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FLUCTUATION INEQUALITIES WITH APPLICATIONS TO CONVERGENCE AND REGULARITY OF STOCHASTIC PROCESSES INDEXED BY $[0,1]^q$.

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Abstract. Extending results of Billingsley and Chentsov, Bickel & Wichura proved some fluctuation inequalities for processes with multi-dimensional time parameter. In the same order of ideas we give here an extension to the case that the marginals of the control measure are not necessarily continuous.

Applications of this results to get some useful convergence criteria for $[0,1]^q$ indexed processes are given, as well as a theorem on regularity of right stochastically continuous processes.

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0. Introduction.

In (1) P. Bickel & M. Wichura prove fluctuation inequalities for processes indexed by a q-dimensional parameter set, extending results of Chentsov and Billingsley, (2), (3). Here we extend their theorem 3 to the case where the marginals of m are not necessarily continuous. Bickel & Wichura (op. cit. pg. 1665, final) announce a possible extension to the case that m depends on n, and the measures m_n converge weakly to a measure with continuous marginals. Our extension has a different character: m will be fixed (independent of n), we will suppose instead that processes in question have independent increments, and the constants that appear in their theorem 1 will depend on m, q, γ and β, in our case. This is the content of point 2. Point 3 is devoted to give applications of the fluctuation inequalities to the convergence of processes indexed by $[0,1]^q$. At point 4 we see an application to the regularity of processes with independent increments over $[0,1]^q$. On this later result it is worthy to say that R. Morkvenas (6), using Dynkin-Kinney's type conditions, proves that all processes with independent increments and stochastically continuous have versions in $D[0,1]^q$. Our Thm. (4.1) is not enclosed in his result because we only impose right stochastic continuity.

1. Definitions and previous results.

Notation is much as in (1). Let q be a positive integer and $T_1, T_2, \ldots, T_q$

subsets of $[0,1]$ each of which contains 0 and 1, and is finite or $[0,1]$. Let $\{X_t\}_{t\in T}$ be a stochastic process indexed by $T = T_1 \times T_2 \times \ldots \times T_q$, with values in a normed space $(E, |\cdot|)$. We suppose X is separable and vanish on the lower boundary of T, $\partial_{inf}T$, i.e. the points of T having some coordinate equal to 0.

For each p, $1 \leqslant p \leqslant q$, and each $t \in T_p$ we define

$$X_t^{(p)} : T_1 \times \ldots \times \hat{T}_p \times \ldots \times T_q \longrightarrow E \quad \text{by}$$

$$X_t^{(p)}(t_1,\ldots,t_{p-1},t_{p+1},\ldots t_q) = X(t_1,\ldots,t_{p-1},t,t_{p+1},\ldots,t_q).$$

If $s \leqslant t \leqslant u$ in T_p, we define

$$m_p(s,t,u)(X) = \min(\|X_t^{(p)} - X_s^{(p)}\|, \|X_u^{(p)} - X_t^{(p)}\|)$$

Where $\|\cdot\|$ is the supremum norm.

<u>Definition (1.1)</u>:

$$M_p''(X) = \sup\{m_p(s,t,u)(X) : s \leqslant t \leqslant u;\ s,t,u \in T_p\}$$

$$M''(X) = \max_{1 \leqslant p \leqslant q} M_p''(X)$$

$$M(X) = \sup\{|X(t)| : t \in T\} \ \square$$

The following proposition is very useful and quite elementary.

<u>Proposition (1.2)</u>: If $1_q = (1,\overset{q}{\ldots},1)$, then

$$M(X) \leqslant \sum_{p=1}^{q} M''_p(X) + |X(1_q)| \leqslant$$

$$\leqslant q.M''(X) + |X(1_q)| \quad \square$$

We say that $B \subset T$ is a block if

$$B = \prod_{p=1}^{q} (s_p, t_p]$$

we also write $B = (s,t]$ where $s = (s_1,\ldots,s_q)$ and $t = (t_1,\ldots,t_q)$.

Denote $X(B)$ the rectangular increment of X over the block B, i.e.:

$$X(B) = \sum_{\epsilon_1=0}^{1} \sum_{\epsilon_2=0}^{1} \cdots \sum_{\epsilon_q=0}^{1} (-1)^{q-\sum_{p=1}^{q}\epsilon_p} X(s_1+\epsilon_1(t_1-s_1),\ldots,s_q+\epsilon_q(t_q-s_q))$$

We say that X has independent increments if $X(B)$ and $X(C)$ are independent random variables whenever B and C are disjoint blocks.

<u>Definition (1.3)</u>: We write $X \in C_i^m(\beta,\gamma)$ if X has independent increments and

$$P\{|X(B)| \geq \lambda\} \leq \lambda^{-\gamma} (m(B))^{\beta}, \quad \forall \lambda > 0$$

for all $B \subset T$, block of T, where γ and β are fixed positive reals, and m is a finite measure over T vanishing over $\partial_{\inf} T$ □

Evidently if $X \in C_i^m(\beta,\gamma)$ then the pair (X,m) belongs to $C(2\beta,2\gamma)$ in the sense of Bickel & Wichura (1).

Theorem (1.4): If $(X,m) \in C(\beta,\gamma)$, i.e. if for all pair of disjoint blocks B,C of T we have

$$P\{|X(B)| \geq \lambda, \; |X(C)| \geq \lambda\} \leq \lambda^{-\gamma} (m(B \cup C))^{\beta}, \quad \forall \lambda > 0$$

then $\forall \lambda > 0$

$$P\{M_p''(X) \geq \lambda\} \leq K_q(\beta,\gamma) \, \lambda^{-\gamma} (m(T))^{\beta}$$

for all p, $1 \leq p \leq q$, and

$$P\{M''(X) \geq \lambda\} \leq L_q(\beta,\gamma) \; \lambda^{-\gamma} (m(T))^{\beta} \; □$$

This is theorem 1 of Bickel and Wichura(1).

Introduction of the following moduli is suggested by the identification

$$D_q = D([0,1]^q; \mathbb{R}) = D([0,1]; D_{q-1}) .$$

Definition (1.5): If $x \in D_q$ and $\delta > 0$ we define

$$w_x''^{(p)}(\delta) = \sup_{\substack{s,t,u \in T_p \\ s \leq t \leq u,\ u-s \leq \delta}} \min(\| x_t^{(p)} - x_s^{(p)} \|, \| x_u^{(p)} - x_t^{(p)} \|)$$

$$w_x''(\delta) = \max_{1 \leq p \leq q} w_x''^{(p)}(\delta) \quad \square$$

In what follows we shall also need the following result on tightness in the space $(D[0,1]^q ; \mathcal{D}_q)$, whose proof may be found in Neuhaus (7).

Theorem (1.6): A sequence $\{P_n\}_{n=1}^{\infty}$ of probability measures on $(D[0,1]^q, \mathcal{D}_q)$ is tight if and only if:

i) For all $\eta > 0$, there exists $a \in \mathbb{R}$ such that

$$P_n\{x : \sup_t |x(t)| > a\} \leq \eta, \quad \text{for all } n \geq 1.$$

ii) For all $\epsilon > 0$, $\eta > 0$, there exist δ, $0 < \delta < 1$, and n_o, such that for $n \geq n_o$

$$P_n \{x : w_x'(\delta) \geq \epsilon \} \leq \eta \quad \square$$

2. Fluctuation inequalities

Theorem (2.1): There exists a constant K, $K = K(q, \beta, \gamma, m(T))$, such that for all process $X \in C_i^m(\beta, \gamma)$, (see Def. (1.3)), is

$$P\{M''_p(X) \geqslant \lambda\} \leqslant K(\lambda^{-4\gamma} \vee \lambda^{-2\gamma})\ (m_p[0,1])^{2\beta} \left(1 - \frac{J_p[0,1]}{m_p[0,1]}\right)^{\beta}$$

for all p, $1 \leqslant p \leqslant q$, where $J_p[0,1]$ is the maximum jump of the distribution function F_{m_p} of the p-th marginal, m_p, of m, and $\vee$ means "the greatest of".

Proof:

Step 1. q = 1 and T finite. Let $0 = t_0 < t_1 < \ldots < t_m = 1$ be the points of T. Define the process

$$Y(u) = \sum_{i=0}^{m-1} X(t_i) I_{[t_i, t_{i+1})}(u) + X(t_m) I_{\{t_m\}}(u)$$

over $[0,1]$. Then, if

$$t_{i-1} \leqslant s < t_i \leqslant t_h \leqslant t < t_{h+1} \leqslant t_k \leqslant u < t_{k+1}$$

$$P\{m(s,t,u)(Y) \geqslant \lambda\} \leqslant \lambda^{-2\gamma} (\sum_{j=i}^{h} m\{t_j\})^{\beta} \ (\sum_{j'=h+1}^{k} m\{t_{j'}\})^{\beta} =$$

$$= \lambda^{-2\gamma} (\sum_{j=i, j'=h+1}^{h,k} m\{t_j\} m\{t_{j'}\})^{\beta} \leqslant$$

$$\leqslant \lambda^{-2\gamma} [(\sum_{j=i}^{k} m\{t_j\} (\sum_{j'=h+1}^{k} m\{t_{j'}\}))^{\beta} \wedge (\sum_{j'=i}^{k} m\{t_{j'}\} (\sum_{j=1}^{h} m\{t_j\}))^{\beta}] \leqslant$$

$$\leqslant \lambda^{-2\gamma} [(m(T) - J_m(T)) \sum_{j=i}^{k} m\{t_j\}]^{\beta} \leqslant$$

$$\leqslant \lambda^{-2\gamma} (m(T) - J_m(T))^{\beta} (\sum_{j=i}^{k-1} 2m\{t_j\} + m\{t_k\} - m\{t_i\})^{\beta} =$$

$$= \lambda^{-2\gamma} (m(T) - J_m(T))^{\beta} \left(\sum_{j=1}^{k-1} 2m\{t_j\} + m\{t_k\} - \sum_{j=1}^{i-1} 2m\{t_j\} - m\{t_i\} \right)^{\beta} =$$

$$= \lambda^{-2\gamma} (m(T) - J_m(T))^{\beta} (F(t_k) - F(t_i))^{\beta} \leqslant$$

$$\leqslant \lambda^{-2\gamma} (m(T) - J_m(T))^{\beta} (F(u) - F(s))^{\beta}$$

where F, continuous, is defined by the relations $F(0) = 0$, $F(t_j) - F(t_{j-1}) = m\{t_j\} + m\{t_{j-1}\}$ and is linear over the intervals $[t_{j-1}, t_j]$.

Hence, the proces Y, together with the measure, m', associated to the distribution function $F' = (m(T) - J_m(T))F$, belongs to $C(\beta, 2\gamma)$. By theorem (1.4) we have

$$P\{M_1''(X) \geqslant \lambda\} = P\{M_1''(Y) \geqslant \lambda\} \leqslant K \lambda^{-2\gamma} (m'(T))^{\beta} =$$

$$= K \lambda^{-2\gamma} (m(T) - J_m(T))^{\beta} (F(1))^{\beta} \leqslant$$

$$\leqslant 2^{\beta} K \lambda^{-2\gamma} (m(T))^{\beta} (m(T) - J_m(T))^{\beta}$$

where we have used $F(1) \leqslant 2m(T)$. This proves the theorem in this case.

<u>Step 2.</u> <u>$q = 1$, $T = [0,1]$, m arbitrary</u>. Let $0 = t_o < t_1 < \ldots < t_m = 1$, and Y the process X restricted to $\{t_o, \ldots, t_m\}$.

Define υ as in step 4, proof of theorem 1, in

Bickel & Wichura (1):

$$\upsilon\{t_j\} = m(t_{j-1}, t_j] \quad \text{if} \quad j \geqslant 1,$$

$$\upsilon\{t_o\} = 0.$$

Then $Y \in C_i^{\upsilon}(\beta,\gamma)$, as a process over $\{t_o, t_1, \ldots t_m\}$. Step 1 now implies

$$P\{M_1''(Y) \geqslant \lambda\} \leqslant \lambda^{-2\gamma} K(\upsilon\{t_o, t_1, \ldots, t_m\})^{2\beta} \left(1 - \frac{J_\upsilon\{t_o, t_1, \ldots, t_m\}}{\upsilon\{t_o, \ldots, t_m\}}\right)^{\beta} =$$

$$= K\ \lambda^{-2\gamma}(m(T))^{2\beta}\left(1 - \frac{J_\upsilon[0,1]}{m[0,1]}\right)^{\beta}.$$

If now we take limit when $m \longrightarrow \infty$, the set $\{t_o, t_1, \ldots, t_m\}$ increasing to a dense subset of $[0,1]$ that contains the points of discontinuity of F, we obtain (by separability):

$$P\{M_1''(Y) \geqslant \lambda\} \leqq K\lambda^{-2\gamma}(m(T))^{2\beta}\left(1 - \frac{J_m[0,1]}{m(T)}\right)^{\beta}.$$

Step 3) $q \geqslant 2$, T and m arbitrary. We know that the theorem is true for $q = 1$. We now will show our result to be true for $p = 1$; for other p the argument is the same.

Like in step 5 of Bickel & Wichura's proof of theorem 1, the key point is that the version for $q = 1$ of our theorem works for the function valued proces $\{X_t^{(1)}\}_{t \in T_1}$. To show this it is enough to find bounds of its increments.

Remember that

$$m_p(s,t,u)(X) = \min\left(\|X_t^{(p)} - X_s^{(p)}\|, \|X_u^{(p)} - X_t^{(p)}\|\right).$$

Let $T^* = T_2 \times \ldots \times T_q$ and $Y = X_t^{(1)} - X_s^{(1)}$ over T^*. Then

$$P\{\|X_t^{(1)} - X_s^{(1)}\| \geq \lambda\} = P\{M(Y) \geq \lambda\} \leq$$

$$\leq P\{(q-1)M''(Y) + |Y(1_{q-1})| \geq \lambda\} \leq$$

$$\leq P\{(q-1)M''(Y) \geq \lambda r_1\} + P\{|Y(1_{q-1})| \geq \lambda r_2\}$$

$$r_1 + r_2 = 1.$$

If B' is a block of T^*, $Y(B') = X((s,t] \times B')$, hence

$$P\{|Y(B')| \geq \lambda\} \leq \lambda^{-\gamma} (m((s,t] \times B'))^{\beta} =: \lambda^{-\gamma} (m^*(B'))^{\beta} .$$

So $Y \in C_i^{m^*}(\beta,\gamma)$, and by thm. (1.4)

$$P\{M''(Y) \geq \lambda r_1 (q-1)^{-1}\} \leq \lambda^{-2\gamma} K_q(\gamma,\beta) r_1^{-2\gamma} (m_1(s,t])^{2\beta} \leq$$

$$\leq \lambda^{-2\gamma} K_q\ (m(T))^{\beta}\ r_1^{-2\gamma}\ (m_1(s,t])^{\beta} .$$

Now if $B = (s,t] \times T^*$

$$Y(1_{q-1}) = X(t,1_{q-1}) - X(s,1_{q-1}) = X(B) .$$

This implies

$$P\{|Y(1_{q-1})| \geq \lambda r_2\} \leq \lambda^{-\gamma} r_2^{-\gamma} (m_1(s,t])^{\beta}.$$

Putting together our inequalities we get finally:

$$P\{\|X_t^{(1)} - X_s^{(1)}\| \geq \lambda\} \leq (r_1^{-2\gamma}(m(T))^{\beta} K_q + r_2^{-\gamma})(\lambda^{-2\gamma} \vee \lambda^{-\gamma})(m_1(s,t])^{\beta}.$$

By the theorem in dimension 1:

$$P\{M_1''(X) \geq \lambda\} \leq (r_{1_o}^{-2\gamma}(m(T))^{\beta} K_q + r_{2_o})(\lambda^{-2\gamma} \vee \lambda^{-4\gamma})(m(T))^{2\beta}\left(1 - \frac{J_1(T_1)}{m_1(T_1)}\right)^{\beta}$$

where r_{1_o} is the solution of the equation $2K_q(m(T))^{\beta}(1-r_1)^{\gamma+1} = r_1^{2\gamma+1}$ over $(0,1)$ and $r_{2_o} = 1 - r_{1_o}$ □

<u>Remark</u>. In step 3 we have in fact used a slight modification of the result of step 2, to cover the $(\lambda^{-2} \vee \lambda^{-\gamma})$ situation. As a referee has pointed out, this proof works only in case $\beta > 1$. As we need this theorem with $\beta = 1$ later on, we remark that an independent easy proof may be given for $\beta > 1/2$ using induction on q and Billingsley (3), Thm 12.6.

3. <u>Convergence of processes indexed by $[0,1]^q$</u>.

<u>Theorem (3.1)</u>: Let $\{X_n\}_{n=1}^{\infty}$ be processes over $T = [0,1]^q$ vanishing on $\partial_{\inf} T$. Suppose that $X_n \in C_i^m(\beta,\gamma)$ for some $\beta > 1/2$, $n = 1,2,\ldots$.

Then:

(3.1.1) $$\lim_{\delta\downarrow o} \limsup_{n\to\infty} P\{ w''_{X_n}(\delta) \geq \epsilon \} = 0$$

for all $\epsilon > 0$:

Proof: It is enough to show that

$$\lim_{\delta\downarrow o} \limsup_{n\to\infty} P\{ w''^{(p)}_{X_n}(\delta) \geq \epsilon \} = 0$$

for all p, $1 \leq p \leq q$, and $\epsilon > 0$.

Let

(3.1.2.) $$w(\sigma,\tau;n) = \sup_{\sigma\leq s\leq t\leq u\leq\tau} \{ \min (\| (X_n)_t^{(p)} - (X_n)_s^{(p)} \| , \| (X_n)_u^{(p)} - (X_n)_t^{(p)} \|) \}$$

Now, an application of theorem (2.1) to the process Y_n, defined over $T^* = [0,1]^{p-1} \times [\sigma,\tau] \times [0,1]^{q-p}$ in such a way that for $\sigma \leq t \leq \tau$

$$(Y_n)_t^{(p)} = (X_n)_t^{(p)} - (X)_\sigma^{(p)},$$

(observe that Y_n vanishes over $\partial_{inf}T^*$, has the same increments that X_n over T^*'s blocks and that, as a consequence, it verifies condition $C_i^{m*}(\beta,\gamma)$ over T^*, where

$$m^*(.) = m(.) - m(. \cap \partial_{inf}T^*)),$$

gives us

$$(3.1.3)\quad P\{w(\sigma,\tau;n) \geq \epsilon\} \leq K\,\epsilon^{-4\gamma}\,(m_p(\sigma,\tau])^{2\beta}\left(1 - \frac{J_p(\sigma,\tau]}{m_p(\sigma,\tau]}\right)^{\beta}$$

for all ϵ, $0 < \epsilon \leq 1$.

If $\delta = 1/2u$ then

$$(3.1.4)\quad \{w''^{(p)}_{X_n}(\delta) \geq \epsilon\} \subset A_1 \cup A_2$$

where

$$A_1 = \bigcup_{i=0}^{u-1} \{w(2i\delta, (2i+2)\delta; n) \geq \epsilon\} \quad \text{and}$$

$$A_2 = \bigcup_{i=0}^{u-2} \{w((2i+1)\delta, (2i+3)\delta; n) \geq \epsilon\}.$$

From (3.1.3) and (3.1.4) we get

$$(3.1.5)\quad P\{w''^{(p)}_{X_n}(\delta) \geq \epsilon\} \leq K\,\epsilon^{-4\gamma}\,(I_1 + I_2)$$

where

$$I_1 = \sum_{i=0}^{u-1} (m_p(2i\delta, (2i+2)\delta])^{2\beta}\left(1 - \frac{J_p(2i\delta, (2i+2)\delta]}{m_p(2i\delta, (2i+2)\delta]}\right)^{\beta}$$

and

$$I_2 = \sum_{i=0}^{u-2} (m_p((2i+1)\delta, (2i+3)\delta])^{2\beta}\left(1 - \frac{J_p((2i+1)\delta, (2i+3)\delta]}{m_p((2i+1)\delta, (2i+3)\delta]}\right)^{\beta}.$$

Now all follow as in Billingsley (5), pg. 133-134 □

Remark: The previous theorems also hold if condition $X \in C_i^m(\beta,\gamma)$ is replaced by: X defined over $(U_1 \times U_2, P_1 \times P_2)$ and for all $u_1 \in U$, $X_t(u_1,.) \in C_i^m(\beta,\gamma)$. We then say that $X \in \tilde{C}_i^m(\beta,\gamma)$.

Theorem (3.2): A sequence, $\{P_n\}_{n=1}^{\infty}$, of probability measures over $(D_q, \mathcal{D}_q)$ is tight if:

i) For all $\eta > 0$, there exists $a \in R$ such that:

$$P_n\{\ x : \sup_t |x(t)| > a\ \} \leqslant \eta,\ n = 1,2....$$

ii) For all positive ϵ, η, there exist δ, $0 < \delta < 1$, and n_o, such that for all $n \geqslant n_o$:

a) $$P_n\{x : w''_x(\delta) \geqslant \epsilon\} \leqslant \eta.$$

b) $$P_n\{x : w_x^{(p)}[0,\delta) \geqslant \epsilon,\ \text{for some } p,\ 1 \leqslant p \leqslant q\} \leqslant \eta.$$

c) $$P_n\{x : w_x^{(p)}[1-\delta,1) \geqslant \epsilon,\ \text{for some } p,\ 1 \leqslant p \leqslant q\} \leqslant \eta.$$

Proof: We show that a), b) and c) imply ii) of thm. (1.6). The argument of Billingsley (2), thm. (14.4), applied to the functions $t \longrightarrow \|x_t^{(p)}\|$, lead to

(3.2.1) $$P_n\{x : w'^{(p)}_x(\delta/2) > \epsilon/q\} \leqslant P_n\{x : w_x^{(p)}[0,\delta) \geqslant \epsilon/6q\} +$$

$+P_n\{x : w_x^{(p)}[1-\delta,1) \geq \epsilon/6q\} + P_n\{x : w_x''^{(p)}(\delta) \geq \epsilon/6q\}$

Now our theorem follows from $w_x'(\delta) \leq \sum_{p=1}^{q} w_x'^{(p)}(\delta)$. □

As an application of the previous results we get the following theorem, which generalizes theorem (2.3) of Giné & Marcus (4), to processes indexed by $[0,1]^q$.

Theorem (3.3): Let $\{X_n\}_{n=1}^{\infty}$, X, be D_q-valued random variables, vanishing on $\partial_{inf}T$, and such that:

i) The finite dimensional distributions of the X_n converge weakly to the corresponding distributions of X.

ii) $X_n \in \tilde{C}_i^m(\beta,\gamma)$, $n = 1,2,\ldots$, for some $\beta > 1/2$, and $\gamma > 0$.

(iii) For all $\epsilon > 0$

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P_n\{x : w_x^{(p)}[1-\delta,1) > \epsilon, \text{ for some } p,\ 1 \leq p \leq q\} = 0$$

Then $\{P_n = L(X_n)\}_{n=1}^{\infty}$ converge weakly to $L(X)$, as a sequence of probability measures on $(D_q, \mathcal{D}_q)$.

Proof: It is an induction on q. We verify conditions i) and ii) of the previous thm.

With respect to ii): a) is a consequence of thm. (3.1), c) is the hypothesis iii). Let us see i).

$$P\{\sup_{t\in T} |x(t)| > a\} = P\{\sup_{t\in[0,1]} \| x_t^{(p)} \| > a\}$$

Hence:

$$P\{\sup_{t\in T} |(X_n)_t| > a\} = P\{\sup_{t\in[0,1]} \| (X_n)_t^{(1)} \| > a\} \leq$$

$$\leq P\{w''^{(1)}_{X_n}(\delta) > 1\} + P\{\max_{1\leq i\leq k} \sup_{t^*\in T_2 x \ldots x T_q} |(X_n)_t^{(1)}(t^*)| > a_o\} \leq$$

$$\leq P\{w''^{(1)}_{X_n}(\delta) > 1\} + \sum_{i=1}^{k} P\{\sup_{t^*\in T_2 x\ldots x T_q} |(X_n)_{t_i}| > a_o\}.$$

Now because of

$$P\{w^{(p')}_{(X_n)^{(1)}_{t_i}}[1-\delta,1) > \epsilon, \text{ for some } p',\ 2\leq p'\leq q\} \leq$$

$$\leq P\{w^{(p)}_{X_n}[1-\delta,1) > \epsilon, \text{ for some } p,\ 1\leq p\leq q\}$$

the processes $(X_n)^{(1)}_{t_i}$ satisfy i), ii) and iii) of our theorem, $i = 1,2,\ldots,k$. By induction hypothesis there exists $\{a_i\}_{i=1}^{k}$ such that

$$P\{\sup_{t^*} |(X_n)^{(1)}_{t_i}| > a_i\} \leq \eta/2k.$$

Given $\eta > 0$, let $\delta > 0$ be such that

$$P\{w''^{(1)}_{X_n}(\delta) > 1\} \leq \eta/2.$$

choose $0 = t_o < t_1 < \ldots \; t_k = 1$ such that $t_i - t_{i-1} < \delta$ and $a_o = \max_{i=1,\ldots k} a_i$.

Then if $a = a_o + 1$:

$$P\{\sup_{t \in T} |(X_n)_t| > a\} \leqslant \eta .$$

This proves that i), ii) and iii) imply i) of thm. (3.2), by induction on q.

It only rests to verify condition b) of ii).

By induction hypothesis $(X_n)_\delta^{(p)} \xrightarrow{w.} (X)_\delta^{(p)}$. Hence, $\|(X_n)_\delta^{(p)}\| \xrightarrow{w} \|(X)_\delta^{(p)}\|$ also. Now observe that

$$\|(X)_\delta^{(p)}\| \xrightarrow[\delta \downarrow o]{Pr.} 0$$

as a consequence of the right continuity of $(X)_t^{(p)}$ and $(X)_o^{(p)} = 0$.

Given positive η and ϵ, let $\delta_o > 0$ be such that if $\delta < \delta_o$

$$P\{\|(X)_\delta^{(p)}\| \geqslant \epsilon\} < \eta/2.$$

Then

$$\limsup_{n \to \infty} P\{\|(X_n)_\delta^{(p)}\| \geqslant \epsilon\} \leqslant P\{\|(X)_\delta^{(p)}\| \geqslant \epsilon\} \leqslant \eta/2.$$

Now from

$$\{x : w_x^{(p)}[0,\delta) \geqslant 4\epsilon\} \subset \{x : w_x''^{(p)}(\delta) \geqslant \epsilon\} \cup \{x : \| x_\delta^{(p)} - x_o^{(p)} \| \geqslant \epsilon\}$$

we get:

$$\limsup_{n \longrightarrow \infty} P_n\{x : w_x^{(p)}[0,\delta) \geqslant 4\epsilon\} \leqslant \eta \ .$$

This proves b) and the theorem. □

In applications quite frequently we don't know that $X \in D_q$. It is then useful to have the following variant of the previous thm., whose proof requires the same argument as above.

<u>Theorem (3.4):</u> Let $\{X_n\}_{n=1}^{\infty}$ be as in thm. (3.3). Suppose:

i) The finite dimensional distributions of X_n are weakly convergent and

$$\lim_{\delta \downarrow o} \limsup_{n \longrightarrow \infty} P_n\{x : \| x_\delta^{(p)} \| \geqslant \epsilon\} = 0$$

for all $\epsilon > 0$ and all p, $1 \leqslant p \leqslant q$.

ii) $X_n \in C_i^m(\beta,\gamma)$, $n = 1,2,\ldots,$ for some $\beta > 1/2$ and $\gamma > 0$.

iii) $\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P_n\{x : w_x^{(p)}[1-\delta,1) > \epsilon, \text{ for some } p,\ 1 \leq p \leq q\} = 0$

for all $\epsilon > 0$.

Then $\{P_n = L(X_n)\}_{n=1}^{\infty}$ is weakly convergent □

4. <u>Regularity of processes with independent increments.</u>

<u>Theorem (4.1)</u>: If $X \in C_i^m(\beta,\gamma)$, where $\beta > 1/2$, $\gamma > 0$, then X has a version with sample paths in $D[0,1]^q$.

<u>Proof:</u> Let $\delta_o < 1/2$. For $t \in [0,1]^q$ define:

$$f_{\delta_o}^i(t) = (t_1, \ldots, t_{i-1}, t_i I_{[\delta_o, 1-\delta_o]}(t_i) + \delta_o I_{[0,\delta_o]}(t_i) +$$

$$(1-\delta_o) I_{(1-\delta_o,1]}(t_i),\ t_{i+1}, \ldots, t_q)$$

for all i, $1 \leq i \leq q$.

$$\overline{f}_{\delta_o}(t) = ((1-2\delta_o)^{-1}(t_1-\delta_o), \ldots, (1-2\delta_o)^{-1}(t_i-\delta_o), \ldots, (1-2\delta_o)^{-1}(t_q-\delta_o))$$

$$f_{\delta_o}(t) = (\overline{f}_{\delta_o} \circ f_{\delta_o}^1 \circ f_{\delta_o}^2 \circ \ldots \circ f_{\delta_o}^q)(t).$$

We first prove that the proces $Y_t = X_{f_{\delta_o}(t)}$ on $[0,1]^q$, has a version with sample paths in $D[0,1]^q$.

Observe that $Y \in C_i^{\tilde{m}}(\beta,\gamma)$, if

$$\tilde{m}(.) = m(\overline{f}_{\delta_o}(. \cap [\delta_o, 1-\delta_o]^q))$$

For each n we define a proces Y_n on $[0,1]^q$, constant over each rectangle of the dyadic net of order n, and equal to the value of Y at "south-west" vertex, i.e.:

$$Y_n(t) = Y((i_1 - 1)2^{-n}, \ldots, (i_q - 1)2^{-n})$$

for all $t \in [(i_1 - 1)2^{-n}, i_1 2^{-n}) x \ldots x [(i_q - 1)2^{-n}, i_q 2^{-n})$, where $1 \leqslant i_1 \leqslant 2^n, \ldots, 1 \leqslant i_q \leqslant 2^n$.

We show that $\{Y_n\}_{n=1}^{\infty}$ is a tight sequence. First, an argument like that in the proof of thm. (3.1) shows that

$$\text{(4.1.1)} \qquad \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P\{w''_{Y_n}(\delta) \geqslant \epsilon\} = 0$$

for all $\epsilon > 0$.

In fact: If Z_n is defined over $T^* = [0,1]^{p-1} x [\sigma, \tau] x$ $x [0,1]^{q-p}$ from Y_n, as in thm. (3.1) Y_n is defined from X_n, $Z_n^{m(2)}$ represents the restriction of Z_n to the diadic net, $T^*_{m(2)}$, of T^*, and $\upsilon^{(m)}$ is defined over $T^*_{m(2)}$ like the υ of step 2, in the proof of thm. (2.1), then:

$$P\{M''_p(Z_n) \geqslant \lambda\} = \lim_{m \to \infty} P\{M''_p(Z_n^{m(2)}) \geqslant \lambda\} \leqslant$$

$$\leqslant \lim_{m \to \infty} K \lambda^{-4\gamma} (\upsilon_p^m(\sigma, \tau])^{2\beta} \left(1 - \frac{J_{\upsilon_p^m}(\sigma, \tau]}{\upsilon_p^m(\sigma, \tau]}\right)^{\beta} =$$

$$= K \lambda^{-4\gamma} \widetilde{m}_p^{2\beta}(\sigma, \tau] \left(1 - \frac{J_p(\sigma, \tau]}{\widetilde{m}_p(\sigma, \tau]}\right)^{\beta}.$$

Hence, $\{Y_n\}$ satisfies (3.1.3), and now all follows as in thm. (3.1)'s proof.

If $2^{-k} < \delta$ and $T_{k(2)}$ denote the set of points of the 2^{-k}-dyadic net in $T = [0,1]^q$, then

$$\sup_{t \in T} |Y_n(t)| \leqslant \max_{t \in T_{k(2)}} |Y_n(t)| + q\, w''_{Y_n}(\delta).$$

Moreover, observe that the variables

$$\max_{t \in T_{k(2)}} |Y_n(t)|, \quad n = k, k+1, \ldots$$

are identically distributed. This, together with (4.1.1) gives condition i) of our thm. (3.2). Besides, $\{Y_n\}_{n=1}^{\infty}$ satisfies b) and c) of ii), thm. (3.2), by construction.

Hence, $\{Y_n\}_{n=1}^{\infty}$ is tight. If W is the weak limit of some subsequence, then it is easy to see that W is a version of Y, looking first at dyadic points, and approaching then any point by dyadics.

The application $\bar{f}_{\delta_o}$ being bijective and continuous between $[\delta_o, 1-\delta_o]^q$ and $[0,1]^q$, and $X_t = Y_{(\bar{f}_{\delta_o})^{-1}(t)}$, the theorem is proved. □

Remarks and comments.

a) It will be very interesting to get a result like thm. (2.1) for processes whose increments are not necessarily independent. I don't know at present how to do this.

b) All previous results extend easily to $[0,\infty)^q$-indexed processes using well known results on $D[0,\infty)^q$ (see B.G. Ivanoff (5)).

c) Using above results and some others, (which constitute my Doctoral Thesis, as presented at the Universitat Autònoma de Barcelona, Spain), we can prove the Central Limit Theorem for processes that admit a representation as stochastic integrals w.r.t. Lévy processes with multidimensional time parameter. This will appear elsewhere.

d) Finally I want to express my indebtness and gratitude to Professor E. Giné, that suggested this problems to me and has given efficient help, whenever needed.

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