

ON KÜNNETH RELATIONS

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Abstract. The aim of this note is to subsume a number of apparently quite distinct results in one general theorem. For a left exact functor $T : R\text{-Mod} \rightarrow \text{Ab}$ and a cochain complex C^* we give a long exact sequence including the canonical map $H^n T \rightarrow TH^n$, where H^n is the n -th cohomology functor. Under the appropriate hypothesis the usual form of the Künneth relation (see [1], chap. VI) is a special case of our long exact sequence (Remark 1.2). Also the latest results of Coelho-Pezennec (see [2]) are contained in this long sequence (Proposition 2.3).

In particular, we obtain a simple proof of the following results of Osofsky on upper bounds of cohomological dimensions (see [7], [8]). If I is a directed set and the cardinal number of it is no greater than $\aleph_m$, then:

1. $l.\text{pd}_{\text{colim}_I R_i} \text{colim}_I M_i \leq \sup_I l.\text{pd}_{R_i} M_i + m + 1$
in the category of modules,
2. $\text{cd}_R \text{colim}_I G_i \leq \sup_I \text{cd}_R G_i + m + 1$ in the category of

groups.

1. Main results. Let R be a ring and let $R\text{-Mod}$ and Ab denote the category of left R -modules and the category of abelian groups, respectively. For a left exact functor $T: R\text{-Mod} \rightarrow \text{Ab}$ denote by T^n the right n -th derived functor of T and for a cochain complex $C^* = (C^n, d^n)$ of R -modules we put $Z^n = \text{Ker } d^n$, $B^n = \text{Im } d^{n-1}$ and $H^n = Z^n/B^n$, where n runs over the integers.

Theorem 1.1. (General theorem). If $T^k C^n = 0$ for $k \geq 1$ and all integers n , then there exists a long exact sequence of abelian groups

$$\begin{aligned} 0 \rightarrow T^1 Z^{n-1} \rightarrow H^n T \rightarrow T H^n \rightarrow T^2 Z^{n-1} \rightarrow \\ \rightarrow T^1 Z^n \rightarrow T^1 H^n \rightarrow T^3 Z^{n-1} \rightarrow \dots \rightarrow \\ \rightarrow T^k Z^n \rightarrow T^k H^n \rightarrow T^{k+2} Z^{n-1} \rightarrow \dots \end{aligned}$$

Proof. The canonical short exact sequence of R -modules $0 \rightarrow Z^n \xrightarrow{i^n} C^n \xrightarrow{d^n} B^{n+1} \rightarrow 0$ yields a long exact sequence of abelian groups

$$\begin{aligned} 0 \rightarrow T Z^n \xrightarrow{T i^n} T C^n \xrightarrow{T d^n} T B^{n+1} \xrightarrow{\alpha_0^n} T^1 Z^n \xrightarrow{T^1 i^n} T^1 C^n \xrightarrow{T^1 d^n} \\ \rightarrow T^1 B^{n+1} \xrightarrow{\alpha_1^n} \dots \xrightarrow{\alpha_{k-1}^n} T^k Z^n \xrightarrow{T^k i^n} T^k C^n \xrightarrow{T^k d^n} T^k B^{n+1} \xrightarrow{\alpha_k^n} \dots \end{aligned}$$

where $\alpha_k^n: T^k B^{n+1} \rightarrow T^{k+1} Z^n$ is the connecting map.

By assumption $T^k C^n = 0$ for $k \geq 1$ and all integers n . So we get a short exact sequence

$$a) \quad 0 \longrightarrow TZ^n \xrightarrow{Ti^n} TC^n \xrightarrow{Td^n} TB^{n+1} \xrightarrow{\alpha_0^n} T^1 Z^n \longrightarrow 0$$

and a family of isomorphisms

$$b) \quad T^{k+1} B^n \xrightarrow[\approx]{\alpha_k^n} T^{k+1} Z^n \quad \text{for } k \geq 1 \text{ and all integers } n.$$

The sequence a) yields an isomorphism

$$a') \quad TB^{n+1} / \text{Im } Td^n \xrightarrow[\approx]{\alpha_0^n} T^1 Z^n.$$

Moreover, the canonical short exact sequence of R-modules $0 \longrightarrow B^n \xrightarrow{j^n} Z^n \xrightarrow{\beta^n} H^n \longrightarrow 0$ yields a long exact sequence of abelian groups

$$c) \quad 0 \longrightarrow TB^n \xrightarrow{Tj^n} TZ^n \xrightarrow{T\beta^n} TH^n \xrightarrow{\gamma_0^n} T^1 B^n \xrightarrow{T^1 j^n} T^1 Z^n \longrightarrow \\ \longrightarrow T^1 \beta^n \xrightarrow{\gamma_1^n} T^1 H^n \xrightarrow{\gamma_2^n} \dots \xrightarrow{\gamma_{k-1}^n} T^k B^n \xrightarrow{T^k j^n} T^k Z^n \xrightarrow{T^k \beta^n} T^k H^n \xrightarrow{\gamma_k^n} \dots,$$

where $\gamma_k^n : T^k H^n \longrightarrow T^{k+1} B^n$ is the usual connecting map. Hence, by b) we obtain the following long exact sequence of abelian groups

$$0 \longrightarrow TZ^n / TB^n \xrightarrow{\overline{T\beta^n}} TH^n \xrightarrow{\alpha_1^{n-1} \circ \gamma_0^n} T^2 Z^{n-1} \xrightarrow{T^1 j^n (\alpha_1^{n-1})^{-1}} T^1 Z^n \xrightarrow{T^1 \beta^n} \\ \longrightarrow T^1 H^n \xrightarrow{\alpha_2^{n-1} \circ \gamma_1^n} \dots \xrightarrow{\alpha_k^{n-1} \circ \gamma_{k-1}^n} T^{k+1} Z^{n-1} \xrightarrow{T^k j^n (\alpha_k^{n-1})^{-1}} T^k Z^n \longrightarrow \\ \longrightarrow T^k \beta^n \xrightarrow{\alpha_{k+1}^{n-1} \circ \gamma_k^n} \dots$$

The functor $T : R\text{-Mod} \rightarrow \text{Ab}$ is left exact, hence $\text{Ker } Td^n = TZ^n$ and the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Im } \overset{0}{\downarrow} Td^{n-1} & \xrightarrow{j'} & \text{Ker } Td^n & \xrightarrow{\beta'} & H^n T \longrightarrow 0 \\
 & & \downarrow \varphi & & \parallel & & \downarrow \psi \\
 0 & \longrightarrow & TB^n & \xrightarrow{Tj^n} & TZ^n & \xrightarrow{\delta} & TZ^n / TB^n \longrightarrow 0 \\
 & & & & & & \downarrow 0
 \end{array}$$

yields $\text{Ker } \psi = \text{coKer } \varphi = TB^n / \text{Im } Td^{n-1}$, by the Snake Lemma. Let $\eta : \text{Ker } \psi \rightarrow H^n T$ be the canonical inclusion. Then, finally, we obtain the long exact sequence

$$\begin{aligned}
 0 \rightarrow T^1 Z^{n-1} & \xrightarrow{\eta \cdot (\alpha_0^{n-1})^{-1}} H^n T \xrightarrow{T\beta^n \circ \psi} TH^n \xrightarrow{\alpha_1^{n-1} \circ \gamma_0^n} T^2 Z^{n-1} \xrightarrow{T^1 j^n \circ (\alpha_1^{n-1})^{-1}} \\
 & \rightarrow T^1 Z^n \xrightarrow{T^1 \beta^n} T^1 H^n \xrightarrow{\alpha_2^{n-1} \circ \gamma_1^n} T^3 Z^{n-1} \xrightarrow{T^2 j^n \circ \alpha_2^{n-1}} \dots \\
 & \xrightarrow{T^k j^n \circ (\alpha_k^{n-1})^{-1}} T^k Z^n \xrightarrow{T^k \beta^n} T^k H^n \xrightarrow{\alpha_{k+1}^n \circ \gamma_k^n} T^{k+2} Z^{n-1} \xrightarrow{T^{k+1} j^n \circ (\alpha_{k+1}^{n-1})^{-1}} \dots \blacksquare
 \end{aligned}$$

As a corollary we get the usual form of the Künneth relation (see [1], chap. VI).

Remark 1.2. If $T^k = 0$ for $k \geq 2$, then $T^1 Z^n \xrightarrow{T^1 \beta^n} T^1 H^n$ and the short sequence

$$0 \longrightarrow T^1 H^{n-1} \xrightarrow{\eta \cdot (\alpha_0^{n-1})^{-1} \circ (T^1 \beta^{n-1})^{-1}} H^n T \xrightarrow{T\beta^n \circ \psi} TH^n \longrightarrow 0$$

is exact.

Moreover, the above Theorem yields the following result.

Corollary 1.3. If the maps $T^k_Z^n \xrightarrow{T^k_\beta^n} T^k_H^n$ induced by the canonical map $Z^n \xrightarrow{\beta^n} H^n$ are left split (i.e. there exists a map $\rho^n : T^k_H^n \rightarrow T^k_Z^n$ such that $\rho^n \circ T^k_\beta^n = \text{id}_{T^k_Z^n}$), then there exists a long exact sequence of abelian groups

$$\begin{aligned} \dots \rightarrow T^{2k+1}_H^{n-k-1} \rightarrow \dots \rightarrow T^{3}_H^{n-2} \rightarrow T^1_H^{n-1} \rightarrow H^n_T \rightarrow TH^n \rightarrow \\ \rightarrow T^{2}_H^{n-1} \rightarrow T^4_H^{n-2} \rightarrow \dots \rightarrow T^{2k}_H^{n-k} \rightarrow \dots \end{aligned}$$

Proof. In virtue of assumption the sequence c) from the proof of Theorem 1.1 determines the short exact sequence

$$0 \rightarrow TZ^n / TB^n \xrightarrow{\overline{T\beta^n}} TH^n \xrightarrow{\gamma_0^n} T^1_B^n \rightarrow 0$$

and the split short exact sequences

$$0 \rightarrow T^k_Z^n \xrightleftharpoons[\rho^n]{T^k_\beta^n} T^k_H^n \xrightleftharpoons[\delta^n]{\gamma_k^n} T^{k+1}_B^n \rightarrow 0$$

for $k \geq 1$ and all integers n .

Hence, using the isomorphisms b) from Theorem 1.1 we obtain the following diagram

$$\begin{array}{c}
\vdots \\
0 \rightarrow T^{2k+3} Z^{n-k-2} \xrightarrow{\delta \frac{n-k-2}{\alpha_{2k+2}}^{-1}} T^{2k+1} H^{n-k-1} \xrightarrow{\rho^{n-k-1}} T^{2k+1} Z^{n-k-1} \rightarrow 0 \\
\vdots \\
0 \rightarrow T^3 Z^{n-2} \xrightarrow{\delta \frac{n-1}{\alpha_2} \frac{n-2}{\alpha_2}^{-1}} T^1 H^{n-1} \xrightarrow{\rho^{n-1}} T^1 Z^{n-1} \rightarrow 0 \\
0 \rightarrow T^1 Z^{n-1} \xrightarrow{\eta \cdot (\alpha_0^{n-1})^{-1}} H^n_T \xrightarrow{\psi} T Z^n / T B^n \rightarrow 0 \\
0 \rightarrow T Z^n / T B^n \xrightarrow{T \beta^n} T H^n \xrightarrow{\alpha_1^{n-1} \gamma_0^n} T^2 Z^{n-1} \rightarrow 0 \\
0 \rightarrow T^2 Z^{n-1} \xrightarrow{T^2 \beta^{n-1}} T^2 H^{n-1} \xrightarrow{\alpha_3^{n-2} \gamma_2^{n-1}} T^4 Z^{n-2} \rightarrow 0 \\
0 \rightarrow T^4 Z^{n-2} \xrightarrow{T^4 \beta^{n-2}} T^4 H^{n-2} \xrightarrow{\alpha_5^{n-3} \gamma_4^{n-2}} T^6 Z^{n-3} \rightarrow 0 \\
\vdots \\
0 \rightarrow T^{2k} Z^{n-k} \xrightarrow{T^{2k} \beta^{n-k}} T^{2k} H^{n-k} \xrightarrow{\alpha_{2k+1}^{n-k} \gamma_{2k}^{n-k}} T^{2k+2} Z^{n-k-1} \rightarrow 0 \\
\vdots
\end{array}$$

Composing the above short exact sequences we obtain the announced long exact sequence of abelian groups. $\square$

2. Applications. Let I be a directed set. It is well known that the functor colim_I is exact. Moreover, if the cardinal number of I is no greater than $\aleph_m$, then $\lim_I^{m+k} = 0$ for $k \geq 2$ (see [4]).

Let $\{R_i, \varphi_{ij}\}_{i,j \in I}$ and $\{M_i, \psi_{ij}\}_{i,j \in I}$ be directed systems of rings and abelian groups respectively, such that

each M_i is a left R_i -module and $\psi_{ij}(r_i m_j) = \varphi_{ij}(r_i) \psi_{ij}(m_j)$ for $r_i \in R_i$, $m_j \in M_j$ and $i < j$.

(Such systems will be called *consistent*).

Then, $M = \operatorname{colim}_I M_i$ is a left $R = \operatorname{colim}_I R_i$ -module and

$M \approx \operatorname{colim}_I R \otimes_{R_i} M_i$ in the category of R -Mod.

For further purposes the following lemmas will be useful.

Lemma 2.1. If each M_i is a (pure) projective R_i -module for all $i \in I$, then

$$\lim_I^n \operatorname{Hom}_{R_i}(M_i, N) \approx \operatorname{Ext}_R^n(\operatorname{colim}_I M_i, N) (\approx \operatorname{Pext}_R^n(\operatorname{colim}_I M_i, N))$$

for any R -module N .

Proof. A directed system $\{M_i, \psi_{ij}\}_{i,j \in I}$ yields an exact sequence of R -modules (see [3], Appendix I)

$$\begin{aligned} \dots \longrightarrow \bigoplus_{i_0 < \dots < i_n} R \otimes_{R_{i_0}} M_{i_0} \longrightarrow \dots \longrightarrow \bigoplus_{i_0 < i_1} R \otimes_{R_{i_0}} M_{i_0} \longrightarrow \bigoplus_{i \in I} R \otimes_{R_{i_0}} M_{i_0} \\ \longrightarrow \operatorname{colim}_I R \otimes_{R_i} M_i \approx \operatorname{colim}_I M_i. \end{aligned}$$

If each M_i is a (pure) projective R_i -module for all $i \in I$ then the above sequence is an R -(pure) projective resolution of $\operatorname{colim}_I M_i$.

Applying the functor $\operatorname{Hom}_R(-, N)$ we obtain the following chain complexes:

$$0 \longrightarrow \text{Hom}_R \left(\bigoplus_{i \in I} R \otimes_{R_i} M_i, N \right) \longrightarrow \text{Hom}_R \left(\bigoplus_{i_0 < i_1} R \otimes_{R_{i_0}} M_{i_0} \right) \longrightarrow \dots$$

$$0 \longrightarrow \prod_{i \in I} \text{Hom}_{R_i} (M_i, N) \longrightarrow \prod_{i_0 < i_1} \text{Hom}_{R_{i_0}} (M_{i_0}, N) \longrightarrow \dots$$

Consequently, $\lim_I^n \text{Hom}_{R_i} (M_i, N) \approx \text{Ext}_R^n (\text{colim}_I M_i, N)$
 $(\approx \text{Pext}_R^n (\text{colim}_I M_i, N)).$

Let F_{M_i} denotes the free R_i -module generated by the elements of M_i , then $F_{\text{colim}_I M} \approx \text{colim}_I F_{M_i}$. Hence, we obtain the following generalization of Lemma 9.5 from [1].

Lemma 2.2. There exist R_i -(pure) projective resolutions $\mathbb{P}_i$ of M_i forming a consistent directed system $\{\mathbb{P}_i, \mathbb{P}_{ij}\}_{i,j \in I}$ such that $\mathbb{P} = \text{colim}_I \mathbb{P}_i$ is an R -(pure) projective resolution of $\text{colim}_I M_i$.

The two lemmas stated above will be used in the sequel.

Let $\{C_*^i, \psi_{ij}\}_{i,j \in I}$ be a consistent directed system of chain complexes such that C_n^i are R_i -modules for all $i \in I$. Put $C_* = \text{colim}_I C_*^i$ and $Z_n^i = \text{coKer } d_n^i$.

Then the following generalization of the Coelho-Pezennec result is a simple consequence of Theorem 1.1 and Lemma 2.1.

Proposition 2.3. (see [2]). If C_n^i are (pure) projective R_i -modules for all integers n , then the following long

sequence

$$\begin{aligned}
0 &\longrightarrow \lim_I^1 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow H^n(C_*, N) \longrightarrow \lim_I H^n(C_*^i, N) \longrightarrow \\
&\longrightarrow \lim_I^2 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow \lim_I^1 \operatorname{Hom}_{R_i}(Z_n^i, N) \longrightarrow \lim_I^1 H^n(C_*^i, N) \longrightarrow \\
&\longrightarrow \lim_I^3 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow \dots \longrightarrow \\
&\longrightarrow \lim_I^k \operatorname{Hom}_{R_i}(Z_n^i, N) \longrightarrow \lim_I^k H^n(C_*^i, N) \longrightarrow \lim_I^{k+2} \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow \\
&\longrightarrow \dots \quad \text{is exact.}
\end{aligned}$$

Moreover, as direct consequences of this Proposition we obtain the results of Osofsky (see [7] and [8]) and Kiełpiński-Simson (see [6]).

Let $l.\operatorname{pd}_R M$ ($l.P.\operatorname{pd}_R M$) denote the left (pure) projective dimension of an R -module M and let $l.\operatorname{gl dim} R$ ($l.P.\operatorname{gl dim} R$) denote the left (pure) global dimension of a ring R .

Corollary 2.4. i) $l.\operatorname{pd}_{\operatorname{colim}_I R_i} \operatorname{colim}_I M_i \leq$

$$\leq \sup_I l.\operatorname{pd}_{R_i} M_i + m + 1$$

$$(l.P.\operatorname{pd}_{\operatorname{colim}_I R_i} \operatorname{colim}_I M_i \leq \sup_I l.P.\operatorname{pd}_{R_i} M_i + m + 1)$$

and

$$ii) l.\operatorname{gl dim} \operatorname{colim}_I R_i \leq \sup_I l.\operatorname{gl dim} R_i + m + 1$$

$$(l.p.gl \dim \operatorname{colim}_I R_i \leq \sup_I l.gl \dim R_i + m + 1).$$

Proof. Applying Proposition 2.3 to the directed system $\{\mathbb{P}_i, \psi_{ij}\}_{i,j \in I}$ of projective resolutions of $\{M_i, \psi_{ij}\}_{i,j \in I}$ given by Lemma 2.2 we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow \lim_I^1 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) &\longrightarrow \operatorname{Ext}_R^n(\operatorname{colim}_I M_i, N) \longrightarrow \lim_I \operatorname{Ext}_{R_i}^n(M_i, N) \longrightarrow \\ &\longrightarrow \lim_I^2 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow \lim_I^1 \operatorname{Hom}_{R_i}(Z_n^i, N) \longrightarrow \lim_I^1 \operatorname{Ext}_{R_i}^n(M_i, N) \longrightarrow \\ &\longrightarrow \lim_I^3 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow \dots \longrightarrow \\ &\longrightarrow \lim_I^k \operatorname{Hom}_{R_i}(Z_n^i, N) \longrightarrow \lim_I^k \operatorname{Ext}_{R_i}^n(M_i, N) \longrightarrow \lim_I^{k+2} \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \longrightarrow \\ &\longrightarrow \dots, \text{ where } R = \operatorname{colim}_I R_i. \end{aligned}$$

Hence, for $n > \sup_I l.pd_{R_i} M_i$ we have the following isomorphisms

$$\lim_I^2 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \approx \lim_I^1 \operatorname{Hom}_{R_i}(Z_n^i, N)$$

.....

$$\lim_I^k \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \approx \lim_I^{k-1} \operatorname{Hom}_{R_i}(Z_n^i, N).$$

Therefore, for $n-k > \sup_I l.pd_{R_i} M_i$

$$\lim_I^1 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) \approx \dots \approx \lim_I^k \operatorname{Hom}_{R_i}(Z_{n-k}^i, N).$$

But $\lim_I^k = 0$ for $k > m + 1$. Consequently,

$$\lim_I^1 \operatorname{Hom}_{R_i}(Z_{n-1}^i, N) = 0 \quad \text{and} \quad \operatorname{Ext}_R^n(\operatorname{colim}_I M_i, N) = 0 \quad \text{for}$$

$n > \sup_I 1.\operatorname{pd}_{R_i} M_i + m + 1$. Hence,

$$1.\operatorname{pd}_{\operatorname{colim}_I R_i} \operatorname{colim}_I M_i \leq \sup_I 1.\operatorname{pd}_{R_i} M_i + m + 1.$$

ii) For any R -module M we have $M = \operatorname{colim}_I M_i$, where $M_i = M$ are R_i -modules for all $i \in I$. Therefore, by i)

$$1.\operatorname{pd}_{\operatorname{colim}_I R_i} M \leq \sup_I 1.\operatorname{pd}_{R_i} M + m + 1 \leq \sup_I 1.\operatorname{gl} \dim R_i + m + 1$$

and hence $1.\operatorname{gl} \dim \operatorname{colim}_I R_i \leq \sup_I 1.\operatorname{gl} \dim R_i + m + 1$.

The analogous results for the left (pure) projective and global dimension are obtained by the same methods. ■

In particular, if $\{G_i, \varphi_{ij}\}_{i,j \in I}$ is a directed system of groups, then for group-rings over a ring R we have $R[\operatorname{colim}_I G_i] \approx \operatorname{colim}_I R[G_i]$.

So, by the above Corollary $\operatorname{pd}_{\operatorname{colim}_I R[G_i]} \Delta R \leq \sup_I \operatorname{pd}_{R[G_i]} \Delta R + m + 1$, where ΔR denotes the trivial module over the appropriate group-ring.

Therefore, we get another result due to Osofsky (see [8])

$$\text{cd}_R \text{colim}_I G_i \leq \sup_I \text{cd}_R G_i + m + 1, \text{ where}$$

cd_R denotes the R -cohomological dimension.

More generally, if $\{C_i \varphi_{ij}\}_{i,j \in I}$ is a directed system of small categories, then using methods similar to those above, we obtain

$$\text{cd}_R \text{colim}_I C_i \leq \sup_I \text{cd}_R C_i + m + 1.$$

Remark 2.5. By results from [5] and [9] we can replace the directed set I by any small category such that the functor colim_I is exact.

Now let R be a hereditary ring and let $\{C_*^i, \psi_{ij}\}_{i,j \in I}$ be a directed system of chain complexes over the category $R\text{-mod}$.

Proposition 2.6. If C_*^i and $C_* = \text{colim}_I C_*^i$ are chain complexes of projective R -modules for all $i \in I$, then the following long sequence

$$\begin{aligned} \dots \rightarrow \lim_I^{2k+1} H^{n-k-1}(C_*^i, N) \rightarrow \dots \rightarrow \lim_I^3 H^{n-2}(C_*^i, N) \rightarrow \\ \rightarrow \lim_I^1 H^{n-1}(C_*^i, N) \rightarrow H^n(C_*, N) \rightarrow \lim_I H^n(C_*^i, N) \rightarrow \\ \rightarrow \lim_I^2 H^{n-1}(C_*^i, N) \rightarrow \lim_I^4 H^{n-2}(C_*^i, N) \rightarrow \dots \rightarrow \lim_I^{2k} H^{n-k}(C_*^i, N) \rightarrow \\ \rightarrow \dots \text{ is exact for all integers } n \text{ and any } R\text{-module } M. \end{aligned}$$

Proof. Because $\{Z^n \text{Hom}_R(C_*^i, N)\}_{i \in I} = \{\text{Hom}_R(C_n^i / B_n^i, N)\}_{i \in I}$
and the sequence

$$0 \longrightarrow H_n C_*^i \longrightarrow C_n^i / B_n^i \longrightarrow C_n^i / Z_n^i \longrightarrow 0 \text{ splits,}$$

therefore

$$\{\text{Hom}_R(C_n^i / B_n^i, N)\}_{i \in I} = \{\text{Hom}_R(H_n C_*^i, N)\}_{i \in I} \oplus \{\text{Hom}_R(C_n^i / Z_n^i, N)\}_{i \in I}$$

and

$$\lim_I^k Z^n \text{Hom}_R(C_*^i, N) = \lim_I^k \text{Hom}_R(H_n C_*^i, N) \oplus \lim_I^k \text{Hom}_R(C_n^i / Z_n^i, N).$$

But C_n^i / Z_n^i are projective R -modules and $\text{colim}_I C_n^i / Z_n^i = C_n / Z_n$
is a projective R -module, so by Lemma 2.1

$$\lim_I^k \text{Hom}_R(C_n^i / Z_n^i, N) = \text{Ext}_R^k(C_n / Z_n, N) = 0 \text{ for } k \geq 1.$$

Moreover, by Universal Coefficient Theorem (see [1] chap. VI) we have natural epimorphisms $H^n(C_*^i, N) \longrightarrow \text{Hom}_R(H_n C_*^i, N)$
for all $i \in I$. Consequently, the map

$$\lim_I^k Z^n \text{Hom}_R(C_*^i, N) = \lim_I^k \text{Hom}_R(H_n C_*^i, N) \longrightarrow \lim_I^k H^n(C_*^i, N)$$

splits and an appropriate long exact sequence is determined by Corollary 1.2. ■

Corollary 2.7. If $\{X_i, \varphi_{ij}\}_{i,j \in I}$ is a directed system
of compact topological spaces, then the cochain functor commutes

with limits. Thus, the following sequence of singular cohomology groups

$$\begin{aligned} 0 \longrightarrow \lim_I^{2n-1} H^0(X_i, A) \longrightarrow \dots \longrightarrow \lim_I^3 H^{n-2}(X_i, A) \longrightarrow \\ \longrightarrow \lim_I^1 H^{n-1}(X_i, A) \longrightarrow H^n(X, A) \longrightarrow \lim_I H^n(X_i, A) \longrightarrow \\ \longrightarrow \lim_I^2 H^{n-1}(X_i, A) \longrightarrow \lim_I^4 H^{n-2}(X_i, A) \longrightarrow \dots \longrightarrow \lim_I^{2n} H^0(X_i, A) \longrightarrow 0 \end{aligned}$$

is exact for any abelian group A , where $X = \operatorname{colim}_I X_i$.

Similary, if $\{G_i, \varphi_{ij}\}_{i,j \in I}$ and $\{C_i, \varphi_{ij}\}_{i,j \in I}$ are directed systems of groups and small categories respectively, then we obtain the appropriate long exact sequence as above.

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