

A THEOREM ON SCHAUDER DECOMPOSITIONS IN BANACH SPACES

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Abstract. In this paper we prove that in a Banach space all Schauder decompositions are shrinking iff all Schauder decompositions are boundedly complete.

1. Definitions and preliminary results

A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis if for every $x \in X$ there exists a unique sequence $(\alpha_n)_{n=1}^{\infty}$ in \mathbb{R} such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$, and this series converges with respect the norm of X . A sequence $(y_n)_{n=1}^{\infty}$ is called a basic sequence if it is a basis of his closed linear span.

A Schauder decomposition of X is a sequence $(X_i)_{i=1}^{\infty}$ of closed subspaces of X such that for every x in X there exists a unique sequence $(x_i)_{i=1}^{\infty}$ with $x_i \in X_i$ for all i and $x = \sum_{i=1}^{\infty} x_i$. Every Schauder decomposition of X is related with a sequence of continuous projections $P_n: X \rightarrow X$ defined by

$$P_n(x) = P_n\left(\sum_{i=1}^{\infty} x_i\right) = \sum_{i=1}^n x_i$$

In all this paper, the linear span of an element $x \in X$ is denoted by $[x]$ and the closed linear span of the subspaces $(X_i)_{i=n}^m$ ($1 \leq n < m \leq \infty$) is denoted by $[X_i]_{i=n}^m$.

The following theorem characterizes the Schauder decompositions and it can be found in [5].

1. Theorem: Let X be a Banach space and $(X_n)_{n=1}^{\infty}$ a sequence of closed subspaces of X . The following are equivalent:

i) $(X_n)_{n=1}^{\infty}$ is a Schauder decomposition of X .

ii) There exists a sequence $(P_n)_{n=1}^{\infty}$ of continuous projections $P_n: X \rightarrow [X_i]_{i=1}^n$ such that $P_n P_m = P_{\min(m,n)}$ and $\lim_{n \rightarrow \infty} P_n(x) = x$ for every x in X .

iii) There exists a sequence $(P_n)_{n=1}^{\infty}$ of continuous projections $P_n: X \rightarrow [X_i]_{i=1}^n$ such that $P_n P_m = P_{\min(m,n)}$ and $(P_n)_{n=1}^{\infty}$ is uniformly bounded.

To $\sup_n \|P_n\|$ is called norm of the decomposition.

A Schauder decomposition $(X_n)_{n=1}^{\infty}$ in a Banach space X is called boundedly complete if for every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$ such that $\sup_n \left\| \sum_{i=1}^n x_i \right\| < \infty$, the sequence $\left(\sum_{i=1}^n x_i \right)_{n=1}^{\infty}$ converges towards an element x in X . And it is called shrinking if for every $x^* \in X$, $\lim_{n \rightarrow \infty} \|x^*\|_n = 0$, where

$$\|x^*\|_n = \sup \{ |x^*(x)| \text{ with } x \in [X_i]_{i=n+1}^{\infty} \text{ and } \|x\| \leq 1 \}.$$

Boundedly complete and shrinking basis and basic sequences are defined in a similar way.

Singer (cf. [6]) has proved that in a Banach space all basic sequences are boundedly complete if and only if all basic sequences are shrinking. Afterwards Zippin (cf. [7] and [3]) proved a similar theorem for Schauder basis of X . Our purpose in this paper is to prove that in a Banach space all Schauder decompositions are boundedly complete iff all Schauder decompositions are shrinking.

If X is a locally bounded F -space, then there exists p ($0 < p < 1$) such that the topology of X is originated by a p -norm. In this case X is called p -Banach space (cf. [1] and [4]). Let X be a p -Banach space such that X separates points of X and let $J: X \longrightarrow X^{**}$ be the canonical imbedding of X into its bidual. We define in X the norm $\|\cdot\|^{**}$:

$$\|x\|^{**} = \|J(x)\| \quad \text{if } x \in X.$$

The Mackey topology of X is originated by this norm (cf. [2]) and it is called the Mackey norm of X . The Mackey completion of X is denoted by $\overline{J(X)}$.

All the above definitions for Banach spaces can be extended to p -Banach spaces.

2. Shrinking and boundedly complete Schauder decomposition.

2. Lemma. Let $(X_n)_{n=1}^{\infty}$ be a Schauder decomposition of a Banach space X and let $(P_n)_{n=1}^{\infty}$ be its sequence of projections. We suppose that each X_n admits a topological decomposition $X_n = Y_n \oplus Z_n$. The following are equivalent:

- i) $(Y_1, Z_1, \dots, Y_n, Z_n, \dots)$ is a Schauder decomposition of X .
- ii) If A_n is the continuous projection from X_n into Y_n , then
 $\sup_n \|A_n\| < \infty$.

Proof: $i \Rightarrow ii$. If $(Q_n)_{n=1}^{\infty}$ is the sequence of projections of $(Y_1, Z_1, \dots, Y_n, Z_n, \dots)$, as $A_n = Q_{2n-1}|_{X_n}$, the statement ii is proved.

ii \Rightarrow i. If $\sup_n \|A_n\| < \infty$, we define

$$Q_{2n} = P_n$$

$$Q_{2n+1} = P_n + A_n(P_n - P_{n-1}) \quad n > 1$$

$$Q_1 = A_1 P_1$$

and thus $(Q_n)_{n=1}^\infty$ is a uniformly bounded sequence of projections which defines the decomposition $(Y_n, Z_n)_{n=1}^\infty$ because of theorem 1. //

Remark that if any of the previous subspaces is 0, it must be taken away in the decomposition.

3. Corollary. Let $(X_n)_{n=1}^\infty$ be a Schauder decomposition of a Banach space X and $(X_n)_{n=1}^\infty$ a normalized sequence in X with $x_n \in X_n$. For every n there exists an hyperplane W_n of X_n such that $([x_1], W_1, \dots, [x_n], W_n, \dots)$ is a Schauder decomposition of X .

Proof: As $\|x_n\| = 1$, we can define $A_n(x) = u_n^*(x)x_n$, where $u_n^* \in X_n^*$ and $u_n^*(x_n) = \|u_n^*\| = 1$. //

4. Lemma. Let X be a Banach space and a Schauder decomposition of the form $([y_1], W_1, \dots, [y_n], W_n, \dots)$ where $(y_n)_{n=1}^\infty$ satisfies $\inf_n \|y_n\| = C > 0$ and $\sup_n \|\sum_{i=1}^n y_i\| = M < \infty$. We define the sequence $(v_n)_{n=1}^\infty$ by $v_n = \sum_{i=1}^n y_i$. Then $([v_1], W_1, \dots, [v_n], W_n, \dots)$ is a Schauder decomposition of X .

Proof: Let $(P_n)_{n=1}^\infty$ be the sequence of projections of $(X_n)_{n=1}^\infty$ and let K be its norm. Each $P_{2n-1} - P_{2n-2}$ (the projection over $[y_n]$) is originated by a $y_n^* \in X^*$ according to

$$(P_{2n-1} - P_{2n-2})(x) = y_n^*(x)y_n \quad \text{if } x \in X.$$

and thus

$$i) y_n^*(y_m) = \delta_{n,m}$$

$$ii) \|y_n^*\| \|y_n\| \leq 2K \quad \text{for every } n, \text{ and}$$

$$iii) y_n^*|_{W_m} = 0 \quad \text{for every } n \text{ and } m.$$

As $\inf_n \|y_n\| = C > 0$, from ii) we obtain that $\sup_n \|y_n^*\| \leq 2K/C$.

Let $(v_n^*)_{n=1}^\infty$ be defined by $v_n^* = y_n^* - y_{n+1}^*$. It is easy to check that

$$v_n^*(v_m) = \delta_{n,m}.$$

We define the sequence of projections by

$$A_{2n}(x) = \sum_{k=1}^n (P_{2k} - P_{2k-1})(x) + \sum_{k=1}^n v_k^*(x)v_k$$

$$A_{2n+1}(x) = A_{2n}(x) + v_{n+1}^*(x)v_{n+1}.$$

Because of the theorem 1 we only need to prove that $(A_n)_{n=1}^\infty$ is uniformly bounded, and, because of the last considerations, it shall be proved if we prove that $\sup_n \|A_{2n}\| < \infty$:

$$\begin{aligned} \|A_{2n}(x)\| &= \|P_{2n}(x) - \sum_{k=1}^n y_k^*(x)y_k + \sum_{k=1}^n (y_k^*(x) - y_{k+1}^*(x))(\sum_{i=1}^k y_i)\| \leq \\ &\leq K \|x\| + \left\| - \sum_{k=1}^n y_k^*(x)y_k + \sum_{k=1}^n (y_k^*(x) - y_{n+1}^*(x))y_k \right\| \leq \\ &\leq K \|x\| + \|y_{n+1}^*\| \|x\| \left\| \sum_{k=1}^n y_k \right\| \leq (K + \frac{2K}{C} M) \|x\|. \end{aligned}$$

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5. Lemma. Let X be a Banach space and $([y_1], w_1, \dots, [y_n], w_n, \dots)$ a Schauder decomposition of X , where $(y_n)_{n=1}^{\infty}$ satisfies $\sup_n \|y_n\| = M < \infty$. We define the sequence $(v_n)_{n=1}^{\infty}$ by $v_1 = y_1$ and $v_n = y_n - y_{n-1}$. Then the following are equivalent:

i) $([v_1], w_1, \dots, [v_n], w_n, \dots)$ is a Schauder decomposition of X .

ii) There exists $x^* \in X^*$ such that

a) $x^*(y_n) = 1$ for every n

b) $x^*|_{w_m} = 0$ for every m

Proof: If, for every n , there is a continuous projection from X into $[v_n]$ parallel to the other subspaces, the existence of $x^* \in X^*$ satisfying a) and b) is necessary. We suppose that there exists such x^* . We define $(y_n^*)_{n=1}^{\infty}$ as in the preceding lemma, and if we consider the sequence

$$v_1^* = x^* \quad \text{and} \quad v_n^* = x^* - \sum_{k=1}^{n-1} y_k^*$$

the orthogonal relations $v_n^*(v_m) = \delta_{n,m}$ hold.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of projections as in the preceding lemma.

We must prove that $\sup_n \|A_n\| < \infty$. For every m , $x^*|_{w_m} = 0$, and hence

$x^*(x) = \sum_{n=1}^{\infty} y_n^*(x)$ for every x in X and so $(v_n^*)_{n=1}^{\infty}$ converges weakly to 0 and $\sup_n \left\| \sum_{k=1}^n y_k \right\| = M_1 < \infty$. Also $\sup_n \|v_n\| \leq 2M$, again we must only prove that $\sup_n \|A_{2n}\| < \infty$:

we have

$$\sum_{k=1}^n v_k^*(x) v_k = x^*(x) + \sum_{k=2}^n x^*(x)(y_k - y_{k-1}) - \sum_{k=2}^n \left[\left(\sum_{i=1}^{k-1} y_i^*(x) \right) (y_k - y_{k-1}) \right] =$$

$$\begin{aligned}
&= x^*(x)y_1 + x^*(x)(y_n - y_1) - \sum_{k=1}^{n-1} y_k^*(x) y_n + \sum_{k=1}^{n-1} y_k^*(x) y_k = \\
&= \sum_{k=1}^n y_k^*(x) y_k + x^*(x) y_n - \sum_{k=1}^n y_k^*(x) y_n,
\end{aligned}$$

and thus

$$A_{2n}(x) = P_{2n}(x) + x^*(x)y_n - \sum_{k=1}^n y_k^*(x)y_n.$$

And finally:

$$\|A_{2n}(x)\| \leq \|P_{2n}\| \|x\| + \|x^*\| \|x\| \|y_n\| + \left\| \sum_{k=1}^n y_k^* \right\| \|x\| \|y_n\|$$

and

$$\|A_{2n}\| \leq K + M \|x^*\| + M_1 M$$

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Now we can prove the main result:

6. Theorem. Let X be a Banach space. The following statements are equivalent:

- i) All Schauder decompositions of X are shrinking
- ii) All Schauder decompositions of X are boundedly complete

Proof. $i \Rightarrow ii$. Let $(X_n)_{n=1}^\infty$ be a non boundedly complete Schauder decomposition of X . There exists then a sequence $(x_i)_{i=1}^\infty$ with $x_i \in X_i$ such that $\sup_n \left\| \sum_{i=1}^n x_i \right\| = 1$ and $\left(\sum_{i=1}^n x_i \right)_{n=1}^\infty$ is not a Cauchy sequence, and thus, there exist ϵ and a strictly increasing sequence $(m_k)_{k=1}^\infty$ such that $\epsilon < \left\| \sum_{i=m_{k-1}+1}^{m_k} x_i \right\| \leq 2$ for every k . We define $Y_k = [X_i]_{i=m_{k-1}+1}^{m_k}$ and $y_k = \sum_{i=m_{k-1}+1}^{m_k} x_i$ if $k \geq 1$. $(Y_k)_{k=1}^\infty$ is a Schauder decomposition of X with $y_k \in Y_k$. Because of the corollary 3, for each k there exists a hyperplane W_k of Y_k such that $([y_1], W_1, \dots$

..., $[y_n], w_n, \dots$) is a Schauder decomposition. Because of the lemma 4, the sequence $(v_k)_{k=1}^\infty$ defined by $v_n = \sum_{i=1}^n y_i$ originates the Schauder decomposition $([v_1], w_1, \dots, [v_n], w_n, \dots)$ which is not shrinking because of $y_1^*(v_k) = 1$ for every $k \geq 1$.

ii \Rightarrow i. Let $(X_n)_{n=1}^\infty$ be a non shrinking Schauder decomposition of X . There exist then $x^* \in X$ with $\|x^*\| = 1$, $\epsilon > 0$, a strictly increasing sequence of index $(m_k)_{k=1}^\infty$ and a sequence $(y_k)_{k=1}^\infty$ with $y_k \in Y_k = [X_i]_{i=m_{k-1}+1}^{m_k}$ such that:

$$a) 1 \leq \|y_n\| \leq 1/\epsilon$$

$$b) x^*(y_n) = 1.$$

We can choose the hyperplane $w_k = Y_k \cap \text{Ker } x^*$ and using the lemma 5, if $v_1 = y_1$ and $v_n = y_n - y_{n-1}$, then $([v_1], w_1, \dots, [v_n], w_n, \dots)$ is a Schauder decomposition of X which is not boundedly complete because of

$$\left\| \sum_{k=2}^n v_k \right\| = \left\| \sum_{k=2}^n (y_k - y_{k-1}) \right\| = \|y_n - y_1\| \leq 2/\epsilon$$

while

$$\|v_k\| = \|y_k - y_{k-1}\| \geq \frac{1}{K} \|y_{k-1}\|$$

where K is the norm of $(X_n)_{n=1}^\infty$.

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With certain modifications, this theorem has an extension to p -Banach spaces (if its dual separates points). The Mackey topology of this spaces plays an important role in this extension. We need before a definition: we shall say that a Schauder decomposition $(X_n)_{n=1}^\infty$ in a p -Banach space is an almost boundedly complete decomposition if for every sequence $(x_n)_{n=1}^\infty$ with $x_n \in X_n$ such that $\sup_n \left\| \sum_{k=1}^n x_k \right\| < \infty$, the sequence $(\sum_{k=1}^n x_k)_{n=1}^\infty$ converges in $(\overline{J(X)}, \|\cdot\|^{**})$.

We must point out that if $(X_n)_{n=1}^{\infty}$ is boundedly complete then it is also almost boundedly complete. Almost boundedly complete basis of X are defined in a similar way.

7. Theorem. Let X be a p -Banach space. The following are equivalent:

- i) All Schauder decompositions of X are shrinking.
- ii) All Schauder decompositions of X are almost boundedly complete.

Proof: Similar to the proof of Theorem 6, and it can be found in [2]

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