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POINTWISE LOCALIZATION AND SHARP WEIGHTED BOUNDS FOR RUBIO DE FRANCIA SQUARE FUNCTIONS

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Abstract: Let $H_\omega f$ be the Fourier restriction of $f \in L^2(\mathbb{R})$ to an interval $\omega \subset \mathbb{R}$. If Ω is an arbitrary collection of pairwise disjoint intervals, the square function of $\{H_\omega f : \omega \in \Omega\}$ is termed the Rubio de Francia square function T^{Ω}_{RF}. This article proves a pointwise bound for T^{Ω}_{RF} by a sparse operator involving local L^2-averages. A pointwise bound for the smooth version of T^{Ω}_{RF} by a sparse square function is also proved. These pointwise localization principles lead to quantified $L^p(w)$, $p > 2$, and weak $L^p(w)$, $p \geq 2$, norm inequalities for T^{Ω}_{RF}. In particular, the obtained weak $L^p(w)$-norm bounds are new for $p \geq 2$ and sharp for $p > 2$. The proofs rely on sparse bounds for abstract balayages of Carleson sequences, local orthogonality, and very elementary time-frequency analysis techniques.

The paper also contains two results related to the outstanding conjecture that T^{Ω}_{RF} is bounded on $L^2(w)$ if and only if $w \in A_1$. The conjecture is verified for radially decreasing even A_1-weights, and in full generality for the Walsh group analogue of T^{Ω}_{RF}.

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