A DUAL CHARACTERIZATION OF
THE $C^1$ HARMONIC CAPACITY
AND APPLICATIONS

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Abstract. The Lipschitz and $C^1$ harmonic capacities $\kappa$ and $\kappa_c$ in $\mathbb{R}^n$ can be considered as high-dimensional versions of the so-called analytic and continuous analytic capacities $\gamma$ and $\alpha$ (respectively).

In this paper we provide a dual characterization of $\kappa_c$ in the spirit of the Garabedian function for the capacity $\alpha$. Using this new characterization, we show that $\kappa(E) = \kappa(\partial_o E)$ for any compact set $E \subset \mathbb{R}^n$, where $\partial_o E$ is the outer boundary of $E$, and we solve an open problem posed by A. Volberg, which consists on estimating from below the Lipschitz harmonic capacity of a graph of a continuous function.

1. Introduction

Let $\text{Lip}^1_{\text{loc}}(\mathbb{R}^n)$ be the set of real-valued locally Lipschitz functions (with exponent 1) on $\mathbb{R}^n$ and $C^1(\mathbb{R}^n)$ the set of real-valued continuously differentiable functions on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$ is a bounded set and

$$U'(E) = \{ \varphi \in \text{Lip}^1_{\text{loc}}(\mathbb{R}^n) : \text{supp}\Delta \varphi \subset E, \nabla \varphi(\infty) = 0 \},$$
$$U'_c(E) = \{ \varphi \in C^1(\mathbb{R}^n) : \text{supp}\Delta \varphi \subset E, \nabla \varphi(\infty) = 0 \},$$

the Lipschitz and $C^1$ harmonic capacities of $E$ are defined by

$$\kappa(E) = \sup \{ \langle 1, \Delta \varphi \rangle : \varphi \in U'(E), \| \nabla \varphi \|_{\infty} \leq 1 \},$$
$$\kappa_c(E) = \sup \{ \langle 1, \Delta \varphi \rangle : \varphi \in U'_c(E), \| \nabla \varphi \|_{\infty} \leq 1 \},$$

where $\langle f, \Delta \varphi \rangle$ means the action of the compactly supported distribution $\Delta \varphi$ on a smooth function $f$, and $\| \nabla \varphi \|_{\infty}$ is the $L^\infty$ norm of $\nabla \varphi$ with respect to the Lebesgue measure in $\mathbb{R}^n$.

In order to deal with the problem of harmonic approximation in the $C^1$-norm, Paramonov introduced in [Pa] the capacities $\kappa$ and $\kappa_c$, and gave a description, in terms of these capacities, of the compact sets $E \subset \mathbb{R}^n$ (with $n \geq 2$) such that any $C^1$ function harmonic in the interior of $E$ can be approximated in the $C^1$-norm by harmonic functions in a neighborhood of $E$.

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The capacities $\kappa$ and $\kappa_c$ can be understood as high-dimensional versions of the so-called analytic and continuous analytic capacities $\gamma$ and $\alpha$ (respectively). Recall that, for a compact set $E \subset \mathbb{C}$,

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. The continuous analytic capacity $\alpha$ has the same definition as $\gamma$ except that one also requires the functions $f$ to be continuous in $\mathbb{C}$ and $|f| \leq 1$ everywhere.

The analytic capacity was first introduced by Ahlfors in [Ah] when he was characterizing the removable compact sets for bounded analytic functions in the plane. The continuous analytic capacity was defined by Vitushkin in [Vi] when he dealt with the problem of rational approximation in the uniform norm on compact sets of the plane. Both capacities have been studied by many authors since then (see [Gt] for a nice survey on results related with $\gamma$ and $\alpha$, and [Da] or [To1] for more recent results). In particular, there exists a dual characterization of $\alpha$ in terms of the Garabedian function that can be stated as follows: let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded domain with smooth boundary. Then, the Garabedian function $\psi$ of $\Omega$ is the one that solves the extremal problem

$$\int_{\partial \Omega} |\psi(z)| ds = \inf \left\{ \int_{\partial \Omega} |h(z)| ds : h \in H^1(\Omega^c), h(\infty) = \frac{1}{2\pi i} \right\},$$

where $ds$ denotes the arc length and $H^1(\Omega^c)$ is the Hardy space of functions $h$ analytic in $\Omega^c \cup \{\infty\}$ such that the subharmonic function $|h(z)|$ has a harmonic majorant. The Garabedian function also satisfies

$$\alpha(\Omega) = \int_{\partial \Omega} |\psi(z)| ds.$$

A classical way to construct the Garabedian function is to use the Hahn-Banach theorem and the F. and M. Riesz theorem. Observe that the quantity $\alpha(\Omega)$ is the norm of the functional $f \mapsto f'(\infty) = \int_{\partial \Omega} f(z) dz$ on the space of continuous functions in $\mathbb{C} \cup \{\infty\}$ that are analytic outside $\Omega$, which is a subspace of the continuous functions on $\partial \Omega$. By the Hahn-Banach theorem one can find a measure $\mu$ supported on $\partial \Omega$, orthogonal to the functions analytic outside $\Omega$, and such that

$$\alpha(\Omega) = \int_{\partial \Omega} |dz + d\mu|.$$

The F. and M. Riesz theorem ensures that, in fact, $dz + d\mu(z) = \psi(z) ds$ for an analytic function $\psi$ that solves the extremal problem (see [Gt, section I.4] for more details).

The aim of this paper is to give a dual characterization of the $C^1$ harmonic capacity $\kappa_c$ in terms of some “Garabedian function” and to use this “function” to deduce some properties of $\kappa_c$ and $\kappa$. This characterization is stated in theorem 3.3 and it is based on the Hahn-Banach theorem, as can be done for the capacity $\alpha$.

Unfortunately, the F. and M. Riesz theorem can not be generalized to higher dimensions in the sense that we need, because the capacity $\kappa_c$ is defined in terms of gradients of harmonic functions and there are examples of
measures orthogonal to those gradients which are not absolutely continuous with respect to the surface measure (see [Ma]). This means that, instead of a “Garabedian function”, we will just have a “Garabedian measure” that minimizes some quantity. We will be able to deduce some geometric properties of that minimal measure by adapting a theorem of B. Gustafsson and D. Khavinson about measures on \( \partial \Omega \) orthogonal to harmonic gradients (see theorem 3.1) and by proving a theorem about restrictions to \( \partial \Omega \) of orthogonal measures on \( \Omega \) (see theorem 3.2).

Many properties of the Lipschitz and \( C^1 \) harmonic capacities were recently proven. The semiaadditivity is a very important one, and it was obtained by A. Volberg in [Vo] for the capacity \( \kappa \). A little bit later, A. Ruiz de Villa and X. Tolsa proved in [Aleix-Tolsa??] that \( \kappa_c \) is also semiaadditive. See [MP] and [Aleix??] for other interesting properties about these capacities.

We will obtain two new results on the capacity \( \kappa \) from theorem 3.3, namely theorem 4.1 and theorem 4.2. The first one states that \( \kappa(E) = \kappa(\partial_o E) \) for any compact set \( E \subset \mathbb{R}^n \), where \( \partial_o E \) denotes the outer boundary of \( E \) (i.e., the boundary of the unbounded component of \( E^c \)). This property is obvious for the capacity \( \gamma \) because of the defining conditions, but this is no longer trivial for \( \kappa \). Evidently, one have \( \kappa(E) \geq \kappa(\partial_o E) \). The difficulties appear when one tries to prove the reverse inequality. Observe that, by Gauss formula,

\[
\langle 1, \Delta \varphi \rangle = \int_{\partial_o V} \nabla \varphi \cdot \eta d\sigma
\]

for any \( \varphi \in U'(E) \), where \( V \) is a sufficiently regular neighborhood of \( E \), and \( \eta \) and \( d\sigma \) are the normal outward unit vector and surface of measure of \( \partial V \), respectively. Suppose, for simplicity, that \( E \) is the closure of a bounded simply connected domain, so \( \partial_o E = \partial E \). One can try to prove that \( \kappa(E) \leq \kappa(\partial E) \) directly from the definition (1.1) and the identity (1.2). The idea is to modify the functions \( \varphi \in U'(E) \) inside \( E \) to obtain functions \( \tilde{\varphi} \in U'(\partial E) \) such that \( \langle 1, \Delta \varphi \rangle = \langle 1, \Delta \tilde{\varphi} \rangle \). The problem is that one cannot ensure that the gradients \( \nabla \tilde{\varphi} \) are bounded by 1 in \( E \).

The second new result that we have obtained is theorem 4.2, where we solve an open problem posed by A. Volberg (private communication). The problem can be stated as follows:

**Problem 1.1.** Let \( f \) be a real continuous function defined on the cube \( Q_0 = [0, d]^{n-1} \subset \mathbb{R}^{n-1} \) and let \( \Gamma = \{(x, f(x)) \in \mathbb{R}^n : x \in Q_0\} \) be the graph of \( f \). Prove that there exists a constant \( C > 0 \) depending only on \( n \) such that \( Cd^{n-1} \leq \kappa(\Gamma) \).

Note that, if \( \text{diam}(\Gamma) \) is comparable to \( d \), problem 1.1 states that \( \kappa(\Gamma) \geq C\text{diam}(\Gamma)^{n-1} \). This is a reasonable analogue of an important result in the area of analytic capacity which says that \( \gamma(E) \geq C\text{diam}(E) \) for any continuum (i.e., compact and connected set) \( E \subset \mathbb{C} \). This classical result on \( \gamma \) is a consequence of the 1/4-theorem of Koebe (see [Ga, theorem 2.1 of chapter VIII]), and a real variable proof was first obtained by P. Jones by using the notion of curvature of a measure (see [Pj, Section 3.5]).

One cannot expect this kind of estimates on \( \kappa(E) \) for any continuum \( E \subset \mathbb{R}^n \), because, for example, a segment in \( \mathbb{R}^3 \) has zero Lipschitz harmonic
capacity. In fact, by using the identity (1.2), it is not difficult to show that 
\( \kappa(E) = 0 \) for any compact set \( E \subset \mathbb{R}^n \) with zero \((n-1)\)-Hausdorff measure. 
So, to obtain a reasonable analogue of the estimate of the analytic capacity of a continuum for the capacity \( \kappa \), one has to restrict himself to continua with positive \((n-1)\)-Hausdorff measure or, in an easier way, to graphs of continuous functions.

The structure of the paper is the following. Section 2 is devoted to the preliminaries, where we will talk about vector measures, Lipschitz and \( \mathcal{C}^1 \) harmonic capacities, and harmonicity at infinity (which includes the exterior Dirichlet and Neumann problems). With these notions, we will be ready to state and prove the dual characterization of \( \kappa_c \), i.e., theorem 3.3. This will be in section 3. Section 4 is devoted to prove the two announced properties of \( \kappa \).

2. Preliminaries

In the whole paper, we assume \( n \geq 2 \). The word smooth means of class \( \mathcal{C}^\infty \), wherever we talk about functions or the boundary of an open set. We write \( \chi_E \) for the characteristic function of a set \( E \subset \mathbb{R}^n \). The letter \( C \) will denote a constant which may be different at different occurrences and which is independent of the relevant variables under consideration.

We denote by \( \mathcal{C}(E) \) the set of real-valued continuous functions defined on a set \( E \subset \mathbb{R}^n \), and by \( \mathcal{C}(E)^n \) the cartesian product of \( n \) spaces \( \mathcal{C}(E) \).

Given a \( \mathcal{C}^\infty \) orientable manifold \( M \) of dimension \( d \leq n \) and \( k \in \mathbb{N} \cup \{\infty\} \), let \( \mathcal{C}^k(M) \) be the set of real-valued differentiable functions in \( M \) such that their partial derivatives (with respect to the local coordinates chosen in \( M \)) of order less than \( k+1 \) exist and are continuous functions in \( M \). In case that \( \partial M \neq \emptyset \), we can take a system of local coordinates

\[ \{U \subset M, y = (y_1, \ldots, y_d)\} \]

such that \( U \cap M = y^{-1}(\{x_d \geq 0\}) \) and \( U \cap \partial M = y^{-1}(\{x_d = 0\}) \), for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). For the points \( p = y^{-1}(x) \in U \cap \partial M \), the partial derivative of a function \( f : y(U) \to \mathbb{R} \) with respect to the coordinate \( x_d \) at a point \( p = (p_1, \ldots, p_{n-1}, 0) \in y(U) \) is defined by the limit (if it exists)

\[
\frac{\partial}{\partial x_d} f \bigg|_p = \lim_{t > 0, t \to 0} \frac{f(p_1, \ldots, p_{n-1}, t) - f(p_1, \ldots, p_{n-1}, 0)}{t}.
\]

When \( M \subset \mathbb{R}^n \), for any function \( \varphi \in \mathcal{C}^1(M) \) we can identify the differential \( d\varphi \) with a vector function in \( \mathcal{C}(M)^n \), which we denote by \( \nabla \varphi \). So, we say that \( \varphi \in \mathcal{C}^1(M) \) if \( \nabla \varphi \in \mathcal{C}(M)^n \). It is an exercise to check that this definition of \( \mathcal{C}^1(M) \) agrees with the classical one in the case that \( M \) is the closure of an open set of \( \mathbb{R}^n \) with smooth boundary, i.e., \( \mathcal{C}^1(M) \) is the set of continuous functions on \( M \) such that their gradients on int\( M \) can be extended to continuous vector functions on \( M \). It also agrees with the definition of \( \mathcal{C}^1(M) \) given in [Pa] or [Wh].

Our typical situation will be that \( M \) is equal to \( U, \overline{U} \) or \( \partial U \), for an open set \( U \subset \mathbb{R}^n \) with smooth boundary.
2.1. Vector measures. The Riesz representation theorem. For a vector function \( f = (f_1, \ldots, f_n) : E \to \mathbb{R}^n \), define
\[
\|f\|_E = \sup\{|f(x)| : x \in E\},
\]
where
\[
|f(x)| = \left(\sum_{i=1}^n (f_i(x))^2\right)^{1/2}.
\]
Clearly, \( C(E)^n \) with the norm \( \| \cdot \|_E \) is a Banach space.

Given a bounded linear functional \( \Lambda \) on \( C(E)^n \) and a subspace \( \mathcal{F} \subset C(E)^n \), define
\[
\|\Lambda\|_F = \sup\{|\Lambda(f)| : f \in \mathcal{F}, \|f\|_E \leq 1\}.
\]
For simplicity, we write \( \|\cdot\|_F \) and \( \|\cdot\|_E \) instead of \( \|\cdot\|_{C(E)^n} \) and \( \|\cdot\|_{C(E)^n} \), respectively (when there is no confusion on what is \( E \)).

Let \( \mathcal{M}(E) \) be the space of finite real Borel measures supported on \( E \) and \( \mathcal{M}(E)^n \) the cartesian product of \( n \) spaces \( \mathcal{M}(E) \). For \( \mu = (\mu_1, \ldots, \mu_n) \in \mathcal{M}(E)^n \), define the variation of \( \mu \) on a subset \( F \subset E \) as
\[
|\mu|(F) = \sup\left\{ \sum_{j=1}^m |\mu(F_j)| : F = \biguplus_{j=1}^m F_j, \ F_j \ \mu_i\text{-measurable} \ \forall i,j \right\},
\]
where
\[
|\mu(F_j)| = \left(\sum_{i=1}^n (\mu_i(F_j))^2\right)^{1/2}.
\]
Finally, define the total variation of \( \mu \) as \( \|\mu\|_E = |\mu|(E) \). It is proved that \( |\mu| \) is a positive and finite measure on \( E \) (see for example [L2, theorem 3.1 of chapter VII]). It is easily seen that \( \| \cdot \|_E \) is a norm on the space \( \mathcal{M}(E)^n \).

Any vector measure \( \mu \in \mathcal{M}(E)^n \) can be considered as a bounded linear functional \( \langle \cdot, \mu \rangle : C(E)^n \to \mathbb{R} \) by putting
\[
\langle f, \mu \rangle = \int f d\mu = \sum_{i=1}^n \int f_i d\mu_i.
\]
On the other hand, the Riesz representation theorem states that any bounded linear functional on \( C(E)^n \) can be represented as \( \langle \cdot, \mu \rangle \) for some vector measure \( \mu \in \mathcal{M}(E)^n \).

**Theorem 2.1** (Riesz representation). The map \( \mu \mapsto \langle \cdot, \mu \rangle \) is an isometric isomorphism of \( \mathcal{M}(E)^n \) onto the space of bounded linear functionals on \( C(E)^n \), so \( \|\mu\|_E = \|\langle \cdot, \mu \rangle\| \) for all \( \mu \in \mathcal{M}(E)^n \).

**Proof.** By the previous comments, it is enough to prove that the map \( \mu \mapsto \langle \cdot, \mu \rangle \) is isometric to obtain the isomorphism. We have to check that for any \( \mu \in \mathcal{M}(E)^n \),
\[
\|\mu\|_E = \sup\{|\langle f, \mu \rangle| : f \in C(E)^n, \|f\|_E \leq 1\}.
\]
We will see first that \( \sup\{|\langle f, \mu \rangle| : f \in C(E)^n, \|f\|_E \leq 1\} \leq \|\mu\|_E \). By density, it is enough to prove that \( |\langle f, \mu \rangle| \leq \|\mu\|_E \) for simple vector functions of the form \( f = (\sum a_m^1 \chi_{F_m}, \ldots, \sum a_m^n \chi_{F_m}) \), where the sums are finite,
the \(a_m^i\)'s are real numbers, the \(F_j\)'s are disjoint subsets of \(E\), and \(\|f\|_E = \sup_m (\sum a_m^i)^{1/2} \leq 1\). By the Cauchy-Schwartz inequality,

\[
\langle f, \mu \rangle = \left| \sum_{i=1}^{n} a_m^i \mu_i(F_m) \right| \leq \sum_{m} \left( \sum_{i=1}^{n} (a_m^i)^2 \sum_{j=1}^{n} (\mu_j(F_m))^2 \right)^{1/2} \\
\leq \sup_m \left( \sum_{i=1}^{n} (a_m^i)^2 \right)^{1/2} \sum_{m} |\mu(F_m)| \leq \|f\|_E |\mu|(E) \leq \|\mu\|_E.
\]

Let us prove now that \(\|\mu\|_E \leq \sup \{ \|\langle f, \mu \rangle \| : f \in C(E)^n, \|f\|_E \leq 1 \}\). Let \(\nu\) be a positive measure such that \(\mu_i\) is absolutely continuous with respect to \(\nu\) for all \(i = 1, \ldots, n\) (for example \(\nu = \sum_{i=1}^{n} |\mu_i|\), where \(|\mu_i|\) is the classical variation of the real measure \(\mu_i\)). Then \(\mu_i = h_i \nu\), where \(h_i\) is a \(\nu\)-measurable function. Observe that, if we put \(h = (h_1, \ldots, h_n)\), then \(\mu = h \nu\) and

\[
(2.1) \quad \|\langle \cdot, \mu \rangle \| = \sup \{ \|\langle f, h \nu \rangle \| : f \in C(E)^n, \|f\|_E \leq 1 \} \leq \int_E |h| d\nu.
\]

Consider the \(\nu\)-measurable vector function \(g\) defined by \(g(x) = h(x)/|h(x)|\) whenever \(h(x) \neq 0\) and \(g(x) = 0\) otherwise. Lusin’s theorem can be adapted to our situation to prove that given \(\varepsilon > 0\) there exists \(f_\varepsilon \in C(E)^n\) with \(\|f_\varepsilon\|_E \leq \|g\|_E \leq 1\) and such that

\[
\left| \int_E (g - f_\varepsilon) d\mu \right| < \varepsilon.
\]

This implies that

\[
\int_E |h| d\nu = \int_E (g \cdot h) d\nu = \int_E g d\mu \\
\leq \left| \int_E (g - f_\varepsilon) d\mu \right| + \left| \int_E f_\varepsilon d\mu \right| \leq \varepsilon + \|\langle \cdot, \mu \rangle \|
\]

for all \(\varepsilon > 0\). This estimate together with (2.1) proves that \(\|\langle \cdot, \mu \rangle \| = \int_E |h| d\nu\). So, to prove that \(\|\mu\|_E \leq \|\langle \cdot, \mu \rangle \|\) it is enough to check that \(|\mu(F)| \leq \int_F |h| d\nu\) for all \(F \subset E\) \(\nu\)-measurable. By a discrete version of Minkowski’s integral inequality,

\[
|\mu(F)| = \left( \sum_{i=1}^{n} \left( \int_F d\mu_i \right)^2 \right)^{1/2} = \left( \sum_{i=1}^{n} \left( \int_F h_i d\nu \right)^2 \right)^{1/2} \\
\leq \int_F \left( \sum_{i=1}^{n} h_i^2 \right)^{1/2} d\nu.
\]

Therefore, \(|\mu(F)| \leq \int_F |h| d\nu\), and the theorem is proved.

\[\square\]

2.2. The Lipschitz and \(C^1\) harmonic capacities. The fundamental solution \(\phi_n\) for the Laplace equation \(\Delta f = 0\) in \(\mathbb{R}^n\) is defined by

\[
\phi_n(x) = \begin{cases} 
  a_n |x|^{-n} & \text{if } n > 2, \\
  a_n \log |x| & \text{if } n = 2,
\end{cases}
\]

where \(a_n\) is a constant which depends on the dimension \(n\).
We stated the definitions of the Lipschitz and $C^1$ harmonic capacities $\kappa$ and $\kappa_c$ in (1.1). The defining conditions for $U'(E)$ and $U'_c(E)$ imply that the functions $\varphi \in U'(E)$ are harmonic in $E^c$ and take the form $\varphi = \phi_n + \Delta \varphi + \text{constant}$, where this last equality is in the sense of distributions; but by the definitions of $\kappa$ and $\kappa_c$ we can suppose that, in fact, $\varphi = \phi_n + \Delta \varphi$. Recall that, if $T$ is a compactly supported distribution, then for each $\psi \in C^\infty(\mathbb{R}^n)$ with compact support, by definition (in view of parity of $\phi_n$),

$$\langle \phi_n * T, \psi \rangle = \langle T, \phi_n * \psi \rangle,$$

where $\phi_n * \psi(x) = \int \phi_n(y)\psi(x - y)dm(y)$ and $m$ is the Lebesgue measure on $\mathbb{R}^n$.

Therefore, if we take

$$U(E) = \{ \varphi \in \text{Lip}_{loc}(\mathbb{R}^n) : \text{supp} \Delta \varphi \subset E, \varphi = \phi_n + \Delta \varphi \},$$

$$U'_c(E) = \{ \varphi \in C^1(\mathbb{R}^n) : \text{supp} \Delta \varphi \subset E, \varphi = \phi_n + \Delta \varphi \},$$

we can redefine the Lipschitz and $C^1$ harmonic capacities by

$$\kappa(E) = \sup \{ (1, \Delta \varphi) : \varphi \in U(E), \| \nabla \varphi \|_\infty \leq 1 \},$$

$$\kappa_c(E) = \sup \{ (1, \Delta \varphi) : \varphi \in U'_c(E), \| \nabla \varphi \|_\infty \leq 1 \}.$$

2.3. Harmonicity outside a compact set and at infinity. Most of this section can be found in [Fo].

**Definition 2.2.** Consider $\mathbb{R}^n \cup \{ \infty \}$ as the Alexandrov’s compactification of $\mathbb{R}^n$. For any set $E \subset \mathbb{R}^n \cup \{ \infty \}$ we define $E^* = \{ x/|x|^2 : x \in E \} \subset \mathbb{R}^n \cup \{ \infty \}$. Given a function $u$ defined on a set $E \subset \mathbb{R}^n \setminus \{ 0 \}$, define the Kelvin transform of $u$ by

$$K_u(x) = |x|^{2-n}u(x/|x|^2), \text{ for } x \in E^*.$$

**Theorem 2.3.** The Kelvin transform is its own inverse. If $V \subset \mathbb{R}^n \setminus \{ 0 \}$ is an open set, then a function $u$ is harmonic in $V$ if and only if $K_u$ is harmonic in $V^*$.

**Definition 2.4.** If $E \subset \mathbb{R}^n$ is compact and $u$ is harmonic in $E^c$, then $u$ is harmonic at $\infty$ provided $K_u$ has a removable singularity at the origin.

**Theorem 2.5.** Suppose that $u$ is harmonic in $E^c$, where $E \subset \mathbb{R}^n$ is compact. Then, the following three conditions are equivalent:

1. $u$ is harmonic at $\infty$.
2. $|u(x)| = o(1)$ as $x \to \infty$ $(n > 2)$, or $|u(x)| = o(\log |x|)$ as $x \to \infty$ $(n = 2)$.
3. $|u(x)| = O(|x|^{2-n})$ as $x \to \infty$.

In particular, any function which is harmonic at infinity vanishes at infinity when $n > 2$ and is bounded when $n = 2$.

**Theorem 2.6** (Exterior Dirichlet problem). Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded domain with smooth boundary. Given $h \in C(\partial \Omega)$ there exists a unique function $u \in C(\overline{\Omega^c})$ such that $u$ is harmonic in $\Omega^c \cup \{ \infty \}$ and $u|_{\partial \Omega} = h$. If $h \in C^\infty(\partial \Omega)$, then $u \in C^\infty(\Omega)$.
Theorem 2.7 (Exterior Neumann problem). Let \( \Omega \subset \mathbb{R}^n \) be the closure of a bounded domain with smooth boundary and \( \eta \) the outward unit normal vector on \( \partial \Omega \). Let \( V_1, \ldots, V_m \) be the bounded connected components of \( \Omega^c \) and \( V_0 \) the un bounded one. Let \( h \in C(\partial \Omega) \) such that

$$\int_{\partial V_i} h d\sigma = 0 \quad \text{for all } i = 1, \ldots, m.$$ 

(1) Assume \( n > 2 \). Then, there exists a function \( u \in C^1(\Omega^c) \) such that \( u \) is harmonic in \( \Omega^c \cup \{\infty\} \) and \( (\nabla u \cdot \eta)|_{\partial \Omega} = h \). The function \( u \) is unique modulo functions which are constant on each bounded connected component of \( \Omega^c \).

(2) Assume \( n = 2 \). Then, there exists a function \( u \in C^1(\Omega^c) \) such that \( u \) is harmonic in \( \Omega^c \cup \{\infty\} \) and \( (\nabla u \cdot \eta)|_{\partial \Omega} = h \) if and only if

$$\int_{\partial V_0} h d\sigma = 0.$$ 

In that case, the function \( u \) is unique modulo functions which are constant on each connected component of \( \Omega^c \).

In both cases, \( u \in C^\infty(\Omega) \) if \( h \in C^\infty(\partial \Omega) \).

Remark 2.8. Theorem 2.7 corresponds with theorem 3.41 of [Fo]. If we do not have the assumption \( \int_{\partial V_0} h d\sigma = 0 \) in theorem 2.7(2), we can still find a function \( u \in C^1(\Omega^c) \) harmonic in \( \Omega^c \) and such that \( (\nabla u \cdot \eta)|_{\partial \Omega} = h \), by looking carefully at the proof of theorem 3.41 in [Fo]. Moreover, \( u \) can be taken as

$$u(x) = \int_{\partial V_0} \log |x - y| u_0(y) d\sigma(y)$$

for all \( x \in V_0 \) and for some \( u_0 \in C(\partial V_0) \) depending on \( h \). But now, \( u \) may not be harmonic at infinity (because it may not be bounded) and we cannot ensure uniqueness in \( \Omega^c \) modulo constant functions. In fact, in proposition 3.35 of [Fo] it is shown that our particular solution \( u \) is harmonic at infinity if and only if \( \int_{\partial V_0} h d\sigma = 0 \).

Lemma 2.9 (Green’s formula). Let \( \Omega \subset \mathbb{R}^n \) be the closure of a bounded domain with smooth boundary and \( \eta \) the outward unit normal vector on \( \partial \Omega \). Let \( u \) and \( v \) be harmonic functions in \( \Omega^c \), \( C^1 \) up to \( \partial \Omega \), and such that

$$|(u(x) \nabla v(x) - v(x) \nabla u(x)) \cdot x| = o(|x|^{2-n})$$

when \( |x| \to \infty \). Then,

$$\int_{\partial \Omega} (\nabla u \cdot \eta) v d\sigma = \int_{\partial \Omega} u(\nabla v \cdot \eta) d\sigma.$$ 

Proof. Let \( B_R \) be the ball centered at the origin with radius \( R \), and take \( R > M \) such that \( \Omega \subset B_{R/2} \). Define \( \Omega_R = B_R \setminus \Omega \) and let \( \eta \) denote also the
inward unit normal vector on $\partial B_R$. By Green’s formula on $\Omega_R$,

$$
\int_{\partial \Omega} (\nabla u \cdot \eta) v d\sigma = \int_{\partial \Omega_R} (\nabla u \cdot \eta) v d\sigma - \int_{\partial B_R} (\nabla u \cdot \eta) v d\sigma
$$

$$
= \int_{\partial \Omega_R} u(\nabla v \cdot \eta) d\sigma - \int_{\partial B_R} (\nabla u \cdot \eta) v d\sigma
$$

$$
= \int_{\partial \Omega} u(\nabla v \cdot \eta) d\sigma + \int_{\partial B_R} (u(\nabla v \cdot \eta) - (\nabla u \cdot \eta)v) d\sigma.
$$

For any $R$ big enough, by the assumption on $u$ and $v$,

$$
\left| \int_{\partial B_R} (u(\nabla v \cdot \eta) - (\nabla u \cdot \eta)v) d\sigma \right| \leq \int_{\partial B_R} |u \nabla v \cdot \eta - v \nabla u \cdot \eta| d\sigma
$$

$$
\leq o(R^{1-n})R^{n-1},
$$

and letting $R \to \infty$ we obtain the desired result. \qed

Remark 2.10. Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded open set with smooth boundary. In the next section, we will need to apply Green’s formula to pairs of functions $\varphi$ and $u$, where $\varphi \in U_c(\Omega)$ and $u$ is a function harmonic in $\Omega^c \cup \{\infty\}$ and continuous up to $\partial \Omega$. For this reason, we give now some estimates on the behavior of $\varphi$ and $u$ near infinity.

Let $\varphi \in U_c(\Omega)$. By applying Gauss formula, we have that for all $x \in \Omega^c$,

$$
\varphi(x) = \int_{\Omega} \phi_n(x - y) \Delta_y \varphi(y) d\mu(y)
$$

$$
= \int_{\Omega} (\operatorname{div}_y(\phi_n(x - y) \nabla_y \varphi(y)) - \nabla_y \phi_n(x - y) \cdot \nabla_y \varphi(y)) d\mu(y)
$$

$$
= \int_{\partial \Omega} \phi_n(x - y) \nabla_y \varphi(y) \cdot \eta(y) d\sigma(y) + \int_{\Omega} \nabla \phi_n(x - y) \cdot \nabla_y \varphi(y) d\mu(y),
$$

where $\eta$ and $d\sigma$ are the outward unit normal vector and surface measure related to $\partial \Omega$.

Assume $n > 2$. By computing the derivatives of $\phi_n$, it is an exercise to see that any $\varphi \in U_c(\Omega)$ satisfies the statement (3) of theorem 2.5, so it is harmonic at infinity. It is proved in [Fo] (proposition 2.73) that any function $u$ harmonic outside $\Omega$ and at infinity satisfies $|\nabla u(x) \cdot x| = O(|x|^{2-n})$, so lemma 2.9 can be applied to the pair $\varphi$ and $u$, and Green’s formula holds in that case.

The case $n = 2$ is a little bit different, because we cannot ensure that a function $\varphi \in U_c(\Omega)$ has the required decay at infinity. We have the estimates $|\varphi(x)| = O(\log |x|)$ and $|\nabla \varphi(x)| = O(|x|^{-1})$ near infinity (and we cannot apply theorem 2.5(2)). For a function $u$ harmonic outside $\Omega$ and at infinity, we still have the estimates $|u(x)| = O(1)$ and $|\nabla u(x) \cdot x| = O(|x|^{-1})$, as can be seen in proposition 2.73 of [Fo]. These estimates are not enough to use lemma 2.9, because $|u(x) \nabla \varphi(x) \cdot x| = O(1)$. But, if $u(\infty) = 0$, then $|u(x)| = O(1)$ and we can still apply lemma 2.9 in that particular case.
3. The heart of the matter

In the whole section, $\Omega \subset \mathbb{R}^n$ will be the closure of a bounded open set with smooth boundary and $\eta$ will denote the outward unit normal vector on $\partial \Omega$.

Consider the following normed spaces related to the compact set $\Omega$:

$$B(\Omega) = \{ f \in C(\Omega) : f = \nabla \varphi, \varphi \in U_c(\Omega) \},$$

$$B(\Omega)^\perp = \{ \mu \in \mathcal{M}(\Omega)^n : \langle f, \mu \rangle = 0 \text{ for all } f \in B(\Omega) \},$$

$$bB(\Omega)^\perp = \{ \mu \in B(\Omega)^\perp : \text{supp}\mu \subset \partial \Omega \},$$

where $B(\Omega)$ is equipped with the norm $\| \cdot \|_\Omega$ and the orthogonal spaces $B(\Omega)^\perp$ and $bB(\Omega)^\perp$ are equipped with the induced norm from $\mathcal{M}(\Omega)^n$. It is easily seen that the norm in $bB(\Omega)^\perp$ induced by the space $\mathcal{M}(\Omega)^n$ coincides with the norm induced by the space $\mathcal{M}(\partial \Omega)^n$.

Let $A(\Omega)$ be the set of smooth vector fields on $\partial \Omega$. For any $g \in A(\Omega)$, let $g_t$ be the tangential component of $g$ and $g_\eta$ the normal one, i.e., $g_\eta = g \cdot \eta$ and $g_t = g - g_\eta \eta$. Denote by $u_g$ the unique harmonic extension of $g_\eta$ to $\Omega^c \cup \{\infty\}$ given by theorem 2.6. Define $A_0(\Omega) = \{ g \in A(\Omega) : u_g(\infty) = 0 \}$. By theorem 2.5, $A_0(\Omega) = A(\Omega)$ for $n > 2$.

For any $g \in A(\Omega)$, the divergence of $g_t$ on $\partial \Omega$ can be defined by its action on smooth and compactly supported functions $\varphi$ as

$$\int_{\partial \Omega} (\text{div}\,g_t) \varphi d\sigma = -\int_{\partial \Omega} g_t \cdot \nabla \varphi d\sigma.$$ 

In the right hand side, we can replace $\nabla \varphi$ by $(\nabla \varphi)_t$, which only depends on the values of $\varphi$ on $\partial \Omega$. This definition agrees with the classical one of divergence in the context of Riemannian manifolds in $\mathbb{R}^n$ (see [Wa]).

The following theorem is a little modification of theorem 3.1 in [GK], where the orthogonal of the space of harmonic gradients inside $\Omega$ is studied. In our case, we need the harmonicity outside $\Omega$. Our proof is almost the same as the one of [GK] and, for completeness, we include all the detailed arguments.

**Theorem 3.1.** Let $g \in A_0(\Omega)$. Then $g d\sigma \in bB(\Omega)^\perp$ if and only if

$$\text{div}\,g_t = \nabla u_g \cdot \eta \text{ on } \partial \Omega.$$ 

Such measures are weak* dense in $bB(\Omega)^\perp$, i.e., for all $\mu \in bB(\Omega)^\perp$ there exists a sequence $\{g^m\}_{m \in \mathbb{N}} \subset A_0(\Omega)$ such that $\text{div}\,g^m_t = \nabla u_{g^m} \cdot \eta$ for all $m$ and

$$\lim_{m \to \infty} \langle f, g^m d\sigma \rangle = \langle f, \mu \rangle$$

for all $f \in C(\partial \Omega)^n$.

**Proof.** Let $g \in A_0(\Omega)$ and take $\nabla \varphi \in B(\Omega)$. Then,

$$\langle \nabla \varphi, g d\sigma \rangle = \int_{\partial \Omega} (\nabla \varphi)_t \cdot g_t d\sigma + \int_{\partial \Omega} (\nabla \varphi \cdot \eta) g_\eta d\sigma$$

$$= -\int_{\partial \Omega} \varphi \text{div} g_t d\sigma + \int_{\partial \Omega} (\nabla \varphi \cdot \eta) u_g d\sigma.$$
The pair \( \varphi \) and \( u_g \) satisfies the statements of lemma 2.9 by remark 2.10, so

\[
\langle \nabla \varphi, gd\sigma \rangle = \int_{\partial \Omega} \varphi(\nabla u_g \cdot \eta - \text{div} g_t) d\sigma.
\]

This integral vanishes for all \( \varphi \) such that \( \nabla \varphi \in B(\Omega) \) if and only if \( \text{div} g_t = \nabla u_g \cdot \eta \) on \( \partial \Omega \).

To prove that such measures \( gd\sigma \) are weak* dense in \( hB(\Omega)^{\perp} \) it is enough to prove that if \( f \in C(\partial \Omega)^n \) and \( \langle f, gd\sigma \rangle = 0 \) for all such \( g \), then there exists \( \psi \in U_c(\Omega) \) with \( \langle \nabla \psi \rangle|_{\partial \Omega} = f \). So, consider \( f \in C(\partial \Omega)^n \) with

\[
0 = \langle f, gd\sigma \rangle = \int_{\partial \Omega} (f_t \cdot g_t + f_\eta g_\eta) d\sigma
\]

for all \( g \in A_0(\Omega) \) such that \( \text{div} g_t = \nabla u_g \cdot \eta \) on \( \partial \Omega \). By Hodge's decomposition theorem on the Riemannian manifold \( \partial \Omega \) (see [MC, lemma 9.1]),

\[
f_t = (\nabla \varphi)_t + h_t,
\]

where \( \varphi \in C^1(\partial \Omega) \) and \( h_t \) is a tangential vector field on \( \partial \Omega \) such that \( \text{div} h_t = 0 \). If we take \( g_\eta = 0 \) and \( g_t = h_t \) in (3.1), we obtain

\[
0 = \langle f, gd\sigma \rangle = \int_{\partial \Omega} ((\nabla \varphi)_t \cdot g_t + h_t \cdot g_t) d\sigma = \int_{\partial \Omega} |h_t|^2 d\sigma,
\]

so \( h_t = 0 \) and \( f_t = (\nabla \varphi)_t \). This implies that

\[
\int_{\partial \Omega} f_t \cdot g_t d\sigma = \int_{\partial \Omega} (\nabla \varphi)_t \cdot g_t d\sigma = -\int_{\partial \Omega} \varphi \text{div} g_t d\sigma = -\int_{\partial \Omega} \varphi(\nabla u_g \cdot \eta) d\sigma,
\]

and then (3.1) takes the form

\[
\int_{\partial \Omega} f_\eta u_g d\sigma = \int_{\partial \Omega} \varphi(\nabla u_g \cdot \eta) d\sigma.
\]

For any hole \( H \) of \( \Omega \) (i.e. \( H \) is a bounded connected component of \( \Omega^c \)), consider a vector field \( g \in A_0(\Omega) \) such that \( g = \eta \) on \( \partial H \) and \( g = 0 \) elsewhere. Then (3.2) shows that \( \int_{\partial H} f_\eta d\sigma = 0 \), so we can apply theorem 2.7 (and also remark 2.8 for \( n = 2 \)) to solve the Neumann problem on the complement of \( \Omega \) with boundary data \( f_\eta \). Let \( \psi \in C^\infty(\Omega^c) \cap C(\Omega^c) \) be a solution such that \( \nabla \psi(\infty) = 0 \) (this is automatically satisfied for \( n > 2 \) using the Kelvin transform and computing the derivatives, and can be assumed for \( n = 2 \) by remark 2.8).

As we did in remark 2.10, it is easily checked that \( u_g \) and \( \psi \) satisfy the statements of lemma 2.9 if \( n \geq 2 \), so we deduce from (3.2) that

\[
\int_{\partial \Omega} (\varphi - \psi)(\nabla u_g \cdot \eta) d\sigma = 0.
\]

This equality holds for all \( u_g \) harmonic in \( \Omega^c \) and smooth up to \( \partial \Omega \) such that there exists \( g \in A_0(\Omega) \) with \( u_g = g \cdot \eta \) and \( \text{div} g_t = \nabla u_g \cdot \eta \) on \( \partial \Omega \). We are going to see that (3.3) implies that \( \varphi - \psi \) is constant on each connected component of \( \partial \Omega \).

Suppose we have a smooth function \( q \) on \( \partial \Omega \) such that \( \int_{\partial \Omega} q d\sigma = 0 \) for each connected component \( S \) of \( \partial \Omega \). Then, by the maximal de Rham cohomology theorem on the Riemannian manifold \( S \) (see [L1], theorem 1.1 of chapter XVIII), the differential form \( qd\sigma \) is exact, i.e., there exists a smooth vector field \( w \) tangent to \( S \) and such that \( \text{div} w = q \). On the other hand, the Neumann problem with boundary data \( q \) can be solved on \( \Omega^c \) by theorem
2.7. Therefore, we can assume that this solution (call it $u_g$) vanishes at infinity. Summarizing, given the function $q$ we have constructed a smooth vector field $w + u_g \cdot \eta \in A_0(\Omega)$ such that $\text{div} w = \nabla u_g \cdot \eta$ on $\partial \Omega$.

So, we deduce from (3.3) that

$$\int_{\partial \Omega} (\varphi - \psi) q d\sigma = 0$$

for all smooth functions $q$ on $\partial \Omega$ such that $\int_S q d\sigma = 0$ for each connected component $S$ of $\partial \Omega$.

Therefore, we deduce from (3.4) that $\varphi - \psi$ is constant on each component of $\partial \Omega$, and we obtain $(\nabla \psi)_t = (\nabla \varphi)_t = f_t$ on $\partial \Omega$. Remember that $\nabla \psi \cdot \eta = f\eta$, so $\nabla \psi = f$ on $\partial \Omega$. As $f \in C(\partial \Omega)^n$ and $\Delta \psi = 0$ in $\Omega^c$, we have $\psi \in C^1(\Omega^c)$ because each coordinate of $\nabla \psi$ must be given by the Poisson integral of the corresponding coordinate of $f$. By the Whitney extension theorem (see [Wh], or [Pa] theorem [8]), we can extend $\psi$ inside $\Omega$ to have $\psi \in C^1(\mathbb{R}^n)$. We have finally obtained $\psi \in \mathcal{U}_c(\Omega)$ and $\nabla \psi|_{\partial \Omega} = f$, so the theorem is proved.

**Proposition 3.2.** If $\mu \in B(\Omega)^\perp$, then $\mu|_{\partial \Omega} \in bB(\Omega)^\perp$.

**Proof.** Let $\nabla \varphi \in B(\Omega)$. We have to check that $\langle \nabla \varphi, \mu|_{\partial \Omega} \rangle = 0$.

Because of $\mu \in B(\Omega)^\perp$,

$$\langle \nabla \varphi, \mu|_{\partial \Omega} \rangle = \int_{\partial \Omega} \nabla \varphi d\mu = \langle \nabla \varphi, \mu \rangle - \int_{\text{int}\Omega} \nabla \varphi d\mu = - \int_{\text{int}\Omega} \nabla \varphi d\mu.$$

Consider a Whitney decomposition $\text{int}\Omega = \bigcup_{i \in \mathbb{N}} Q_i$, where $\{Q_i\}_{i \in \mathbb{N}}$ are disjoint cubes such that $2Q_i \subset \text{int}\Omega$ for all $i \in \mathbb{N}$ and the family $\{2Q_i\}_{i \in \mathbb{N}}$ has finite overlap of order $M$, and consider a partition of unity subordinated to this decomposition: let $\{\psi_i\}_{i \in \mathbb{N}}$ be a family of $C^\infty$ functions such that $0 \leq \psi_i \leq 1$, $\|\nabla \psi_i\| \leq C/\text{diam}(Q_i)$ and $\text{supp} \psi_i \subset \frac{3}{2} Q_i$ for each $i \in \mathbb{N}$, so that $\sum_{i \in \mathbb{N}} \psi_i = 1$ in $\text{int}\Omega$. Then $\varphi = \sum_{i \in \mathbb{N}} \psi_i \varphi$ in $\text{int}\Omega$ and at most $M$ terms are non zero in the last sum for all $x \in \text{int}\Omega$.

Observe that

$$\int_{\text{int}\Omega} \nabla (\psi_i \varphi) d\mu = \langle \nabla (\psi_i \varphi), \mu \rangle = 0,$$

because $\text{supp}(\psi_i \varphi) \subset \text{int}\Omega$ and $\nabla (\psi_i \varphi) \in B(\Omega)$. Hence, for $N \in \mathbb{N}$

$$\int_{\text{int}\Omega} \nabla \varphi d\mu = \int_{\text{int}\Omega} \nabla \varphi d\mu - \sum_{i=1}^N \int_{\text{int}\Omega} \nabla (\psi_i \varphi) d\mu$$

$$= \int_{\text{int}\Omega} \nabla \varphi d\mu - \sum_{i=1}^N \int_{\text{int}\Omega} (\varphi \nabla \psi_i + \psi_i \nabla \varphi) d\mu$$

$$= \int_{\text{int}\Omega} \nabla \varphi \left(1 - \sum_{i=1}^N \psi_i\right) d\mu + \int_{\text{int}\Omega} \sum_{i=1}^N \varphi \nabla \psi_i d\mu = I + II.$$

By dominated convergence, $|I| \to 0$ as $N \to \infty$.

In order to estimate $|II|$, define the set

$$\Omega_N = \{x \in \text{int}\Omega : \exists i > N \text{ with } x \in \text{supp} \psi_i\}.$$
Clearly, $\Omega_{N+1} \subset \Omega_N$ and $\bigcap_{N \in \mathbb{N}} \Omega_N = \emptyset$. Because of the finite overlap of the squares $2Q_i$, we also have
\begin{equation}
\sum_{i=1}^{N} \nabla \psi_i(x) = 0 \quad \text{for all} \quad x \in \text{int} \Omega \setminus \Omega_N.
\end{equation}

Let $x_i$ be the center of $Q_i$. Using that $\nabla \psi_i \in B(\Omega)$, that $\text{supp} \psi_i \subset \text{int} \Omega$ and equation (3.5), we deduce
\begin{equation}
II = \int_{\Omega_N} \sum_{i=1}^{N} (\varphi(x) - \varphi(x_i)) \nabla \psi_i(x) d\mu(x).
\end{equation}

For all $x \in \Omega_N$,
\begin{align*}
\left| \sum_{i=1}^{N} (\varphi(x) - \varphi(x_i)) \nabla \psi_i(x) \right| &\leq \sum_{i=1}^{N} |\varphi(x) - \varphi(x_i)||\nabla \psi_i(x)| \\
&\leq \sum_{i=1}^{N} \|\nabla \varphi\|_{\Omega} |x - x_i| |\nabla \psi_i(x)|.
\end{align*}

If $x \in \text{supp} \psi_i$, then $|\nabla \psi_i(x)||x - x_i| \leq C$. Because of the finite overlap of the cubes $2Q_i$, each $x \in \Omega_N$ belongs to the support of at most $M$ functions $\psi_i$, so the last sum is less than or equal to $CM\|\nabla \varphi\|_\Omega$ (which does not depend on $x \in \Omega_N$). This implies that
\begin{equation}
|II| \leq CM\|\nabla \varphi\|_\Omega |\mu|(\Omega_N).
\end{equation}

Now, $\lim_{N \to \infty} |\mu|(\Omega_N) = |\mu|(\bigcap_{N \in \mathbb{N}} \Omega_N) = 0$, so $|II| \to 0$ when $N \to \infty$. This completes the proof. \hfill $\Box$

**Theorem 3.3.**
\[ \kappa_c(\Omega) = \min \left\{ \|\eta d\sigma_e + \mu\|_{\partial_0 \Omega} : \mu \in B(\Omega)^+, \text{supp} \mu \subset \partial_0 \Omega \right\}, \]
where $d\sigma_e$ is the surface measure of $\partial_0 \Omega$, i.e., the restriction of $d\sigma$ to $\partial_0 \Omega$.

**Proof.** By definition,
\begin{equation}
\kappa_c(\Omega) = \sup \{(1, \Delta \varphi) : \varphi \in U_c(\Omega), \|\nabla \varphi\| \leq 1\}.
\end{equation}

Using Gauss formula, for each $\varphi \in U_c(\Omega)$ we have
\begin{align*}
(1, \Delta \varphi) &= \int_{\partial \Omega} \nabla \varphi \cdot \eta d\sigma \\
&= \int_{\partial_0 \Omega} \nabla \varphi \cdot \eta d\sigma + \int_{\partial \Omega \setminus \partial_0 \Omega} \nabla \varphi \cdot \eta d\sigma = \int_{\partial_0 \Omega} \nabla \varphi \cdot \eta d\sigma_e,
\end{align*}
because each connected component of $\partial \Omega \setminus \partial_0 \Omega$ is the boundary of a smooth bounded domain where $\varphi$ is harmonic.

Define the functional $\Phi : C(\Omega)^n \to \mathbb{R}$ as $\Phi(f) = \int_{\partial \Omega} f \cdot \eta d\sigma_e$. Then (3.6) becomes
\begin{align*}
\kappa_c(\Omega) &= \sup \{||\Phi(\nabla \varphi)| : \nabla \varphi \in B(\Omega), \|\nabla \varphi\| \leq 1\} \\
&= \sup \{||\Phi(\nabla \varphi)| : \nabla \varphi \in B(\Omega), \|\nabla \varphi\|_{\Omega} \leq 1\} = \|\Phi\|_{B(\Omega)},
\end{align*}
because $|\nabla \varphi|$ is subharmonic when $\varphi$ is harmonic, and the maximum principle can be applied.
Clearly, a functional \( \tilde{\Phi} \) on \( C(\Omega)^n \) is an extension of \( \Phi \mid_{B(\Omega)} \) if and only if 
\[ \Psi := \tilde{\Phi} - \Phi \] 
is orthogonal to \( B(\Omega) \), and for all such extensions \( \| \Phi \|_{B(\Omega)} \leq \| \tilde{\Phi} \| \).
Hence,
\[
\| \Phi \|_{B(\Omega)} = \min \left\{ \| \tilde{\Phi} \mid_{B(\Omega)} : \tilde{\Phi} \text{ is an extension of } \Phi \mid_{B(\Omega)} \text{ to } C(\Omega)^n \right\}
\]
\[
= \min \left\{ \| \Phi + \Psi \| : \Psi \text{ is orthogonal to } B(\Omega) \right\}
\]
\[
= \min \left\{ \| \eta \sigma_e + \mu \|_\Omega : \mu \in B(\Omega)^\perp \right\},
\]
where we have used by Hahn-Banach’s theorem in the first equality, and Riesz’s representation theorem 2.1 in the third one.

To summarize, for every measure \( \nu \) supported on \( \Omega \),
\[
\| \nu \|_{\partial \Omega} = \| \nu \|_{\partial \Omega} \leq \| \nu \|_{\Omega},
\]
so by proposition 3.2,
\[
\min \left\{ \| \eta \sigma_e + \mu \|_\Omega : \mu \in B(\Omega)^\perp \right\} = \min \left\{ \| \eta \sigma_e + \mu \|_{\partial \Omega} : \mu \in bB(\Omega)^\perp \right\}.
\]
It only remains to check that the last minimum is attained on a measure \( \mu \in bB(\Omega)^\perp \) with \( \text{supp} \mu \subset \partial \Omega \).

By theorem 3.1, for every \( \mu \in bB(\Omega)^\perp \) we can find a sequence \( h_n \in A_0(\Omega) \cap bB(\Omega)^\perp \) that tends weakly* to \( \mu \). Since the connected components of \( \partial \Omega \) are a finite number of disjoint compact sets, we see that \( h_n \chi_S \) tends weakly* to \( \mu \mid_S \) for every connected component \( S \) of \( \partial \Omega \). In particular, \( h_n \chi_{\partial \Omega} \) tends weakly* to \( \mu \mid_{\partial \Omega} \) and, by theorem 3.1, \( h_n \chi_{\partial \Omega} \in bB(\Omega)^\perp \) for all \( n \), so \( \mu \mid_{\partial \Omega} \in bB(\Omega)^\perp \). We also clearly have \( \| \eta \sigma_e + \mu \|_{\partial \Omega} \leq \| \eta \sigma_e + \mu \|_{\partial \Omega} \).

To summarize, for every measure \( \mu \in B(\Omega)^\perp \) with \( \text{supp} \mu \subset \partial \Omega \) we have seen that \( \mu \mid_{\partial \Omega} \in bB(\Omega)^\perp \) and that
\[
\| \eta \sigma_e + \mu \|_{\partial \Omega} \leq \| \eta \sigma_e + \mu \|_{\partial \Omega},
\]
so the theorem is proved. \( \square \)

4. Some consequences of theorem 3.3

**Theorem 4.1.** For every compact set \( E \subset \mathbb{R}^n \), \( \kappa(E) = \kappa(\partial E) \).

**Proof.** Since \( \kappa \) is non decreasing set function, \( \kappa(E) \geq \kappa(\partial E) \).

In order to prove the converse inequality, let \( \{ \Omega_m^1 \}_{m \in \mathbb{N}} \) be a sequence of closures of smooth neighborhoods of \( E \) collapsing to \( E \), i.e.
\[
E \subseteq \Omega_{m+1}^1 \subseteq \text{int} \Omega_m^1, \quad \bigcap_{m \in \mathbb{N}} \Omega_m^1 = E.
\]

Denote by \( V \) the unbounded connected component of \( E^c \) and consider the bounded open set \( V' = \overline{V}^c \). Take an increasing sequence of open sets \( \{ V_m' \}_m \) such that
\[
V_m' \subset V', \quad V_m^c \subset V_{m+1}, \quad \partial V_m' \subset \Omega_m^1, \quad \bigcup_{m \in \mathbb{N}} V_m' = V'
\]
and with smooth boundary, for all \( m \). Finally, define the sequence
\[
\{ \Omega_m^2 := \Omega_m^1 \setminus V_m' \}_{m \in \mathbb{N}}.
\]
By construction, the sequence \( \{ \Omega_m^2 \}_{m \in \mathbb{N}} \) is a decreasing sequence of closures of bounded open sets with smooth boundary. Observe also that \( \partial_\nu \Omega_m^1 = \partial_\nu \Omega_m^2 \) for all \( m \in \mathbb{N} \).

Now, if \( x \in \partial_\nu E = \partial V \), then \( x \in \Omega_m^2 \) for all \( m \in \mathbb{N} \). On the other hand, if \( x \in \Omega_m^1 \cap \Omega_m^1 \) for all \( m \in \mathbb{N} \), then \( x \in \partial V \cap E \), so \( x \in \partial_\nu E \). Therefore, we have proved that \( \bigcap_{m \in \mathbb{N}} \Omega_m^2 = \partial_\nu E \).

By [Pa] lemma 2.2(7),

\[
\lim_{m \to \infty} \kappa(\Omega_m^1) = \kappa(E),
\]
\[
\lim_{m \to \infty} \kappa(\Omega_m^2) = \kappa(\partial_\nu E).
\]

Let \( \eta_m \) be the outward unit normal vector on \( \partial_\nu \Omega_m^1 \) and \( d\sigma_m \) the surface measure on \( \partial_\nu \Omega_m^1 \). It is easy to see that a measure \( \mu \) supported in \( \partial_\nu \Omega_m^1 \) belongs to \( B(\Omega_m^1)^\perp \) if and only if it belongs to \( B(\Omega_m^2)^\perp \). Applying theorem 3.3,

\[
\kappa_c(\Omega_m^1) = \min \left\{ \| \eta_m d\sigma_m + \mu \|_{\partial_\nu \Omega_m^1} : \mu \in B(\Omega_m^1)^\perp, \text{supp} \mu \subset \partial_\nu \Omega_m^1 \right\}
\]
\[
= \min \left\{ \| \eta_m d\sigma_m + \mu \|_{\partial_\nu \Omega_m^2} : \mu \in B(\Omega_m^2)^\perp, \text{supp} \mu \subset \partial_\nu \Omega_m^2 \right\}
\]
\[
= \kappa_c(\Omega_m^2).
\]

By definition, \( \kappa_c \) is also monotone and \( \kappa_c \leq \kappa \) and by [Pa] lemma 2.2(1), both capacities coincide on open sets. So, we have

\[
\kappa(\Omega_{m+1}^1) \leq \kappa(\text{int} \Omega_m^1) = \kappa_c(\text{int} \Omega_m^1) \leq \kappa_c(\Omega_m^1) = \kappa_c(\Omega_m^2) \leq \kappa(\Omega_m^2).
\]

The theorem is proved by letting \( m \) tend to infinity.

Theorem 4.2. Let \( f \) be a real continuous function defined on the cube \( Q_0 = [0,d]^{n-1} \subset \mathbb{R}^{n-1} \) and let \( \Gamma = \{ (x,f(x)) \in \mathbb{R}^n : x \in Q_0 \} \) be the graph of \( f \). Then, there exists a constant \( C > 0 \) depending only on \( n \) such that

\[
Cd^{n-1} \leq \kappa(\Gamma).
\]

Proof. The proof is based on theorem 4.1 and the semiadditivity of \( \kappa \). Starting from \( Q_0 \), consider a decomposition of \( \mathbb{R}^{n-1} \) into cubes \( Q_i \) of side length \( d \) and with disjoint interiors. By doing reflections with respect to the sides of the cubes \( Q_i \), we can extend \( f \) to a function \( \tilde{f} \) continuous on \( \mathbb{R}^{n-1} \) and such that \( \tilde{f} \) in \( Q_i \) is a reflection of \( f \) in \( Q_0 \).

Let \( Q^m \) be a cube in \( \mathbb{R}^{n-1} \) of side length \( md \) made by the union of \( m^{n-1} \) cubes \( Q_i \). Let \( \Gamma_m \) be the graph of the function \( \tilde{f} \) on \( Q^m \), i.e.,

\[
\Gamma_m = \{ (x,\tilde{f}(x)) \in \mathbb{R}^n : x \in Q^m \},
\]

and consider its translation

\[
\Gamma_m' = \{ (x,\tilde{f}(x) + 4\|f\|_{Q_0}) \in \mathbb{R}^n : x \in Q^m \}.
\]

Clearly, the sets \( \Gamma_m \) and \( \Gamma_m' \) do not intersect. Moreover, they are separated by the set \( P_m = \{ (x,2\|f\|_{Q_0}) : x \in Q^m \} \).

Finally, let \( E_m \) be the region enclosed by \( \Gamma_m \), \( \Gamma_m' \) and the \( 2(n-1) \) pieces of vertical hyperplanes of \( \mathbb{R}^n \) that join the endpoints of \( \Gamma_m \) and \( \Gamma_m' \). Roughly speaking, \( E_m \) is a kind of \( n \)-dimensional rectangle which has \( \Gamma_m \) as the
bottom side and $\Gamma^t_m$ as the top side. By construction, $E_m$ is a compact set that contains $P_m$, so

$$\kappa(E_m) \geq \kappa(P_m) = C(md)^{n-1}$$

by [Pa] lemma 2.2(8). Applying theorem 4.1, the countable semiadditivity of $\kappa$ and [Pa] lemma 2.2(8), we have

$$\kappa(E_m) = \kappa(\partial_o E_m) \leq C(\kappa(\Gamma_m^t) + \kappa(\Gamma_m^t) + 20(n-1)\|f\|_{Q_0}(md)^{n-2}).$$

By the construction of $\Gamma_m$ and the countable semiadditivity of $\kappa$, $\kappa(\Gamma_m^t) = \kappa(\Gamma_m^t) \leq Cm^{n-1}\kappa(\Gamma)$, so (4.2) becomes

$$\kappa(E_m) \leq C'(2m^{n-1}\kappa(\Gamma) + 20(n-1)\|f\|_{Q_0}(md)^{n-2}).$$

Combining (4.1) and (4.3), we get

$$\kappa(\Gamma) \geq Cd^{n-1},$$

and the theorem is proved.

Remark 4.3. One can show that $\kappa(E) \geq C\text{diam}(E)$ for any continuum $E \subset \mathbb{R}^2$ by using the same ideas as in the proof of theorem 4.2. One starts by choosing two points $a,b \in E$ such that $\text{diam}(E) = |b-a|$ and assuming that these points belong to the real axis in $\mathbb{R}^2$. Then, one extends the set $E$ by symmetries along the real axis as we did before. The rest of the proof remains the same.

The inequality $\kappa(E) \geq C\text{diam}(E)$ for a continuum $E \subset \mathbb{R}^2$ was stated as an open question in problem 2.6 of [Pa] and was first proved by P. Jones by using the notion of curvature of a measure and other capacities called $\gamma_+$ and $\kappa_+$ (see [Pj], [To2], and [Vo]). We have proved it by a different method.

References


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