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A NON-TRIVIAL VARIANT OF HILBERT'S INEQUALITY, AND AN APPLICATION TO THE NORM OF THE HILBERT MATRIX ON THE HARDY–LITTLEWOOD SPACES

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Abstract: Hilbert's inequality for non-negative sequences states that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} {a_m}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} {b_n}^q\right)^{\frac{1}{q}},$$

where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. This implies that the norm of the Hilbert matrix as an operator on the sequence space ℓ^p equals $\frac{\pi}{\sin\frac{\pi}{p}}$.

In this article we prove the non-trivial variant

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}$$

of Hilbert's inequality, and we use it to prove that the norm of the Hilbert matrix as an operator on the Hardy–Littlewood space K^p equals $\frac{\pi}{\sin\frac{\pi}{p}}$, where K^p consists of all functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ analytic in the unit disk with $\|f\|_{K^p}^p = \sum_{m=0}^{\infty} (m+1)^{p-2} |a_m|^p < \infty$. We also see that $\frac{\pi}{\sin\frac{\pi}{p}}$ is the norm of the Hilbert matrix on the space ℓ_{p-2}^p of sequences (a_m) with $\|(a_m)\|_{\ell_{p-2}^p}^p = \sum_{m=1}^{\infty} m^{p-2} |a_m|^p < \infty$.

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Key words: Hilbert's inequality, Hilbert matrix, Hardy–Littlewood spaces.