CHARACTERIZATION OF THE ATOMIC SPACE $H^1$ FOR NON DOUBLING MEASURES IN TERMS OF A GRAND MAXIMAL OPERATOR

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Abstract. Let $\mu$ be a Radon measure on $\mathbb{R}^d$, which may be non doubling. The only condition that $\mu$ must satisfy is the size condition $\mu(B(x, r)) \leq C r^n$, for some fixed $0 < n \leq d$. Recently, the author introduced spaces of type $\text{BMO}(\mu)$ and $H^1(\mu)$ with properties similar to ones of the classical spaces $\text{BMO}$ and $H^1$ defined for doubling measures. These new spaces proved to be useful to study the $L^p(\mu)$ boundedness of Calderón-Zygmund operators without assuming doubling conditions.

In this paper a characterization of this new atomic Hardy space $H^1(\mu)$ in terms of a maximal operator $M_\Phi$ is given. It is shown that $f$ belongs to $H^1(\mu)$ if and only if $f \in L^1(\mu)$, $\int f \, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$, as in the usual doubling situation.

1. Introduction

The aim of this paper is to characterize the atomic Hardy space $H^1_{atb}(\mu)$ introduced in [To3] in terms of a grand maximal operator. Throughout all the paper $\mu$ will be a (positive) Radon measure on $\mathbb{R}^d$ satisfying the growth condition

\[ \mu(B(x, r)) \leq C_0 r^n \quad \text{for all } x \in \text{supp}(\mu), \quad r > 0, \]

where $n$ is some fixed number with $0 < n \leq d$. We do not assume that $\mu$ is doubling ($\mu$ is said to be doubling if there exists some constant $C$ such that $\mu(B(x, 2r)) \leq C \mu(B(x, r))$ for all $x \in \text{supp}(\mu)$, $r > 0$).

The doubling condition on $\mu$ is an essential assumption in most results of classical Calderón-Zygmund theory. Nevertheless, recently it has been shown that many results in this theory also hold without the doubling assumption. For example, in [To1] a $T(1)$ theorem and weak $(1, 1)$ estimates for the Cauchy transforms are obtained. For general Calderón-Zygmund operators (CZO’s) a $T(1)$ theorem in [NTV1], and weak $(1, 1)$ estimates and Cotlar’s inequality in [NTV2] are proved. A $T(b)$ is also given in [NTV3].

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For more results, see [MMNO], [NTV4], [OP], [To2], [To3], [To4] and [Ve], for example.

In [To3] some variants of the classical spaces $BMO(\mu)$ and $H^1(\mu)$ are introduced. These variants are denoted by $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$ respectively. There, it is shown that many of the properties fulfilled by $BMO(\mu)$ and $H^1(\mu)$ when $\mu$ is doubling are also satisfied by $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$ without assuming $\mu$ doubling. For example, the functions from $RBMO(\mu)$ fulfill a John-Nirenberg type inequality (see Section 5 for the precise statement of this inequality), $RBMO(\mu)$ is the dual of $H^{1,\infty}_{atb}(\mu)$, CZO’s which are bounded in $L^2(\mu)$ are also bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into $RBMO(\mu)$ and, on the other hand, any operator which is bounded from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$ and from $L^\infty(\mu)$ into $RBMO(\mu)$ is bounded in $L^p(\mu)$, $1 < p < \infty$.

Let us remark that if $\mu$ is non doubling and one defines $BMO(\mu)$ and the atomic space $H^{1,\infty}_{at}(\mu) \equiv H^1(\mu)$ exactly as in the classical doubling situation (see [GR], [Jo] or [St], for instance), then these spaces still fulfill some of the properties stated above [MMNO]. However a basic one fails: CZO’s may be bounded in $L^2(\mu)$ but not from $H^{1,\infty}_{at}(\mu)$ into $L^1(\mu)$ or from $L^\infty(\mu)$ into $BMO(\mu)$ (see [Ve] and [MMNO]). For this reason, if one wants to study the $L^p$-boundedness of CZO’s, the spaces $BMO(\mu)$ and $H^{1,\infty}_{at}(\mu)$ are not appropriate. This is the main reason for the introduction of $RBMO(\mu)$ and $H^{1,\infty}_{atb}(\mu)$ in [To3].

Before stating our main result, we need some notation and terminology. By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube centered at some point in supp($\mu$) with sides parallel to the axes. Its side length is denoted by $l(Q)$ and its center by $z_Q$. Given $\rho > 0$, we denote by $\rho Q$ the cube concentric with $Q$ with side length $\rho l(Q)$. Recall that a function $f \in L^1_{loc}(\mu)$ belongs to the classical space $H^{1,\infty}_{at}(\mu)$ if it can be written as $f = \sum_{i} \lambda_i a_i$, where $\lambda_i \in \mathbb{R}$ are numbers such that $\sum_{i} |\lambda_i| < \infty$ and $a_i$ are functions called \textit{atoms} such that

1. there exists some cube $Q_i$ such that supp($a_i$) $\subset Q_i$,
2. $\int a_i d\mu = 0$,
3. $\|a_i\|_{L^\infty(\mu)} \leq \mu(Q_i)^{-1}$.

In order to recall the precise definition of $H^{1,\infty}_{atb}(\mu)$ we have to introduce the coefficients $K_{Q,R}$. Given two cubes $Q \subset R$, we set

$$K_{Q,R} = 1 + \int_{Q \backslash Q} \frac{1}{|x - z_Q|^n} d\mu(x),$$

where $Q_R$ is the smallest cube concentric with $Q$ containing $R$.

For a fixed $\rho > 1$, a function $b \in L^1_{loc}(\mu)$ is called an \textit{atomic block} if

1. there exists some cube $R$ such that supp($b$) $\subset R$,
2. $\int b d\mu = 0$,
3. there are functions $a_j$ supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \sum_{j=1}^{\infty} \lambda_j a_j$, and 

$$\|a_j\|_{L^\infty(\mu)} \leq (\mu(\rho Q_j) K_{Q_j,R})^{-1}.$$ 

We denote 

$$|b|_{H_{atb}^{1,\infty}(\mu)} = \sum_j |\lambda_j|$$

(to be rigorous, we should think that $b$ is not only a function, but a ‘structure’ formed by the function $b$, the cubes $R$ and $Q_j$, the functions $a_j$, etc.). Then, we say that $f \in H_{atb}^{1,\infty}(\mu)$ if there are atomic blocks $b_i$ such that

$$(1.2) \quad f = \sum_{i=1}^{\infty} b_i,$$

with $\sum_i |b_i|_{H_{atb}^{1,\infty}(\mu)} < \infty$ (notice that this implies that the sum in (1.2) converges in $L^1(\mu)$). The $H_{atb}^{1,\infty}(\mu)$ norm of $f$ is

$$\|f\|_{H_{atb}^{1,\infty}(\mu)} = \inf \sum_i |b_i|_{H_{atb}^{1,\infty}(\mu)},$$

where the infimum is taken over all the possible decompositions of $f$ in atomic blocks.

The definition of $H_{atb}^{1,\infty}(\mu)$ does not depend on the constant $\rho > 1$. The $H_{atb}^{1,\infty}(\mu)$ norms for different choices of $\rho > 1$ are equivalent. Nevertheless, for definiteness, we will assume $\rho = 2$ in the definition.

Compare the definitions of the spaces $H_{at}^{1,\infty}(\mu)$ and $H_{atb}^{1,\infty}(\mu)$: In $H_{at}^{1,\infty}(\mu)$ the cancellation condition 2 and the size condition 3 are imposed over the atoms $a_j$. On the other hand, in $H_{atb}^{1,\infty}(\mu)$ the cancellation condition 2 is imposed over the atomic blocks $b_i$, and the size condition 3 is satisfied by the “components” $a_{i,j}$ of $b_i$ separately for each $j$. It is not difficult to check that $H_{at}^{1,\infty}(\mu) = H_{atb}^{1,\infty}(\mu)$ if $\mu(B(x,r)) \approx r$ for all $x \in \text{supp}(\mu)$, $r > 0$ (the notation $A \approx B$ means that there exists some constant $C > 0$ such that $C^{-1} A \leq B \leq C A$, that is $A \lesssim B \lesssim A$). If the latter condition does not hold, then $H_{at}^{1,\infty}(\mu)$ may be different from $H_{atb}^{1,\infty}(\mu)$, even when $\mu$ is doubling (see [To3]).

Now we are going to introduce the “grand” maximal operator $M_\Phi$, which is the main tool in our characterization of $H_{atb}^{1,\infty}(\mu)$.

**Definition 1.1.** Given $f \in L^1_{loc}(\mu)$, we set

$$M_\Phi f(x) = \sup_{\varphi \sim x} \left| \int f \varphi d\mu \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

1. $\|\varphi\|_{L^1(\mu)} \leq 1$,
2. $0 \leq \varphi(y) \leq \frac{1}{|y-x|^n}$ for all $y \in \mathbb{R}^d$, and
\( |\varphi'(y)| \leq \frac{1}{|y - x|^n + 1} \) for all \( y \in \mathbb{R}^d \).

In this paper we will prove the following result.

**Theorem 1.2.** A function \( f \) belongs to \( H_{atb}^{1,\infty}(\mu) \) if and only if \( f \in L^1(\mu) \), \( \int f \, d\mu = 0 \) and \( M_\Phi f \in L^1(\mu) \). Moreover, in this case

\[
\|f\|_{H_{atb}^{1,\infty}(\mu)} \approx \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.
\]

Theorem 1.2 can be considered as a version for non doubling measures of some results that are already known in more classical situations. When \( \mu \) is the Lebesgue measure on the real line, a characterization of \( H_{at}^{1,\infty}(\mu) \) such as the one of Theorem 1.2 was proved by Coifman [Co]. This result was extended to the Lebesgue measure on \( \mathbb{R}^d \) by Latter [La]. Let us remark that in these cases, in the definition of \( M_\Phi \), for each \( x \) it is enough to take the supremum over functions \( \varphi_{x,r}, r > 0 \), of the form

\[
\varphi_{x,r}(y) = \frac{1}{r^d} \psi \left( \frac{y - x}{r} \right),
\]

where \( 0 \neq \psi \in \mathcal{S} \) is some fixed function.

If

\[
\mu(B(x, r)) \approx r^n \quad \text{for all } x \in \text{supp}(\mu), \ r > 0,
\]
then \( \text{supp}(\mu) \) is a homogeneous space in the sense of [CW]. For general homogeneous spaces satisfying (1.3), Coifman, Meyer and Weiss showed that there exists a description of \( H_{at}^{1,\infty}(\mu) \) in terms of a grand maximal operator (see [CW] for this result and for the detailed definition of homogeneous spaces). They observed that a proof of this description by Carleson [Ca] using the duality \( H_{at}^{1,\infty}(\mu) - \text{BMO}(\mu) \) in the case where \( \mu \) is the Lebesgue measure on \( \mathbb{R}^n \) can be easily extended to the more general situation of homogeneous spaces.

For a measure \( \mu \) on \( \mathbb{R}^d \) which is doubling but which may not satisfy (1.3), Macías and Segovia ([MS1], [MS2]) obtained a characterization of \( H_{at}^{1,\infty}(\mu) \) by means of a grand maximal operator too (see also [Uc]). They showed that if \( \mu \) is doubling, then taking a suitable quasimetric one can assume that (1.3) holds. Their result applies not only to doubling measures on \( \mathbb{R}^d \), but to more general homogeneous spaces. On the other hand, since \( H_{at}^{1,\infty}(\mu) \) may be different from \( H_{atb}^{1,\infty}(\mu) \) if \( \mu \) is a doubling measure on \( \mathbb{R}^d \) which does not satisfy (1.3), the result of Macías and Segovia (in the precise case that we are considering) cannot be derived as a particular instance of Theorem 1.2.

The absence of any regularity condition on \( \mu \), apart from the size condition (1.1), makes impossible to extend the classical arguments to the present situation without major changes. We will not consider any quasimetric on \( \mathbb{R}^d \) different from the Euclidean distance and we are not able to reduce our case to a situation where (1.3) holds.
Let us remark that the results of [Co], [La], [MS1] and [MS2] concern not only the Hardy space $H^1$ but also the Hardy spaces $H^p$, with $0 < p < 1$. However, it is not possible to extend our proof of Theorem 1.2 to $0 < p < 1$ because we have obtained it by duality (following the same approach as Carleson [Ca]).

The paper is organized as follows. In Section 2 we deal with some preliminary questions. In Section 3 we show that the grand maximal operator $M_\Phi$ is bounded from $H^1_{atb}(\mu)$ into $L^1(\mu)$, which proves the “only if” part of Theorem 1.2 (the easy implication). In the remaining sections of the paper we prove the other implication. In Section 4 we explain how this can be proved by duality. A suitable version for our purposes of John-Nirenberg inequality if obtained in Section 5. In Section 6 some kind of dyadic cubes are constructed, and in the following section a suitable approximation of the identity adapted to the measure $\mu$ is obtained. Section 8 contains a construction which is the core of the proof of the “if” part of Theorem 1.2. Finally, Section 9 is an Appendix where we prove a density result which is necessary in the proof by duality of the “if” part of Theorem 1.2.

2. Preliminaries

The letter $C$ will be used for constants that may change from one occurrence to another. Constants with subscripts, such as $C_1$, do not change in different occurrences.

We will assume that the constant $C_0$ in (1.1) has been chosen big enough so that for all the cubes $Q \subset \mathbb{R}^d$ we have

(2.1) $\mu(Q) \leq C_0 \ell(Q)^n$.

Given a function $f \in L^{1}_{loc}(\mu)$, we denote by $m_Q f$ the mean of $f$ over $Q$ with respect to $\mu$, i.e. $m_Q f = \frac{1}{\mu(Q)} \int_Q f \, d\mu$.

**Definition 2.1.** Given $\alpha > 1$ and $\beta > \alpha^n$, we say that the cube $Q \subset \mathbb{R}^d$ is $(\alpha, \beta)$-doubling if $\mu(\alpha Q) \leq \beta \mu(Q)$.

**Remark 2.2.** As shown in [To3], due to the fact that $\mu$ satisfies the growth condition (1.1), there are a lot “big” doubling cubes. To be precise, given any point $x \in \text{supp}(\mu)$ and $c > 0$, there exists some $(\alpha, \beta)$-doubling cube $Q$ centered at $x$ with $\ell(Q) \geq c$. This follows easily from (1.1) and the fact that $\beta > \alpha^n$.

On the other hand, if $\beta > \alpha^d$, then for $\mu$-a.e. $x \in \mathbb{R}^d$ there exists a sequence of $(\alpha, \beta)$-doubling cubes $\{Q_k\}_k$ centered at $x$ with $\ell(Q_k) \to 0$ as $k \to \infty$. So there are a lot of “small” doubling cubes too.

For definiteness, if $\alpha$ and $\beta$ are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-doubling cube.

Now we are going to recall the definition of $RBMO(\mu)$. In fact, in Section 2 of [To3] several equivalent definitions are given. Maybe the easiest one is the following. Let $f \in L^{1}_{loc}(\mu)$. We say that $f \in RBMO(\mu)$ if there exists
some constant $C_1$ such that for any doubling cube $Q$

$$\int_Q |f - m_Q f| \, d\mu \leq C_1 \mu(Q)$$

and

$$|m_Q f - m_R f| \leq C_1 K_{Q,R}$$

for any two doubling cubes $Q \subset R$.

The best constant $C_1$ is the $\text{RBMO}(\mu)$ norm of $f$, that we denote as $\|f\|_{\ast}$.

Given any pair of constants $0 < \alpha, \beta$, with $\beta > \alpha^n$, if in the definition of $\text{RBMO}(\mu)$ we ask (2.2) and (2.3) to hold for $(\alpha, \beta)$-doubling cubes (instead of doubling cubes), we will get the same space $\text{RBMO}(\mu)$, with an equivalent norm $[\text{To3}]$. In fact, $\text{RBMO}(\mu)$ can be defined also without talking about doubling cubes: Given some fixed constant $\rho > 1$, $f \in \text{RBMO}(\mu)$ if and only if there exists a collection of numbers $\{f_Q\}_Q$ (i.e. for each cube $Q$ some number $f_Q$) and some constant $C_2$ such that

$$\int_Q |f(x) - f_Q| \, d\mu(x) \leq C_2 \mu(\rho Q)$$

for any cube $Q \subset \mathbb{R}^d$ and,

$$|f_Q - f_R| \leq C_2 K_{Q,R}$$

for any two cubes $Q \subset R$.

The best constant $C_2$ is comparable to the $\text{RBMO}(\mu)$ norm of $f$ given by (2.2) and (2.3).

Recall that given two cubes $Q \subset R$, $Q_R$ stands for the smallest cube concentric with $Q$ containing $R$. Without assuming $Q \subset R$, we will denote by $Q_R$ the smallest cube concentric with $Q$ containing $Q$ and $R$.

**Definition 2.3.** Consider two cubes $Q, R \subset \mathbb{R}^d$ (we do not assume $Q \subset R$ or $R \subset Q$). We denote

$$\delta(Q, R) = \max \left( \int_{Q \setminus Q} \frac{1}{|x - z_Q|^n} \, d\mu(x), \int_{R \setminus R} \frac{1}{|x - z_R|^n} \, d\mu(x) \right).$$

Notice that $\ell(Q_R) \approx \ell(R_Q) \approx \ell(Q) + \ell(R) + \text{dist}(Q, R)$, and if $Q \subset R$, then $R_Q = R$ and $\ell(Q) = \ell(Q_R) \leq 2\ell(R)$.

It is clear that if $Q \subset R$, then $K_{Q,R} = 1 + \delta(Q, R)$. Quite often we will treat points $x \in \text{supp}(\mu)$ as if they were cubes (with $\ell(x) = 0$). So for $x, y \in \text{supp}(\mu)$ and some cube $Q$, the notations $\delta(x, Q)$ and $\delta(x, y)$ make sense. In some way, they are particular cases of Definition 2.3. Of course, it may happen $\delta(x, Q) = \infty$ or $\delta(x, y) = \infty$.

In the following lemma we show that $\delta(\cdot, \cdot)$ satisfies some very useful properties.

**Lemma 2.4.** The following properties hold:

(a) If $\ell(Q) \approx \ell(R)$ and $\text{dist}(Q, R) \ll \ell(Q)$, then $\delta(Q, R) \leq C$. In particular, $\delta(Q, \rho Q) \leq C_0 2^n \rho^n$ for $\rho > 1$.

(b) Let $Q \subset R$ be concentric cubes such that there are no doubling cubes of the form $2^k Q$, $k \geq 0$, with $Q \subset 2^k Q \subset R$. Then, $\delta(Q, R) \leq C_3$. 

(c) If $Q \subset R$, then
\[ \delta(Q, R) \leq C \left( 1 + \log \frac{\ell(R)}{\ell(Q)} \right). \]

(d) If $P \subset Q \subset R$, then
\[ |\delta(P, R) - [\delta(P, Q) + \delta(Q, R)]| \leq \varepsilon_0. \]
That is, with a different notation, $\delta(P, R) = \delta(P, Q) + \delta(Q, R) \pm \varepsilon_0$. If $P$ and $Q$ are concentric, then $\varepsilon_0 = 0$.

(e) For $P, Q, R \subset \mathbb{R}^d$,
\[ \delta(P, R) \leq C_4 + \delta(P, Q) + \delta(Q, R). \]

The constants that appear in (b), (c), (d) and (e) depend on $C_0, n, d$. The constant $C$ in (a) depends, further, on the constants that are implicit in the relations $\approx, \lesssim$.

Proof. The estimates in (a) are immediate. The proof of (b) is also an easy estimate, which can be found in [To3, Lemma 2.1], for example. The arguments for (c) are also quite standard. We leave the proof for the reader.

Let us see that (d) holds. If $P$ and $Q$ are concentric, the identity $\delta(P, R) = \delta(P, Q) + \delta(Q, R)$ is a direct consequence of the definition. In case $P$ and $Q$ are not concentric we have to make some calculations:
\[ \delta(P, R) = \delta(P, P) + \int_{P \cap P_Q} \frac{1}{|y - z_P|^n} \, d\mu(y) \]
\[ = \delta(P, Q) + \int_{P \cap P_Q} \frac{1}{|y - z_P|^n} \, d\mu(y). \]

So we must show that
\[ S := \left| \int_{P \cap P_Q} \frac{1}{|y - z_P|^n} \, d\mu(y) - \delta(Q, R) \right| \leq C. \]

We set
\[ S \leq \int_{P \cap P_Q} \frac{1}{|y - z_Q|^n} \, d\mu(y) + \int_{P \cap \Delta_Q R} \left( \frac{1}{|y - z_P|^n} + \frac{1}{|y - z_Q|^n} \right) \, d\mu(y) \]
\[ + \int_{R \setminus P_Q} \left| \frac{1}{|y - z_P|^n} - \frac{1}{|y - z_Q|^n} \right| \, d\mu(y) \]
\[ = S_1 + S_2 + S_3. \]

The integral $S_2$ is easily estimated above by some constant $C$, since $|y - z_P|, |y - z_Q| \leq C \ell(R)$ for $y \in P \Delta Q R$. An analogous calculation yields $S_1 \leq C$. For $S_3$ we have
\[ S_3 \leq C \int_{|y - z_Q| \geq \ell(Q)/2} \frac{|z_P - z_Q|}{|y - z_Q|^{n+1}} \, d\mu(y) \leq C \frac{|z_P - z_Q|}{\ell(Q)} \leq C, \]
and we are done with (d).

We leave the proof of (e) for the reader too. \hfill \Box

Notice that if we set $D(Q, R) = 1 + \delta(Q, R)$ for $Q \neq R$ and $D(Q, Q) = 0$, then $D(\cdot, \cdot)$ is a quasidistance on the set of cubes, by (e) in the preceding lemma.

From (a) and the fact that $Q_R$ and $R_Q$ have comparable sizes and $Q_R \cap R_Q \neq \emptyset$, we get that $Q_R$ and $R_Q$ are close in the quasimetric $D(\cdot, \cdot)$. Also, if we denote by $Q$ the smallest doubling cube of the form $2^k Q$, $k \geq 0$, by (b) we know that $Q$ is not far from $Q$ (using again the quasimetric $D$). So $Q$ and $\hat{Q}$ may have very different sizes, but we still have $D(Q, \hat{Q}) \leq C$.

In Remark 2.2 we have explained that there a lot of big and small doubling cubes. In the following lemma we state a more precise result about the existence of small doubling cubes in terms of $\delta(\cdot, \cdot)$.

**Lemma 2.5.** There exists some (big) constant $\eta > 0$ depending only on $C_0$, $n$ and $d$ such that if $R_0$ is some cube centered at some point of $\text{supp}(\mu)$ and $\alpha > \eta$, then for each $x \in R_0 \cap \text{supp}(\mu)$ such that $\delta(x, 2R_0) > \alpha$ there exists some doubling cube $Q \subset 2R_0$ centered at $x$ satisfying

$$|\delta(Q, 2R_0) - \alpha| \leq \varepsilon_1,$$

where $\varepsilon_1$ depends only on $C_0$, $n$ and $d$ (but not on $\alpha$).

**Proof.** Let $Q_1$ be the biggest cube centered at $x$ with side length $2^{-k} \ell(R_0)$, $k \geq 1$, such that $\delta(Q_1, 2R_0) \geq \alpha$. Then, $\delta(2Q_1, 2R_0) < \alpha$. Otherwise, $k = 1$ and since $\ell(Q_1) = \ell(R_0)/2$ and $\ell(Q_1, R_0) \leq 4 \ell(R_0)$ we get

$$\delta(Q_1, 2R_0) \leq \int_{\ell(Q_1)/2 < |y-x|, y \in Q_1, R_0} \frac{1}{|y-x|^n} d\mu(y) \leq C_0^2 \frac{8^n \ell(R_0)^n}{\ell(Q_1)^n} = C_0 16^n,$$

which contradicts the choice of $Q_1$, assuming $\eta > C_0 16^n$.

Now we have $\delta(Q_1, 2R_0) \leq \alpha + \delta(Q_1, 2Q_1) \leq \alpha + C_0 16^n$. Thus

$$|\delta(Q_1, 2R_0) - \alpha| \leq C_0 16^n.$$

Let $Q$ be the smaller doubling cube of the form $2^k Q_1$, $k \geq 0$. Then $\delta(Q_1, Q) \leq C_3$. Also, $\ell(Q) \leq \ell(R_0)$. Otherwise, $R_0 \subset 3Q$ and

$$\delta(Q_1, 2R_0) \leq \delta(Q_1, 3Q) = \delta(Q_1, Q) + \delta(Q, 3Q) \leq C_3 + 6^n C_0.$$

This is not possible if we assume $\eta > C_3 + 6^n C_0$.

Now $Q$ satisfies the required properties, since it is doubling, it is contained in $2R_0$, and

$$|\delta(Q, 2R_0) - \alpha| \leq |\delta(Q, 2R_0) - \delta(Q_1, 2R_0)| + |\delta(Q_1, 2R_0) - \alpha| \leq \delta(Q, Q_1) + C_0 16^n \leq C_3 + C_0 16^n =: \varepsilon_1.$$

\hfill \Box
As in (d) of Lemma 2.4, instead of (2.4), often we will write \( \delta(Q, 2R_0) = \alpha \pm \varepsilon_1 \).

Notice that by (e) and (a) of Lemma 2.4, we get
\[
|\delta(Q, R_0) - \alpha| \leq |\delta(Q, 2R_0) - \alpha| + |\delta(Q, 2R_0) - \delta(Q, R_0)| \\
\leq \varepsilon_1 + \delta(R_0, 2R_0) + C_4 \\
\leq \varepsilon_1 + C + C_4 := \varepsilon'_1.
\]

However we prefer the estimate (2.4), because we have \( Q \subset 2R_0 \) but \( Q \not\subset R_0 \), in general. So the cube \( 2R_0 \), in some sense, is a more appropriate reference.

Results analogous to the ones in Lemma 2.5 can be stated about the existence of cubes \( Q \) centered at some point \( x \in R_0 \) with \( Q \supset R_0 \), but since we will not need this fact below, we will not show any precise result of this kind.

If \( Q \subset R \) are doubling cubes and \( f \in RBMO(\mu) \), then \( |m_Q f - m_R f| \leq (1 + \delta(Q, R)) \|f\|_* \). Without assuming \( Q \subset R \), we have a similar result:

**Proposition 2.6.** Let \( Q, R \subset \mathbb{R}^d \) be doubling cubes. If \( f \in RBMO(\mu) \), then
\[
|m_Q f - m_R f| \leq (C + 2 \delta(Q, R)) \|f\|_*.
\]

**Proof.** Suppose, for example, \( \ell(R_0) \geq \ell(Q) \). Then, \( Q \subset 3R_Q \).

Let \( 3\tilde{R}_Q \) be the smallest doubling cube of the form \( 2^k 3R_Q \), \( k \geq 0 \). We have
\[
\delta(R, 3\tilde{R}_Q) = \delta(R, R_Q) + \delta(R_Q, 3\tilde{R}_Q) \leq \delta(R, Q) + C.
\]

Thus
\[
(2.5) \quad |m_R f - m_{3\tilde{R}_Q} f| \leq (1 + C + \delta(R, Q)) \|f\|_*.
\]

We also have
\[
\delta(Q, 3\tilde{R}_Q) \leq C + \delta(Q, 3R_Q) + \delta(3R_Q, 3\tilde{R}_Q) \leq C + \delta(Q, Q_R) + \delta(Q_R, 3R_Q).
\]

Since \( Q_R \) and \( R_Q \) have comparable sizes, \( \delta(Q_R, 3R_Q) \leq C \), and so
\[
\delta(Q, 3\tilde{R}_Q) \leq C + \delta(Q, R).
\]

Therefore,
\[
(2.6) \quad |m_Q f - m_{3\tilde{R}_Q} f| \leq (1 + C + \delta(Q, R)) \|f\|_*.
\]

By (2.5) and (2.6), the proposition follows. \( \square \)

### 3. The easy implication of Theorem 1.2

In this section we will prove the “only if” part of Theorem 1.2.

**Lemma 3.1.** The operator \( M_\Phi \) is bounded from \( H_{alb}^{1,\infty}(\mu) \) into \( L^1(\mu) \).
Proof. Let \( b = \sum \lambda_i a_i \) be an atomic block supported on some cube \( R \), with \( \lambda_i \in \mathbb{R} \), where \( a_i \) are functions supported on cubes \( Q_i \subset R \) such that \( \| a_i \|_{\infty} \leq ((1 + \delta(Q_i, R)) \mu(2Q_i))^{-1} \). We will show that \( \| M \Phi b \|_{L^1(\mu)} \leq C \sum_i |\lambda_i| \).

First we will estimate the integral \( \int_{\mathbb{R}^d \setminus 2R} M \Phi b \ d\mu \). For \( x \in \mathbb{R}^d \setminus 2R \) and \( \varphi \sim x \), since \( \int b \ d\mu = 0 \), we have

\[
\left| \int b \varphi \ d\mu \right| = \left| \int b(y) (\varphi(y) - \varphi(z_R)) \ d\mu(y) \right| \leq C \int |b(y)| \frac{\ell(R)}{|x - z_R|^{n+1}} \ d\mu(y).
\]

Thus

\[
\int_{\mathbb{R}^d \setminus 2R} M \Phi b \ d\mu \leq C \| b \|_{L^1(\mu)} \int_{\mathbb{R}^d \setminus 2R} \frac{\ell(R)}{|x - z_R|^{n+1}} \ d\mu(x)
\]

(3.2)

(3.1)

Now we will show that

\[
\int_{2R} M \Phi a_i \ d\mu \leq C,
\]

and we will be done. If \( x \in 2Q_i \) and \( \varphi \sim x \), then

\[
\left| \int a_i \varphi \ d\mu \right| \leq C \| a_i \|_{L^{\infty}(\mu)} \| \varphi \|_{L^1(\mu)} \leq C \| a_i \|_{L^{\infty}(\mu)}.
\]

(3.3)

So,

\[
\int_{2Q_i} M \Phi a_i \ d\mu \leq C \| a_i \|_{L^{\infty}(\mu)} \mu(2Q_i) \leq C.
\]

For \( x \in 2R \setminus 2Q_i \) and \( \varphi \sim x \), we have

\[
\left| \int a_i \varphi \ d\mu \right| \leq C \| a_i \|_{L^1(\mu)} \frac{1}{|x - z_{Q_i}|^n}.
\]

Therefore,

\[
\int_{2R \setminus 2Q_i} M \Phi a_i \ d\mu \leq C \| a_i \|_{L^1(\mu)} \int_{2R \setminus 2Q_i} \frac{1}{|x - z_{Q_i}|^n} \ d\mu(x)
\]

(3.4)

(3.2)

and (3.3) follows. \( \square \)

4. An approach by duality for the other implication

We have to show that if \( f \in L^1(\mu) \), \( \int f \ d\mu = 0 \) and \( M \Phi f \in L^1(\mu) \), then \( f \in H_{\text{atb}}^{1,\infty}(\mu) \). We will obtain this result by duality, following the ideas of Carleson [Ca]. So we will prove
Lemma 4.1 (Main Lemma). Let $f \in RBMO(\mu)$ with compact support and $\int f \, d\mu = 0$. There exist functions $h_m \in L^\infty(\mu)$, $m \geq 0$, such that

\begin{equation}
 f(x) = h_0(x) + \sum_{m=1}^{\infty} \int \varphi_{y,m}(x) h_m(y) \, d\mu(y),
\end{equation}

with convergence in $L^1(\mu)$ where, for each $m \geq 1$, $\varphi_{y,m} \sim y$, and

\begin{equation}
 \sum_{m=0}^{\infty} |h_m| \leq C \|f\|_*.
\end{equation}

Let us see that from this lemma the “if” part of Theorem (1.2) follows. Consider $f \in L^1(\mu)$ such that $\int f \, d\mu = 0$ and $M\Phi f \in L^1(\mu)$. Assume first that $f \in L^\infty(\mu)$ and has compact support. In this case, $f \in H_{atb}^{1,\infty}(\mu)$ and so we only have to estimate the norm of $f$.

Since $RBMO(\mu)$ is the dual of $H_{atb}^{1,\infty}(\mu)$ [To3], given $f \in L^1(\mu)$, by the Hahn-Banach theorem we have

$$\|f\|_{H_{atb}^{1,\infty}(\mu)} = \sup_{\|g\|_* \leq 1} \langle f, g \rangle.$$

Since $\int f \, d\mu = 0$, we can assume that $g$ has compact support and $\int g \, d\mu = 0$. Then, applying the Main Lemma to $g$ we get

$$\langle f, g \rangle \leq \int f h_0 \, d\mu + \left| \sum_{m=1}^{\infty} \int \varphi_{y,m}(x) h_m(y) f(x) \, d\mu(x) \, d\mu(y) \right|.$$

Since $|\int \varphi_{y,m}(x) f(x) \, d\mu(x)| \leq M\Phi f(y)$, we have

$$\langle f, g \rangle \leq \|f\|_{L^1(\mu)} \|h_0\|_{L^\infty(\mu)} + \sum_{m=1}^{\infty} \int M\Phi f(y) |h_m(y)| \, d\mu(y)$$

$$\leq \|f\|_{L^1(\mu)} \|h_0\|_{L^\infty(\mu)} + \|M\Phi f\|_{L^1(\mu)} \left\| \sum_{m=1}^{\infty} |h_m| \right\|_{L^\infty(\mu)}$$

$$\leq C \left( \|f\|_{L^1(\mu)} + \|M\Phi f\|_{L^1(\mu)} \right) \|g\|_*.$$

That is, $\|f\|_{H_{atb}^{1,\infty}(\mu)} \leq C \left( \|f\|_{L^1(\mu)} + \|M\Phi f\|_{L^1(\mu)} \right)$.

In the general case where we don’t know a priori that $f \in H_{atb}^{1,\infty}(\mu)$, we can consider a sequence of functions $f_n$ bounded with compact support such that $\int f_n \, d\mu = 0$, $f_n \to f$ in $L^1(\mu)$ and $\|M\Phi (f - f_n)\|_{L^1(\mu)} \to 0$, and then we apply the usual arguments. The existence of such a sequence is showed in Lemma 9.1, in the Appendix.

The rest of the paper, with the exception of the Appendix, is devoted to the proof of the Main Lemma.
5. The inequality of John-Nirenberg

In [To3] it is shown that the functions of the space $\text{RBMO}(\mu)$ satisfy a John-Nirenberg type inequality. Let us state the precise result.

**Theorem 5.1.** Let $Q \subset \mathbb{R}^d$ be a doubling cube. If $f \in \text{RBMO}(\mu)$, then

$$
\mu\{x \in Q : |f - m_Q f| > \lambda\} \leq C_5 \mu(Q) \exp\left(\frac{-C_6 \lambda}{\|f\|_*}\right), \quad \lambda > 0,
$$

where $C_5, C_6 > 0$ are constants that only depend on $C_0, n, d$.

In the proof of the Main Lemma we will need a version of the above inequality which appears to be stronger (although it is equivalent). In this section we will state and prove this new version of John-Nirenberg inequality.

**Definition 5.2.** Given a doubling cube $Q$, we denote by $Z(Q, \lambda)$ the set of points $x \in Q$ such that any doubling cube $P$ with $x \in P$ and $\ell(P) \leq \ell(Q)/4$ satisfies $|m_P f - m_Q f| \leq \lambda$.

In other other words, $Q \setminus Z(Q, \lambda)$ is the subset of $Q$ such that for some doubling cube $P$ with $x \in P$ and $\ell(P) \leq \ell(Q)/4$ we have

$$
|m_P f - m_Q f| > \lambda.
$$

**Proposition 5.3.** Let $Q \subset \mathbb{R}^d$ be a doubling cube. If $f \in \text{RBMO}(\mu)$, then

$$
\mu(Q \setminus Z(Q, \lambda)) \leq C'_5 \mu(Q) \exp\left(\frac{-C'_6 \lambda}{\|f\|_*}\right), \quad \lambda > 0.
$$

where $C'_5, C'_6 > 0$ are constants that only depend on $C_0, n, d$.

**Proof.** The arguments are quite standard. For any $x \in Q \setminus Z(Q, \lambda)$ there exists some cube $P_x$ which contains $x$, with $\ell(P_x) \leq \ell(Q)/4$ and such that $|m_{P_x} f - m_Q f| > \lambda$. Then by Besicovitch’s Covering Theorem, there are points $x_i \in Q \setminus Z(Q, \lambda)$ such that

$$
Q \setminus Z(Q, \lambda) \subset \bigcup_{i} 2P_i,
$$

and so that the cubes $2P_i$, $i = 1, 2, \ldots$, form an almost disjoint family. Observe that the Covering Theorem of Besicovitch cannot be applied to the cubes $P_x$ (they are non centered), however we have applied it to the cubes $2P_x$, which are non centered too, but fulfil the condition

$$
x \in \frac{1}{2} 2P_x.
$$

That is, the point $x$ is “far” from the boundary of $2P_x$. Under this condition, Besicovitch’s Covering Theorem also holds.
Since, for each \(i\), \(\ell(P_i) \leq \ell(Q)/4\) and \(P_i \cap Q \neq \emptyset\), it is easily seen that \(2P_i \subset \frac{7}{4}Q\). Then,

\[
\mu(Q \setminus Z(Q, \lambda)) \leq \sum_i \mu(2P_i) \\
\leq \sum_i \int_{P_i} \exp(|f(x) - m_\mathcal{Q}f|k) \exp(-\lambda k) \, d\mu(x) \\
\leq C \int_{\frac{7}{4}Q} \exp(|f(x) - m_\mathcal{Q}f|k) \exp(-\lambda k) \, d\mu(x),
\]

where \(k\) is some constant that will be fixed below. Now, we have

\[
\exp(|f(x) - m_\mathcal{Q}f|k) \leq \exp\left(|f(x) - m_{\frac{7}{4}Q}f|k\right) \\
\leq \exp\left(|f(x) - m_{\frac{7}{4}Q}f|k\right) \exp\left(C \|f\|_* k\right).
\]

The last inequality follows from \(|m_{\frac{7}{4}Q}f - m_\mathcal{Q}f| \leq C \|f\|_*\) (notice that the cube \(\frac{7}{4}Q\) is \((\frac{7}{4}, 2^{d+1})\)-doubling).

Therefore, by Theorem 5.1 (which also holds for cubes that are \((\frac{7}{4}, 2^{d+1})\)-doubling instead of \((2, 2^{d+1})\)-doubling, with constants \(\tilde{C}_1\) and \(\tilde{C}_2\) instead of \(C_1\) and \(C_2\)) we have

\[
\mu(Q \setminus Z(Q, \lambda)) \leq C \exp\left((-\lambda k) \exp(C \|f\|_* k) \int_{\frac{7}{4}Q} \exp\left(|f(x) - m_{\frac{7}{4}Q}f|k\right) \, d\mu(x) \\
= C \exp(-\lambda k) \exp(C \|f\|_* k) \\
\times \int_0^\infty \mu\left\{x \in \frac{7}{4}Q : \exp\left(|f(x) - m_{\frac{7}{4}Q}f|k\right) > t\right\} \, dt \\
\leq C \mu(\frac{7}{4}Q) \exp(-\lambda k) \exp(C \|f\|_* k) \int_0^\infty \tilde{C}_1 \exp\left(\frac{-\tilde{C}_2 \log t}{k \|f\|_*}\right) \, dt.
\]

So if we choose \(k := \tilde{C}_2/2\|f\|_*\), we get

\[
\mu(Q \setminus Z(Q, \lambda)) \leq C \mu(\frac{7}{4}Q) \exp\left(\frac{-\tilde{C}_2 \lambda}{2\|f\|_*}\right) \leq C \mu(Q) \exp\left(\frac{-\tilde{C}_2 \lambda}{2\|f\|_*}\right).
\]

\[\square\]

6. The “dyadic” cubes

In [Ca], Carleson proves a result analogous to the one stated in the Main Lemma for \(\mu\) being the Lebesgue measure on \(\mathbb{R}^d\). He uses dyadic cubes of side length \(2^{-mA}\), where \(A\) is some big positive integer. In our proof, we will also consider some cubes which will play the role of the dyadic cubes with side length \(2^{-mA}\) of Carleson. In this section we will introduce these new
“dyadic” cubes and we will show some of the properties that they satisfy and that will be needed in the proof of the Main Lemma.

As in [Ca], we will take some big positive integer $A$ whose precise value will be fixed after knowing or choosing several additional constants. In particular, we assume that $A$ is much bigger than the constants $\varepsilon_0$, $\varepsilon_1$ and $\eta$ of Section 2.

**Definition 6.1.** Suppose that the support of the function $f$ of the Main Lemma is contained in a doubling cube $R_0$. Let $m \geq 1$ be some fixed integer and $x \in \text{supp}(\mu) \cap R_0$. If $\delta(x, 2R_0) > mA$, we denote by $Q_{x,m}$ a doubling cube (with $Q_{x,m} > 0$) such that

$$|\delta(Q_{x,m}, 2R_0) - mA| \leq \varepsilon_1.$$  

(6.1)

Also, $D'_m = \{Q_{i,m}\}_{i \in I'_m}$, is a subfamily with finite overlap of the cubes $Q_{x,m}$, such that each cube $Q_{i,m} = Q_{y_i,m}$ is centered at some point $y_i \in \text{supp}(\mu) \cap R_0$ if $\delta(y_i, 2R_0) > mA$, and

$$\{x \in \text{supp}(\mu) \cap R_0 : \delta(x, 2R_0) > mA\} \subset \bigcup_{i \in I'_m} Q_{i,m}$$  

(this family exists because of Besicovitch’s Covering Theorem).

If $\delta(x, 2R_0) \leq mA$, we set $Q_{x,m} = \{x\}$. We denote by $D''_m$ the family of cubes $Q_{x,m} \equiv \{x\}$ such that $\delta(x, 2R_0) \leq mA$ and $x \notin \bigcup_{i \in I'_m} Q_{i,m}$. We set $D_m = D'_m \cup D''_m$.

The cubes $Q_{x,m}$, $x \in \text{supp}(\mu) \cap R_0$ (not necessarily from the family $D_m$) are called cubes of the $m$-th generation.

Obviously, the whole family of cubes in $D_m$ has also finite overlap. Notice that if $x$ is a point in $\text{supp}(\mu)$ such that $\delta(x, 2R_0) = \infty$, then $\ell(Q_{x,m}) > 0$ for all $m \geq 1$. Otherwise, there exists some $m_0$ such that $\ell(Q_{x,m}) = 0$ for all $m \geq m_0$.

It is easily seen that if $A$ is big enough, then $\ell(Q_{x,m+1}) \leq \ell(Q_{x,m})/10$ (a more precise version of this result will be proved in Lemma 6.3 below). So $\ell(Q_{x,m}) \rightarrow 0$ as $m \rightarrow \infty$.

If $A$ is much bigger than $\varepsilon_1$ and $Q_{x,m} \neq \{x\}$, then $\delta(Q_{x,m}, 2R_0) \approx mA$. However, the estimate (6.1) is much sharper. This will very useful in our construction.

**Lemma 6.2.** Assume that $P$ and $Q$ are cubes contained in $2R_0$ whose centers are in $R_0$. Let $S$ be a cube such that $P, Q \subset S \subset 2R_0$.

(a) If $|\delta(P, 2R_0) - \delta(Q, 2R_0)| \leq \beta$, then

$$|\delta(P, S) - \delta(Q, S)| \leq \beta + 2\varepsilon_0.$$  

(b) If $|\delta(P, S) - \delta(Q, S)| \leq \beta$, then

$$|\delta(P, 2R_0) - \delta(Q, 2R_0)| \leq \beta + 2\varepsilon_0.$$  

In particular, this lemma can be applied to cubes $P$ and $Q$ belonging to the same generation $m$, with $\beta = 2\varepsilon_1$ (assuming $\ell(P), \ell(Q) \neq 0$).
Proof. Both statements are a straightforward consequence of (d) in Lemma 2.4, since
\[ \delta(P, 2R_0) = \delta(P, S) + \delta(S, 2R_0) \pm \varepsilon_0 \]
and
\[ \delta(Q, 2R_0) = \delta(Q, S) + \delta(S, 2R_0) \pm \varepsilon_0. \]
\[ \square \]

The constants \( \varepsilon_0 \) and \( \varepsilon_1 \) should be understood as upper bounds for some “errors” and deviations of our construction from the classical dyadic lattice.

We will need the following result too.

**Lemma 6.3.** Assume that \( A \) is big enough. There exists some \( \gamma > 0 \) such that if \( Q_{x,m} \cap Q_{y,m+1} \neq \emptyset, \ x, y \in \text{supp}(\mu) \), then \( \ell(Q_{y,m+1}) \leq 2^{-\gamma A} \ell(Q_{x,m}) \).

Proof. We can assume \( Q_{y,m+1} \neq \{y\} \). Let \( B > 1 \) be some fixed constant. If \( \ell(Q_{y,m+1}) > B^{-1} \ell(Q_{x,m}) \), then \( Q_{x,m} \subset 3BQ_{y,m+1} \). So, if \( R_x \) is a cube centered at \( x \) with side length \( 6B \ell(Q_{y,m+1}) \), we have \( Q_{x,m}, Q_{y,m+1} \subset R_x \).

By (c) of Lemma 2.4 we get
\[ \delta(Q_{y,m+1}, R_x) \leq C \left( 1 + \log \left( \frac{\ell(R_x)}{\ell(Q_{y,m+1})} \right) \right) \leq C (1 + \log B). \]

Since
\[ \delta(Q_{y,m+1}, 2R_0) = \delta(Q_{y,m+1}, R_x) + \delta(R_x, 2R_0) \pm \varepsilon_0, \]
if we set \( B = 2^{-\gamma A} \), we obtain
\[ \delta(R_x, 2R_0) > (m + 1) A - \varepsilon_1 - \varepsilon_0 - C(1 + \gamma A \log 2). \]
Then for \( \gamma \) small enough we have
\[ \delta(R_x, 2R_0) > (m + 1) A - \varepsilon_1 - \varepsilon_0 - C - \frac{1}{2} A > mA + \varepsilon_1. \]
This implies \( \delta(Q_{x,m}, 2R_0) > mA + \varepsilon_1 \), which is not possible. \[ \square \]

As a consequence, we obtain

**Lemma 6.4.** Assume that \( A \) is big enough. If \( x, y \in \text{supp}(\mu) \) are such that \( Q_{x,m} \cap Q_{y,m+k} \neq \emptyset \) (with \( k \geq 1 \)), then \( \ell(Q_{y,m+k}) \leq 2^{-\gamma A k} \ell(Q_{x,m}) \).

Proof. By the previous lemma, \( \ell(Q_{y,j+1}) \leq 2^{-\gamma A j} \ell(Q_{y,j}) \) and \( \ell(Q_{y,m+1}) \leq 2^{-\gamma A} \ell(Q_{x,m}) \). This gives \( \ell(Q_{y,m+1}) \leq 2^{-\gamma A k} \ell(Q_{x,m}) \). \[ \square \]

7. AN APPROXIMATION OF THE IDENTITY

The proof of the Main Lemma will be constructive. At the level of cubes of generation \( m \) we will construct a function \( h_m \) yielding the “potential”
\[ U_m(x) = \int \varphi_{y,m}(x) h_m(y) \, d\mu(y) \]
(to be precise, instead of one function \( h_m \), for each \( m \) we will have \( N \) functions \( h^1_m, \ldots, h^N_m \), but this is a rather technical detail that we can skip now).
We need to use functions \( \varphi \) would have some information, at least, about the behaviour under dilations. 

This is not our case. The measure \( \mu \) is not invariant by translations and we don’t know how it behaves under dilations (notice that if \( \mu \) were doubling, we would have some information, at least, about the behaviour under dilations). We need to use functions \( \varphi_{y,m} \) such that \( \| \varphi_{y,m} \|_{L^1(\mu)} = 1 \) (or at least equal to some value close to 1). So \( \varphi_{y',m} \) cannot be obtained as a translation of \( \varphi_{y,m} \) for \( y' \neq y \), neither as a dilation of \( \varphi_{y',k}, k \neq m \). In this section we will show how these problems can be overcome.

We denote 
\[
\sigma := 10\varepsilon_0 + 10\varepsilon_1 + 12^{n+1}C_0.
\]

We introduce two new constants \( \alpha_1, \alpha_2 > 0 \) whose precise value will be fixed below. For the moment, let us say that \( \varepsilon_0, \varepsilon_1, C_0, \sigma \ll \alpha_1 \ll \alpha_2 \ll A \).

**Definition 7.1.** Let \( y \in \text{supp}(\mu) \). We denote by \( Q_{y,m}, \hat{Q}_{y,m}^1, Q_{y,m}^2, \hat{Q}_{y,m}^2, Q_{y,m}^3 \) some doubling cubes (with positive side length) centered at \( y \) such that

\[
\begin{align*}
\delta(Q_{y,m}, 2R_0) &= mA \pm \varepsilon_1, \\
\delta(Q_{y,m}^1, 2R_0) &= mA - \alpha_1 \pm \varepsilon_1, \\
\delta(Q_{y,m}^2, 2R_0) &= mA - \alpha_1 - \sigma \pm \varepsilon_1, \\
\delta(Q_{y,m}^3, 2R_0) &= mA - \alpha_1 - \alpha_2 \pm \varepsilon_1, \\
\delta(\hat{Q}_{y,m}^1, 2R_0) &= mA - \alpha_1 - \alpha_2 - 2\sigma \pm \varepsilon_1.
\end{align*}
\]

(7.1)

By Lemma 2.5 we know that if \( \delta(y, 2R_0) > mA \), then all the cubes \( Q_{y,m}, \hat{Q}_{y,m}^1, Q_{y,m}^2, \hat{Q}_{y,m}^2, Q_{y,m}^3 \) exist. Otherwise only some (or none) of them may exist. If any of these cubes does not exists, we let this cube be the point \( \{y\} \).

Notice that we can only assume that the estimates in (7.1) hold for the cubes \( Q \) which are different from \( \{y\} \) (i.e. with \( \ell(Q) > 0 \)). So if \( \hat{Q}_{y,m}^1 = \{y\} \), say, then, we only know that \( \delta(\hat{Q}_{y,m}^1, 2R_0) \leq mA - \alpha_1 - \sigma + \varepsilon_1 \).

**Lemma 7.2.** Let \( y \in \text{supp}(\mu) \). If we choose the constants \( \alpha_1, \alpha_2 \) and \( A \) big enough, we have

\[
Q_{y,m} \subset Q_{y,m}^1 \subset \hat{Q}_{y,m}^1 \subset Q_{y,m}^2 \subset \hat{Q}_{y,m}^2 \subset Q_{y,m}^3 \subset Q_{y,m-1}.
\]

(7.2)
Proof. Notice first that for $\alpha_1$, $\alpha_2$ and $A$ big enough, then the numbers that appear in the right hand side of the estimates in (7.1) form an strictly decreasing sequence. That is,
\begin{align*}
  mA - \epsilon_1 &> mA - \alpha_1 + \epsilon_1, \\
  mA - \alpha_1 - \epsilon_1 &> mA - \alpha_1 - \sigma + \epsilon_1, \\
  mA - \alpha_1 - \sigma - \epsilon_1 &> mA - \alpha_1 - \alpha_2 + \epsilon_1, \\
  mA - \alpha_1 - \alpha_2 - \epsilon_1 &> mA - \alpha_1 - \alpha_2 - \sigma + \epsilon_1, \\
  mA - \alpha_1 - \alpha_2 - \sigma - \epsilon_1 &> mA - \alpha_1 - \alpha_2 - 2\sigma + \epsilon_1, \\
  mA - \alpha_1 - \alpha_2 - 2\sigma - \epsilon_1 &> (m - 1)A + \epsilon_1.
\end{align*}

Let us check the inclusion $\hat{Q}_{y,m}^1 \subset Q_{y,m}^2$, for example. Suppose first that $Q_{y,m}^2 \neq \emptyset$, then
\[ \delta(Q_{y,m}^2, 2R_0) = mA - \alpha_1 - \alpha_2 \pm \epsilon_1. \]
If $\hat{Q}_{y,m}^1 = \{y\}$, the inclusion is obvious. Otherwise,
\[ \delta(\hat{Q}_{y,m}^1, 2R_0) = mA - \alpha_1 - \sigma \pm \epsilon_1. \]
Then $\delta(\hat{Q}_{y,m}^1, 2R_0) > \delta(Q_{y,m}^2, 2R_0)$, and so $\hat{Q}_{y,m}^1 \subset Q_{y,m}^2$. Assume now $Q_{y,m}^2 = \{y\}$. Then,
\[ \delta(y, 2R_0) \leq mA - \alpha_1 - \alpha_2 + \epsilon_1. \]
In this case there is not any cube $\hat{Q}_{y,m}^1$ satisfying
\[ \delta(\hat{Q}_{y,m}^1, 2R_0) = mA - \alpha_1 - \sigma \pm \epsilon_1, \]
and so, by our convention, $\hat{Q}_{y,m}^1 = \{y\}$. That is, the inclusion holds in any case.

The other inclusions are proved in a similar way. \hfill \Box

For a fixed $m$, the cubes $Q_{y,m}^1$ may have very different sizes for different $y$’s. The same happens for the cubes $Q_{y,m}^2$. Nevertheless, in the following lemma we show that we still have some kind of regularity. This regularity property will be essential for our purposes.

**Lemma 7.3.** Let $x, y$ be points in $\text{supp}(\mu)$. Then,
\begin{enumerate}
  \item If $Q_{x,m}^1 \cap Q_{y,m}^1 \neq \emptyset$, then $Q_{x,m}^1 \subset \hat{Q}_{y,m}^1$, in particular $x \in \hat{Q}_{y,m}^1$.
  \item If $Q_{x,m}^2 \cap Q_{y,m}^2 \neq \emptyset$, then $Q_{x,m}^2 \subset \hat{Q}_{y,m}^2$, in particular $x \in \hat{Q}_{y,m}^2$.
\end{enumerate}

So, although we cannot expect to have the equivalence
\[ y \in Q_{x,m}^1 \iff x \in Q_{y,m}^1, \]
we still have something quite close to it, because the cubes $Q_{x,m}^1$ and $\hat{Q}_{x,m}^1$ are close one each other in the quasimetric $D(\cdot, \cdot)$, since $\delta(Q_{x,m}^1, \hat{Q}_{x,m}^1)$ is small (at least in front of $A$). Of course, the same idea applies if we change 1 by 2 in the superscripts of the cubes.
Proof of Lemma 7.3. Let us proof the statement (a). The second statement is proved in an analogous way. Let \( x, y \) be as in (a). If \( \ell(Q^1_{y,m}) > \ell(Q^1_{x,m}) \) (in particular, \( Q^1_{y,m} \neq \{ y \} \)), then \( Q^1_{x,m} \subset 3Q^1_{y,m} \subset \hat{Q}^1_{y,m} \) (the latter inclusion holds provided \( \delta(\hat{Q}^1_{y,m}, 2R_0) < \delta(Q^1_{y,m}, 2R_0) - 6^n C_0 \)).

Assume now \( \ell(Q^1_{y,m}) \leq \ell(Q^1_{x,m}) \). If \( Q^1_{x,m} = \{ x \} \), then \( x = y \) and the result is trivial. If \( Q^1_{x,m} \neq \{ x \} \), we denote by \( P_y \) a cube centered at \( y \) with side length \( 3\ell(Q^1_{x,m}) \). Then, \( Q^1_{x,m} \subset P_y \subset 6Q^1_{x,m} \) and so \( \delta(Q^1_{x,m}, P_y) \leq 12^n C_0 \). Thus

\[
\delta(P_y, 2R_0) \geq \delta(Q^1_{x,m}, 2R_0) - \delta(Q^1_{x,m}, P_y) - \varepsilon_0 \geq \delta(Q^1_{x,m}, 2R_0) - 12^n C_0 - \varepsilon_0 \geq m A - \alpha_1 - \sigma + \varepsilon_1.
\]

Therefore, \( \hat{Q}^1_{y,m} \neq \{ y \} \) and \( \hat{Q}^1_{y,m} \supset P_y \supset Q^1_{x,m} \). \( \square \)

Now we are going to define the functions \( \varphi_{y,m} \). First we introduce the auxiliary functions \( \psi_{y,m} \).

**Definition 7.4.** For any \( y \in \text{supp}(\mu) \cap 2R_0 \), the function \( \psi_{y,m} \) is a function such that

1. \( 0 \leq \psi_{y,m}(x) \leq \min\left(\frac{4}{\ell(Q^1_{y,m})^n}, \frac{1}{|y-x|^n}\right) \),
2. \( \psi_{y,m}(x) = \frac{1}{|x-y|^n} \) if \( x \in \hat{Q}^2_{y,m} \setminus Q^1_{y,m} \),
3. \( \text{supp}(\psi_{y,m}) \subset Q^1_{y,m} \),
4. \( |\psi_{y,m}(x)| \leq C_{12} \min\left(\frac{1}{\ell(Q^1_{y,m})^{n+1}}, \frac{1}{|y-x|^{n+1}}\right) \).

It is not difficult to check that such a function exists if we choose \( C_{12} \) big enough. We have to take into account that \( 2\hat{Q}^2_{y,m} \subset Q^3_{y,m} \). This is due to the fact that \( \delta(\hat{Q}^2_{y,m}, 2\hat{Q}^2_{y,m}) \leq 4^n C_0 < \delta(\hat{Q}^2_{y,m}, \hat{Q}^3_{y,m}) \) if \( \ell(\hat{Q}^2_{y,m}) \neq 0 \).

In the definition of \( \psi_{y,m} \), if \( Q^1_{y,m} = \{ y \} \), then one must take \( 1/\ell(\hat{Q}^2_{y,m}) = \infty \). If \( \hat{Q}^2_{y,m} = \{ y \} \), then we set \( \psi_{y,m} \equiv 0 \). This choice satisfies the conditions for the definition of \( \psi_{y,m} \) stated above.

Choosing \( \alpha_2 \) big enough, the largest part of the \( L^1(\mu) \) norm of \( \psi_{y,m} \) will come from the integral over \( Q^2_{y,m} \setminus \hat{Q}^1_{y,m} \). We state this in a precise way in the following lemma.

**Lemma 7.5.** There exists some constant \( \varepsilon_2 \) depending on \( n, d, C_0, \varepsilon_0, \varepsilon_1 \) and \( \sigma \) (but not on \( \alpha_1, \alpha_2 \) nor \( A \)) such that if \( Q^1_{y,m} \neq \{ y \} \), then

\[
\|\psi_{y,m}\|_{L^1(\mu)} - \alpha_2 \leq \varepsilon_2 \tag{7.3}
\]

and

\[
\|\psi_{y,m}\|_{L^1(\mu)} - \int_{Q^2_{y,m} \setminus \hat{Q}^1_{y,m}} \frac{1}{|y-x|^n} d\mu(x) \leq \varepsilon_2. \tag{7.4}
\]
The proof of this result is an easy calculation that we will skip. A direct consequence of it is

$$\lim_{\alpha_2 \to \infty} \frac{1}{\alpha_2} \int_{Q_{y,m} \setminus \hat{Q}_{y,m}} \frac{1}{|y-x|^n} \, d\mu(x) = 1$$

for \( y \in \text{supp}(\mu) \) such that \( \delta(y,2R_0) > mA \).

**Definition 7.6.** Let \( w_{i,m} \) be the weight function defined for \( y \in \bigcup_{i \in I} Q_{i,m} \) (these are the cubes of \( D_m \) with \( \ell(Q_{i,m}) > 0 \)) by

$$w_{i,m}(y) = \frac{\chi_{Q_{i,m}}(y)}{\sum_{j \in I} \chi_{Q_{j,m}}(y)}.$$

If \( y \in \text{supp}(\mu) \cap 2R_0 \) belongs to some cube \( Q_{i,m} \) centered at some point \( y_i \), with \( \ell(Q_{i,m}) > 0 \), then we set

$$\varphi_{y,m}(x) = \alpha_2^{-1} \sum_i w_{i,m}(y) \psi_{y_i,m}(x).$$

If \( y \) does not belong to any cube \( Q_{i,m} \) with \( \ell(Q_{i,m}) > 0 \) (this implies \( \delta(y,2R_0) \leq mA \) and \( Q_{y,m} = \{y\} \)), then we set

$$\varphi_{y,m}(x) = \alpha_2^{-1} \psi_{y,m}(x).$$

Setting \( w_{i,m}(y) = \chi_{Q_{i,m}}(y) \) if \( \ell(Q_{i,m}) = 0 \), we can write

$$\varphi_{y,m}(x) = \alpha_2^{-1} \sum_i w_{i,m}(y) \psi_{y_i,m}(x),$$

for any \( y \).

Let us remark that a more natural definition for \( \varphi_{y,m} \) would have been the choice \( \varphi_{y,m}(x) = \alpha_2^{-1} \psi_{y,m}(x) \) for all \( y \). However, as we shall see, for some of the arguments in the proof of the Main Lemma below (in Subsection 8.2), the choice of Definition 7.6 is better.

In order to study some of the properties of the functions \( \varphi_{y,m} \), we need to introduce some additional notation.

**Definition 7.7.** Given \( x \in \text{supp}(\mu) \), we denote by \( \hat{Q}_{x,m}^3 \) a doubling cube centered at \( x \) such that \( \delta(\hat{Q}_{x,m}^3,2R_0) = mA - \alpha_1 - 2\sigma \pm \epsilon_1 \). Also, we denote by \( \hat{Q}_{x,m}^1 \) and \( \hat{Q}_{x,m}^1 \) some doubling cubes centered at \( x \) such that

$$\delta(\hat{Q}_{x,m}^1,2R_0) = mA - \alpha_1 + \sigma \pm \epsilon_1,$$

$$\delta(\hat{Q}_{x,m}^1,2R_0) = mA - \alpha_1 + 2\sigma \pm \epsilon_1$$

(the idea is that the symbols \( \hat{^} \) and \( \hat{^} \) are inverse operations, modulo some small errors). If any of the cubes \( \hat{Q}_{x,m}^1, \hat{Q}_{x,m}^1, \hat{Q}_{x,m}^3, \hat{Q}_{x,m}^3, \hat{Q}_{x,m}^3 \) does not exist, then we let it be the point \( x \).
So, when \( \delta(x, 2R_0) \) is big enough, one should think that \( \hat{Q}^3_{x,m} \) is a cube a little bigger than \( \check{Q}^3_{x,m} \), while \( Q^1_{x,m} \) is a little smaller than \( Q^1_{x,m} \). Also, \( Q^1_{x,m} \) is a little smaller than \( \check{Q}^1_{x,m} \), but still much bigger than \( Q_{x,m} \).

**Lemma 7.8.** Let \( x, y \in \text{supp}(\mu) \). For \( \alpha_1 \) and \( \alpha_2 \) big enough, we have:

(a) If \( x \in Q_{x_0,m} \) and \( y \notin \check{Q}^3_{x_0,m} \), then \( \varphi_{y,m}(x) = 0 \). In particular, \( \varphi_{y,m}(x) = 0 \) if \( y \notin \check{Q}^3_{x,m} \).

(b) If \( y \in \check{Q}^1_{x,m} \), then \( \varphi_{y,m}(x) \leq C \frac{\alpha_2^{-1}}{\ell(\check{Q}^1_{x,m})} \).

(c) Let \( \varepsilon_3 > 0 \) be an arbitrary constant. If \( \alpha_1 \) is big enough (depending on \( \varepsilon_3, C_0, n, d \) but not on \( \alpha_2 \)), then

\[
\varphi_{y,m}(x) \leq \frac{\alpha_2^{-1}(1 + \varepsilon_3/2)}{|y - x|^n} \quad \text{if } y \notin \check{Q}^1_{x,m},
\]

and

\[
\varphi_{y,m}(x) \geq \frac{\alpha_2^{-1}(1 - \varepsilon_3/2)}{|y - x|^n} \quad \text{if } y \in Q^2_{x,m} \setminus \check{Q}^1_{x,m}.
\]

(d) If \( x \in Q_{x_0,m} \), then

\[
|\varphi'_{y,m}(x)| \leq C \alpha_2^{-1} \min \left( \frac{1}{\ell(\hat{Q}^1_{x_0,m})^{n+1}}, \frac{1}{|y - x|^{n+1}} \right).
\]

Notice that, in Definition 7.4 of the functions \( \psi_{y,m} \), the properties that define these functions are stated with respect to cubes centered at \( y \) \( (Q^1_{y,m}, Q^2_{y,m}, Q^3_{y,m}...) \). In this lemma some analogous properties are stated, but these properties have to do with cubes centered at \( x \) or containing \( x \) \( (Q_{x_0,m}, Q^1_{x,m}, Q^2_{x,m}, Q^3_{x,m}...) \).

**Proof.**

(a) Let \( x_0 \in \text{supp}(\mu) \) and \( x \in Q_{x_0,m} \). If \( \varphi_{y,m}(x) \neq 0 \), there exists some \( i \) with \( y \in Q_i,m \equiv Q_{y,m} \) and \( x \in Q^3_{y,m} \). Then \( Q^3_{x_0,m} \cap Q^3_{y,m} \neq \emptyset \) and so \( y \in Q^3_{y,m} \subset \check{Q}^3_{x_0,m} \) (as in Lemma 7.3).

(b) Let \( y \in \check{Q}^1_{x,m} \) and let \( y_i \) be such that \( y \in Q_{y_i,m} \). We know that

\[
\varphi_{y_i,m}(x) \leq C \alpha_2^{-1} \frac{1}{\ell(Q_{y_i,m})^{n+1}}.
\]

So we are done if we see that \( \ell(Q^1_{y_i,m}) \geq \ell(\check{Q}^1_{x,m}) \).

As in Lemma 7.3, we have

\[
y \in \check{Q}^1_{x,m} \Rightarrow \hat{Q}^1_{y_i,m} \cap \check{Q}^1_{x,m} \neq \emptyset \Rightarrow \hat{Q}^1_{x,m} \subset Q^1_{y_i,m}.
\]

Thus \( \ell(\check{Q}^1_{x,m}) \leq \ell(Q^1_{y_i,m}) \).
(c) Let us see the first inequality. If \( y \notin \hat{Q}_{x,m}^1 \) and \( y \) belongs to some cube \( Q_{y,m} \) with \( \ell(Q_{y,m}) > 0 \), then \( x \notin \hat{Q}_{y,m}^1 \) because otherwise, as in Lemma 7.3, we would get \( \hat{Q}_{y,m}^1 \subset Q_{x,m}^1 \). However, since we assume \( \alpha_1 \gg \sigma \), the cube \( \hat{Q}_{y,m}^1 \) is bigger than \( Q_{y,m} \) and contains \( y \). So \( y \in Q_{x,m}^1 \), which is a contradiction.

Since \( x \notin \hat{Q}_{y,m}^1 \) and this cube is much bigger than \( Q_{y,m} \), if \( \alpha_1 \) is big enough we get

\[
\frac{\alpha_2^{-1}}{|y_i - x|^n} \leq \frac{\alpha_2^{-1} (1 + \varepsilon_3)}{|y - x|^n}.
\]

As this holds for all \( i \) with \( w_{i,m}(y) \neq 0 \), we obtain

\[
\varphi_{y,m}(x) \leq \frac{\alpha_2^{-1}(1 + \varepsilon_3)}{|y - x|^n}.
\]

This inequality also holds if \( \ell(Q_{y,m}) = 0 \) with \( \varepsilon_3 = 0 \), since in this case \( y_i = y \).

We consider now the second inequality in (c). Let \( y \in \text{supp}(\mu) \) be such that \( y \in Q_{x,m}^2 \setminus \hat{Q}_{x,m}^1 \). If \( y \in Q_{y,m} \) with \( \ell(Q_{y,m}) > 0 \) for some \( i \), by Lemma 7.3 we get \( x \in \hat{Q}_{y,m}^2 \setminus Q_{y,m}^1 \). Since this is satisfied for all \( i \) such that \( w_{i,m}(y) \neq 0 \),

\[
\varphi_{y,m}(x) = \sum_i w_{i,m}(y) \frac{\alpha_2^{-1}}{|y_i - x|^n}.
\]

If \( \alpha_1 \) has been chosen big enough, then \( \ell(Q_{y,m}) \gg \ell(Q_{y,m}) \) and one has

\[
\frac{\alpha_2^{-1}}{|y_i - x|^n} \geq \frac{\alpha_2^{-1}(1 - \varepsilon_3/2)}{|y - x|^n}.
\]

Thus

\[
(7.5) \quad \varphi_{y,m}(x) \geq \frac{\alpha_2^{-1}(1 - \varepsilon_3/2)}{|y - x|^n}.
\]

If \( y \in Q_{x,m}^2 \setminus \hat{Q}_{x,m}^1 \) and \( y \in Q_{i,m} \) with \( \ell(Q_{i,m}) = 0 \), then by Lemma 7.3 we also get \( x \in \hat{Q}_{y,m}^2 \setminus Q_{y,m}^1 \) (in particular \( \hat{Q}_{y,m}^2 \neq \{y\} \)). Then (7.5) holds in this case too (with \( \varepsilon_3 = 0 \)).

(d) Suppose first that \( y \in Q_{x,0,m}^1 \). In this case we must show that

\[
|\varphi_{y,m}'(x)| \leq C \frac{\alpha_2^{-1}}{\ell(Q_{x,0,m}^1)^n+1}.
\]

Let \( y_i \) be such that \( y \in Q_{y,m} \). We know that

\[
|\varphi_{y,m}'(x)| \leq C \frac{\alpha_2^{-1}}{\ell(Q_{y,m}^1)^n+1}.
\]
By the definition of $\varphi_y(x)$, it is enough to see that $\ell(Q_{y,m}^1) \geq \ell(\tilde{Q}_{x_0,m}^1)$. This follows from the inclusion $Q_{y,m}^1 \supset \tilde{Q}_{x_0,m}^1$, which holds because $y \in Q_{y,m}^1 \cap \tilde{Q}_{x_0,m}^1$ and then we can apply Lemma 7.3 (in fact, a slight variant of Lemma 7.3).

Suppose now that $y \notin \tilde{Q}_{x_0,m}^1$. It is enough to show that

$$|\varphi_{y,m}'(x)| \leq C \frac{\alpha_2^{-1}}{|y - x|^{n+1}}.$$

Let $y_i$ be such that $y \in Q_{y_i,m}$. By definition we have

$$|\varphi_{y_i,m}'(x)| \leq C \frac{\alpha_2^{-1}}{|y_i - x|^{n+1}}.$$

We are going to see that

$$|y - y_i| \leq |y - x|/2. \tag{7.6}$$

Assume $|y - y_i| > |y - x|/2$. Then, since $x \in \frac{1}{2} \tilde{Q}_{x_0,m}^1$ (for $\alpha_1$ big enough),

$$\ell(Q_{y,m}^1) > C^{-1} |y - x| \geq C^{-1} \ell(\tilde{Q}_{x_0,m}^1). \tag{7.7}$$

Notice that from the first inequality in (7.7) we get $\text{dist}(x,Q_{y_i,m}) \leq C \ell(Q_{y,m}^1)$. In this situation we have $\tilde{Q}_{x_0,m}^1 \subset C Q_{y,m} \subset \tilde{Q}_{y_i,m}^1$. This is not possible, since by Lemma 7.3 we would have $\tilde{Q}_{x_0,m}^1 \supset \tilde{Q}_{y_i,m}^1$, and then we would get $\tilde{Q}_{x_0,m}^1 = \tilde{Q}_{y_i,m}^1$. This would imply $x_0 = y_i$ and also $x_0 = y_i = \tilde{Q}_{x_0,m}^1 = \tilde{Q}_{y_i,m}^1$, and then $y = y_i$ which is a contradiction because we are assuming that (7.6) does not hold.

So (7.6) is true and $|y_i - x| \approx |y - x|$. Thus

$$|\varphi_{y_i,m}'(x)| \leq C \frac{\alpha_2^{-1}}{|y - x|^{n+1}}.$$

Since this holds for any $i$ such that $y \in Q_{y_i,m}$, we get

$$|\varphi_{y,m}'(x)| \leq C \frac{\alpha_2^{-1}}{|y - x|^{n+1}}.$$

□

Some of the estimates in the preceding lemma will be used to prove next result, which was one of our main goals in this section.

**Lemma 7.9.** For any $\varepsilon_3 > 0$, if $\alpha_1$ and $\alpha_2$ are big enough, for all $x \in \text{supp}(\mu)$ we have

$$\int \varphi_{y,m}(x) d\mu(y) \leq 1 + \varepsilon_3. \tag{7.8}$$
If \( x \in \text{supp}(\mu) \) is such that there exists some cube \( Q \in D_m \) with \( Q \ni x \) and \( \ell(Q) > 0 \) (in particular if \( \delta(x, 2R_0) > mA \)), then

\[
(7.9) \quad 1 - \epsilon_3 \leq \int \varphi_{y,m}(x) \, d\mu(y)
\]

Let us observe that if \( \mu \) were invariant by translations and \( \varphi_{y,m}(x) = \varphi_m(y - x) \), then (7.8) and (7.9) would hold with \( \epsilon_3 = 0 \) (choosing \( \|\varphi_{y,m}\|_{L^1(\mu)} = 1 \)).

**Proof.** Let us see (7.9) first. So we assume that there exist some cube \( Q_{i,m} \in D_m \) containing \( x \) with \( \ell(Q_{i,m}) > 0 \). Since \( x \in Q_{i,m} \subset \tilde{Q}_{i,m} \), we have \( \tilde{Q}_{i,m} \subset Q_{i,m} \). In particular, \( \ell(Q_{i,m}^1) > 0 \). By Lemma 7.5 and the second inequality of (c) in Lemma 7.8 we get

\[
\int \varphi_{y,m}(x) \, d\mu(y) \geq \int_{Q_{i,m}^2 \setminus Q_{i,m}^1} \varphi_{y,m}(x) \, d\mu(y)
\]

\[
\geq \int_{Q_{i,m}^2 \setminus Q_{i,m}^1} \frac{\alpha_2^{-1} (1 - \epsilon_3/2)}{|y - x|^n} \, d\mu(y)
\]

\[
\geq \alpha_2^{-1} \left( \alpha_2 - 2\epsilon_2 \right) (1 - \epsilon_3/2).
\]

(7.11)

So (7.9) holds if we take \( \alpha_2 \) big enough.

Consider now (7.8). By (a) in Lemma 7.8 have

\[
\int \varphi_{y,m}(x) \, d\mu(y) = \int_{\tilde{Q}_{i,m}} \varphi_{y,m}(x) \, d\mu(y).
\]

Thus we can write

\[
(7.10) \quad \int \varphi_{y,m}(x) \, d\mu(y) = \int_{\tilde{Q}_{i,m}^2 \setminus \tilde{Q}_{i,m}^1} \varphi_{y,m}(x) \, d\mu(y) + \int_{\tilde{Q}_{i,m}^1} \varphi_{y,m}(x) \, d\mu(y).
\]

Let us estimate the first integral on the right hand side of (7.10). Using the first inequality in (c) of Lemma 7.8 we obtain

\[
\int_{\tilde{Q}_{i,m}^2 \setminus \tilde{Q}_{i,m}^1} \varphi_{y,m}(x) \, d\mu(y) \leq \int_{\tilde{Q}_{i,m}^2 \setminus \tilde{Q}_{i,m}^1} \frac{\alpha_2^{-1} (1 + \epsilon_3/2)}{|y - x|^n} \, d\mu(y)
\]

\[
= \delta(\tilde{Q}_{i,m}^1, \tilde{Q}_{i,m}^3) \alpha_2^{-1} \left(1 + \epsilon_3/2\right)
\]

\[
\leq \alpha_2^{-1} \left( \alpha_2 + 4\sigma + 2\epsilon_1 \right) (1 + \epsilon_3/2).
\]

(7.11)

Let us consider the last integral in (7.10) (only in the case \( \tilde{Q}_{i,m}^1 \neq \{x\} \)). By (b) in Lemma 7.8 we have

\[
(7.12) \quad \int_{\tilde{Q}_{i,m}^1} \varphi_{y,m}(x) \, d\mu(y) \leq \int_{\tilde{Q}_{i,m}^1} \frac{C \alpha_2^{-1}}{\ell(Q_{i,m}^1)^n} \, d\mu(y) \leq C C_0 \alpha_2^{-1}.
\]

From (7.11) and (7.12) we get (7.8).
8. Proof of the Main Lemma

8.1. The argument. As stated above, $A$ is a large positive integer that will be fixed at the end of the proof. We assume that the support of $f$ is contained in some doubling cube $R_0$, and for each integer $m \geq 1$ we consider the family $D_m$ of “dyadic” cubes $Q_{i,m}, i \in I_m$, introduced in Definition 6.1, and we set $D = \bigcup_{m \geq 1} D_m$. Recall that the elements of $D$ may be cubes with side length 0, i.e. points.

For each $m$ we will construct functions $g_m$ and $b_m$. The function $g_m$ will be supported on a subfamily $D^G_m$ of the cubes in $D_m$. On the other hand, $b_m$ will be supported on a subfamily $D^B_m$ of the cubes in $D_m$. We set $D^G = \bigcup_{m \geq 1} D^G_m$ and $D^B = \bigcup_{m \geq 1} D^B_m$. The cubes in $D^G$ will be called good cubes and the ones in $D^B$ bad cubes (let us remark that in the family $D_m$, in general, there are also cubes which are neither good nor bad).

From $g_m$ and $b_m$, we will obtain the following potentials:

\[
U^G_m(x) = \int \varphi_{y,m}(x) g_m(y) \, d\mu(y),
\]

\[
U^B_m(x) = \int \varphi_{y,m}(x) b_m(y) \, d\mu(y),
\]

\[
U_m(x) = U^G_m(x) + U^B_m(x).
\]

This potentials will be successively subtracted from $f$. We will set

\[
f_{m+1}(x) = f(x) - \sum_{j=1}^m U_j(x) = f_m(x) - U_m(x)
\]

and

\[
h_0 = f - \sum_{m=1}^{\infty} U_m = \lim_{m \to \infty} f_m.
\]

The support of the functions $g_m$, $b_m$, $U^G_m$, $U^B_m$ will be contained in $2R_0$.

By induction we will show that the functions $g_m$, $b_m$, $U_m$ and $f_m$ fulfil the following properties:

(a) $|g_m|, |b_m| \leq C_8 A \|f\|_s$.

(b) $|m_Q f_{m+1}| \leq A \|f\|_s$ if $Q \in D_m$ and $\ell(Q) > 0$.

(c) If $g_m \not\equiv 0$ on $Q$, $Q \in D_m$, with $\ell(Q) > 0$, then $|m_Q f_{m+1}| \leq \frac{7}{20} A \|f\|_s$.

(d) If $Q \in D_m$ and $|m_Q f_m| \leq \frac{8}{20} A \|f\|_s$, then $U_m \equiv 0$ and $g_m \equiv b_m \equiv 0$ on $Q$.

(e) If $Q \in D_m$ and $\delta(Q, 2R_0) \leq (m - \frac{1}{16}) A$ (so $\ell(Q) = 0$), then $U_m \equiv 0$ and $g_m \equiv b_m \equiv 0$ on $Q$.

Finally, we will see that our construction satisfies the following properties too:
(f) If $\delta(x, 2R_0) < \infty$, then $|h_0(x)| \leq C_3 A \|f\|_*$, and if $Q \in D_m$ and $\ell(Q) = 0$, then $|m_{Qf_{m+1}}| = |f_{m+1}(z_Q)| \leq C_9 A \|f\|_*$

(g) For each $m$, there are functions $y_1^m, \ldots, y_N^m$ such that

\(\text{(g.1)} \quad U_m^G(x) = \sum_{p=1}^{N} \int_\varphi_{y,m}^p(x) g_p^m(y) d\mu(y)\), where $\varphi_{y,m}^p$ is defined below.

\(\text{(g.2)} \quad |g_p^m| \leq 2C_8 A \|f\|_* \text{ for } p = 1, \ldots, N,\)

\(\text{(g.3)} \quad \text{The functions } \sum_{p=1}^{N} |g_p^m| \text{ have disjoint supports for different } m.\)

(h) The family of cubes $D^B$ that support the functions $b_m$, $m \geq 1$, satisfies the following Carleson packing condition for each cube $R \in D_m$ with $\ell(R) > 0$:

\[
\sum_{Q: Q \cap R \neq \emptyset} \mu(Q) \leq C \mu(R).
\]

Let us remark that if some cube $Q$ coincides with a point $\{x\}$, then we set $m_{Qf_m} = f_m(x)$. Also, the notation for the sum in (h) is an abuse of notation. This sum has to be understood as

\[
\sum_{Q: Q \cap R \neq \emptyset, \ |Q| > 2R} \mu(Q) \equiv \sum_{Q: \ell(Q) > 0, \ |Q| > 2R} \mu(Q) + \sum_{k>m} \mu\{x \in 2R : \{x\} \in D^B_k\}.
\]

On the other hand, the number $N$ that appears in (g) is the number of disjoint families of cubes given in the Covering Theorem of Besicovitch, which only depends only on $d$.

The functions $\varphi_{y,m}^p$ of (g) are defined as follows. We set $D_m = D^1_m \cup \cdots \cup D^N_m$, where each subfamily $D^p_m$ is disjoint (recall that the cubes of $D_m$ originated from Besicovitch's Covering Theorem). Then we set

$\varphi_{y,m}^p(x) = \varphi_{y,m}(x)$

if $y \in Q_{i,m}^p$ with $Q_{i,m}^p \in D^p_m$, and $\varphi_{y,m}(x) = 0$ if there does not exist any cube of the subfamily $D^p_m$ containing $y$.

First we will show that if there exist functions $g_m$ and $b_m$ satisfying (a)–(h) then the Main Lemma follows, and later we will show the existence of these functions.

It is not difficult to check that if (4.1) and (4.2) hold, then the sum of (8.1) converges in $L^1_{loc}(\mu)$ (this is left to the reader). Since the support of all the functions involved is contained in $2R_0$, the convergence is in $L^1(\mu)$.

Let us see now that if (b) and (f) hold, then $\|h_0\|_{L^\infty(\mu)} \leq C A \|f\|_*$. Because of (f), we only have to see that $|h_0(x)| \leq C A \|f\|_*$ for $x \in \text{supp}(\mu)$ such that $\delta(x, 2R_0) = \infty$. In this case, if $Q \in D_k$ is such that $x \in Q$, then $\ell(Q) > 0$. 


For \( \lambda > 0 \), we denote
\[
F_0 = \{ x \in \text{supp}(\mu) : \delta(x, 2R_0) = \infty, |h(x)| > A \|f\|_* + \lambda \}.
\]
Given \( k, k_0 \geq 1 \), take \( m_k \) such that \( \|f_{m_k} - h_0\|_{L^1(\mu)} \leq 2^{-k-k_0} \), and set
\[
F_1 = \{ x \in \text{supp}(\mu) : \exists Q \in \bigcup_k D_{m_k-1}, x \in Q, |m_Q(h_0)| > A \|f\|_* + \lambda \}.
\]
By the Lebesgue differentiation theorem, we have \( \mu(F_0 \setminus F_1) = 0 \) (this theorem can be applied to the cubes \( Q \in D \) which are non centered because they are doubling). Consider \( Q \in D_{m_k-1} \) such that \( |m_Q(h_0)| > A \|f\|_* + \lambda \).
Since \( |m_Q(f_{m_k})| \leq A \|f\|_* \) for \( Q \in D_{m_k-1} \) [by (b)] and
\[
|m_Q(h_0)| \leq |m_Q(h_0 - f_{m_k})| + |m_Q(f_{m_k})|,
\]
we deduce that \( |m_Q(h_0 - f_{m_k})| > \lambda \). Thus, if \( x \in F_1 \), then \( |m_Q(h_0 - f_{m_k})| > \lambda \) for some dyadic cube which contains \( x \) and some \( k \geq 1 \). By the usual arguments (using the fact that our dyadic cubes are doubling),
\[
\mu(F_0) \leq \mu(F_1) \leq C \sum_k \frac{|h_0 - f_{m_k}|_{L^1(\mu)}}{\lambda} \leq C \sum_k \frac{2^{-k-k_0}}{\lambda} = C \frac{2^{-k_0}}{\lambda}.
\]
Since this holds for any integer \( k_0 \), we have \( \mu(F_0) = 0 \) for all \( \lambda > 0 \). Thus, \( |h(x)| \leq A \|f\|_* \) for \( \mu \)-a.e. \( x \in \text{supp}(\mu) \) such that \( \delta(x, 2R_0) = \infty \).

Observe that the functions \( g_p^m \) in (g.1) originate the same potential as \( g_m \).
In fact, they will be constructed modifying slightly the function \( g_m \) in such a way that they are supported in disjoint sets for different \( m \)'s. By (g.2) we have
\[
\sum_m \sum_{p=1}^N |g_p^m| \leq 2N C_8 A \|f\|_*.
\]

The supports of the functions \( b_m \) may be not disjoint. To solve this problem, we will construct “corrected” versions \( (b_p^m, p = 1, \ldots, N) \) of \( w_{i,m} b_m \). Moreover, as in the case of \( g_m \), the modifications will be made in such a way that the potentials \( U_{m}^B \) will not change.

### 8.2. The “correction” of \( b_m \)

We assume that the functions \( b_m, m \geq 1 \), have been obtained and they satisfy (a)–(h). We will start the construction of some new functions (the corrected versions of \( w_{i,m} b_m \)) in the small cubes, and then we will go over the cubes from previous generations. However, since there is an infinite number of generations, we will need to use a limiting argument.

For each \( j \) we can write the potential originated by \( b_j \) as
\[
U_j^B(x) = \sum_{i \in I_j} \varphi_{y_i,j}(x) \int w_{i,j}(y) b_j(y) d\mu(y).
\]
For a fixed $m \geq 1$ we are going to define functions $v^m_{i,j}$, for $j = m, m-1, \ldots, 1$, and all $i \in I_j$. The functions $v^m_{i,j}$ will satisfy

\[(8.3) \quad \text{supp}(v^m_{i,j}) \subset Q_{i,j},\]

where $Q_{i,j} \in \mathcal{D}^B_j$, the sign of $v^m_{i,j}$ will be constant on $Q_{i,j}$, and

\[(8.4) \quad \int v^m_{i,j}(y) \, d\mu(y) = \int w_{i,j}(y) b_j(y) \, d\mu(y).\]

Moreover, we will also have

\[(8.5) \quad \sum_{j=1}^{m} \sum_{i \in I_j} |v^m_{i,j}| \leq C_{10} A \|f\|_*.\]

We set $v^m_{i,m}(y) = w_{i,m}(y) b_m(y)$ for all $i \in I_m$. Assume that we have obtained functions $v^m_{i,m}, v^m_{i,m-1}, \ldots, v^m_{i,k+1}$ for all the $i$’s, fulfilling (8.3), (8.4), and such that

\[\sum_{j=k+1}^{m} \sum_{i \in I_j} |v^m_{i,j}| \leq B A \|f\|_*,\]

where $B$ is some constant that will be fixed below. We are going to construct $v^m_{i,k}$ now.

Let $Q_{i_0,k} \in \mathcal{D}_k$ be some fixed cube from the $k$-th generation. Assume first that $Q_{i_0,k}$ is not a single point. Since the cubes in the family $\mathcal{D}^B$ satisfy the packing condition (8.2), for any $t > 0$ we get

\[
\mu\left\{ y \in Q_{i_0,k} : \sum_{j=k+1}^{m} \sum_{i \in I_j} |v^m_{i,j}(y)| > t \right\} \\
\leq \frac{1}{t} \sum_{j=k+1}^{m} \sum_{i \in I_j} \int_{Q_{i_0,k}} |v^m_{i,j}(y)| \, d\mu(y) \\
\leq \frac{1}{t} \sum_{j=k+1}^{m} \sum_{i \in I_j} \int_{Q_{i_0,k}} |w_{i,j}(y) b_j(y)| \, d\mu(y) \\
\leq \frac{C_8 A \|f\|_*}{t} \sum_{Q : Q \cap Q_{i_0,k} \neq \emptyset} \frac{\mu(Q)}{t} \mu(Q_{i_0,k}).
\]

Therefore, if we choose $t = 2C_{12} A \|f\|_*$ and we denote

\[V^m_{i_0,k} = \left\{ y \in Q_{i_0,k} : \sum_{j=k+1}^{m} \sum_{i \in I_j} |v^m_{i,j}(y)| \leq t \right\},\]

we have $\mu(V^m_{i_0,k}) \geq \frac{1}{2} \mu(Q_{i_0,k})$. If we set $v^m_{i_0,k} = c^m_{i_0,k} \chi_{V^m_{i_0,k}}$, where $c^m_{i_0,k} \in \mathbb{R}$ is such that (8.4) holds for $i = i_0$, then

\[|c^m_{i_0,k}| \leq \frac{1}{\mu(V^m_{i_0,k})} \int |w_{i,k}(y) b_k(y)| \, d\mu(y) \leq 2C_8 A \|f\|_*.
\]
By the finite overlap of the cubes in $D_k^B$, we get

$$\sum_{i_0: Q_{i_0,k} \in D_k^B \cap Q_{i_0,k} \neq 0} |v_{i_0,k}^m| \leq C_B 2 C_8 A \|f\|_*,$$

where $C_B$ is the overlap constant in the Covering Theorem of Besicovitch. Now if we take $B := 2 C_B C_8 + 2 C_{12}$, we will have

$$(8.6) \quad \sum_{i_0: Q_{i_0,k} \in D_k^B \cap Q_{i_0,k} \neq 0} |v_{i_0,k}^m| + \sum_{j=k+1}^m \sum_{i \in I_j} |v_{i,j}^m| \leq B A \|f\|_*.$$  

In case $Q_{i_0,k}$ is a single point $\{y\}$, then we set $v_{i_0,k}^m(y) = w_{i_0,k}(y) b_k(y) = b_k(y)$. All the cubes of the generations $k+1, \ldots, m$ that intersect $Q_{i_0,k} \equiv \{y\}$ coincide with $\{y\}$ by Lemma 6.3. From (e) we get that $b_{k+1}(y) = b_{k+2}(y) = \cdots = 0$, which is the same as saying that $v_{i_0,k+1}^m(y) = v_{i_0,k+2}^m(y) = \cdots = 0$ for all $i$. So we have

$$(8.7) \quad \sum_{j=k+1}^m \sum_{i \in I_j} |v_{i,j}^m(y)| = |b_k(y)| \leq C_8 A \|f\|_* \leq B A \|f\|_*.$$  

From (8.6) and (8.7) we get

$$\sum_{j=k+1}^m \sum_{i \in I_j} |v_{i,j}^m| \leq B A \|f\|_*.$$  

Operating in this way, the functions $v_{i,j}^m$, $j = m, m-1, \ldots, 1$, $i \in I_j$, will satisfy the conditions (8.3), (8.4) and (8.5) (with $C_{11} = B$).

Now we can take a subsequence $\{m_k\}_k$ such that for all $i \in I_1$ (i.e. for all the cubes of the first generation) the functions $\{v_{i,1}^{m_k}\}_k$ converge weakly in $L^\infty(\mu)$ to some function $v_{i,1} \in L^\infty(\mu)$. Let us remark that the sequence $\{m_k\}_k$ can be chosen independently of $i$ since, by the Besicovitch’s Covering Theorem, there is a bounded number $N$ of subfamilies $D_1^1, \ldots, D_1^N$ of $D_1$ such that each subfamily $D_1^p$ is disjoint. If we denote by $D_1^{p,B}$ the subfamily of bad cubes of $D_1^p$, we can write

$$\sum_{i \in I_1} v_{i,1}^{m_k} = \sum_{p=1}^N \sum_{i: Q_{i,1} \in D_1^{p,B}} v_{i,1}^{m_k},$$

and we can choose $\{m_k\}_k$ such that, for each $p$, $\sum_{i: Q_{i,1} \in D_1^{p,B}} v_{i,1}^{m_k}$ converges weakly to $\sum_{i: Q_{i,1} \in D_1^{p,B}} v_{i,1}$.

In a similar way, we can consider another subsequence of $\{m_k\}_j$ of $\{m_k\}_k$ such that for all $i \in I_2$ the functions $\{v_{i,2}^{m_k}\}_j$ converge weakly in $L^\infty(\mu)$ to some function $v_{i,2} \in L^\infty(\mu)$. Going on with this process, we will obtain
functions $v_{i,j}$, $j \geq 1$, that satisfy (8.3), (8.4) (without the superscript $m$) and

\begin{equation}
\sum_{j=1}^{\infty} \sum_{i \in I_j} |v_{i,j}| \leq C_{11} A \|f\|_*.
\end{equation}

Also, we have

\[ U_j^B(x) = \sum_{i \in I_j} \varphi_{y,j}(x) \int v_{i,j}(y) \, d\mu(y). \]

We denote $D_m^p = D_m^p \cap D_m^B$ and

\[ b_m^p(y) = \sum_{i : Q_{i,m} \in D_m^B} v_{i,m}(y). \]

Recall also that $\varphi_{y,m}^p(x) = \varphi_{y,m}(x)$ if $y \in Q_{i,m}$ with $Q_{i,m} \in D_m^p$, and $\varphi_{y,m}^p(x) = 0$ if there does not exist any cube of the subfamily $D_m^p$ containing $y$. Then we have

\[ U_m^B(x) = \sum_{p=1}^{N} \int \varphi_{y,m}^p(x) b_m^p(y) \, d\mu(y). \]

Now we set $h_m^p = g_m^p + b_m^p$, and we get

\[ f(x) = h_0(x) + \sum_{p=1}^{N} \sum_{m=1}^{\infty} \int \varphi_{y,m}^p(x) h_m^p(y) \, d\mu(y), \]

with $C \varphi_{y,m} \sim y$ for some constant $C > 0$, and

\[ |h_0| + \sum_{p=1}^{N} \sum_{m=1}^{\infty} |h_m^p| \leq C A \|f\|_*, \]

and the Main Lemma follows, by (g) and (8.8).

8.3. The construction of $g_m$ and $b_m$. In this subsection we will construct inductively functions $g_m$ and $b_m$ satisfying the properties (a)–(e). We will check in Subsection 8.4 that these functions fulfil (f)–(h) too.

Assume that $g_1, \ldots, g_{m-1}$ and $b_1, \ldots, b_{m-1}$ have been constructed and they satisfy (a)–(e). Let $\Omega_m$ be the set of points $x \in \text{supp}(\mu)$ with $\delta(x, 2R_0) > m A$ such that there exists some $Q \in D_m$, $\ell(Q) > 0$, with $Q \ni x$ and $|m_Q f_m| \geq \frac{3}{4} A$. For each $x \in \Omega_m$, we consider a doubling cube $S_{x,m}$ centered at $x$ such that $\delta(S_{x,m}, 2R_0) = m A - \alpha_1 - \alpha_2 - \alpha_3 \pm \varepsilon_1$, where $\alpha_3$ is some big constant with $10 \alpha_2 < \alpha_3 \ll A$, whose precise value will be fixed below. One has to think that $S_{x,m}$ is much bigger than $Q_{x,m}^2$ but much smaller than $Q_{x,m-1}$ (observe that all these cubes have positive side length).

Now we take a Besicovitch covering of $\Omega_m$ with cubes of type $S_{x,m}$, $x \in \Omega_m$:

\[ \Omega_m \subset \bigcup_j S_{j,m}, \]
where $S_{j,m}$ stands for $S_{x_j,m}$, with $x_j \in \Omega_m$. We say that a cube $Q \in \mathcal{D}_m$ is
good (i.e. $Q \in \mathcal{D}_m^G$) if
\[ Q \subset \bigcup_j 3^{2S_{j,m}}, \]
and we say that it is bad (i.e. $Q \in \mathcal{D}_m^B$) if it is not good and
\[ Q \subset \bigcup_j 2^{2S_{j,m}}. \]
Both good and bad cubes are contained in $\bigcup_j 2^{2S_{j,m}}$. Roughly speaking,
the difference between good and bad cubes is that bad cubes may be sup-
ported near the boundary of $\bigcup_j 2^{2S_{j,m}}$, while the good ones are far from the
boundary.

Now we define $g_m$ and $b_m$:
\[ g_m = \sum_{i : Q_{i,m} \in \mathcal{D}_m^G} w_{i,m} m_{Q_{i,m}}(f_m), \]
\[ b_m = \sum_{i : Q_{i,m} \in \mathcal{D}_m^B} w_{i,m} m_{Q_{i,m}}(f_m). \]
Because there is some overlapping among the cubes in $\mathcal{D}_m$, we have used
the weights $w_{i,m}$ in the definition of these functions. However one should
think that $g_m$ and $b_m$ are approximations of the mean of $f$ over the cubes
of $\mathcal{D}_m^G$ and $\mathcal{D}_m^B$, respectively.

The following remark will be useful.

**Claim 1.** Let $Q_{h,m} \in \mathcal{D}_m$ be such that either $g_m \neq 0$, $b_m \neq 0$ or $U_m \neq 0$
on $Q_{h,m}$. Then there exists some $j$ such that $\hat{Q}_{h,m}^3 \subset 4S_{j,m}$ and so $Q_{h,m} \subset 4S_{j,m}$.

**Proof.** In the first two cases $Q_{h,m} \cap 2S_{j,m} \neq \emptyset$ for some $j$. In the latter case,
by (a) of Lemma 7.8 and our construction, there exists some $j$ such that
$\hat{Q}_{h,m}^3 \cap 2S_{j,m} \neq \emptyset$.

So in any case $\hat{Q}_{h,m}^3 \cap 2S_{j,m} \neq \emptyset$ for some $j$. Arguing as in Lemma 6.3,
for $\alpha_3$ big enough, it is easily checked that $\ell(\hat{Q}_{h,m}^3) \leq \ell(S_{j,m})/4$, and so
$\hat{Q}_{h,m}^3 \subset 4S_{j,m}$. \qed

Let us see now that (e) is satisfied.

**Claim 2.** If $Q \in \mathcal{D}_m$ and $\delta(Q, 2R_0) \leq (m - \frac{1}{10}) A$ (so $\ell(Q) = 0$), then
$U_m \equiv g_m \equiv b_m \equiv 0$ on $Q$ and $Q \not\in \mathcal{D}_m^G \cup \mathcal{D}_m^B$. 

First we will prove the first statement. By Claim 2, we know that
\[
\delta(x, 2R_0) = \delta(x, 4S_{j,m}) + \delta(4S_{j,m}, 2R_0) + \varepsilon_0 \\
\geq \delta(4S_{j,m}, 2R_0) - \varepsilon_0 \\
\geq \delta(S_{j,m}, 2R_0) - 8^n C_0 - \varepsilon_0 > \left( m - \frac{1}{10} \right) A.
\]

The following estimate will be necessary in many steps of our construction.

**Claim 3.** Let \( Q \) be some cube of the \( m \)-th generation and \( x, y \in 2Q \). Then, if \( g_1, \ldots, g_m \) and \( b_1, \ldots, b_m \) satisfy (a), then
\[
\sum_{k=1}^{m} |U_k(x) - U_k(y)| \leq \frac{A}{10^0} \|f\|_*.
\]

We postpone the proof of Claim 3 until Subsection 8.5. Let us see that (a) holds.

**Claim 4.** If \( Q \in \mathcal{D}_m^G \cup \mathcal{D}_m^B \), then \( |m_Q f_m| \leq C_9 A \|f\|_* \). Also, \( |g_m|, |b_m| \leq C_8 A \|f\|_* \).

**Proof.** First we will prove the first statement. By Claim 2, we know that \( \delta(Q, 2R_0) > (m - \frac{1}{10}) A \). Let \( R \in \mathcal{D}_{m-1} \) be such that \( Q \cap R = \emptyset \). We must have \( \ell(R) > 0 \). Otherwise, \( Q \equiv R \) and \( \delta(R, 2R_0) > (m - \frac{1}{10}) A > (m - 1) A + \varepsilon_1 \), which is not possible.

Since \( \ell(Q) \leq \ell(R)/10 \), we have \( Q \subset 2R \). We know \( |m_R f_m| \leq A \|f\|_* \) because (b) holds for \( m - 1 \). By Claim 3 (for \( m - 1 \) and \( R \)) we get
\[
|m_Q f_m| \leq |m_R f_m| + |m_Q f_m - m_R f_m| \\
\leq |m_R f_m| + |m_Q f - m_R f| + |m_Q \left( \sum_{k=1}^{m-1} U_k \right) - m_R \left( \sum_{k=1}^{m-1} U_k \right)| \\
\leq C A \|f\|_* + |m_Q f - m_R f|.
\]
The term \( |m_Q f - m_R f| \) is also bounded above by \( C A \|f\|_* \) because \( Q \) and \( R \) are doubling, \( f \in RBMO(\mu) \), and it is easily checked that \( \delta(Q, R) \leq C A \).

The estimates on \( g_m \) and \( b_m \) follow from from the definition of these functions and the estimate \( |m_Q f_m| \leq C_9 A \|f\|_* \) for \( Q \in \mathcal{D}_m^G \cup \mathcal{D}_m^B \). □

Let us prove (d) now.

**Claim 5.** If \( Q \in \mathcal{D}_m \) and \( |m_Q f_m| \leq \frac{8}{20} A \|f\|_* \), then \( U_m \equiv 0 \) and \( g_m \equiv b_m \equiv 0 \) on \( Q \).

**Proof.** Suppose that \( Q \equiv Q_{h,m} \in \mathcal{D}_m \) is such that either \( g_m \neq 0 \), \( b_m \neq 0 \) or \( U_m \neq 0 \) on \( Q_{h,m} \). By Claim 1 we have \( Q_{h,m} \subset 4S_{j,m} \) for some \( j \). By construction, the center of \( S_{j,m} \) belongs to some cube \( Q_{i,m} \) with \( |m_{Q_{i,m}} f_m| \geq \frac{8}{20} A \|f\|_* \).
\( \frac{3}{4} A \| f \|_s \). It is easily seen that \( \delta(Q_{h,m}, 4S_{j,m}) \), \( \delta(Q_{i,m}, 4S_{j,m}) \leq C' + \alpha_1 + \alpha_2 + \alpha_3 \). Thus

\[
|m_{Q_{i,m}}f - m_{Q_{h,m}}f| \leq (C' + 2\alpha_1 + 2\alpha_2 + 2\alpha_3) \| f \|_s.
\]

Since \( Q_{i,m} \) and \( Q_{h,m} \) are contained in a common cube of the generation \( m - 1 \), by Claim 3 we get

\[
|m_{Q_{i,m}}f_m - m_{Q_{h,m}}f_m| \leq |m_{Q_{i,m}}f - m_{Q_{h,m}}f| + \left| m_{Q_{i,m}}\left( \sum_{k=1}^{m-1} U_k \right) - m_{Q_{h,m}}\left( \sum_{k=1}^{m-1} U_k \right) \right|
\]

\[
\leq (C' + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + A/100) \| f \|_s
\]

\[
\leq \frac{1}{10} A \| f \|_s,
\]

and so

\[
|m_{Q_{h,m}}f_m| \geq \left( \frac{3}{4} - \frac{1}{10} \right) A \| f \|_s > \frac{8}{20} A \| f \|_s.
\]

\[\square\]

The statement (c) is a consequence of the fact that if \( Q \in D_m^G \), then \( Q \) is far from the boundary of \( \bigcup_j 2S_{j,m} \). Then \( U_m \) is very close to \( m_Q f_m \) on \( Q \), since we only integrate over cubes of \( D_m^G \cup D_m^B \) in order to obtain \( U_m(x) \) for \( x \in Q \). On the other hand, if \( Q \in D_m^B \), this argument does not work because \( Q \) may be near the boundary of \( \bigcup_j 2S_{j,m} \), and so it may happen that we integrate on some cubes from \( D_m \setminus (D_m^G \cup D_m^B) \) for obtaining \( U_m(x), x \in Q \).

Let us see (c) in detail.

**Claim 6.** If \( Q \in D_m^G \) and \( \ell(Q) > 0 \), then \( |m_{Q_{i,m}}f_{m+1}| \leq \frac{7}{20} A \| f \|_s \).

**Proof.** Consider \( Q_{i,m} \in D_m^G \). We want to see that \( U_m \) is very close to \( m_{Q_{i,m}}f_m \) on this cube. By (a) of Lemma 7.8 we have to deal with the cube \( \hat{Q}_{i,m} \).

Let us see that if \( P \in D_m \) is such that \( P \cap \hat{Q}_{i,m} \neq \emptyset \), then \( P \in D_m^G \cup D_m^B \).

Notice that \( P \subset \hat{Q}_{i,m} \). Now, by the definition of good cubes, there exists some \( j \) such that \( Q_{i,m} \cap 2S_{j,m} \neq \emptyset \), which implies \( \hat{Q}_{i,m} \cap 2S_{j,m} \neq \emptyset \). For \( \alpha_3 \) big enough, we have \( \ell(\hat{Q}_{i,m}) < \ell(S_{j,m}) \), and then \( \hat{Q}_{i,m} \subset 2S_{j,m} \). So \( P \in D_m^G \cup D_m^B \).

Let us estimate the term

\[
\sup_{y \in \hat{Q}_{i,m}} |(g_m(y) + b_m(y)) - m_{Q_{i,m}}f_m|.
\]

Recall that

\[
g_m(y) + b_m(y) = \sum_{h: Q_{h,m} \in D_m^G \cup D_m^B} w_{h,m}(y) m_{Q_{h,m}}f_m.
\]
By the arguments above, if \( y \in \hat{Q}^3_{i,m} \) and \( w_{h,m}(y) \neq 0 \), then \( Q_{h,m} \) has been chosen for supporting \( g_m \) or \( b_m \), i.e. \( Q_{h,m} \in D_m \cup D_m^G \). Then,
\[
g_m(y) + b_m(y) - m_{Q_{i,m}}f_m = \sum_{h: Q_{h,m} \in D_m} w_{h,m}(y)(m_{Q_{h,m}}f_m - m_{Q_{i,m}}f_m).
\]

By Claim 3 we obtain
\[
|m_{Q_{h,m}}f_m - m_{Q_{i,m}}f_m| \leq \frac{1}{100} A \|f\|_* + |m_{Q_{h,m}}f - m_{Q_{i,m}}f| \\
\leq \left( \frac{1}{100} A + C + 2 \delta(Q_{h,m}, Q_{i,m}) \right) \|f\|_* \\
\leq \frac{1}{50} A \|f\|_*
\]

(we have used that \( \delta(Q_{h,m}, Q_{i,m}) \leq C \), with \( C \) depending on \( \alpha_1, \alpha_2 \)). Then we get
\[
(8.9) \quad |g_m(y) + b_m(y) - m_{Q_{i,m}}f_m| \leq \frac{1}{50} A \|f\|_*.
\]

For \( x \in Q_{i,m} \), we have
\[
|U_m(x) - m_{Q_{i,m}}f_m| \leq \left| U_m(x) - m_{Q_{i,m}}f_m \int \varphi_{y,m}(x) d\mu(y) \right| \\
+ |m_{Q_{i,m}}f_m| \left| 1 - \int \varphi_{y,m}(x) d\mu(y) \right|.
\]

(8.10)

Let us estimate the first term on the right hand side. By (8.9) and (7.8) we obtain
\[
\left| U_m(x) - m_{Q_{i,m}}f_m \int \varphi_{y,m}(x) d\mu(y) \right| \\
= \left| \int_{\hat{Q}^3_{i,m}} \varphi_{y,m}(x)(g_m(y) + b_m(y) - m_{Q_{i,m}}f_m) d\mu(y) \right| \\
\leq (1 + \varepsilon_3) \frac{1}{50} A \|f\|_*.
\]

On the other hand, by (7.8), (7.9) and Claim 4, the second term on the right hand side of (8.10) is bounded above by \( \varepsilon_3 C_8 A \|f\|_* \). Thus we have
\[
|m_{Q_{i,m}}f_{m+1}| \leq \left( (1 + \varepsilon_3) \frac{1}{50} + \varepsilon_3 C_8 \right) A \|f\|_* \leq \frac{7}{20} A \|f\|_*,
\]

if we choose \( \varepsilon_3 \) small enough.

Now we are going to show that (b) also holds.

Claim 7. If \( Q \in D_m \) and \( \ell(Q) > 0 \), then \( |m_Q f_{m+1}| \leq A \|f\|_* \).
Proof. If \( Q \in D_m^G \), we have already seen that \(|m_Q f_{m+1}| \leq \frac{7}{20} A \|f\|_*\).

If \( Q \in D_m \setminus D_m^G \), then \( Q \cap \bigcup_j S_{j,m} = \emptyset \) (because \( \ell(Q) \ll \ell(S_{j,m}) \) and \( Q \not\subseteq \bigcup_j \frac{3}{2} S_{j,m} \)). By construction, we have

\[
|m_Q f_m| \leq A \|f\|_*.
\]

If \( U_m \equiv 0 \) on \( Q \), then \(|m_Q f_{m+1}| = |m_Q f_m| \leq \frac{3}{4} A \|f\|_*\).

Now we consider the case \( Q \equiv Q_{h,m} \cap \bigcup_j S_{j,m} = \emptyset \) such that \( U_m \not\equiv 0 \) on \( Q \). By Claim 1 there exists some \( j \) with \( \hat{Q}_{3h,m} \subset 4S_{j,m} \). Recall that by (a) of Lemma 7.8, if \( x \in Q_{h,m} \), we have
\[
U_m(x) = \int_{\hat{Q}_{h,m}} \varphi_{y,m}(x) (g_m(y) + b_m(y)) d\mu(y).
\]

So if \( \varphi_{y,m}(x) \neq 0 \) and \( y \in Q_{i,m} \), we have \( Q_{i,m} \cap \hat{Q}_{3h,m} \neq \emptyset \). Therefore, \( Q_{i,m} \subset \hat{Q}_{3h,m} \). Then,
\[
\delta(Q_{i,m}, Q_{h,m}) \leq C + \delta(Q_{i,m}, \hat{Q}_{h,m}) + \delta(Q_{h,m}, \hat{Q}_{3h,m}) \leq C + 2\alpha_1 + 2\alpha_2 \leq \frac{A}{400}.
\]

Therefore, \(|m_{Q_{i,m}} f - m_{Q_{h,m}} f| \leq \frac{A}{400} \|f\|_*\). By Claim 1 we get
\[
|m_{Q_{i,m}} f - m_{Q_{h,m}} f| \leq |m_{Q_{i,m}} f - m_{Q_{h,m}} f| + |m_{Q_{i,m}} f - m_{Q_{i,m}} f_m| + |m_{Q_{i,m}} f_m - m_{Q_{h,m}} f_m|
\]
\[
\leq \frac{1}{10} A \|f\|_*.
\]

Recall also that, by (d),
\[
|m_{Q_{h,m}} f_m| \geq \frac{8}{20} A \|f\|_*.
\]

From the definition of \( g_m, b_m \) and (8.12), (8.13), we derive that \( m_{Q_{h,m}} f \) and \( U_m(x) \) have the same sign.

On the other hand, from (8.11) and (8.12) we get
\[
|m_{Q_{i,m}} f_m| \leq \frac{34}{40} A \|f\|_*\.
\]

So by the definition of \( g_m \) and \( b_m \) we have
\[
\|g_m + b_m\|_{L^\infty(\mu)} \leq \frac{34}{40} A \|f\|_*\,
\]
and by (7.8) we obtain
\[
|U_m(x)| \leq \frac{34}{40} A \|f\|_* \int \varphi_{y,m}(x) d\mu(y) \leq (1 + \varepsilon_3) \frac{34}{40} A \|f\|_* \leq A \|f\|_*
\]
(assuming \( \varepsilon_3 \) small enough). By (8.11), (8.14) and since \( m_{Q_{h,m}} f \) and \( U_m(x) \) have the same sign, (b) holds also in this case. \( \square \)
Therefore, (a)–(e) are satisfied.

8.4. Proof of (f), (g) and (h). The statement (f) is a direct consequence of the following.

Claim 8. If \( \delta(x,2R_0) < \infty \), and if \( Q = \{x\} \in D_m \) (i.e. \( \ell(Q) = 0 \)), then \( h_0(x) = f_{m+1}(x) \) and \( |h_0(x)| \leq C_9 A \|f\|_* \).

Proof. Take \( m \) such that \((m-1)A < \delta(x,2R_0) \leq mA\). By (e) we get \( U_{m+k}(x) = 0 \) for \( k \geq 1 \). Therefore, \( f_{m+1}(x) = f_{m+2}(x) = \cdots = h_0(x) \). By (a) we have

\[
|f_{m+1}(x)| \leq |f_m(x)| + |U_m(x)| \leq |f_m(x)| + 2 C_3 (1 + \varepsilon_3) A \|f\|_*.
\]

So we only have to estimate \( |f_m(x)| \).

Take \( Q_{i,m-1} \in D_{m-1} \) with \( x \in Q_{i,m-1} \). Since \( \ell(Q_{i,m-1}) > 0 \), by (h) we have \( |m_{Q_{i,m-1}} f_m| \leq A \|f\|_* \). Applying Claim 3 we get

\[
|m_{Q_{i,m-1}} f_{m+1} - f_m(x)| \leq |m_{Q_{i,m-1}} f - f(x)| + A \|f\|_*
\]

\[
\leq C \left( 1 + \delta(x,Q_{i,m-1}) + \frac{A}{100} \right) \|f\|_*.
\]

It is easily checked that \( \delta(x,Q_{i,m-1}) \leq A + \varepsilon_0 + \varepsilon_1 \). Then we get \( |f_m(x)| \leq C_2 A \|f\|_* \). \( \qed \)

Now we turn our attention to (g). Given some good cube \( Q_{i,m} \in D_m^G \) with \( \ell(Q_{i,m}) > 0 \), we denote

\[
Z_{i,m} := Z(Q_{i,m}, A \|f\|_*/30)
\]

(see Definition 5.2; roughly speaking \( Z_{i,m} \) is the part of \( Q_{i,m} \) where \( f \) does not oscillate too much with respect to \( m_{Q_{i,m}} f \)). If \( Q_{i,m} \in D_m^G \) and \( \ell(Q_{i,m}) = 0 \), we set \( Z_{i,m} = Q_{i,m} \). The set \( Z_{i,m} \) has a very nice property:

Claim 9. Let \( k > m \) and \( Q_{i,m} \in D_m^G \). If \( P \in D_k \) is such that \( P \cap Z_{i,m} \neq \emptyset \), then \( g_k \equiv b_k \equiv 0 \) on \( P \) and \( P \notin D_k^G \cup D_k^B \).

Proof. Consider first the case \( \ell(Q_{i,m}) = 0 \). If \( P \in D_k \) is such that \( P \cap Q_{i,m} \neq \emptyset \), then \( \ell(P) \leq \ell(Q_{i,m})/10 = 0 \) and so \( P = Q_{i,m} \). Therefore,

\[
\delta(P,2R_0) \leq mA \leq \left( k - \frac{1}{10} \right) A.
\]

By (e), we get \( b_k \equiv g_k \equiv 0 \) on \( P \).

Assume now \( \ell(Q_{i,m}) > 0 \). Let \( x \in P \cap Z_{i,m} \). From the definition of \( Z_{i,m} \), we have

\[
|m_{Q_{i,m}} f - m_S f| \leq A \|f\|_*/30
\]

for any \( S \in D_{m+j} \), \( j \geq 1 \), with \( x \in S \). Also, by Claim 6 we have

\[
|m_{Q_{i,m}} f_{m+1} | \leq \frac{7}{20} A \|f\|_*.
\]
Consider now $P_{m+1} \in D_{m+1}$ with $x \in P_{m+1}$. Observe that $\ell(P_{m+1}) \leq \ell(Q_{i,m})/10$ and $P_{m+1} \subset 2Q_{i,m}$. We have
\[
|m_{P_{m+1}} f_{m+1}| \leq |m_{Q_{i,m}} f_{m+1}| + |m_{Q_{i,m}} f_{m+1} - m_{P_{m+1}} f_{m+1}|
\leq \frac{7}{20} A \|f\|_\ast + |m_{Q_{i,m}} f - m_{P_{m+1}} f|
\leq |m_{Q_{i,m}} \left( \sum_{k=1}^m U_k \right) - m_{P_{m+1}} \left( \sum_{k=1}^m U_k \right)|.
\]
By (8.15) and Claim 3 we obtain $|m_{P_{m+1}} f_{m+1}| \leq \frac{8}{20} A \|f\|_\ast$. By (d), on $P_{m+1}$ we have $g_{m+1} \equiv b_{m+1} \equiv 0$ and also $U_{m+1} \equiv 0$. Thus,
\[
f_{m+2} \equiv f_{m+1}
\]
on any cube $P_{m+1} \in D_{m+1}$ containing $x$.

Take now $P_{m+2} \in D_{m+2}$ with $x \in P_{m+2}$. On this cube $f_{m+2} \equiv f_{m+1}$, and then we have
\[
|m_{P_{m+2}} f_{m+2}| \leq |m_{Q_{i,m}} f_{m+1}| + |m_{Q_{i,m}} f_{m+1} - m_{P_{m+2}} f_{m+1}|
\leq \frac{7}{20} A \|f\|_\ast + |m_{Q_{i,m}} f - m_{P_{m+2}} f|
\leq |m_{Q_{i,m}} \left( \sum_{k=1}^m U_k \right) - m_{P_{m+2}} \left( \sum_{k=1}^m U_k \right)|.
\]
Again by (d), we get $g_{m+2} \equiv b_{m+2} \equiv U_{m+2} \equiv 0$ on $P_{m+2}$. Thus, $f_{m+3} = f_{m+1}$ on $P_{m+2}$.

Going on, we will obtain $g_{m+j} \equiv b_{m+j} \equiv U_{m+j} \equiv 0$ for all $j \geq 1$ on any cube $P_{m+j} \in D_{m+j}$ containing $x$.

As a consequence of Claim 9, $Z_{i,m}$ is a good place for supporting $g_m$. If, for each $m$, $g_m$ were supported on $\bigcup Z_{i,m}$, then the supports of $g_m$, $m \geq 1$, would be disjoint for different $m$’s. This is the idea that Carleson used in [Ca].

So we are going to make some “corrections” according to this argument. We have
\[
U^G_m(x) = \sum_{i \in I_m} \varphi_{y_i,m}(x) \int w_{i,m}(y) g_m(y) \, d\mu(y).
\]
For each $Q_{i,m}$ with $\ell(Q_{i,m}) > 0$ we set
\[
u_{i,m}(y) = \int w_{i,m} g_m \, d\mu \cdot \frac{\chi_{Z_{i,m}}(y)}{\mu(Z_{i,m})}.
\]
If $\ell(Q_{i,m}) = 0$, we set $\nu_{i,m}(y) = w_{i,m}(y) g_m(y) \equiv g_m(y)$ (we do not change anything in this case). Then $U^G_m$ can be written as
\[
U^G_m(x) = \sum_{i \in I_m} \varphi_{y_i,m}(x) \int \nu_{i,m}(y) \, d\mu(y).
\]
As in the case of $U^B_m$ in Subsection 8.2, if we set $\mathcal{D}^G_m = \mathcal{D}^{1G}_m \cup \cdots \cup \mathcal{D}^{NG}_m$ where each subfamily $\mathcal{D}^{pG}_m$ is disjoint, we can write $U^G_m$ in the following way:

$$U^G_m(x) = \sum_{p=1}^{N} \int \varphi^p_{g,y,m}(x) g^p_m(y) \, d\mu(y)$$

with

$$g^p_m(y) = \sum_{i : Q_{i,m} \in \mathcal{D}^{pG}_m} u_{i,m}(y)$$

and

$$\varphi^p_{g,y,m}(x) = \varphi_{y,m}(x)$$

if $y \in Q_{i,m}$ and $Q_{i,m} \in \mathcal{D}^{pG}_m$.

By Proposition 5.3, if $A$ is big enough we have $\mu(Z_{i,m}) \geq \mu(Q_{i,m})/2$ (if $\ell(Q_{i,m}) > 0$). Then it easily checked that $\|u_{i,m}\|_{L^\infty(\mu)} \leq 2 \|g_m\|_{L^\infty(\mu)}$ for all $i$. Thus, from (a), (g.2) follows. Moreover, because of Claim 9, (g.3) also holds.

One of the differences between our construction and Carleson’s one is that, because of the regularity of Lebesgue measure, Carleson can treat the bad cubes in a way very similar to the way for the good ones. We have not been able to operate as Carleson. However, as it has been shown in Subsection 8.2, the packing condition (8.2) is also a good solution. Let us prove that this condition is satisfied.

**Claim 10.** For any $R \in \mathcal{D}_m$ with $\ell(R) > 0$, the bad cubes satisfy the packing condition

$$\sum_{Q : Q \cap R \neq \emptyset} \mu(Q) \leq C \mu(R).$$

**Proof.** Let $k > m$ be fixed. We are going to estimate the sum

$$\sum_{Q : Q \cap R \neq \emptyset} \mu(Q).$$

Let $Q \in \mathcal{D}^B_k$ be such that $Q \cap R \neq \emptyset$. Since $Q$ is a bad cube, there exists some $j$ such that $2S_{j,k} \cap Q \neq \emptyset$. Then we have $Q \subset 4S_{j,k}$. Since $A \gg \alpha_1 + \alpha_2 + \alpha_3$ and $4S_{j,k} \cap R \neq \emptyset$, we get $\ell(S_{j,k}) \leq \ell(R)/20$, and so $4S_{j,k} \subset 2R$.

By the finite overlapping of the cubes $Q$ in $\mathcal{D}_k$, we have

$$\sum_{Q : Q \cap R \neq \emptyset \atop Q \in \mathcal{D}^B_k} \mu(Q) \leq C \mu\left( \bigcup_{j : S_{j,k} \subset 2R} 2S_{j,k} \right) \leq C \sum_{j : S_{j,k} \subset 2R} \mu(2S_{j,k}) \leq C \sum_{j : S_{j,k} \subset 2R} \mu(S_{j,k}).$$
Now, from the construction of $g_k^p$, it is easy to check that $\mu(S_{j,k}) \leq C \mu\left(\sum_{p=1}^{N} |g_k^p| \neq 0 \right)$). This fact and the bounded overlapping of the cubes $S_{j,k}$ give \[
\sum_{Q, Q \cap R \neq \emptyset, Q \in D_k^B} \mu(Q) \leq C \mu\left(2R \cap \left\{\sum_{p=1}^{N} |g_k^p| \neq 0 \right\}\right).
\] Summing over $k > m$, as the supports of the functions $g_k^p$ are disjoint for different $k$'s, we obtain \[
\sum_{Q, Q \cap R \neq \emptyset, Q \in D_k^B, k > m} \mu(Q) \leq C \sum_{k > m} \mu\left(2R \cap \left\{\sum_{p=1}^{N} |g_k^p| \neq 0 \right\}\right) \leq C \mu(2R) \leq C \mu(R).
\]

8.5. Proof of Claim 3. We only need to check that \[
\sum_{k=1}^{m} C_8 A\int_{\mathbb{R}^d} |\varphi_{z,k}(x) - \varphi_{z,k}(y)| \, d\mu(z) \leq \frac{A}{100}.
\] Let $x_0 \in \text{supp}(\mu)$ be such that $x, y \in 2Q_{x_0,m}$. Obviously, we can assume $\ell(Q_{x_0,m}) > 0$. For each $k \leq m$ we set \[
\int |\varphi_{z,k}(x) - \varphi_{z,k}(y)| \, d\mu(z) = \int_{\mathbb{R}^d \setminus \tilde{Q}_{x_0,k}^1} + \int_{\tilde{Q}_{x_0,k}^1} = I_{1,k} + I_{2,k}.
\] Let us estimate the integrals $I_{1,k}$. Notice that if $x, y \in 2Q_{x_0,m}$, then $x, y \in 2Q_{x_0,k} \subset \frac{1}{2} \tilde{Q}_{x_0,k}^1$. Thus $|x - z| \approx |y - z| \approx |x_0 - z|$ for $z \in \mathbb{R}^d \setminus \tilde{Q}_{x_0,k}^1$. So by (d) of Lemma 7.8 we have \[
I_{1,k} \leq C \alpha_2^{-1} \int_{\mathbb{R}^d \setminus \tilde{Q}_{x_0,k}^1} \frac{|x - y|}{|x - z|^{n+1}} \, d\mu(z)
\leq C \alpha_2^{-1} \frac{\ell(Q_{x_0,m})}{\ell(Q_{x_0,k}^1)}.
\] (8.16)

In case $k > m$, by Lemma 6.4 we get \[
I_{1,k} \leq C \alpha_2^{-1} \frac{\ell(Q_{x_0,m})}{\ell(Q_{x_0,k}^1)} \leq C_{13} \alpha_2^{-1} 2^{-\gamma(m-k)} A.
\]

Therefore, \[
(8.17) C_8 A \sum_{k=1}^{m} I_{1,k} \leq C_8 \alpha_2^{-1} A \sum_{k=1}^{m-1} 2^{-\gamma(m-k)} A + C_8 C_{13} \alpha_2^{-1} A \frac{\ell(Q_{x_0,m})}{\ell(Q_{x_0,k}^1)}.
\] The first sum on the right hand side is $\leq C \alpha_2^{-1} A 2^{-\gamma A}$, and for $A$ big enough and $\alpha_2 > 1$ is $\leq 1 \leq A/400$. The second term on the right hand side
is also $\leq A/400$ if we choose $\alpha_2$ big enough (or $\alpha_1$ big enough since then $\ell(\tilde{Q}_{x_0,m}) \gg \ell(\tilde{Q}^1_{x_0,m})$). Thus

$$C_8 A \sum_{k=1}^{m} I_{1,k} \leq \frac{A}{200}.$$ 

We consider now the integrals $I_{2,k}$. By Lemma 7.8,

$$|\varphi'(u)| \leq C \frac{\alpha_2^{-1}}{\ell(Q^1_{x_0,k})^{n+1}}$$

for all $u \in Q_{x_0,k}$. Therefore,

$$I_{2,k} \leq C \frac{\alpha_2^{-1}}{\ell(Q^1_{x_0,k})^{n+1}} \int_{\tilde{Q}^1_{x_0,k}} |x-y| d\mu(z) \leq C \frac{\alpha_2^{-1}}{\ell(Q^1_{x_0,k})} \ell(Q^1_{x_0,k}).$$

This is the same estimate that we have obtained for $I_{1,k}$ in (8.16), and then we also have

$$C_8 A \sum_{k=1}^{m} I_{2,k} \leq \frac{A}{200},$$

if we choose $A$ and $\alpha_2$ (or $\alpha_1$) big enough. \qed

9. Appendix

In this section we will prove the following result, which is used in Section 4 to show that Theorem 1.2 follows from the Main Lemma.

Lemma 9.1. Consider $f \in L^1(\mu)$ with $\int f \, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Then there exists a sequence of functions $f_k, k \geq 1$, bounded with compact support such that $\int f_k \, d\mu = 0$, $f_k \to f$ in $L^1(\mu)$ and $\|M_\Phi (f - f_k)\|_{L^1(\mu)} \to 0$.

So if we consider the space

$$H^1_\Phi(\mu) = \{ f \in L^1(\mu) : \int f \, d\mu = 0, M_\Phi f \in L^1(\mu) \},$$

with norm $\|f\|_{H^1_\Phi(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}$, then Lemma 9.1 asserts that functions in $H^1_\Phi(\mu)$ which are bounded and have compact support are dense in $H^1_\Phi(\mu)$. In particular, $H^1_\Phi(\mu) \cap H^{1,\infty}_{\text{atb}}(\mu)$ is dense in $H^1_\Phi(\mu)$.

In this section we will assume that the center of any cube $Q$ may be any point of $\mathbb{R}^d$, not necessarily belonging to $\text{supp}(\mu)$. As in the previous sections, the sides of the cubes are parallel to the axes and they are closed.

Let us introduce some additional notation. For $\rho > 1$, we set

$$M_{(\rho)} f(x) = \sup_{Q \ni x \mu(\rho Q)} \frac{1}{\mu Q} \int_Q |f| \, d\mu.$$ 

This non centered maximal operator is bounded above by the operator defined as

$$M^{(\rho)} f(x) = \sup_{\rho^{-1}Q \ni x \mu(Q)} \frac{1}{\mu(Q)} \int_Q |f| \, d\mu.$$
This is the version of the Hardy-Littlewood operator that one obtains taking supremaums over cubes $Q$ which may be non centered at $x$ but such that $x \in \rho^{-1}Q$. Recall that since $0 < \rho^{-1} < 1$, one can apply Besicovitch’s Covering Theorem and then one gets that $M(\rho)$ is of weak type $(1,1)$ and bounded in $L^p(\mu)$, $p \in (1,\infty)$. As a consequence, $M(\rho)$ is also of weak type $(1,1)$ and bounded in $L^p(\mu)$, $p \in (1,\infty)$.

Remark 9.2 (Whitney covering). Let $\Omega \subset \mathbb{R}^d$ be open, $\Omega \neq \mathbb{R}^d$. Then $\Omega$ can be decomposed as $\Omega = \bigcup_{i \in I} Q_i$, where $Q_i$, $i \in I$, are cubes with disjoint interiors, with $20Q_i \subset \Omega$ and such that, for some constants $\beta > 20$ and $D \geq 1$, $\beta Q_k \cap \Omega^c \neq \emptyset$ and for each cube $Q_k$ there are at most $D$ cubes $Q_i$ with $10Q_k \cap 10Q_i \neq \emptyset$ (in particular, the family of cubes $\{10Q_i\}_{i \in I}$ has finite overlapping).

In [To3] a decomposition of Calderón-Zygmund type adapted for non doubling measures was introduced. This decomposition was used to prove an interpolation theorem between $(H^{1}_{\text{loc}}(\mu), L^1(\mu))$ and $(L^{\infty}(\mu), \text{RBMO}(\mu))$. In [To4] it was shown that this decomposition was also useful for proving that CZO’s bounded in $L^2(\mu)$ are of weak type $(1,1)$ too, as in the doubling case (this result had been proved previously in [NTV2] using different techniques). To prove Lemma 9.1 we will use the following variant of the Calderón-Zygmund decomposition of [To3].

Lemma 9.3. Let $f \in L^1(\mu)$ with $\int f \, d\mu = 0$ and $M_{\Phi}f \in L^1(\mu)$. For any $\lambda > 0$, let $\Omega_{\lambda} = \{x \in \mathbb{R}^d : M_{\Phi}f(x) > \lambda\}$. Then $\Omega_{\lambda}$ is open and $|f| \leq 2^{d+1}\lambda$ $\mu$-a.eq. in $\mathbb{R}^d \setminus \Omega_{\lambda}$. Moreover, if we consider a Whitney decomposition of $\Omega_{\lambda}$ into cubes $Q_i$ (as in Remark 9.2), then we have:

(a) For each $i$ there exists a function $w_i \in C^\infty(\mathbb{R}^d)$ with supp$(w_i) \subset \frac{3}{2}Q_i$, $0 \leq w_i \leq 1$, $\|w_i\|_{\infty} \leq C \ell(Q_i)^{-1}$ such that $\sum_i w_i(x) = 1$ if $x \in \Omega_{\lambda}$.

(b) For each $i$, let $R_i$ be the smallest $(6,6^{n+1})$-doubling cube of the form $6^kQ_i$, $k \geq 1$, with $R_i \cap \Omega_{\lambda}^c \neq \emptyset$. Then there exists a family of functions $\alpha_i$ with supp$(\alpha_i) \subset R_i$ satisfying

\begin{align}
\int \alpha_i \, d\mu &= \int f \, w_i \, d\mu,
\end{align}

\begin{align}
\|\alpha_i\|_{L^{\infty}(\mu)} \mu(R_i) \leq C \|\alpha_i\|_{L^1(\mu)}
\end{align}

and

\begin{align}
\sum_i |\alpha_i| \leq B \lambda
\end{align}

(where $B$ is some constant).

(c) $f$ can be written as $f = g + b$, with

\begin{align}
g = f \left(1 - \sum_i w_i\right) + \sum_i \alpha_i
\end{align}
and

\[ b = \sum_i (f w_i - \alpha_i), \]

and then \( \|g\|_{L^\infty(\mu)} \leq C \lambda \) and \( \text{supp}(b) \subset \Omega_\lambda \).

**Proof.** The set \( \Omega_\lambda \) is open because \( M(2) \) is lower semicontinuous. Since for \( \mu \)-a.e. \( x \in \mathbb{R}^d \) there exists a sequence of \( (2, 2^{d+1}) \)-doubling cubes centered at \( x \) with side length tending to zero, it follows that for \( \mu \)-a.e. \( x \in \mathbb{R}^d \) such that \( |f(x)| > 2^{d+1} \lambda \) there exists some \( (2, 2^{d+1}) \)-doubling cube \( Q \) with \( \int_Q |f| \, d\mu/\mu(Q) > 2^{d+1} \lambda \) and so \( M(2)f(x) > \lambda \).

The existence of the functions \( w_i \) of (a) is a standard known fact. The assertion (c) follows from the other statements in the lemma. So the only question left is the statement (b).

Notice that, since \( R_i \cap \Omega_\lambda^c \neq \emptyset \), we have

\[ (9.4) \int_{R_i} |f| \, d\mu \leq \lambda \mu(2R_i) \]

for each \( i \).

To construct the functions \( \alpha_j \) we would like to start by the smallest cube \( R_i \), and go on with the bigger cubes \( R_j \) following an order of non decreasing sizes. Since in general there does not exist a cube \( R_i \) with minimal side length in the family \( \{R_i\}_{i=1}^\infty \), we will have to modify a little the argument. For each fixed \( N \) we will construct functions \( \alpha_i^N, 1 \leq i \leq N \), with \( \text{supp}(\alpha_i^N) \subset R_i \), satisfying (9.1), (9.2) and (9.3). Finally, applying weak limits when \( N \to \infty \), we will get the functions \( \alpha_i \).

The functions \( \alpha_i^N \) that we will construct will be of the form \( \alpha_i^N = a_i^N \chi_{A_i^N} \), with \( a_i^N \in \mathbb{R} \) and \( A_i^N \subset R_i \). To avoid a complicate notation, suppose that the cubes \( R_i, 1 \leq i \leq N \), satisify \( \ell(R_i) \leq \ell(R_{i+1}) \) (we can assume this because we are taking a finite number of cubes). We set \( A_1^N = R_1 \) and

\[ a_1^N = a_1^N \chi_{R_1}, \]

where the constant \( a_1^N \) is chosen so that \( \int_{Q_1} f w_1 \, d\mu = \int a_1^N \, d\mu \).

Suppose that \( a_1^N, a_2^N, \ldots, a_{k-1}^N \) (for some \( k \leq N \)) have been constructed, satisfy (9.1) and \( \sum_{i=1}^{k-1} |\alpha_i| \leq B \lambda \), where \( B \) is some constant (which will be fixed below).

Let \( R_{s_1}, \ldots, R_{s_m} \) be the subfamily of cubes \( R_i, 1 \leq i \leq k-1, \) such that \( R_{s_i} \cap R_k \neq \emptyset \). As \( \ell(R_{s_i}) \leq \ell(R_k) \) (because of the non decreasing sizes of \( R_i \)), we have \( R_{s_i} \subset 3R_k \). Taking into account that for \( i = 1, \ldots, k-1 \)

\[ \int |\alpha_i^N| \, d\mu \leq \int |f w_i| \, d\mu \]
by (9.1), and using that \( R_k \) is \((6, 6^{n+1})\)-doubling and (9.4), we get
\[
\sum_j \int_{R_{sj}} |\alpha^N_{s_j}| \, d\mu \leq \sum_j \int |f w_{sj}| \, d\mu \\
\leq C \int_{3R_k} |f| \, d\mu \leq C\lambda \mu(6R_k) \leq C_{14} \lambda \mu(R_k).
\]
Therefore,
\[
\mu \left\{ \sum_j |\alpha^N_{s_j}| > 2C_{14} \lambda \right\} \leq \frac{\mu(R_k)}{2}.
\]
So we set
\[
A^N_k = R_k \cap \left\{ \sum_j |\alpha^N_{s_j}| \leq 2C_{14} \lambda \right\},
\]
and then
\[
\mu(A^N_k) \geq \frac{\mu(R_k)}{2}.
\]
The constant \( a^N_k \) is chosen so that for \( \alpha^N_k = a^N_k \chi_{A^N_k} \) we have \( \int \alpha^N_k \, d\mu = \int f w_k \, d\mu \). Then we obtain
\[
|a^N_k| \leq \frac{1}{\mu(A^N_k)} \int |f w_k| \, d\mu \leq \frac{2}{\mu(R_k)} \int |f w_k| \, d\mu \\
\leq \frac{2}{\mu(R_k)} \int \frac{1}{2} R_k |f| \, d\mu \leq C_{15} \lambda
\]
(this calculation also applies to \( k = 1 \)). Thus,
\[
|\alpha^N_k| + \sum_j |\alpha^N_{s_j}| \leq (2C_{14} + C_{15}) \lambda.
\]
If we choose \( B = 2C_{14} + C_{15} \), (9.3) follows for the cubes \( R_1, \ldots, R_n \).

Now it is easy to check that (9.2) also holds. Indeed we have
\[
\|\alpha^N_i\|_{L^\infty(\mu)} \mu(R_i) \leq C |\alpha^N_i| \mu(A^N_i) = C \int_{Q_i} f w_i \, d\mu \leq C \|\alpha^N_i\|_{L^1(\mu)}.
\]
Finally, taking weak limits in the weak-* topology of \( L^\infty(\mu) \), one easily obtains the required functions \( \alpha_i \). The details are left to reader. A similar argument can be found in the proof of Lemma 7.3 of [To3]. \( \square \)

Using the decomposition above we can prove Lemma 9.1 partially. This will be the first step of its proof.

**Lemma 9.4.** The subspace \( H^1(\Phi) \cap L^\infty(\mu) \) is dense in \( H^1(\Phi) \).

**Proof.** Given \( f \in H^1(\Phi) \), for each integer \( k \geq 0 \), we consider the generalized Calderón-Zygmund decomposition of \( f \) given in the preceding lemma, with \( \lambda = 2^k \). We will adopt the convention that all the elements of that decomposition will carry the subscript \( k \). Thus we write \( f = g_k + b_k \), as in (c) of Lemma 9.1. We know that \( g_k \) is bounded and satisfies \( \int g_k \, d\mu = 0 \) (because \( \int b_k \, d\mu = 0 \)). We will show that \( g_k \to f \) in \( L^1(\mu) \) and \( \|M_{\Phi}(g_k - f)\|_{L^1(\mu)} \to 0 \) as \( k \to \infty \) too.
It is not difficult to check that $b_k$ tends to 0 in $L^1(\mu)$. Indeed, if we set $\Omega_k = \{ M(2) f(x) > 2^k \},$ then $\mu(\Omega_k) \to 0$ as $k \to \infty$, because $f \in L^1(\mu)$. Thus

$$\int |b_k| \, d\mu \leq 2 \sum_i \int |f_{w_i,k}| \, d\mu \leq C \int \Omega_k \, |f| \, d\mu \xrightarrow{k \to \infty} 0,$$

and so $g_k \to f$ in $L^1(\mu)$.

Let us see that $\|M\Phi b_k\|_{L^1(\mu)} \to 0$ as $k \to \infty$. We denote $b_{i,k} = f_{w_i,k} - \alpha_{i,k}$. Then we have

$$\|M\Phi b_{i,k}\|_{L^1(\mu)} \leq \sum_i \|M\Phi b_{i,k}\|_{L^1(\mu)}.$$

The estimates for each term $\|M\Phi b_{i,k}\|_{L^1(\mu)}$ are (in part) similar to the ones in Lemma 3.1 for estimating $M\Phi$ over atomic blocks. We write

$$\|M\Phi b_{i,k}\|_{L^1(\mu)} \leq \int_{\mathbb{R}^d \setminus 2R_{i,k}} M\Phi b_{i,k} \, d\mu + \int_{2R_{i,k}} M\Phi(f_{w_i,k}) \, d\mu + \int_{2R_{i,k}} M\Phi \alpha_{i,k} \, d\mu$$

(9.5)

Taking into account that $f_{b_{i,k}} \, d\mu = 0$, it is easily seen that

$$\int_{\mathbb{R}^d \setminus 2R_{i,k}} M\Phi b_{i,k} \, d\mu \leq \|b_{i,k}\|_{L^1(\mu)} \leq C \|f_{w_i,k}\|_{L^1(\mu)}$$

(the calculations are similar to the ones in (3.1) and (3.2)).

Let us consider the last term on the right hand side of (9.5) now. By (9.1) and (9.2) we get

$$\int_{2R_{i,k}} M\Phi \alpha_{i,k} \, d\mu \leq \|\alpha_{i,k}\|_{L^\infty(\mu)} \mu(2R_{i,k}) \, d\mu \leq C \|f_{w_i,k}\|_{L^1(\mu)}.$$

We split the second integral on the right hand side of (9.5) as follows:

$$\int_{2R_{i,k}} M\Phi(f_{w_i,k}) \, d\mu = \int_{2R_{i,k} \setminus 2Q_{i,k}} + \int_{2Q_{i,k}}.$$

As in (3.4), we have

$$\int_{2R_{i,k} \setminus 2Q_{i,k}} M\Phi(f_{w_i,k}) \, d\mu \leq C \|f_{w_i,k}\|_{L^1(\mu)} \int_{2R_{i,k} \setminus 2Q_{i,k}} \frac{1}{|x - 2Q_{i,k}|^n} \, d\mu(x) \leq C \|f_{w_i,k}\|_{L^1(\mu)} (1 + \delta(Q_{i,k}, R_{i,k})) \leq C \|f_{w_i,k}\|_{L^1(\mu)}.$$

Finally we have to deal with $\int_{2Q_{i,k}} M\Phi(f_{w_i,k}) \, d\mu$. Consider $x \in 2Q_{i,k}$ and $\varphi \sim x$. Then

$$\int_{2Q_{i,k}} M\Phi(f_{w_i,k}) \, d\mu = \left| \int (\varphi_{w_i,k}) f \, d\mu \right| \leq C \|M\Phi f(x),$$

(9.6)
because $C \varphi w_{i,k} \sim x$ for some constant $C > 0$. Indeed, for $y \in \mathbb{R}^d$ we have

$$0 \leq w_{i,k} \varphi(y) \leq \frac{1}{|y - x|^n}$$

and

$$|(\varphi w_{i,k})'(y)| \leq |\varphi'(y) w_{i,k}(y)| + |\varphi(y) w'_{i,k}(y)| \leq \frac{1}{|y - x|^{n+1}} + \frac{C}{|y - x|^n} |w'_{i,k}(y)|.$$

Recall also that $|w'_{i,k}(y)| \leq C|Q_i^{-1} - Q_i|$ and $\text{supp}(w_{i,k}) \subset 2Q_{i,k}$. Then we get $|w'_{i,k}(y)| \leq C|y - x|^{-1}$ for all $y \in \mathbb{R}^d$. Thus $|(\varphi w_{i,k})'(y)| \leq C|y - x|^{-n-1}$.

So (9.6) holds and then

$$\int_{2Q_{i,k}} M_{\Phi}(f w_{i,k}) \, d\mu \leq C \int_{2Q_{i,k}} M_{\Phi}f \, d\mu.$$

When we gather the previous estimates, we obtain

$$\|M_{\Phi}b_{i,k}\|_{L^1(\mu)} \leq C \|f w_{i,k}\|_{L^1(\mu)} + C \int_{2Q_{i,k}} M_{\Phi}f \, d\mu.$$

Taking into account the finite overlap of the cubes $2Q_{i,k}$ (recall that they are Whitney cubes covering $\Omega_k$), we get

$$\|M_{\Phi}b_k\|_{L^1(\mu)} \leq C \int_{\Omega_k} (|f| + M_{\Phi}f) \, d\mu \xrightarrow{k \to \infty} 0,$$

and we are done. \hfill \Box

**Proof of Lemma 9.1.** Take $f \in H^1_{\Phi}(\mu) \cap L^\infty(\mu)$. Consider the infinite increasing sequence of the cubes $Q_k = 4^{\mathbb{N}} [-1,1]^d$ that are $(4, 4^{n+1})$-doubling. Let $w$ be a $C^\infty$ function such that $\chi_{[-1,1]^d}(x) \leq w(x) \leq \chi_{[-2,2]^d}(x)$ for all $x$. We denote $w_k(x) = w(4^{-k}x)$ (so $\chi_{Q_k}(x) \leq w_k(x) \leq \chi_{4Q_k}(x)$) and we set

$$f_k = w_k f - \frac{\chi_{Q_k}}{\mu(Q_k)} \int w_k \, f \, d\mu.$$

It is clear that $f_k$ is bounded, has compact support and converges to $f$ in $L^1(\mu)$ as $k \to \infty$. We will prove that

$$\|M_{\Phi}(f - f_k)\|_{L^1(\mu)} \leq C \left| \int w_k \, f \, d\mu \right| + C \int_{\mathbb{R}^d \setminus 4Q_k} M_{\Phi}f \, d\mu + \int_{4Q_k} M_{\Phi}(1 - w_k) \, f \, d\mu.$$

Finally we will show that the terms on the right hand side of (9.7) tend to 0 as $k \to \infty$ and we will be done.

Let us consider first the integral of $M_{\Phi}(f - f_k)$ over $\mathbb{R}^d \setminus 4Q_k$. We set

$$\int_{\mathbb{R}^d \setminus 4Q_k} M_{\Phi}(f - f_k) \, d\mu \leq \int_{\mathbb{R}^d \setminus 4Q_k} M_{\Phi}f \, d\mu + \int_{\mathbb{R}^d \setminus 4Q_k} M_{\Phi}f_k \, d\mu.$$
We only have to estimate the last integral on the right hand side. Take $x \in \mathbb{R}^d \setminus 4Q_k$, $\varphi \sim x$ and let $y_0 \in 2Q_k$ be the point where $\varphi$ attains its minimum over $2Q_k$ (recall that we assume $\varphi \geq 0$ and $\varphi \in C^1$). We denote $c_k = \int w_k f \, d\mu/\mu(Q_k)$ and then we set

$$
\int f_k \, \varphi \, d\mu = \int f(y) (\varphi(y) - \varphi(y_0)) \, d\mu(y) = \int w_k(y) f(y) (\varphi(y) - \varphi(y_0)) \, d\mu(y) - c_k \int_{Q_k} (\varphi(y) - \varphi(y_0)) \, d\mu(y) = I_1 - I_2.
$$

Let us consider the function $\psi(y) = w_k(y) (\varphi(y) - \varphi(y_0))$. This function satisfies

$$
0 \leq \psi(y) \leq \varphi(y)
$$

and

$$
|\psi'(y)| \leq |w_k(y) \varphi'(y)| + |w_k'(y)| |\varphi(y) - \varphi(y_0)| \leq \frac{1}{|y - x|^{n+1}} + C \frac{\ell(Q_k)}{|y - x|^{n+1}} = C \frac{1}{|y - x|^{n+1}}.
$$

Therefore $C \psi \sim x$ for some constant $C > 0$ and so $|I_1| \leq C \Phi f(x)$. For $I_2$ we use a cruder estimate:

$$
|I_2| \leq C |c_k| \mu(Q_k) \frac{\ell(Q_k)}{|y_0 - x|^{n+1}}.
$$

Thus we obtain

$$
M \Phi f_k(x) \leq C M \Phi f(x) + C |c_k| \mu(Q_k) \frac{\ell(Q_k)}{|y_0 - x|^{n+1}}.
$$

Since

$$
\int_{\mathbb{R}^d \setminus 4Q_k} \frac{1}{|y - x|^{n+1}} \, d\mu(x) \leq C \ell(Q_k)^{-1},
$$

we get

$$
\int_{\mathbb{R}^d \setminus 4Q_k} M \Phi f_k \, d\mu \leq C \int_{\mathbb{R}^d \setminus 4Q_k} M \Phi f \, d\mu + C |c_k| \mu(Q_k) = C \int_{\mathbb{R}^d \setminus 4Q_k} M \Phi f \, d\mu + C \left| \int w_k f \, d\mu \right|.
$$

(9.8)

Now we have to deal with $\int_{4Q_k} M \Phi (f - f_k) \, d\mu$. For $x \in 4Q_k$ we write

$$
M \Phi (f - f_k)(x) \leq M \Phi ((1 - w_k) f)(x) + M \Phi \left( \frac{|c_k|}{\mu(Q_k)} \chi_{Q_k} \right)(x).
$$

Since $M \Phi \chi_{Q_k}(x) \leq 1$ and $Q_k$ is $(4, 4^{n+1})$-doubling, we get

$$
\int_{4Q_k} M \Phi \left( \frac{|c_k|}{\mu(Q_k)} \chi_{Q_k} \right) \, d\mu(x) \leq C |c_k| = C \left| \int w_k f \, d\mu \right|.
$$

(9.10)

From (9.8), (9.9) and (9.10) we derive (9.7).
Now we have to see that the terms on the right hand side of (9.7) tend to 0 as \( k \to \infty \). Since \( f, M_\Phi f \in L^1(\mu) \), by the dominated convergence theorem
\[
\lim_{k \to \infty} \left| \int w_k f \, d\mu \right| + \int_{\mathbb{R}^d \setminus 4Q_k} M_\Phi f \, d\mu = 0.
\]
Let us turn our attention to the third term on the right hand side of (9.7). Take \( x \in 4Q_k \) and \( \varphi \sim x \). It is easily seen that \( C w_k \varphi \sim x \) for some constant \( C > 0 \). So we get
\[
\chi_{4Q_k}(x) M_\Phi ((1 - w_k(x)) f)(x) \leq C M_\Phi f(x).
\]
Therefore, if we show that \( \chi_{4Q_k}(x) M_\Phi ((1 - w_k(x)) f)(x) \) tends to 0 point-wise as \( k \to \infty \), we will be done by a new application of the dominated convergence theorem.

For a fixed \( x \in \mathbb{R}^d \), let \( k_0 \) be such that \( x \in \frac{1}{2} Q_k \) for \( k \geq k_0 \). Notice that if \( \varphi \sim x \) and \( y \not\in Q_k \), then \( |\varphi(y)| \leq C/\ell(Q_k)^n \). Thus
\[
\left| \int \varphi(x)(1 - w_k(x)) f(x) \, d\mu(x) \right| \leq \|f\|_{L^1(\mu)} \|(1 - w_k) \varphi\|_{L^\infty(\mu)} \leq C \frac{\|f\|_{L^1(\mu)}}{\ell(Q_k)^n}.
\]
Then we get
\[
\chi_{4Q_k}(x) M_\Phi ((1 - w_k(x)) f)(x) \leq C \frac{\|f\|_{L^1(\mu)}}{\ell(Q_k)^n} \xrightarrow{k \to \infty} 0.
\]

\[\square\]

**REFERENCES**


In terms of a maximal operator


