_3x^{3/2}$, we obtain

$$P_3 = y^4 - \tfrac{5\sqrt{11}}{2}x^{\frac{3}{2}}y^3 - \tfrac{3\sqrt{11}}{2}x^{\frac{5}{2}}y^2 + xy^3 + \tfrac{33}{2}x^3y^2 + x^4y + \tfrac{11^{\frac{3}{2}}}{2}x^{\frac{9}{2}}y - \tfrac{121}{2}x^6 + x^7 \\ + (-xy^3 - x^2y^2 + 3\sqrt{11}x^{\frac{5}{2}}y^2 + 2\sqrt{11}x^{\frac{7}{2}}y - 33x^4y - 8x^5 + 11^{\frac{3}{2}}x^{\frac{11}{2}})y_1,$$

whose element E_3 is, in this case, of the same length as $E_{2,3/2}$ but contains a new point with non-integral x-coordinates: $(5/2, 2)$. As $e_1 = 3$ is a characteristic exponent, there must be at least one such point in the cloud, as we shall show in Lemma 7. The element $E_{3,4/2}$ corresponding to a_4 is the side joining $(4, 1)$ and $(6, 0)$. The initial polynomial is

$$\Phi_{3,4/2}(C) = -15C - \frac{121}{2},$$

whose unique root is, certainly, $a_4 = -\frac{121}{30}$.

(iv) Finally, after the substitution corresponding to a_4, we obtain P_4, whose Newton polygon is also depicted.

Notice that the Newton polygon $\mathcal{N}(P_{k-1})$ coincides with $\mathcal{N}(P_l)$ for $l \geq k$ from $\mathrm{Top}(E_{k-1,k/n})$ up, as per Lemma 2. Also, $E_{k,k/n} = E_{l,k/n}$ for $l \geq k$, as in the diagram corresponding to $\mathcal{N}(P_4)$.

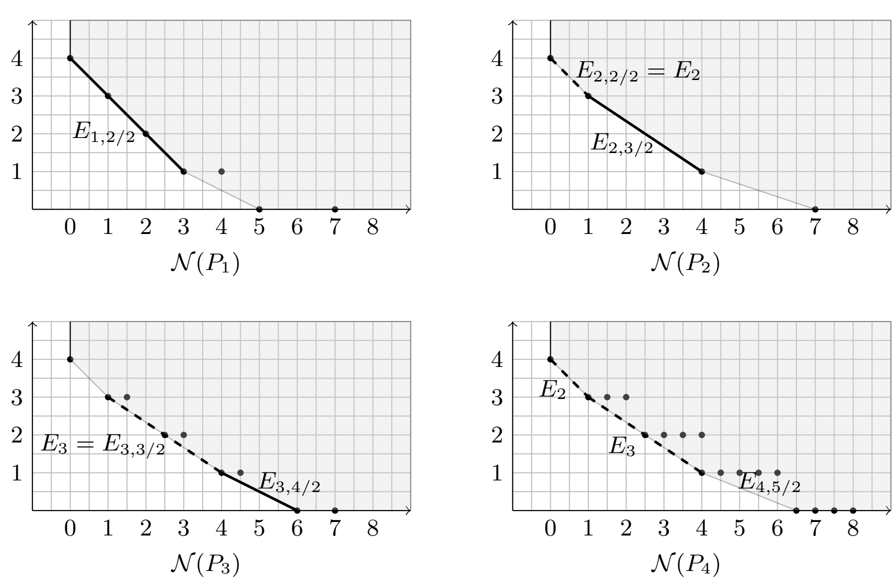

Figure 2. Newton polygons and relevant elements for P_0 in (7). Notice that $E_{k,k/n}$ is later referred to as E_k.

2.4. Characteristic exponents. A remarkable property of the characteristic exponents of $s(x)$ in terms of the Newton polygon is that each one (except possibly the last one), say k/n, gives rise to a *whole side* in $\mathcal{N}(P_k)$ and, by Lemma 2, in $\mathcal{N}(P_l)$ for $l > k$. This fact, already noted in (iii) of Subsection 2.3, and which we now prove, is essential to find our bounds.

Lemma 6. *Assume that P is 1-covered, i.e. the initial equation has only integer exponents, and let $s(x)$ be a solution as above. If $k = e_\ell$ for some $\ell = 1, \dots, g-1$, then the Newton polygon of P_k (and, by Lemma 2, of P_j for $j > k$) has a side of co-slope k/n; that is, E_k is indeed a side, not just a vertex. If e_g is not dicritical, then the result holds also for $k = e_g$.*

Proof: Assume $k = e_\ell$ with $\ell \leq g$. Let $\Phi(C) = \Phi_{k-1,k/n}(C)$ be the corresponding initial polynomial. By recurrence, $\mathcal{C}(P_{k-1})$ is included in $\frac{1}{r_1 \cdots r_{\ell-1}}\mathbb{Z}\times\mathbb{Z}$, so that all the points have abscissa with denominator at most $r_1 \cdots r_{\ell-1}$. Let (ι, t) be the topmost vertex of $E_{k-1,k/n}$, corresponding to the terms $A^{k-1}_{\iota t} x^\iota y^t + B^{k-1}_{\iota t} x^{\iota+o_\sigma} y^{t-1} y_1$. In particular, $t = \operatorname{Top}(E_{k-1,k/n})$. Performing the substitution $y = y + a_k x^{k/n}$ at the terms of P_{k-1} corresponding to the point (ι, t), we get

$$
\begin{aligned}
A^{k-1}_{\iota t} x^\iota (y + a_k x^{k/n})^t &+ B^{k-1}_{\iota t} x^{\iota+o_\sigma} (y + a_k x^{k/n})^{t-1} (y_1 + \sigma(a_k x^{k/n})) \\
&= A^{k-1}_{\iota t} x^\iota y^t + B^{k-1}_{\iota t} x^{\iota+o_\sigma} y^{t-1} y_1 + (tA^{k-1}_{\iota t} + \delta_{k/n} B^{k-1}_{\iota t}) a_k x^{\iota+k/n} y^{t-1} \\
&\qquad + (t-1) B^{k-1}_{\iota t} a_k x^{\iota+k/n+o_\sigma} y^{t-2} y_1 + \cdots ,
\end{aligned}
$$

where the dots indicate terms whose ordinates in $\mathcal{N}(P_k)$ are strictly less than $t-1$. Recall that $\delta_{k/n}$ is either k/n or $q^{k/n}$, depending on P being a differential or q-difference equation. Note that $(\iota+k/n, t-1)$ does not belong to $\mathcal{C}(P_{k-1})$ since $\iota+k/n \notin \frac{1}{r_1\cdots r_{\ell-1}}\mathbb{Z}$. Moreover, (ι, t) is the only point in $\mathcal{C}(P_{k-1})$ contributing to $(\iota+k/n, t-1)$ when we perform the substitution. Thus, it suffices to show that either $(t-1)B^{k-1}_{\iota t} a_k \neq 0$ or $tA^{k-1}_{\iota t} + \delta_{k/n} B^{k-1}_{\iota t} \neq 0$ to obtain $(\iota + k/n, t-1) \in \mathcal{C}(P_k)$.

Notice that $a_k \neq 0$ because e_ℓ is a characteristic exponent, so that it appears explicitly in $s(x)$. If $t > 1$, then either $B^{k-1}_{\iota t} \neq 0$, so that $(t-1)B^{k-1}_{\iota t} a_k$ is not zero, or $(tA^{k-1}_{\iota t} + \delta_{k/n} B^{k-1}_{\iota t}) = tA^{k-1}_{\iota t}$ is not zero. Thus, we can consider $t = 1$ and $A^{k-1}_{\iota 1} + \delta_{k/n} B^{k-1}_{\iota 1} = 0$ from now on. In particular we obtain $A^{k-1}_{\iota 1} \neq 0 \neq B^{k-1}_{\iota 1}$.

We claim that $t = 1$ and $A^{k-1}_{\iota 1} + \delta_{k/n} B^{k-1}_{\iota 1} = 0$ imply that e_ℓ is dicritical and $\ell = g$, finishing the proof. As $t = 1$, the condition $A^{k-1}_{\iota 1} + \delta_{k/n} B^{k-1}_{\iota 1} = 0$ implies that $\Phi(C) \equiv (A^{k-1}_{\iota 1} + \delta_{k/n} B^{k-1}_{\iota 1})C + \Phi(0) \equiv \Phi(0)$. Since $\Phi(a_k) = 0$, we deduce that $\Phi(C) \equiv 0$. Thus, $k = e_\ell$ is dicritical. Also, since for $k' \geq k$ we have

$$
1 = \operatorname{Top}(E_{k-1,k/n}) = \operatorname{Top}(E_k) \geq \operatorname{Top}(E_{k'}) > 0
$$

by Lemma 4 and Remark 5, we deduce $\operatorname{Top}(E_{k'}) = 1$ for $k' \geq k$. We claim that $k'/n \in \frac{1}{r_1\cdots r_\ell}\mathbb{Z}$ for any $k' \geq k$ such that $a_{k'} \neq 0$. This property implies $\ell = g$ by the definition of characteristic exponents.

Let us show the claim. By construction, the property is satisfied by k. Assume it holds for any $k \leq k' < k'_0$ and suppose, aiming at contradiction, that $k'_0/n \notin \frac{1}{r_1\cdots r_\ell}\mathbb{Z}$ and $a_{k'_0} \neq 0$. Since $\delta_{k/n} \neq \delta_{k'_0/n}$, we get

$$
A^{k'_0-1}_{\iota 1} + \delta_{k'_0/n} B^{k'_0-1}_{\iota 1} = A^{k-1}_{\iota 1} + \delta_{k'_0/n} B^{k-1}_{\iota 1} \neq A^{k-1}_{\iota 1} + \delta_{k/n} B^{k-1}_{\iota 1} = 0
$$

by Lemma 2. Note that $\mathcal{C}(P_{k'_0-1}) \subset \frac{1}{r_1\cdots r_\ell}\mathbb{Z}\times\mathbb{Z}$ as a consequence of the induction hypothesis. This property, together with $\iota \in \frac{1}{r_1\cdots r_\ell}\mathbb{Z}$ and $k'_0/n \notin \frac{1}{r_1\cdots r_\ell}\mathbb{Z}$, implies $A^{k'_0-1}_{\iota+k'_0/n\,0} = 0$. Since

$$\Phi_{k'_0-1,k'_0/n}(C) = (A^{k'_0-1}_{\iota 1} + \delta_{k'_0/n}B^{k'_0-1}_{\iota 1})C + A^{k'_0-1}_{\iota+k'_0/n\,0},$$

0 is the unique root of $\Phi_{k'_0-1,k'_0/n}$, contradicting $\Phi_{k'_0-1,k'_0/n}(a_{k'_0}) = 0$. $\square$

The proof of the previous result implies that the Newton polygon after a substitution corresponding to a characteristic exponent has points in a grid with a different scale in the variable x. Namely:

Lemma 7. *Assume that P is 1-covered. Let $k \in \mathbb{Z}_{>0}$, and let r be the minimum integer such that $\mathcal{C}(P_{k-1}) \subset \frac{1}{r}\mathbb{Z}\times\mathbb{Z}$. If $\mathrm{Top}(E_k) > 1$, then k is a characteristic exponent if and only if the cloud of points of P_k is not included in $\frac{1}{r}\mathbb{Z}\times\mathbb{Z}$.*

The fact that $\mathrm{Top}(E_k) > 1$ corresponds to the case $t > 1$ in the previous proof, which guarantees that the point $(\iota + k/n, t-1)$ belongs to $\mathcal{C}(P_k)$, and by definition, $\iota+k/n \notin \frac{1}{r}\mathbb{Z}$. This hypothesis is necessary: the solutions $y = Cx^{m/n}$ of the differential equation $P \equiv my - nxy' = 0$, which has the single point $(0,1)$ in its cloud, leave the cloud of points invariant after the substitution $y = y + Cx^{m/n}$.

2.5. Decomposing the initial polynomial. By Lemma 1, the coefficient a_k is always a root of the corresponding initial polynomial $\Phi_{k-1,k/n}(C)$, which is the basis of Newton's technique. This means that $\mathrm{Bot}(E_k) \geq 1$ for all k; see Remark 5. However, we can obtain much more information about the element E_k and the equation P_k if we study the transformation of $E_{k-1,k/n}$ into E_k as a parametric family depending on a complex parameter C, that is, by studying the substitution $y = y + Cx^{k/n}$ instead of $y = y + a_k x^{k/n}$. This is, in the differential case, similar to studying the whole exceptional divisor corresponding to $x^{k/n}$ and the singularities and regular points of the strict transform of the foliation given by P at the corresponding exceptional divisor. This study is carried out by means of the *k-th initial form*, which gathers all that information. It also provides an additive decomposition of $\Phi_{k-1,k/n}(C)$ into two terms: one corresponding to the algebraic part of P_k, and the other to the one with y_1. This decomposition and its consequences are key in further results. We follow [**5**] in the definition, leaving $s(x)$ implicit.

Fix $k \geq 1$ from now in this section, and let

$$P_k(C) := P_{k-1}(x, y + Cx^{k/n}, y_1 + \sigma(Cx^{k/n}))$$

be the k-th substitution with a complex parameter C instead of a_k. Writing

$$P_k(C) = A^k(C)(x,y) + B^k(C)(x,y)y_1 = \sum A^k_{\iota j}(C)x^\iota y^j + B^k_{\iota j}(C)x^{\iota+o_\sigma}y^{j-1}y_1,$$

the property $P_k(a_k) = P_k$ implies $A^k_{\iota j}(a_k) = A^k_{\iota j}$ and $B^k_{\iota j}(a_k) = B^k_{\iota j}$ for any (ι, j), where $A^k_{\iota j}$ and $B^k_{\iota j}$ were defined in Definition 6. Notice also that $P_k(0) = P_{k-1}$, $A^k_{\iota j}(0) = A^{k-1}_{\iota j}$, and $B^k_{\iota j}(0) = B^{k-1}_{\iota j}$.

Definition 13. The *k-th initial form* of P for $s(x)$ is the polynomial in C given by the expression:

$$\mathrm{In}_k(C) = \sum_{(\iota,j)\in L_{k/n}(P_{k-1})} A^k_{\iota j}(C)x^\iota y^j + B^k_{\iota j}(C)x^{\iota+o_\sigma}y^{j-1}y_1.$$

From bottom to top, $\mathrm{In}_k(C)$ can be rewritten as

$$\mathrm{In}_k(C)=A^k_{\nu\,0}(C)x^\nu+\sum_{j=1}^{t} A^k_{\nu-jk/n\,j}(C)x^{\nu-jk/n}y^j+B^k_{\nu-jk/n\,j}(C)x^{\nu-jk/n+o_\sigma}y^{j-1}y_1, \tag{8}$$

where $t=\mathrm{Top}(E_{k-1,k/n})=\mathrm{Top}(E_k)$ and $\nu=\bar{\iota}+tk/n$ if $(\bar{\iota},t)$ is the topmost vertex of $E_{k-1,k/n}$. For the sake of simplicity, as k is fixed throughout all this section, we set, for the remainder of this section, $\iota_j:=\nu-jk/n$, and write:

$$\mathrm{In}_k(C)=A^k_{\iota_0\,0}(C)x^\nu+\sum_{j=1}^{t} A^k_{\iota_j\,j}(C)x^{\nu-jk/n}y^j+B^k_{\iota_j\,j}(C)x^{\nu-jk/n+o_\sigma}y^{j-1}y_1,$$

because we are mostly interested in $A_{\iota_j j}(C)$ and $B_{\iota_j j}(C)$, and their pairwise relations, for $j=1,\dots,t$.

The following two polynomials decompose $\Phi_{k-1,k/n}(C)$ into two parts: one corresponding to the terms without y_1 in $\mathrm{In}_k(C)$, and the other to those with y_1. They will be key in many later computations:

$$\alpha_k(C)=\sum_{j=0}^{t} A^k_{\iota_j\,j}(0)C^j=\sum_{j=0}^{t} A^{k-1}_{\iota_j\,j}C^j,$$

$$\beta_k(C)=\sum_{j=1}^{t} B^k_{\iota_j\,j}(0)C^{j-1}=\sum_{j=1}^{t} B^{k-1}_{\iota_j\,j}C^{j-1}.$$

By definition, the initial polynomial $\Phi_{k-1,k/n}(C)$ satisfies

$$\Phi_{k-1,k/n}(C)=\alpha_k(C)+\delta_{k/n}C\beta_k(C).$$

The following result is the basis of the relevance of this decomposition [**5**, cf. equation (1)]. Recall that $t=\mathrm{Top}(E_{k-1,k/n})=\mathrm{Top}(E_k)$ in this subsection:

Lemma 8. *With the notation above, let $f^{(r)}(C)$ denote $\frac{\partial^r f}{\partial C^r}(C)$ for any function $f(C)$. Then:*

$$A^k_{\iota_0\,0}(C)=\Phi_{k-1,k/n}(C), \tag{9}$$

$$A^k_{\iota_j\,j}(C)=\frac{1}{j!}\Phi^{(j)}_{k-1,k/n}(C)-\delta_{k/n}\frac{1}{(j-1)!}\beta_k^{(j-1)}(C),\quad j=1,\dots,t, \tag{10}$$

$$B^k_{\iota_j\,j}(C)=\frac{1}{(j-1)!}\beta_k^{(j-1)}(C),\qquad j=1,\dots,t, \tag{11}$$

$$A^k_{\iota_j\,j}(C)+\delta_{k/n}B^k_{\iota_j\,j}(C)\equiv\frac{1}{j!}\Phi^{(j)}_{k-1,k/n}(C),\qquad j=1,\dots,t. \tag{12}$$

Specifically, $A^k_{\iota_t t}(C)=A^{k-1}_{\iota_t t}$ and $B^k_{\iota_t t}(C)=B^{k-1}_{\iota_t t}$ are independent of C (this is already known by Lemma 2), and finally, $A^k_{\iota_0 0}(C)=0$ if and only if $\Phi_{k-1,k/n}(C)=0$.

The following proof is rather technical but its gist is to transform P_{k-1} and $P_k(C)$ into new equations whose elements $E_{k-1,k/n}$ correspond to a vertical side of the same height. When written like this, the argument becomes a direct application of Taylor's formula.

Proof: Define the polynomial

$$I(y,y_1)=\mathrm{In}_k(0)=A^k_{\iota_0\,0}(0)x^\nu+\sum_{j=1}^{t} A^k_{\iota_j\,j}(0)x^{\nu-jk/n}y^j+B^k_{\iota_j\,j}(0)x^{\nu-jk/n+o_\sigma}y^{j-1}y_1.$$

Rather than compute $\mathrm{In}_k(C)$ directly, we will do so in three steps. Each step is an algebraic change of indeterminates that does not correspond to a differential or q-differential change of indeterminate, but the composition of the three does. First, we substitute y by $x^{k/n}y$ and y_1 by $\delta_{k/n}x^{k/n-o_\sigma}y_1$ and notice how $\alpha_k(y)$ and $\beta_k(y)$ can be used to rewrite the result:

$$\begin{aligned} I_1(y,y_1) &:= I(x^{k/n}\,y, \delta_{k/n}\,x^{k/n-o_\sigma}\,y_1) \\ &= A^k_{\iota_0\,0}(0)x^\nu \\ &\quad + \sum_{j=1}^{t} A^k_{\iota_j\,j}(0)x^{\nu-j\,k/n}x^{j\,k/n}y^j + B^k_{\iota_j\,j}(0)x^{\nu-jk/n}x^{(j-1)k/n}y^{j-1}\delta_{k/n}\,x^{k/n}\,y_1 \\ &= x^\nu\cdot\{\alpha_k(y)+\delta_{k/n}\,\beta_k(y)\,y_1\}. \end{aligned}$$

Then we translate by C both y and y_1 and expand $\alpha(C+y)$ and $\beta(C+y)$ using Taylor's formula:

$$\begin{aligned} I_2(y,y_1) &:= I_1(C+y,C+y_1) = x^\nu(\alpha_k(C+y)+\delta_{k/n}\,\beta_k(C+y)(C+y_1)) \\ &= x^\nu\left(\sum_{j=0}^{t}\frac{1}{j!}\alpha_k^{(j)}(C)\,y^j + \delta_{k/n}(C+y_1)\sum_{j=0}^{t-1}\frac{1}{j!}\beta_k^{(j)}(C)y^j\right) \\ &= x^\nu\left(\sum_{j=0}^{t}\frac{1}{j!}(\alpha_k^{(j)}(C)+\delta_{k/n}\,C\,\beta_k^{(j)}(C))y^j+\delta_{k/n}\sum_{j=1}^{t}\frac{1}{(j-1)!}\beta_k^{(j-1)}(C)y^{j-1}y_1\right). \end{aligned}$$

Finally, we undo the first transformation:

$$\begin{aligned} \mathrm{In}_k(C) &= I_2(x^{-k/n}\,y, \delta_{k/n}^{-1}\,x^{-k/n+o_\sigma}\,y_1) \\ &= x^\nu\left(\sum_{j=0}^{t}\frac{1}{j!}(\alpha_k^{(j)}(C)+\delta_{k/n}\,C\,\beta_k^{(j)}(C))\,x^{-j\,k/n}y^j\right. \\ &\qquad\left. + \delta_{k/n}\sum_{j=1}^{t}\frac{1}{(j-1)!}\beta_k^{(j-1)}(C)x^{-(j-1)k/n}y^{j-1}\delta_{k/n}^{-1}x^{-k/n+o_\sigma}y_1\right) \\ &= \sum_{j=0}^{t}\frac{1}{j!}(\alpha_k^{(j)}(C)+\delta_{k/n}\,C\,\beta_k^{(j)}(C))\,x^{\nu-j\,k/n}y^j \\ &\quad + \sum_{j=1}^{t}\frac{1}{(j-1)!}\beta_k^{(j-1)}(C)x^{\nu-jk/n+o_\sigma}y^{j-1}y_1. \end{aligned}$$

From the last equalities we can already infer:

$$\begin{aligned} A^k_{\iota_0\,0}(C) &= \alpha_k(C)+\delta_{k/n}\,C\,\beta_k(C) = \Phi_{k-1,k/n}(C), \\ A^k_{\iota_j\,j}(C) &= \frac{1}{j!}(\alpha_k^{(j)}(C)+\delta_{k/n}\,C\,\beta_k^{(j)}(C)), && j=1,\dots,t, \\ B^k_{\iota_j\,j}(C) &= \frac{1}{(j-1)!}\beta_k^{(j-1)}(C), && j=1,\dots,t. \end{aligned}$$

By induction on j, we obtain

$$\Phi^{(j)}_{k-1,k/n}(C) = \alpha^{(j)}_k(C) + \delta_{k/n}\, C\, \beta^{(j)}_k(C) + j\, \delta_{k/n}\, \beta^{(j-1)}_k(C),$$

whence

$$A^k_{\iota_j\, j}(C) = \frac{1}{j!}\Phi^{(j)}_{k-1,k/n}(C) - \frac{\delta_{k/n}}{(j-1)!}\beta^{(j-1)}_k(C),$$

as desired. Equation (12) is an immediate consequence of equations (10) and (11). $\square$

Recall that $P_k = P_k(a_k) = P_{k-1}(x, y + a_k x^{k/n}, y_1 + \sigma(a_k x^{k/n}))$. In the following result, we see how the information about $\mathrm{In}_k(C)$ provided by Lemma 8 allows us to better understand the properties of the particular case $C = a_k$ and more precisely of the element E_k obtained after the k-th substitution.

Corollary 9. *Let $b = \mathrm{Bot}(E_k)$. With the notation of Lemma 8, the following statements hold:*

(i) *In any case, the multiplicity of a_k as a root of $\beta_k(C)$ is at least $b - 1$.*
(ii) *If $\Phi_{k-1,k/n}(C) \not\equiv 0$ (non-dicritical case), then the multiplicity of a_k as a root of $\Phi_{k-1,k/n}(C)$ is at least b.*
(iii) *If $\Phi_{k-1,k/n}(C) \equiv 0$ (dicritical case), then*

$$A^k_{\iota_j\, j}(C) + \delta_{k/n} B^k_{\iota_j\, j}(C) = 0 \tag{13}$$

for all $C \in \mathbb{C}$ and $j = 1, \ldots, \mathrm{Top}(E_k)$, and in particular for $C = a_k$.

Proof: The last statement is an immediate consequence of equation (12) and the dicritical condition $\Phi_{k-1,k/n}(C) \equiv 0$.

There are no points in E_k with ordinate less than b by definition. Since

$$\beta^{(j-1)}_k(a_k) = (j-1)!\, B^k_{\iota_j\, j}(a_k) = 0 \tag{14}$$

for $j = 0, \ldots, b-1$ by equation (11), we deduce statement (i). Moreover, we obtain $\Phi^{(j)}_{k-1,k/n}(C) = 0$ for any $0 \leq j \leq b-1$ by equation (12). Statement (ii) follows. $\square$

2.6. An excursus on dicritical elements. In the arguments to come, the appearance of a dicritical characteristic exponent is problematic. In the setting of differential equations, this mirrors the fact that dicritical divisors appearing in the reduction of singularities of a holomorphic 1-form "complicate" the combinatorial structure of the residues and indices associated with the exceptional divisors (see [**3**], for instance).

Consider the family of analytic branches $s'_{k,c}(x)$ given by the k-truncation of $s(x)$ with a_k replaced by a complex parameter c, with $c \neq 0$:

$$s'_{k,c}(x) = \sum_{i=1}^{k-1} a_i x^{i/n} + c x^{k/n}.$$

This family of curves admits a common desingularization in a sequence π_k of point blow-ups, ending at an exceptional divisor which, for the sake of economy, we shall call D_k. Each non-zero $c \in \mathbb{C}$ corresponds to a point $Q_c \in D_k$, and through Q_c passes a single non-singular curve $\overline{\Gamma}_c$, transverse to D_k, such that the parametrization of $\pi_k(\overline{\Gamma}_c)$ coincides with $s'_{k,c}(x)$.

Let $P = A(x,y) + B(x,y)y_1$ be differential, and $\mathcal{F}$ be the foliation associated with $\omega = A(x,y)\,dx + B(x,y)\,dy$. Let $\mathcal{F}_k$ be the strict transform of $\mathcal{F}$ by π_k. In the non-dicritical case, i.e. $\Phi_{k-1,k/n}(C) \not\equiv 0$, the divisor D_k is invariant by $\mathcal{F}_k$. The roots c of $\Phi_{k-1,k/n}(C)$ correspond precisely with the singular points Q_c of $\mathcal{F}_k$ in D_k.

The dicritical case, where $\Phi_{k-1,k/n}(C) \equiv 0$, is totally different. Indeed, it is easy to prove that for those $c \in \mathbb{C}^*$ such that $\mathrm{Bot}(E_k) = 1$, there exists a power series $s_{k,c}(x)$

in $\mathbb{C}[[x^{\frac{1}{n}}]]$, that coincides with $s'_{k,c}(x)$ up to order $\leq k/n$, which is a solution of P (see [**7**], for instance). In such a case, the $\mathcal{F}$-invariant curve Γ_c, whose parametrization is $s_{k,c}(x)$, satisfies that its strict transform $\tilde{\Gamma}_c$ by π_k intersects D_k at Q_c, Q_c is a regular point of $\mathcal{F}_k$, and $\tilde{\Gamma}_c$ is a smooth curve transverse to D_k. However, there may be special points in D_k, that is, values of c, where $\mathcal{F}_k$ is either singular or tangent to D_k, and these are the ones we need to control, as a_k may be one of them. It is these points which present a challenge. The following straightforward consequence of Lemma 8 asserts that these challenging points are finite in number.

Corollary 10. *Assume $E_{k-1,k/n}$ is dicritical. Consider the point $(\iota_1, 1)$ of the line $L_{k/n}(P_k)$ of height 1. Then*

$$B^k_{\iota_1 1}(C) \not\equiv 0.$$

As a consequence, if $\operatorname{Bot}(E_k) > 1$*, then a_k is a root of the polynomial $B^k_{\iota_1 1}(C)$.*

Proof: If $B^k_{\iota_1 1}(C) \equiv 0$, then $\beta_k(C) \equiv 0$ by equation (11). We obtain $B^k_{\iota_t t} = 0$, where $(\iota_t, t) = \operatorname{Top}(E_k)$, by applying again equation (11). Since (ι_t, t) is a vertex of P_k, we deduce $A^k_{\iota_t t} \neq 0$. As $E_{k-1,k/n}$ is dicritical, and $\delta_{k/n} \neq 0$, Corollary 9(iii) implies that both $A^k_{\iota_t t}$ and $B^k_{\iota_t t}$ are non-zero, contradicting $B^k_{\iota_t t} = 0$. □

Finally, by their nature, dicritical exponents impose strict relations between the coefficients in $A(x, y)$ and those in $B(x, y)$ belonging to the corresponding element. This is precisely item (iii) in Corollary 9: whenever $A^k_{\iota j}$, $B^k_{\iota j}$ come from a dicritical element $E_{k-1,k/n}$, then $A^k_{\iota j} = -\delta_{k/n} B^k_{\iota j}$. This equality will play an essential role in all our arguments in the dicritical case. Among many other things, it will prevent the existence of immediately consecutive dicritical elements (Lemma 13), and, even more, the existence of consecutive elements $E_k, E_{k+1}, \ldots, E_l$ with E_k, E_l dicritical, and the ones in between satisfying a “bad” property (Lemma 20). Remarkably, this latter fact is true for differential equations and *most* q-difference equations, and this is the only point on which the two types of equations differ. We shall gather the differential and these generic q-difference equations under the concept of *reasonable* equations (Definition 17).

In summary, dicriticalness is, in some sense, a complication but its very nature imposes conditions which can be taken advantage of to overcome most of said complication.

3. Main result

We show Theorem A, Corollary A, and Theorem B in this section. As a starting point, let P be a 1-covered equation – that is, with only integral exponents – and let

$$s(x) = \sum_{i \geq 1} a_i x^{i/n}$$

be a solution of P in Puiseux form. Let $e_1, \ldots, e_g$ be the characteristic exponents of $s(x)$ and $r_1, \ldots, r_g$ be the characteristic factors.

Before proceeding, we provide some definitions and notation which will simplify the statements.

Definition 14. For an integer $k \geq 1$, the *factor corresponding to* k, denoted by ρ_k, is defined as follows:

(i) If k is not a characteristic exponent, then $\rho_k = 1$,
(ii) otherwise, if k is equal to the characteristic exponent e_l, then $\rho_k = r_l$.

That is, $\rho_{e_l} = r_l$ and $\rho_k = 1$ if k is not a characteristic exponent.

The main relation between $\mathrm{Top}(E_k)$ and $\mathrm{Bot}(E_k)$ – notice that this is *after* the substitution of the term $a_k x^{k/n}$ – when k is a characteristic exponent is:

Proposition 11. *Let $k \in \mathbb{N}$. We have*

(i) *If $E_{k-1,k/n}$ is non-dicritical or $\rho_k = 1$, then*

$$\mathrm{Bot}(E_k) \leq \frac{\mathrm{Top}(E_k)}{\rho_k}.$$

(ii) *If $E_{k-1,k/n}$ is dicritical, then*

$$\mathrm{Bot}(E_k) \leq \frac{\mathrm{Top}(E_k)}{\rho_k} + \frac{\rho_k - 1}{\rho_k},$$

or, what amounts to the same, $\mathrm{Top}(E_k) \geq \rho_k \,\mathrm{Bot}(E_k) - (\rho_k - 1)$.

Both inequalities are sharp.

Examples of equality are, for the non-dicritical case, the algebraic equation $P \equiv y^n - x = 0$, where $k = 1$, $\rho_k = n$, $\mathrm{Bot}(E_k) = 1$, and $\mathrm{Top}(E_k) = n$, and for the dicritical case, $P \equiv py - nxy_1 = 0$ in the differential setting and $P \equiv q^{p/n}y - y_1 = 0$ in the q-algebraic one, where $\gcd(p, n) = 1$, $k = p$, $\rho_k = n$, and $\mathrm{Bot}(E_k) = \mathrm{Top}(E_k) = 1$.

Proof of Proposition 11: Since $\mathrm{Bot}(E_k) \leq \mathrm{Top}(E_k)$, the result is obvious if $\rho_k = 1$. Thus, we can assume that $k = e_l$ for some characteristic exponent e_l. For brevity, let $\Phi(C) = \Phi_{k-1,k/n}(C)$ be the initial polynomial of the element $E_{k-1,k/n}$. The argument hinges on Lemma 8, but there are two cases.

Non-dicritical case. This means that $\Phi(C) \not\equiv 0$. Let $\overline{h}$ be the multiplicity of 0 as a root of $\Phi(C)$ ($\overline{h}$ may be 0). As e_l is a characteristic exponent, Lemma 6 implies that E_k is indeed a side of the Newton polygon $\mathcal{N}(P_k)$, of co-slope k/n. Write, as in Lemma 8, $\Phi(C) = \alpha_k(C) + \delta_{k/n} C \beta_k(C)$. As $\Phi(C) \not\equiv 0$, it has degree less than or equal to $\mathrm{Top}(E_{k-1,k/n}) = \mathrm{Top}(E_k)$.

Let $(\iota, \mathrm{Top}(E_{k-1,k/n}))$ be the topmost vertex of $E_{k-1,k/n}$, which is also the topmost vertex of E_k by Lemma 2. We claim that any $(\iota', j') \in \mathcal{C}(P_{k-1}) \cap E_{k-1,k/n}$, with $j' \leq \mathrm{Top}(E_{k-1,k/n})$, satisfies that $s := \mathrm{Top}(E_{k-1,k/n}) - j'$ is a multiple of r_l. Note that $E_{k-1,k/n}$ and E_k both have co-slope $\frac{e_l}{n} = \frac{p_l}{r_1 \cdots r_l}$, where $\gcd(p_l, r_l) = 1$ by equation (4). We have

$$\iota' = \iota + s\frac{k}{n} = \iota + s\frac{p_l}{r_1 \cdots r_l}.$$

As (ι', j') and $(\iota, \mathrm{Top}(E_{k-1,k/n}))$ belong to $L_{k/n}(P_{k-1})$, then ι' and ι belong to $\frac{\mathbb{Z}}{r_1 \cdots r_{l-1}}$. It follows that

$$s\frac{p_l}{r_l} = r_1 \cdots r_{l-1}(\iota' - \iota) \in \mathbb{Z}$$

and thus s is of the form $s = r_l\, r$ for some $r \in \mathbb{Z}_{\geq 0}$. As a consequence, $A^{k-1}_{\iota' j'}$ and $B^{k-1}_{\iota' j'}$ are 0 except possibly when $j' = \mathrm{Top}(E_{k-1,k/n}) - r_l\, r$ for $r \in \mathbb{Z}_{\geq 0}$. Thus, $\Phi(C)$ can be written as

$$\Phi(C) = C^{\overline{h}}\overline{\Phi}(C^{r_l}),$$

that is, $\Phi(C)$ is, except for the factor $C^{\overline{h}}$, a polynomial in C^{r_l}. This implies that any root of $\Phi(C)$ different from 0 has multiplicity at most $(\deg(\Phi(C)) - \overline{h})/r_l$. Let m be the multiplicity of a_k as a root of $\Phi(C)$. Since e_l is a characteristic exponent, $a_k \neq 0$, so that $m \leq (\deg(\Phi(C)) - \overline{h})/r_l$. Corollary 9 states that $m \geq \mathrm{Bot}(E_k)$, as $E_{k-1,k/n}$ is non-dicritical. Thus, as $\mathrm{Top}(E_k) = \mathrm{Top}(E_{k-1,k/n})$, we obtain

$$\mathrm{Bot}(E_k) \leq m \leq \frac{\deg(\Phi(C)) - \overline{h}}{r_l} \leq \frac{\mathrm{Top}(E_k)}{r_l} = \frac{\mathrm{Top}(E_k)}{\rho_k}, \tag{15}$$

as desired.

Dicritical case. Denote by (ι_j, j) the point of ordinate j in $L_{k/n}(P_{k-1})$. If $\Phi(C) \equiv 0$, which is the dicritical condition, write $0 = \Phi(C) = \alpha_k(C) + \delta_{k/n} C\beta_k(C)$. Denote $b = \mathrm{Bot}(E_k)$. We have $A^k_{\iota_b b} + \delta_{k/n} B^k_{\iota_b b} = 0$ by equation (13). Since $b = \mathrm{Bot}(E_k)$, at least one of them is non-vanishing and hence $A^k_{\iota_b b} \neq 0$ and $B^k_{\iota_b b} \neq 0$. By definition, $B^k_{\iota_j j}(a_k) = 0$ for $j = 1, \ldots, b-1$, so that by (11) in Lemma 8 or equation (14) we have $\beta_k^{(j-1)}(a_k) = 0$ for $j = 1, \ldots, b-1$. Moreover, we have $\beta_k^{(b-1)}(a_k) \neq 0$ since $B^k_{\iota_b b} \neq 0$. Thus, a_k is a root of multiplicity precisely $b-1$ of $\beta_k(C)$. Let $\overline{h} \geq 1$ be the multiplicity of 0 as a root of $C\beta_k(C)$. Considering that r_l is a novel factor of the denominator of k/n, the same argument as in the previous case shows that both $\alpha_k(C)/C^{\overline{h}}$ and $C\beta_k(C)/C^{\overline{h}}$ are, indeed, polynomials in C^{r_l}. On the other hand, dicriticalness also gives that $\beta_k(C)$ has degree equal to $\mathrm{Top}(E_k) - 1$. Thus, we get

$$\mathrm{Bot}(E_k) - 1 = b - 1 \leq \frac{\deg(C\beta_k(C)) - \overline{h}}{r_l} = \frac{\mathrm{Top}(E_k) - \overline{h}}{r_l},$$

from which it follows that

$$\mathrm{Bot}(E_k) \leq \frac{\mathrm{Top}(E_k)}{r_l} + \frac{r_l - \overline{h}}{r_l} \leq \frac{\mathrm{Top}(E_k)}{\rho_k} + \frac{\rho_k - 1}{\rho_k},$$

as desired. The last inequality is an equality if $\overline{h} = 1$, i.e. if $\beta_k(0) \neq 0$. □

Corollary 12. *If $E_{k-1,k/n}$ is non-dicritical and the equality $\mathrm{Bot}(E_k) = \mathrm{Top}(E_k)/\rho_k$ holds, then $\Phi_{k-1,k/n}(C)$ is of the form $\Phi_{k-1,k/n}(C) = u(C^{\rho_k} - a_k^{\rho_k})^{\mathrm{Bot}(E_k)}$ for some $u \in \mathbb{C}^*$.*

Proof: First assume $\rho_k = 1$. Since the degree of $\Phi_{k-1,k/n}(C)$ is at most $\mathrm{Top}(E_k)$ and a_k is a root of $\Phi_{k-1,k/n}(C)$ of multiplicity at least $\mathrm{Bot}(E_k)$ (Corollary 9), it follows that $\Phi_{k-1,k/n}(C) = u(C - a_k)^{\mathrm{Bot}(E_k)}$ for some $u \in \mathbb{C}^*$.

Now assume $\rho_k > 1$ and consider the notation in the proof of Proposition 11. The equality $\mathrm{Bot}(E_k) = \mathrm{Top}(E_k)/\rho_k$ implies by (15) that $\overline{h} = 0$ and $\deg(\Phi(C)) = \mathrm{Top}(E_k)$. As a consequence, $\Phi(C)$ is a polynomial in C^{ρ_k} with a non-vanishing root a_k of multiplicity $\deg(\Phi(C))/\rho_k$. Thus $\Phi(C) = u(C^{\rho_k} - a_k^{\rho_k})^{\mathrm{Bot}(E_k)}$ for some $u \in \mathbb{C}^*$. □

3.1. Proof of Theorem A. Denote $\mu = \mathrm{ord}(s(x))$ and let $k_0 = n\,\mu$. Since the coefficients a_i of $s(x)$ are zero for $i = 1, \ldots, k_0 - 1$, we have that $P = P_{k_0-1}$ and

$$H(P) \geq H(P, s(x)) = \mathrm{Top}(E_{P,\mu}) = \mathrm{Top}(E_{k_0-1,k_0/n}) = \mathrm{Top}(E_{k_0}).$$

We split the proof into two cases.

Assume $d = 0$, that is, there are no dicritical characteristic exponents. We obtain $\mathrm{Top}(E_{k_0}) \geq \prod_{j=1}^{g} r_j \geq 1$ by applying Proposition 11 iteratively from $j \geq k_0$, and the fact that $\mathrm{Top}(E_{j,(j+1)/n}) \leq \mathrm{Bot}(E_{j,j/n})$ for any j, as in Lemma 4.

Now suppose $d \geq 1$. We get $\mathrm{Top}(E_{k_0}) \geq r_1 \cdots r_{i_1-1}\,\mathrm{Top}(E_{e_{i_1}})$ by successive applications of the non-dicritical case of Proposition 11. Using again Proposition 11, now in the dicritical case, we get

$$r_1 \cdots r_{i_1-1}\,\mathrm{Top}(E_{e_{i_1}}) \geq r_1 \cdots r_{i_1}\,\mathrm{Bot}(E_{e_{i_1}}) - (r_1 \cdots r_{i_1} - r_1 \cdots r_{i_1-1}), \tag{16}$$

which gives, taking into account that $\mathrm{Bot}(E_{e_{i_1}}) \geq \mathrm{Top}(E_{e_{i_1}+1})$,

$$\mathrm{Top}(E_{k_0}) \geq r_1 \cdots r_{i_1}\,\mathrm{Top}(E_{e_{i_1}+1}) - (r_1 \cdots r_{i_1} - r_1 \cdots r_{i_1-1}).$$

We obtain the desired bound

$$\mathrm{Top}(E_{k_0}) \geq \prod_{j=1}^{g} r_j - \sum_{k=1}^{d} \left(\prod_{j=1}^{i_k} r_j - \prod_{j=1}^{i_k-1} r_j \right)$$

by iterating this argument. □

3.2. Bound without assumptions on dicritical exponents. We shall see next that dicritical elements cannot be immediately consecutive, which will provide us with a bound in which the dicritical elements play no role.

Lemma 13. *Let $Q = (\kappa, b)$ be the bottom point of the element E_k. Assume $E_{k-1,k/n}$ is dicritical. Then the point Q does not belong to any later dicritical element. As a consequence, if k is a characteristic exponent, then either the next characteristic exponent e_ℓ is non-dicritical or the element E_{e_ℓ} satisfies*

$$\mathrm{Top}(E_{e_\ell}) \leq b - 1.$$

Proof: Let $A^k_{\kappa b}$, $B^k_{\kappa b}$ be the coefficients of $P_k = A^k(x,y) + B^k(x,y)y_1$ corresponding to the point Q. As $E_{k-1,k/n}$ is dicritical, $\Phi_{k-1,k/n}(C) \equiv 0$, and by equation (13) applied to $j = b$ and $C = a_k$,

$$A^k_{\kappa b} + \delta_{k/n} B^k_{\kappa b} = 0,$$

so that $\delta_{k/n} = -A^k_{\kappa b}/B^k_{\kappa b}$. We claim that Q does not belong to a dicritical element $E_{m-1,m/n}$ with $m > k$. By Lemma 2 and $b \geq \mathrm{Top}(E_{k,(k+1)/n})$, $A^k_{\kappa b} = A^m_{\kappa b}$ and $B^k_{\kappa b} = B^m_{\kappa b}$ for $m > k$. If $E_{m-1,m/n}$ were dicritical, we should have

$$A^k_{\kappa b} + \delta_{m/n} B^k_{\kappa b} = 0,$$

which is a contradiction. Here we use the condition $|q| \neq 1$ in the case of q-difference equations, so that $q^{m/n} \neq q^{k/n}$. The result now follows straightforwardly. □

The first consequence of this lemma is:

Corollary 14. *If e_ℓ, $e_{\ell+1}$ are two consecutive dicritical characteristic exponents, then*

$$\mathrm{Top}(E_{e_{\ell+1}}) \leq \mathrm{Bot}(E_{e_\ell}) - 1 < \frac{\mathrm{Top}(E_{e_\ell})}{r_\ell}.$$

As a consequence, if $e_\ell, e_{\ell+1}, \dots, e_{\ell+p}$ is a sequence of consecutive dicritical characteristic exponents with $p \geq 1$, then

$$\mathrm{Top}(E_{e_{\ell+p}}) < \frac{\mathrm{Top}(E_{e_\ell})}{r_\ell \cdots r_{\ell+p-1}}. \tag{17}$$

We can now proceed to prove Corollary A.

Proof of Corollary A: In Theorem A, we can improve the inequality as follows: we say that e_ℓ is a *terminally dicritical* characteristic exponent if e_ℓ is dicritical and either $\ell = g$ or $e_{\ell+1}$ is non-dicritical. By equation (17) in Corollary 14, the argument giving equation (16) in the proof of Theorem A can be restricted to terminally dicritical exponents. Hence, if $\ell_1, \dots, \ell_s$ is the sequence of indices of terminally dicritical exponents,

$$\begin{aligned}\mathrm{Top}(E_{k_0}) &\geq \prod_{j=1}^{g} r_j - \sum_{k=1}^{s}\left(\prod_{j=1}^{\ell_k} r_j - \prod_{j=1}^{\ell_k - 1} r_j\right) \\ &= \left(\prod_{j=1}^{g} r_j - \prod_{j=1}^{\ell_s} r_j\right) + \sum_{k=2}^{s}\left(\prod_{j=1}^{\ell_k - 1} r_j - \prod_{j=1}^{\ell_{k-1}} r_j\right) + \prod_{j=1}^{\ell_1 - 1} r_j,\end{aligned} \tag{18}$$

where all the terms in parentheses are non-negative.

If $\ell_s < g$, i.e. the last characteristic exponent is non-dicritical, then

$$\operatorname{Top}(E_{k_0}) \geq \prod_{j=1}^{g} r_j - \prod_{j=1}^{\ell_s} r_j \geq \prod_{j=1}^{g} r_j - \prod_{j=1}^{g-1} r_j \geq \prod_{j=1}^{g-1} r_j > \prod_{j=1}^{g-1} r_j - \prod_{j=1}^{g-2} r_j$$

because $r_g \geq 2$. Otherwise, if $\ell_s = g$ and $s = 1$, then (18) becomes

$$\operatorname{Top}(E_{k_0}) \geq \prod_{j=1}^{g} r_j - \left(\prod_{j=1}^{g} r_j - \prod_{j=1}^{g-1} r_j \right) = \prod_{j=1}^{g-1} r_j > \prod_{j=1}^{g-1} r_j - \prod_{j=1}^{g-2} r_j.$$

When $\ell_s = g$ and $s > 1$ then we reason as follows: all the terms in the inner summation in (18) are non-negative, and the largest one is the one with $k = s$ because $r_j \geq 2$. Thus, by keeping just this term, we obtain

$$\left(\prod_{j=1}^{g} r_j - \prod_{j=1}^{\ell_s} r_j \right) + \sum_{k=2}^{s} \left(\prod_{j=1}^{\ell_k - 1} r_j - \prod_{j=1}^{\ell_{k-1}} r_j \right) + \prod_{j=1}^{\ell_1 - 1} r_j$$
$$\geq \left(\prod_{j=1}^{g} r_j - \prod_{j=1}^{g} r_j \right) + \prod_{j=1}^{g-1} r_j - \prod_{j=1}^{\ell_{s-1}} r_j + \prod_{j=1}^{\ell_1 - 1} r_j > \prod_{j=1}^{g-1} r_j - \prod_{j=1}^{g-2} r_j$$

because $\ell_{s-1} \leq g - 2$. □

Corollary A is the best we can say in general for any kind of covered equation. However, for *generic* or *contracting* q-difference equations and for general differential equations, we can be more precise. The genericity condition for q-difference equations we shall state becomes clear after Lemmas 15 and 17 below.

3.3. Reasonable equations: relations between consecutive dicritical elements. As we explained above, the dicritical property not only affects the initial polynomial $\Phi_{k-1,k/n}(C)$ but also creates relations between the coefficients $A^k_{\iota j}$ and $B^k_{\iota j}$ falling on a point (ι, j) in the dicritical element E_k. These relations, which in some sense bring to mind the concept of *residues* or *indices* for a singular holomorphic foliation along a non-singular separatrix, will be key to discern what kind of covered equations admit an even sharper bound of $H(P)$ in terms of the factors $r_1, \ldots, r_g$. Hence the name given to the following concept.

Definition 15. Consider the element E_k with topmost vertex (ι, t) and lowest vertex (κ, b). Using (8), we define the k-th *top* and *bottom residues* as

$$\overline{\operatorname{Res}}_k = \frac{A^k_{\iota t}(0)}{B^k_{\iota t}(0)} = \frac{A^{k-1}_{\iota t}}{B^{k-1}_{\iota t}}, \quad \underline{\operatorname{Res}}_k = \frac{A^k_{\kappa b}(a_k)}{B^k_{\kappa b}(a_k)}$$

respectively. By convention, $*/0 = \infty$ (as $0/0$ does not happen by definition).

The following result gives a necessary condition for the inequality in case (i) of Proposition 11 being an equality, which is the worst case for our bounds.

Lemma 15. *Let $\overline{\operatorname{Res}}_k$ and $\underline{\operatorname{Res}}_k$ be the k-th top and bottom residues, respectively. Assume that the initial polynomial $\Phi(C) = \Phi_{k-1,k/n}(C)$ is of the form $\Phi(C) = u(C^{\rho_k} - a_k^{\rho_k})^{\operatorname{Bot}(E_k)}$, where $\Phi(C)$ has degree $\operatorname{Top}(E_k)$, ρ_k is as in Definition 14, and u is a non-zero constant (as a consequence, $E_{k-1,k/n}$ is non-dicritical). Then*

$$\underline{\operatorname{Res}}_k = \rho_k \overline{\operatorname{Res}}_k + (\rho_k - 1)\delta_{k/n}$$

if $\overline{\operatorname{Res}}_k \neq \infty$, or $\underline{\operatorname{Res}}_k = \infty$ otherwise.

Proof: Let (ι, t) be the top point of E_k and (κ, b) its bottom one. Before proceeding, notice that $u = A^k_{\iota t}(0)+\delta_{k/n}B^k_{\iota t}(0)$. First assume $\rho_k = 1$. Our hypothesis implies $t = b$ and hence $\underline{\mathrm{Res}}_k = \overline{\mathrm{Res}}_k$ as desired. From now on we assume $\rho_k > 1$ and hence $a_k \neq 0$.

Analogously to the proof of the non-dicritical case of Proposition 11, we obtain that all the points in $E_{k-1,k/n}$ are of the form $(\iota + s\rho_k k/n, t - s\rho_k)$ for some $s \in \mathbb{Z}_{\geq 0}$. Moreover, since $\Phi(0) \neq 0$, it follows that t is of the form $t = m\rho_k$ for some $m \in \mathbb{Z}_{\geq 0}$. Thus, if (ι, j) is a point in $E_{k-1,k/n}$, then $j = t-s\rho_k = (m-s)\rho_k$ for some $s \geq 0$. Thus, $B^{k-1}_{\iota_j,j} \neq 0$, where $\iota_j = \iota+(m\rho_k-j)k/n$, implies that j is a multiple of ρ_k. Therefore, the polynomial $C\beta_k(C)$ is in fact a polynomial in C^{ρ_k} of degree at most t. By Corollary 9, as a_k is a root of multiplicity at least $b-1$ of $\beta_k(C)$, then either $\beta_k(C) = 0$ or $C\beta_k(C) = vC^{\rho_k}(C^{\rho_k} - a_k^{\rho_k})^{b-1}$ for $v = B^k_{\iota t}(0)$. In the first case $\overline{\mathrm{Res}}_k = \infty$ and certainly $\underline{\mathrm{Res}}_k = \infty$ as well. From now on we assume $\beta_k(C) = vC^{\rho_k-1}(C^{\rho_k} - a_k^{\rho_k})^{b-1}$ with $v \neq 0$. By (11) in Lemma 8, we obtain

$$B^k_{\kappa b}(a_k) = \frac{1}{(b-1)!}\beta_k^{(b-1)}(a_k),$$

which gives, for ξ a primitive ρ_k-th root of unity,

$$B^k_{\kappa b}(a_k) = \frac{B^k_{\iota t}(0)}{(b-1)!}a_k^{\rho_k-1}(b-1)!\prod_{j=1}^{\rho_k-1}(a_k - \xi^j a_k)^{b-1} = B^k_{\iota t}(0)a_k^{(\rho_k-1)b}\prod_{j=1}^{\rho_k-1}(1-\xi^j)^{b-1}.$$

On the other hand, for the same ξ, we have

$$\frac{1}{b!}\Phi^{(b)}(a_k) = u\prod_{j=1}^{\rho_k-1}(a_k - \xi^j a_k)^b = ua_k^{(\rho_k-1)b}\prod_{j=1}^{\rho_k-1}(1-\xi^j)^b.$$

Applying equations (10) and (11) of Lemma 8, and the two equalities above, we get

$$\frac{A^k_{\kappa b}(a_k)}{B^k_{\kappa b}(a_k)} = \frac{\Phi^{(b)}(a_k)}{b!B^k_{\kappa b}(a_k)} - \delta_{k/n} = \frac{ua_k^{(\rho_k-1)b}\prod_{j=1}^{\rho_k-1}(1-\xi^j)^b}{B^k_{\iota t}(0)a_k^{(\rho_k-1)b}\prod_{j=1}^{\rho_k-1}(1-\xi^j)^{b-1}} - \delta_{k/n}.$$

And, as $u = A^k_{\iota t}(0) + \delta_{k/n}B^k_{\iota t}(0)$, we obtain

$$\underline{\mathrm{Res}}_k = \frac{A^k_{\kappa b}(a_k)}{B^k_{\kappa b}(a_k)} = \frac{A^k_{\iota t}(0) + \delta_{k/n}B^k_{\iota t}(0)}{B^k_{\iota t}(0)}\prod_{j=1}^{\rho_k-1}(1-\xi^j) - \delta_{k/n}.$$

But since $\prod_{j=1}^{\rho_k-1}(1-\xi^j) = (C^{\rho_k}-1)'(1)$, we have $\prod_{j=1}^{\rho_k-1}(1-\xi^j) = \rho_k$, so that

$$\underline{\mathrm{Res}}_k = \rho_k\overline{\mathrm{Res}}_k + (\rho_k-1)\delta_{k/n}$$

as desired. □

Corollary 16. *Assume that either $E_{k-1,k/n}$ is dicritical or* $\mathrm{Bot}(E_k) = \mathrm{Top}(E_k)/\rho_k$. *Then*

$$\underline{\mathrm{Res}}_k = \rho_k\overline{\mathrm{Res}}_k + (\rho_k-1)\delta_{k/n}. \tag{19}$$

Furthermore:

(i) *If P is a differential equation and $\overline{\mathrm{Res}}_k$ is a real number with $\overline{\mathrm{Res}}_k \geq -k/n$, then $\underline{\mathrm{Res}}_k \geq -k/n$.*

(ii) *If P is a q-difference equation with $q\in\mathbb{R}^+\setminus\{1\}$, then $\overline{\mathrm{Res}}_k \geq -q^{k/n}$ (resp. $\overline{\mathrm{Res}}_k \leq -q^{k/n}$) if and only if $\underline{\mathrm{Res}}_k \geq -q^{k/n}$ (resp. $\underline{\mathrm{Res}}_k \leq -q^{k/n}$).*

Proof: If $E_{k-1,k/n}$ is dicritical, then all the coefficients of $\Phi_{k-1,k/n}(C) = \alpha_k(C) + \delta_{k/n} C\beta_k(C)$ are 0 and equation (13) gives:

$$A^k_{\iota_j j} = -\delta_{k/n} B^k_{\iota_j j}$$

for each $(\iota_j, j) \in E_k$, and $j = 0, \ldots, \mathrm{Top}(E_{k-1,k/n})$, so that (19) is trivial in this case, with $\overline{\mathrm{Res}}k = \underline{\mathrm{Res}}_k = -\delta_{k/n}$. When $E_{k-1,k/n}$ is not dicritical, the result follows from the fact that $\mathrm{Bot}(E_k) = \mathrm{Top}(E_k)$ if $\rho_k = 1$, and from Lemma 15 if $\rho_k > 1$. The other results follow trivially from that equality. □

Our next goal is to improve the lower bound provided by Corollary A. Indeed, we can obtain a better lower bound in the case of differential equations and, for q-difference equations, when q is not a root of a certain subset of algebraic equations, the so-called unreasonable equations, which we shall describe below.

Definition 16. A polynomial $Q(z) = z^m + u_{m-1}z^{m-1} + \cdots + u_0 \in \mathbb{R}[z]$ is *improper* if $u_0, \ldots, u_{m-1} \in \mathbb{R}^+$ and $1 \leq u_{m-1} \leq u_{m-2} \leq \cdots \leq u_0$.

Lemma 17. *Any complex root z_0 of an improper polynomial $Q(z)$ satisfies $|z_0| \geq 1$ and $|z_0| \leq \max\left(\frac{u_0}{u_1}, \ldots, \frac{u_{m-2}}{u_{m-1}}, \frac{u_{m-1}}{1}\right)$.*

Proof: It is obvious that no positive real number can be a root of $Q(z)$. Denote $u_m = 1$. We have

$$(z-1)Q(z) = (z^{m+1} - z^m) + \sum_{j=0}^{m-1} u_j(z^{j+1} - z^j) = z^{m+1} + \sum_{j=1}^{m}(u_{j-1} - u_j)z^j - u_0.$$

Given $z_0 \in \mathbb{C}$ with $|z_0| < 1$, we obtain

$$\left| z_0^{m+1} + \sum_{j=1}^{m}(u_{j-1} - u_j)z_0^j \right| < 1 + (u_{m-1} - u_m) + \cdots + (u_0 - u_1) = u_0,$$

so z_0 is not a root of $Q(z)$.

Let us prove the other inequality. Denote $r = \max_{0 \leq j < m-1}(u_j/u_{j+1})$. It suffices to prove that $\tilde{Q}(z) := z^m Q(r/z)/u_0$ has no roots of modulus less than 1. Since $\tilde{Q}(z)$ is improper, the result is a consequence of the first part of the proof. □

Corollary 16 on top and bottom residues has an important consequence on chains of non-dicritical elements that follow a dicritical one. We want to apply Proposition 11 repeatedly without a priori information on the dicritical exponents in order to obtain a sharp bound for $H(P, s(x))$, and hence $H(P)$ (Theorem B). To this end, it is important to study situations in which the iterative use of Proposition 11 and Lemma 4 gives us poor lower bounds. This is the goal of the following result in which we analyze a "worst case scenario" for a pair of dicritical elements $E_{k-1,k/n}$ and $E_{l,(l+1)/n}$ that are separated by a chain of non-dicritical elements $E_{k,(k+1)/n}, E_{k+1,(k+2)/n}, \ldots, E_{l-1,l/n}$. We shall also see that it can only happen in very special cases of q-difference equations. This motivates Definition 17, of reasonable equations.

Proposition 18. *Assume that $E_{k-1,k/n}$ is a dicritical element and that for $j = k+1, \ldots, l$, any element E_j satisfies* $\mathrm{Bot}(E_j) = \mathrm{Top}(E_j)/\rho_j$ *and* $\mathrm{Top}(E_j) = \mathrm{Bot}(E_{j-1})$. *Assume also that if $\rho_j > 1$, then $E_{j-1,j/n}$ is non-dicritical for $j = k+1, \ldots, l$. If the bottom of E_l belongs to a dicritical element, then the equation P is q-differential and, denoting $s = q^{1/n}$, we have*

$$s^{\bar{l}-k} + \rho_{\bar{l}}\, s^{\bar{l}-k-1} + \rho_{\bar{l}-1}\,\rho_{\bar{l}}\, s^{\bar{l}-k-2} + \cdots + \rho_{k+1}\cdots\rho_{\bar{l}} = 0 \tag{20}$$

for some $\bar{l} \geq l$, which implies that $q^{1/n}$ is the root of an improper polynomial.

Proof: By hypothesis, the bottom of E_l belongs to a first dicritical element, $E_{\bar{l},(\bar{l}+1)/n}$ with $\bar{l} \geq l$, so that $\mathrm{Bot}(E_l) = \mathrm{Top}(E_{\bar{l},(\bar{l}+1)/n}) = \mathrm{Top}(E_{\bar{l}+1})$. This implies that we have $\mathrm{Bot}(E_j) = \mathrm{Top}(E_j)$ for any $l < j \leq \bar{l}$ and $\mathrm{Top}(E_j) = \mathrm{Bot}(E_{j-1})$ for any $l < j \leq \bar{l}+1$. Moreover, $\mathrm{Bot}(E_j) = \mathrm{Top}(E_j)$ implies $\rho_j = 1$ for any $l < j \leq \bar{l}$ by the first case of Proposition 11. This property, together with the hypotheses, provide $\mathrm{Bot}(E_j) = \mathrm{Top}(E_j)/\rho_j$ for any $k < j \leq \bar{l}$.

If P is a differential equation, then the top and bottom residues of E_k satisfy $\overline{\mathrm{Res}}_k = \underline{\mathrm{Res}}_k = -k/n$ because of the dicritical condition and equation (13). Now, $\overline{\mathrm{Res}}_k \geq -k/n$ implies $\overline{\mathrm{Res}}_{k+1} = \underline{\mathrm{Res}}_k \geq -k/n > -(k+1)/n$, where the first equality is a consequence of $\mathrm{Top}(E_{k+1}) = \mathrm{Bot}(E_k)$. An iterative application of Corollary 16 provides the inequality $\underline{\mathrm{Res}}_j \geq -j/n$ for $j = k, \dots, \bar{l}$. This prevents the bottom of each E_j, $k \leq j \leq \bar{l}$, from being dicritical at some later step, as $\underline{\mathrm{Res}}_j$ should be $-m/n$ for some $m > j$ otherwise. Thus, P cannot be a differential equation.

We can then assume that P is a q-difference equation. Denoting $s = q^{1/n}$, let us show, by induction, that

$$\underline{\mathrm{Res}}_j = s^k(s-1)(s^{j-k} + \rho_j s^{j-k-1} + \rho_{j-1}\rho_j s^{j-k-2} + \cdots + \rho_{k+1}\cdots\rho_j) - s^{j+1} \tag{21}$$

for any $k \leq j \leq \bar{l}$, from which the result follows because $\underline{\mathrm{Res}}_{\bar{l}} = -s^{\bar{l}+1}$ by $\mathrm{Top}(E_{\bar{l}+1}) = \mathrm{Bot}(E_{\bar{l}})$ and the dicriticalness of $E_{\bar{l},(\bar{l}+1)/n}$. Equation (21) reads

$$\underline{\mathrm{Res}}_k = s^k(s-1) - s^{k+1} = -s^k$$

in the base case $j = k$. In this case, as $E_{k-1,k/n}$ is dicritical, we get

$$\overline{\mathrm{Res}}_k = \underline{\mathrm{Res}}_k = -\delta_{k/n} = -s^k$$

by (13), so that equation (21) holds indeed for $j = k$. Suppose equation (21) holds for some $j < \bar{l}$. We have $\overline{\mathrm{Res}}_{j+1} = \underline{\mathrm{Res}}_j$ because $\mathrm{Top}(E_{j+1}) = \mathrm{Bot}(E_j)$. Since $\mathrm{Bot}(E_{j+1}) = \mathrm{Top}(E_{j+1})/\rho_{j+1}$, Corollary 16 gives the equality

$$\underline{\mathrm{Res}}_{j+1} = \rho_{j+1}\overline{\mathrm{Res}}_{j+1} + (\rho_{j+1} - 1)s^{j+1}.$$

Thus,

$$\underline{\mathrm{Res}}_{j+1} = \rho_{j+1}\underline{\mathrm{Res}}_j + (\rho_{j+1} - 1)s^{j+1},$$

so that the induction hypothesis on j gives, then, by direct substitution

$$\underline{\mathrm{Res}}_{j+1} = \rho_{j+1}(s^k(s-1)(s^{j-k} + \rho_j s^{j-k-1} + \cdots + \rho_{k+1}\cdots\rho_j) - s^{j+1}) + (\rho_{j+1}-1)s^{j+1}.$$

Inserting the common factor ρ_{j+1} into the parenthesis starting with s^{j-1}, we get

$$\begin{aligned}\underline{\mathrm{Res}}_{j+1} &= s^k(s-1)(\rho_{j+1}s^{j-k} + \rho_j\rho_{j+1}s^{j-k-1} + \cdots + \rho_{k+1}\cdots\rho_{j+1}) \\ &\qquad - \rho_{j+1}s^{j+1} + \rho_{j+1}s^{j+1} - s^{j+1} \\ &= s^k(s-1)(\rho_{j+1}s^{j-k} + \rho_j\rho_{j+1}s^{j-k-1} + \cdots + \rho_{k+1}\cdots\rho_{j+1}) - s^{j+1}.\end{aligned}$$

Finally, introducing the zero expression $s^{j+1-k} - s^{j+1-k}$ into the second parenthesis and operating the second term, we get

$$\begin{aligned}\underline{\mathrm{Res}}_{j+1} = s^k(s-1)(s^{j+1-k} + \rho_{j+1}s^{j-k} + \rho_j\rho_{j+1}s^{j-k-1} + \cdots + \rho_{k+1}\cdots\rho_{j+1}) \\ - s^{j+2} + s^{j+1} - s^{j+1},\end{aligned}$$

which is equation (21) for $j+1$. □

Definition 17. We say that P is a *reasonable* equation if it is a differential equation or if it is a q-difference equation such that $q^{1/n}$ is not a solution of any equation of the form (20).

Remark 19. The following cases provide reasonable q-difference equations; notice that $q \in \mathbb{R}^+$ does not imply $q^{1/n} \in \mathbb{R}^+$, as one might have chosen $\log(q)$ to be non-real:

(i) If q is a positive real number, different from 1, and $q^{1/n} \in \mathbb{R}^+$,
(ii) or $|q| < 1$,
(iii) or $|q|^{\frac{1}{n}} > \max(r_1, \dots, r_g)$,
(iv) or q is a trascendental number over $\mathbb{Q}$.

This is a direct consequence of Proposition 18 and Lemma 17 since a non-reasonable equation only happens if $q^{1/n}$ is a root of an improper polynomial. So in order for a q-difference equation to be reasonable, q just needs to avoid a countable subset of the annulus $\{1 < |q|^{1/n} \leq \max(r_1, \dots, r_g)\}$.

The next result is what makes reasonable equations interesting: whenever there are two dicritical elements, any "fastidious" term $(\rho_k - 1)/\rho_k$ between them coming from item (ii) of Proposition 11 disappears.

Lemma 20. *Let P be a 1-covered reasonable equation. Assume the following conditions hold:*

(i) *The elements $E_{k-1,k/n}$ and $E_{l-1,l/n}$ are dicritical, and*
(ii) *any characteristic exponent e_j with $k < e_j < l$ is non-dicritical.*

Then

$$\mathrm{Top}(E_l) < \frac{\mathrm{Top}(E_k)}{\rho_k \cdots \rho_{l-1}}.$$

Proof: If $\mathrm{Top}(E_j) < \mathrm{Bot}(E_{j-1})$ for some $j \in \{k+1, \dots, l\}$, then, applying Proposition 11 iteratively, we obtain:

$$\mathrm{Top}(E_j) \leq \mathrm{Bot}(E_{j-1}) - 1 \leq \frac{\mathrm{Bot}(E_k)}{\rho_{k+1} \cdots \rho_{j-1}} - 1 \leq \frac{\frac{\mathrm{Top}(E_k)}{\rho_k} + \frac{\rho_k - 1}{\rho_k}}{\rho_{k+1} \cdots \rho_{j-1}} - 1,$$

which gives

$$\mathrm{Top}(E_j) \leq \frac{\mathrm{Top}(E_k)}{\rho_k \cdots \rho_{j-1}} + \frac{\rho_k - 1}{\rho_k \cdots \rho_{j-1}} - 1 < \frac{\mathrm{Top}(E_k)}{\rho_k \cdots \rho_{j-1}},$$

and as all characteristic exponents strictly between j and l are non-dicritical, Proposition 11 gives once more the result.

If $\mathrm{Top}(E_j) = \mathrm{Bot}(E_{j-1})$ for all $j \in \{k+1, \dots, l\}$, there are two possibilities:

- If $\mathrm{Bot}(E_j) < \frac{\mathrm{Top}(E_j)}{\rho_j}$ for some j with $k < j < l$, we get $\mathrm{Bot}(E_j) \leq \frac{\mathrm{Top}(E_j) - 1}{\rho_j}$ and the same argument as above gives

$$\mathrm{Top}(E_{j+1}) \leq \mathrm{Bot}(E_j) \leq \frac{\mathrm{Top}(E_j) - 1}{\rho_j} \leq \frac{\mathrm{Top}(E_k)}{\rho_k \cdots \rho_j} + \frac{\rho_k - 1}{\rho_k \cdots \rho_j} - \frac{1}{\rho_j} < \frac{\mathrm{Top}(E_k)}{\rho_k \cdots \rho_j}$$

  and as the characteristic exponents between $j+1$ and l are non-dicritical, the result follows.
- If $\mathrm{Bot}(E_j) = \mathrm{Top}(E_j)/\rho_j$ for all $j \in \{k+1, \dots, l-1\}$, then Proposition 18 implies that $\mathrm{Top}(E_l)$ cannot be equal to $\mathrm{Bot}(E_{l-1})$, so that this case cannot happen, and we are done. ☐

3.4. Proof of Theorem B. In order to prove Theorem B, we need to be able to control the elements of P corresponding to exponents k/n for $k/n < \operatorname{ord}(s(x))$: despite their coefficients being 0, the elements $E_{k-1,k/n}$ could, in principle, be dicritical, preventing the desired bound from holding. This is tackled in the next technical result, from which Theorem B follows trivially, setting $m = \operatorname{ord}(s(x))n$.

Proposition 21. *Let $s(x)$ be a Puiseux solution of genus g of the reasonable equation $P = 0$ and fix $1 \leq m \leq \operatorname{ord}(s(x))n$. Then*

$$\operatorname{Top}(E_{P,m/n}) \geq r_1 \cdots r_{g-1}. \tag{22}$$

The inequality is strict unless, possibly, all the following conditions hold: e_g/n is the unique dicritical exponent of $s(x)$ in $\mathbb{Z}_{\geq m}/n$, $\operatorname{Bot}(E_{e_g}) = \operatorname{Top}(E_{e_g}) = 1$, and for any $m \leq j < e_g$, one has $\operatorname{Bot}(E_j) = \operatorname{Top}(E_j)/\rho_j$ and $\operatorname{Bot}(E_j) = \operatorname{Top}(E_{j+1})$.

Proof: Since all coefficients of $s(x)$ of index less than m vanish, we have that $P = P_{m-1}$ and $\operatorname{Top}(E_m) = \operatorname{Top}(E_{m-1,m/n}) = \operatorname{Top}(E_{P,m/n})$. Recall that, for any element E_k, we have $\operatorname{Bot}(E_k) \geq 1$ by Remark 5. If there are no dicritical characteristic exponents, then the result follows from Proposition 11, as

$$1 \leq \operatorname{Bot}(E_{e_g}) \leq \frac{\operatorname{Top}(E_m)}{r_1 \cdots r_g}.$$

Otherwise, let e_ℓ be the dicritical characteristic exponent with greatest index. By Proposition 11, we know that

$$\operatorname{Bot}(E_{e_\ell}) \leq \frac{\operatorname{Top}(E_{e_\ell})}{r_\ell} + \frac{r_\ell - 1}{r_\ell}$$

and an iterative use of the same proposition, Lemma 20, and Corollary 14, if needed, gives

$$\operatorname{Top}(E_{e_\ell}) \leq \frac{\operatorname{Top}(E_m)}{r_1 \cdots r_{\ell-1}}.$$

Combining both inequalities, we obtain

$$\operatorname{Bot}(E_{e_\ell}) \leq \frac{\operatorname{Top}(E_m)}{r_1 \cdots r_\ell} + \frac{r_\ell - 1}{r_\ell}.$$

Finally, using Proposition 11 for $\operatorname{Bot}(E_{e_\ell})$, taking into account that there are no more dicritical exponents,

$$\operatorname{Bot}(E_{e_\ell}) \geq \operatorname{Bot}(E_{e_g}) r_{\ell+1} \cdots r_g \geq r_{\ell+1} \cdots r_g.$$

Thus, we obtain

$$\operatorname{Top}(E_m) \geq \operatorname{Bot}(E_{e_g}) r_1 \cdots r_g - \frac{r_\ell - 1}{r_\ell} r_1 \cdots r_\ell$$

and (22) follows from $\operatorname{Bot}(E_{e_g}) \geq 1$ and $r_g \geq 2$.

Assume that the strict inequality does not hold, that is, $\operatorname{Top}(E_m) = r_1 \cdots r_{g-1}$. This implies that $\ell = g$ and $\operatorname{Bot}(E_{e_g}) = 1$. There cannot be more dicritical exponents other than e_g/n in $\{m, \ldots, e_g\}/n$, since Lemma 20 provides strict inequalities. Moreover, we get $\operatorname{Top}(E_{e_g}) = r_g \operatorname{Bot}(E_{e_g}) - (r_g - 1) = r_g - (r_g - 1) = 1$. Finally, an iterative use of Proposition 11 gives $\operatorname{Top}(E_j)/\rho_j = \operatorname{Bot}(E_j)$ and hence $\operatorname{Bot}(E_j) = \operatorname{Top}(E_{j+1})$ for any $m \leq j < e_g$. ☐

Taking into account that $r_i \geq 2$ for $i = 1, \ldots, g$, we get:

Corollary 22. *With the same notation as in Theorem A,*

$$g \leq 1 + \log_2(H(P, s(x))) \leq 1 + \log_2(H(P)),$$

and if $\operatorname{ord}(s(x)) \geq 1$*, then* $g \leq 1 + \log_2(\nu_0(P) + 1)$*.*

4. Multiplicity and height

In the case of singular holomorphic foliations (see [**18**] for instance), given a Pfaffian 1-form $\omega = A(x, y)\, dx + B(x, y)\, dy$ with $A(x, y), B(x, y) \in \mathbb{C}\{x, y\}$ satisfying $A(0, 0) = B(0, 0) = 0$ and $\gcd(A(x, y), B(x, y)) = 1$ that defines a germ of holomorphic foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$, the *multiplicity* of $\mathcal{F}$ at 0 is

$$\nu_0(\mathcal{F}) = \min\{\operatorname{ord}_{(x,y)}(A(x, y)), \operatorname{ord}_{(x,y)}(B(x, y))\},$$

whose value does not depend on the coordinates (x, y) and is at least 1. In [**8**], we proved, using geometric arguments, that, if Γ is an invariant irreducible curve, which in generic coordinates has characteristic exponents $e_1, \ldots, e_g$ and Puiseux factors $r_1, \ldots, r_g$, then the inequality

$$\nu_0(\mathcal{F}) \geq r_1 \cdots r_{g-1}$$

holds. The natural question is whether we can obtain this result using just the Newton polygon. The answer is *yes*. Set $P = A(x, y) + B(x, y)y_1$ and let $s(x)$ be the Puiseux series expansion corresponding to Γ. Since Γ is an integral curve of ω, then $s(x)$ is a solution of the differential equation $P = 0$. After a linear change of coordinates, we can assume that Γ is not tangent to $x = 0$; in particular we obtain $\operatorname{ord}_x(s(x)) \geq 1$. Furthermore, after a generic linear change of coordinates we can also assume the following facts, depending on whether or not $E_{P,1}$ is dicritical, which corresponds to $\mathcal{F}$ having an invariant branch tangent to any generic line in $(\mathbb{C}^2, 0)$:

(i) Either $\Phi_{P,1}(C) \equiv 0$, or in other words, every line through the origin is an invariant curve of $A_\nu(x, y)\, dx + B_\nu(x, y)\, dy = 0$, where A_ν and B_ν are the homogeneous components of degree $\nu = \nu_0(\mathcal{F})$ of A and B respectively, this being the dicritical case.

(ii) Or $x = 0$ is not an invariant curve of $A_\nu(x, y)\, dx + B_\nu(x, y)\, dy = 0$, or equivalently $\deg(\Phi_{P,1}(C)) = \nu + 1$, the non-dicritical case.

Those assumptions do not change $\nu_0(\mathcal{F})$. We have the inequality

$$\nu_0(P) = \nu_0(\mathcal{F}) \geq \operatorname{Top}(E_{P,1}) - 1$$

by definition, and by Proposition 21,

$$\nu_0(\mathcal{F}) \geq \operatorname{Top}(E_{P,1}) - 1 \geq r_1 \cdots r_{g-1} - 1.$$

We now prove that the trailing -1 can be removed from the inequality, just using arguments from the Newton construction.

Proof of Corollary B: The foliation $\mathcal{F}$ is singular, so that $\nu_0(\mathcal{F}) \geq 1$ and the result holds for $g \leq 1$. We assume henceforward that $g > 1$.

Assume, aiming at contradiction, that $\operatorname{Top}(E_{P,1}) = r_1 \cdots r_{g-1}$. By Proposition 21 we infer that e_g/n is the unique dicritical exponent in $\mathbb{Z}_{\geq 1}/n$ and the element $E_{e_g-1, e_g/n}$ is dicritical. Since the exponent 1 is not a characteristic exponent the element $E_{P,1} = E_n$ of co-slope 1 is non-dicritical and, by assumption (ii), there is a point with abscissa -1 in E_n, which is indeed $(-1, \nu_0(\mathcal{F})+1)$. As its abscissa is -1, it is the topmost vertex of E_n, and we have $\overline{\operatorname{Res}}_n = 0$. Moreover, we have $\operatorname{Bot}(E_j) = \operatorname{Top}(E_{j+1})$ and hence $\underline{\operatorname{Res}}_j = \overline{\operatorname{Res}}_{j+1}$ for any $n \leq j < e_g$ by Proposition 21. Furthermore, we get

$\mathrm{Bot}(E_{e_g}) = \mathrm{Top}(E_{e_g}) = 1$, $\mathrm{Bot}(E_j) = \mathrm{Top}(E_j)/\rho_j$, and $\mathrm{Bot}(E_j) = \mathrm{Top}(E_{j+1})$ for any $n \leq j < e_g$. Applying again Proposition 21 and Corollary 16, we obtain

$$\overline{\mathrm{Res}}_{j+1} = \underline{\mathrm{Res}}_j = \rho_j \overline{\mathrm{Res}}_j + (\rho_j - 1)\delta_{j/n}$$

for any $n \leq j < e_g$. Since $\overline{\mathrm{Res}}_n = 0$, we obtain $\overline{\mathrm{Res}}_j \geq 0$ for any $n \leq j \leq e_g$ by induction. As $\overline{\mathrm{Res}}_{e_g} \geq 0$, the top vertex of E_{e_g} cannot belong to a dicritical element, contradicting that $E_{e_g-1,e_g/n}$ is dicritical. □

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