GROWTH ESTIMATES FOR CAUCHY INTEGRALS OF MEASURES AND RECTIFIABILITY

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Abstract. We show that if μ is a finite Radon measure on the complex plane such that

$$\mathcal{C}_*\mu(z) = \sup_{\varepsilon>0} |\mathcal{C}_\varepsilon\mu(z)| = \sup_{\varepsilon>0} \left| \int_{|\xi-z|>\varepsilon} \frac{1}{\xi - z}\, d\mu(\xi) \right| < \infty$$

for μ-a.e. $z \in \mathbb{C}$, then μ must be the addition of some point masses, plus some measure absolutely continuous with respect to arc length on countably many rectifiable curves, plus another measure with zero linear density. We also prove that the same conclusion holds if instead of the condition $\mathcal{C}_*\mu(z) < \infty$ μ-a.e. one assumes $\mathcal{C}_\varepsilon\mu(z) = o\big(\mu(B(z,\varepsilon))/\varepsilon\big)$ as $\varepsilon \to 0+$ μ-a.e.

1. Introduction

The *Cauchy transform* of a finite positive Borel measure μ on the complex plane is defined by

$$\mathcal{C}\mu(z) = \int \frac{1}{\xi - z}\, d\mu(\xi).$$

This definition does not make sense, in general, for $z \in \operatorname{supp}(\mu)$, although one can easily see that the integral above is convergent for a.e. $z \in \mathbb{C}$ with respect to Lebesgue measure. This is the reason why one introduces the *ε-truncated Cauchy transform* of μ, for $\varepsilon > 0$, $z \in \mathbb{C}$:

$$\mathcal{C}_\varepsilon\mu(z) = \int_{|\xi-z|>\varepsilon} \frac{1}{\xi - z}\, d\mu(\xi), \tag{1.1}$$

and one considers the principal value p.v.$\mathcal{C}\mu(z) = \lim_{\varepsilon\to 0} \mathcal{C}_\varepsilon\mu(z)$, whenever the limit exists.

Many problems in complex analysis involve the study of the relationship between the geometric properties of μ and the function $\mathcal{C}\mu$, which is analytic in $\mathbb{C} \setminus \operatorname{supp}(\mu)$ (see [Da2] and [To4] for two recent examples). In this paper we show that if μ is a positive finite Radon measure on the complex plane such that

$$\mathcal{C}_*\mu(z) = \sup_{\varepsilon>0} \left| \int_{|\xi-z|>\varepsilon} \frac{1}{\xi - z}\, d\mu(\xi) \right| < \infty$$

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for μ-a.e. $z \in \mathbb{C}$, then μ must be the addition of some point masses, plus some measure absolutely continuous with respect to arc length on countably many rectifiable curves, plus another measure with zero linear density. This solves a question raised by Pertti Mattila about ten years ago (see for example [Ma2, Section 1]). We also prove that the same conclusion holds if one replaces the assumption $\mathcal{C}_*\mu(z) < \infty$ μ-a.e. by $\mathcal{C}_\varepsilon\mu(z) = o\big(\mu(B(z,\varepsilon))/\varepsilon\big)$ as $\varepsilon \to 0+$ μ-a.e.

To state our results in detail we need to introduce some notation and terminology. The one dimensional Hausdorff measure (i.e. arc length measure) is denoted by $\mathcal{H}^1$, and a Dirac delta at $z \in \mathbb{C}$ by δ_z. The upper and lower linear densities of μ at z are defined, respectively, by

$$\Theta^*_\mu(z) = \limsup_{r\to 0} \frac{\mu(B(z,r))}{2r}, \qquad \Theta_{*,\mu}(z) = \liminf_{r\to 0} \frac{\mu(B(z,r))}{2r}.$$

When both densities coincide, we set $\Theta_\mu(z) := \Theta^*_\mu(z) = \Theta_{*,\mu}(z)$, and we call it linear density.

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x,y,z) = \frac{1}{R(x,y,z)},$$

where $R(x,y,z)$ is the radius of the circumference passing through x, y, z (with $c(x,y,z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we set $c(x,y,z) = 0$. Given a positive Radon measure μ, its *curvature*, introduced by Melnikov [Me], is given by

$$c^2(\mu) = \iiint c(x,y,z)^2\, d\mu(x)d\mu(y)d\mu(z). \tag{1.2}$$

Melnikov and Verdera [MeV] found that there is a close relationship between the curvature of a measure and its Cauchy transform. Namely they showed that if μ has linear growth, i.e. there exists some constant C_0 such that $\mu(B(x,r)) \le C_0 r$ for all $x \in \mathbb{C}$, $r > 0$, then

$$\|\mathcal{C}_\varepsilon\mu\|^2_{L^2(\mu)} = \frac{1}{6} c^2_\varepsilon(\mu) + E_\varepsilon, \tag{1.3}$$

where $c^2_\varepsilon(\mu)$ is an ε-truncated version of $c^2(\mu)$ (defined as in the right hand side of (1.2), but with the triple integral over $\{(x,y,z) \in \mathbb{C}^3 : |x-y|, |y-z|, |x-z| > \varepsilon\}$), and E_ε is an "error term" such that $|E_\varepsilon| \lesssim \mu(\mathbb{C})$.

Our first result is the following.

Theorem 1.1. *Let μ be a finite Radon measure on the complex plane and set $E = \{z \in \operatorname{supp}(\mu) : \mathcal{C}_*\mu(z) < \infty\}$. Then $\mu_{|E}$ can be decomposed as $\mu_{|E} = \mu_d + \mu_r + \mu_0$, where*

$$\mu_d = \sum_i a_i \delta_{z_i}$$

for some $a_i > 0$ and $z_i \in \mathbb{C}$,

$$\mu_r = \sum_i g_i \mathcal{H}^1_{|\Gamma_i}$$

for some rectifiable curves Γ_i and non negative functions $g_i \in L^1(\mathcal{H}^1_{|\Gamma_i})$, and

$$\mu_0 = \sum_i \sigma_i,$$

where σ_i are measures with finite curvature such that $\Theta_{\sigma_i}(z) = 0$ for σ_i-a.e. $z \in \mathbb{C}$.

An interesting case of this result is stated in next corollary.

Corollary 1.2. *Let μ be a finite Radon measure on the complex plane such that $\Theta^*_\mu(z) > 0$ for μ-a.e. $z \in \mathbb{C}$. If $\mathcal{C}_*\mu(z) < \infty$ at μ-a.e. $z \in \mathbb{C}$, then μ can be decomposed as $\mu = \mu_d + \mu_r$, where $\mu_d = \sum_i a_i \delta_{z_i}$ for some $a_i > 0$ and $z_i \in \mathbb{C}$, and $\mu_r = \sum_i g_i \mathcal{H}^1_{|\Gamma_i}$ for some rectifiable curves Γ_i and non negative functions $g_i \in L^1(\mathcal{H}^1_{|\Gamma_i})$.*

In particular, under the assumptions of the corollary, μ is concentrated on (i.e. vanishes out of) a countable union of rectifiable curves.

Several related results have been obtained previously by other authors. In [Ma2] Mattila showed that if $\Theta_{*,\mu}(z) > 0$ and p.v.$\mathcal{C}\mu(z)$ exists μ a.e. on $\mathbb{C}$, then μ is concentrated on a countable union of rectifiable curves. Notice that our assumptions $\Theta^*_\mu(z) > 0$ and $\mathcal{C}_*\mu(z) < \infty$ in Corollary 1.2 are *both* weaker than the ones in Mattila's result.

In [To2], using some deep results from [NTV1], Theorem 1.1 and its corollary were proved under the additional assumption $\Theta^*_\mu(z) < \infty$ μ-a.e. (obviously, this assumption implies that then μ_d vanishes). That is to say, in Theorem 1.1 one asks $\Theta^*_\mu(z) + \mathcal{C}_*\mu(z) < \infty$ μ-a.e., and in the corollary $\mathcal{C}_*\mu(z) < \infty$ and $0 < \Theta^*_\mu(z) < \infty$.

Jones and Poltoratski [JP] proved recently, among other things, that if μ is supported on a line (and more generally on a $\mathcal{C}^1$ curve) and $\mathcal{C}_*\mu(z) < \infty$ μ-a.e., then μ equals a countable collection of point masses plus some measure absolutely continuous with respect to arc length. Observe that this result is implied by Corollary 1.2, because any measure μ supported on a line satisfies $\Theta^*_\mu(z) > 0$ μ-a.e. In [JP] it is also shown that if one does not assume μ to be supported on a line and instead one asks the same conditions as Mattila in [Ma2] (i.e. $\Theta_{*,\mu}(z) > 0$ and p.v.$\mathcal{C}\mu(z)$ exists μ a.e. on $\mathbb{C}$), then the conclusion of our corollary holds: μ equals a countable collection of point masses plus some measure absolutely continuous with respect to arc length on a rectifiable set.

In higher dimensions, Mattila and Preiss obtained results analogous to the ones in [Ma2] (although with some extra assumption that we will not detail). For this result one has to replace the Cauchy transform by n-dimensional Riesz transforms in $\mathbb{R}^d$, and rectifiable sets by n-dimensional rectifiable sets, linear densities by n-dimensional densities, etc.

See [MaM], [Ve], [To1] and [TVe] for other connected results.

The main difficulty to prove Theorem 1.1 consists in proving that if $\mathcal{C}_*\mu(z) < \infty$ μ-a.e. on $\mathbb{C}$, then on the set $\{z : \Theta^*_\mu(z) = \infty\}$ μ must be discrete (i.e. the addition of countably many point masses). Once this is proved, one can argue as in [To2].

The same techniques used for Theorem 1.1 also yield the following result, proved by Jones and Poltoratski [JP] when μ is supported on a $\mathcal{C}^1$ curve.

Theorem 1.3. *Let μ be a finite Radon measure on the complex plane and set $E = \big\{x \in \operatorname{supp}(\mu) : \mathcal{C}_\varepsilon\mu(x) = o\big(\mu(B(x,\varepsilon)/\varepsilon\big)$ as $\varepsilon \to 0+\big\}$. Then $\mu_{|E}$ can be decomposed as $\mu_{|E} = \mu_d + \mu_r + \mu_0$, with μ_d, μ_r, and μ_0 as in Theorem 1.1.*

Let us remark that if one assumes μ to be supported on a $\mathcal{C}^1$ curve, like in [JP], then Theorem 1.1 follows easily from the preceding result and the fact that μ is absolutely continuous with respect to arc length on the set $\{x : \Theta_{*,\mu}(x) < \infty\}$. In the general case $\operatorname{supp}(\mu) \subset \mathbb{C}$, the latter assertion fails, and then Theorem 1.1 is not a direct consequence of Theorem 1.3.

The arguments in this paper rely on the relationship between the Cauchy transform and curvature of measures mentioned above (in particular, Léger's theorem [Lé] is used here) and so they do not extend to higher dimensions. We also use the so called "suppressed kernels" introduced by Nazarov, Treil and Volberg ([NTV1], [Vo]), as well as some ideas and techniques from [Ma2] and [MaP].

The plan of the paper is the following. Next section deals with some preliminary questions. In Section 3 we state the Main Lemma of this paper, which contains the main technical difficulties for the proofs of Theorems 1.1 and 1.3. Section 4, 5 and 6 are devoted to the proof of the Main Lemma. From this result, we deduce Theorem 1.1 in Section 7, and Theorem 1.3 in Section 8.

2. Preliminaries

As usual, in the paper the letter C stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as C_1, retain their value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. So $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

Recall that we say that a positive Radon measure μ has linear growth if there exists some constant C_0 such that $\mu(B(x,r)) \leq C_0 r$ for all $x \in \mathbb{C}$, $r > 0$. The average (linear) density of μ on a set $A \subset \mathbb{C}$ is

$$\Theta_\mu(A) = \frac{\mu(A)}{\operatorname{diam}(A)}. \tag{2.1}$$

Abusing notation, we will also write $\Theta(A)$ instead of $\Theta_\mu(A)$, when μ is a fixed measure. We also set

$$P(x,r) = \int \frac{r}{r^2 + |x-y|^2}\, \mu(y)$$

and

$$P_2(x,r) = \int \frac{r^3}{(r^2 + |x-y|^2)^2}\, \mu(y).$$

Observe that $P(x,r)$ is a "Poisson type" integral. $P(x,r)$ and $P_2(x,r)$ can be considered as smooth versions of $\Theta(B(x,r))$.

We consider the following ε-regularized version of the Cauchy transform of μ:

$$\widetilde{\mathcal{C}}_\varepsilon \mu(x) = \int \frac{\overline{x-y}}{|x-y|^2 + \varepsilon^2}\, d\mu(y). \tag{2.2}$$

We also set

$$\widetilde{\mathcal{C}}_\varepsilon^\ell \mu(x) = \int_{|x-y|\leq \ell} \frac{\overline{x-y}}{|x-y|^2 + \varepsilon^2}\, d\mu(y).$$

We denote by μ_{at} the atomic part of μ. That is, $\mu_{at} = \mu_{|A}$, where $A = \{x : \mu(\{x\}) > 0\}$.

In the paper, by a square we mean a closed square with sides parallel to the axes. The side length of a square Q is denoted by $\ell(Q)$. For $a > 0$, aQ denotes the square concentric with Q and side length $a\ell(Q)$. Also, the notation $Q(x,\ell)$ stands for a square centered at x and side length 2ℓ.

Given $a, b > 1$, we say that Q is (a,b)-doubling if $\mu(aQ) \leq b\mu(Q)$. When we don't specify the constants a, b and we say that Q is doubling, we mean that Q is $(2,b)$-doubling, where b is some absolute constant.

Now we need to introduce the notion of *squares with thin boundaries*, as David in [Da1]:

Definition 2.1. Given a fixed Radon measure μ, a square Q has *thin boundaries* if

$$\mu\big\{x \in \mathbb{C} : \operatorname{dist}(x, \partial Q) \leq t\ell(Q)\big\} \leq C_1 t \mu(2Q) \tag{2.3}$$

for all t with $0 \leq t \leq 1/4$.

The constant C_1 in (2.3) is fixed, and it is not difficult to prove that if C_1 is chosen big enough, then there exist a lot of squares with thin boundaries. The precise result is the following.

Lemma 2.2. *Let μ be a Radon measure on $\mathbb{C}$. Suppose that the constant C_1 in (2.3) is big enough. Given any square Q, there exists a square $\widetilde{Q}$ with thin boundaries, concentric with Q, such that $Q \subset \widetilde{Q} \subset \frac{11}{10}Q$.*

For the proof we refer the reader to [NTV2, Lemma 2.8]. Actually, in this reference the result is proved for balls instead of squares, but the same arguments can be applied to squares. The required constant C_1 does not depend on the precise measure μ. It is an absolute constant.

Of course, we will assume that C_1 is big enough so that the lemma holds.

3. The main lemma

Roughly speaking, in order to prove Theorem 1.1, the main difficulty that we have to solve consists in showing that if $\mathcal{C}_*\mu$ is small at many points, then the linear density of μ cannot grow too much in a big part of μ (i.e. μ must have linear growth in a big piece of μ), except when one finds point masses. Then, in the places where μ has linear growth, one can argue as in [To2].

The main difficulties of the proof are concentrated in the next technical result. Sections 4-6 are devoted to its proof. In Section 7 it is shown how Theorem 1.1 follows from it, and in Section 8 we prove Theorem 1.3 using a slight variant of this result.

Main Lemma 3.1. *Let μ be a finite Radon measure on $\mathbb{C}$. Let R_0 be a square with thin boundaries such that $\Theta_\mu(\lambda R_0) \leq 100\Theta_\mu(\frac{1}{2}R_0)$ for all $\lambda > 1$. Let ε_0 and τ be two (small) positive constants. Suppose that*

$$\mu\big\{x \in 10R_0 : \mathcal{C}_*\mu(x) > \varepsilon_0\Theta(R_0)\big\} \leq \tau\mu(R_0).$$

If ε_0 and τ are small enough, then there exists some positive constant η such that one of the following holds:

(a) *either $\mu_{at}(9R_0) \geq \eta\,\mu(R_0)$, or*
(b) *there exists $E \subset R_0$ with $\mu(E) \geq \eta\,\mu(R_0)$ such that $\mu_{|E}$ has linear growth and $c^2(\mu_{|E}) < \infty$.*

Let us remark that in case (b) one deduces that $\mu_{|E}$ has linear growth with constant $C(\varepsilon_0,\tau)\Theta(R_0)$ and that $c^2(\mu_{|E}) \leq C(\varepsilon_0,\tau)\Theta(R_0)^2\mu(E)$.

Notice that the squares R_0 and $\frac{1}{2}R_0$ in the Main Lemma are (a,b)-doubling for all $a > 0$, with b depending on a, because of the assumption $\Theta_\mu(\lambda R_0) \leq 100\Theta_\mu(\frac{1}{2}R_0)$ for all $\lambda > 1$.

The relationship between the Cauchy transform and curvature of measures (see (1.3)) plays a key role in the proof of the Main Lemma. Recall also that there is a strong connection between the notion of curvature and the β's from Jones' traveling salesman theorem [Jo]. The relationship with Favard length is an open problem (see Section 6 of [Ma3], for example).

4. Curvature in terms of suppressed Cauchy transforms

In this section we will obtain formulae analogous to (1.3) in terms of the so called *suppressed Cauchy transform* of μ. Given a positive 1-Lipschitz function Φ, consider the *suppressed kernel* introduced in [NTV1] (see also [Vo]):

$$k_\Phi(x,y) = \frac{\overline{x-y}}{|x-y|^2 + \Phi(x)\Phi(y)},$$

and let

$$\mathcal{C}_\Phi^\ell\mu(x) = \int_{|x-y|\leq\ell} k_\Phi(x,y)\,d\mu(y).$$

We will also set

$$\widetilde{k}_\Phi(x,y) = \frac{\overline{x-y}}{|x-y|^2+\Phi(x)^2},$$

and

$$\widetilde{\mathcal{C}}^\ell_\Phi\mu(x) = \int_{|x-y|\le\ell} \widetilde{k}_\Phi(x,y)\,d\mu(y).$$

For $\ell=\infty$, we write $\mathcal{C}^\ell_\Phi\mu(x) =: \mathcal{C}_\Phi\mu(x)$ and $\widetilde{\mathcal{C}}^\ell_\Phi\mu(x) =: \widetilde{\mathcal{C}}_\Phi\mu(x)$. Observe that when $\Phi\equiv\varepsilon$ both $\widetilde{\mathcal{C}}_\Phi\mu$ and $\mathcal{C}_\Phi\mu$ coincide with $\widetilde{\mathcal{C}}_\varepsilon\mu$ (see (2.2)).

It is easy to check that $k_\Phi(x,y)$ and $\widetilde{k}_\Phi(x,y)$ are Calderón-Zygmund kernels. Notice that $k_\Phi(x,y)$ is antisymmetric, but not $\widetilde{k}_\Phi(x,y)$. Because of this fact the formula relating curvature and Cauchy transform works better for $\mathcal{C}_\Phi\mu$ than for $\widetilde{\mathcal{C}}_\Phi\mu$. However, the kernel $\widetilde{k}_\Phi(x,y)$ has a nice feature which $k_\Phi(x,y)$ lacks: it can be put as a convex combination of the truncated kernels $\chi_{|x-y|>\varepsilon}/(x-y)$, and so it easily follows that

$$|\widetilde{\mathcal{C}}^\ell_\Phi\mu(x)| \lesssim \mathcal{C}_*\mu(x).$$

The following easy estimate will be used often without further comment:

$$|x-y|^2+\Phi(x)\Phi(y) \approx |x-y|^2+\Phi(x)^2+\Phi(y)^2.$$

As a consequence, we have

$$|k_\Phi(x,y)| \lesssim \frac{1}{|x-y|+\Phi(x)+\Phi(y)}.$$

Also, it is not difficult to check that

$$|\nabla_x k_\Phi(x,y)| + |\nabla_y k_\Phi(x,y)| \lesssim \frac{1}{|x-y|^2+\Phi(x)^2+\Phi(y)^2}. \tag{4.1}$$

We denote

$$D_\Phi(x,y,z) = \big(|x-y|^2+\Phi(x)\Phi(y)\big)\big(|x-z|^2+\Phi(x)\Phi(z)\big)\big(|y-z|^2+\Phi(y)\Phi(z)\big). \tag{4.2}$$

Given Radon measures μ_1, μ_2, μ_3, we set

$$c^{2,\ell}_\Phi(\mu_1,\mu_2,\mu_3) = \iiint_{\substack{|x-y|\le\ell\\ |x-z|\le\ell\\ |y-z|\le\ell}} \frac{16S(x,y,z)^2}{D_\Phi(x,y,z)}\,d\mu_1(x)d\mu_2(y)d\mu_3(z),$$

where $S(x,y,z)$ stands for the area of the triangle with vertices x,y,z. When $\Phi\equiv0$, $c^{2,\ell}_\Phi(\mu)$ coincides with the curvature of μ ℓ-truncated from above. We have

Lemma 4.1. *Suppose that $\Phi(x)\ge\varepsilon$ for all $x\in\mathbb{C}$ and some $\varepsilon>0$. We have*

$$\sum_\sigma \int_{\mathbb{C}} \big(\overline{\mathcal{C}^\ell_\Phi\mu_{\sigma_1}}\big)\big(\mathcal{C}^\ell_\Phi\mu_{\sigma_2}\big)\,d\mu_{\sigma_3} = c^{2,\ell}_\Phi(\mu_1,\mu_2,\mu_3) + E_1 + E_2, \tag{4.3}$$

where σ runs over all the permutations of $\{1,2,3\}$, and

$$E_1 = \sum_\sigma \iiint_{\substack{|x_{\sigma_1}-x_{\sigma_2}|\leq\ell\\ |x_{\sigma_1}-x_{\sigma_2}|\leq\ell\\ |x_{\sigma_2}-x_{\sigma_3}|>\ell}} \overline{k_\Phi(x_{\sigma_1},x_{\sigma_2})}\, k_\Phi(x_{\sigma_1},x_{\sigma_3})\, d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3), \tag{4.4}$$

and

$$E_2 = \iiint \frac{\sum_\sigma (x_{\sigma_1}-x_{\sigma_2})(\overline{x_{\sigma_1}-x_{\sigma_3}})\Phi(x_{\sigma_2})\Phi(x_{\sigma_3})}{D_\Phi(x_1,x_2,x_3)}\, d\mu_1(x_1)d\mu_2(x_2)d\mu_3(x_3). \tag{4.5}$$

Proof. We have

$$\begin{aligned}\sum_\sigma &\overline{k_\Phi(x_{\sigma_1}-x_{\sigma_2})}k_\Phi(x_{\sigma_1}-x_{\sigma_3})\\ &= \frac{\sum_\sigma (x_{\sigma_1}-x_{\sigma_2})(\overline{x_{\sigma_1}-x_{\sigma_3}})\big[|x_{\sigma_2}-x_{\sigma_3}|^2+\Phi(x_{\sigma_2})\Phi(x_{\sigma_3})\big]}{D_\Phi(x_1,x_2,x_3)}\\ &= \frac{16\,S(x_1,x_2,x_3)^2}{D_\Phi(x_1,x_2,x_3)} + \frac{\sum_\sigma (x_{\sigma_1}-x_{\sigma_2})(\overline{x_{\sigma_1}-x_{\sigma_3}})\Phi(x_{\sigma_2})\Phi(x_{\sigma_3})}{D_\Phi(x_1,x_2,x_3)},\end{aligned}$$

since $\sum_\sigma (x_{\sigma_1}-x_{\sigma_2})(\overline{x_{\sigma_1}-x_{\sigma_3}})|x_{\sigma_2}-x_{\sigma_3}|^2 = 16S(x,y,z)^2$. Then, arguing as in [MeV], the lemma follows. The details are left for the reader. $\square$

Next lemma deals with the error term E_1:

Lemma 4.2. *Let R be a doubling square in $\mathbb{C}$. Suppose that any square P centered at some point x_P in $5R\cap\operatorname{supp}(\mu)$ with $\Phi(x_P)\leq\ell(P)\leq\ell(R)$ satisfies $\Theta(P)\leq B\Theta(R)$. If we choose $\mu_1=\mu_2=\mu$ and $\mu_3=\mu_{|R}$ in the preceding lemma, then there exists $\ell\in[\ell(R)/2,\ell(R)]$ such that*

$$|E_1| = |E_1(\ell)| \lesssim B\,\Theta(R)^2\mu(R). \tag{4.6}$$

The fact that in the preceding estimate the constant B appears without a power 2 (unlike $\Theta(R)$) will be important in the subsequent arguments.

Proof. It is easy to check that for any $\ell\in[\ell(R)/2,\ell(R)]$ the integrand in the definition of E_1 vanishes if any of the points x_1,x_2,x_3 is in $\mathbb{C}\setminus 5Q$. So we have

$$|E_1| \lesssim \iiint_{\substack{|x-y|\leq\ell\\ |z-x|\leq\ell\\ |y-z|>\ell\\ x,y,z\in 5R}} \frac{1}{(|y-x|+\Phi(y))(|z-x|+\Phi(z))}\, d\mu(x)d\mu(y)d\mu(z).$$

By symmetry,

$$|E_1| \lesssim \iiint_{\substack{|x-y|\leq|x-z|\leq\ell\\ |y-z|>\ell\\ x,y,z\in 5R}} \cdots$$

Since $|x-z|\approx\ell\approx\ell(R)$ in the domain of integration, we get

$$|E_1| \lesssim \frac{1}{\ell(R)} \iiint_{\substack{|x-y|\leq|x-z|\leq\ell\\ |y-z|>\ell\\ x,y,z\in 5R}} \frac{1}{|y-x|+\Phi(y)}\, d\mu(x)d\mu(y)d\mu(z).$$

Since for x, y, z as above we have $|y-z| \leq |x-y| + \ell$, we infer

$$|E_1| \lesssim \frac{1}{\ell(R)} \iint_{\substack{|y-z|>\ell \\ y,z\in 5R}} \left(\int_{|y-z|-\ell \leq |y-x| \leq \ell} \frac{1}{|y-x| + \Phi(y)} \, d\mu(x) \right) d\mu(y) d\mu(z).$$

Splitting the annulus $\{x \in \mathbb{C} : |y-z| - \ell \leq |y-x| \leq \ell\}$ into smaller annuli, taking into account that $\mu(B(x,r)) \lesssim B\Theta(R) r$ for $\Phi(y) \lesssim r \lesssim \ell$, we get

$$\int_{|y-z|-\ell \leq |y-x| \leq \ell} \frac{1}{|y-x| + \Phi(y)} \, d\mu(x) \lesssim B\Theta(R) \log \frac{2\ell}{|y-z| - \ell}.$$

Thus

$$|E_1(\ell)| = |E_1| \lesssim \frac{B\Theta(R)}{\ell(R)} \iint_{\substack{|y-z|>\ell \\ y,z\in 5R}} \log \frac{2\ell}{|y-z| - \ell} d\mu(y) d\mu(z).$$

To find an appropriate ℓ satisfying (4.6), we integrate the preceding estimate with to respect to ℓ in $[\ell(R)/2, \ell(R)]$. We obtain

$$\begin{aligned} \int_{\ell(R)/2}^{\ell(R)} |E_1(\ell)| d\ell &\lesssim \frac{B\Theta(R)}{\ell(R)} \int_{\ell(R)/2}^{\ell(R)} \iint_{\substack{|y-z|>\ell \\ y,z\in 5R}} \log \frac{2\ell}{|y-z| - \ell} \, d\mu(y) d\mu(z) \, d\ell \\ &\leq \frac{B\Theta(R)}{\ell(R)} \iint_{\substack{|y-z|>\ell(R)/2 \\ y,z\in 5R}} \int_{\ell(R)/2}^{|y-z|} \log \frac{2\ell}{|y-z| - \ell} \, d\ell \, d\mu(y) d\mu(z). \end{aligned}$$

Since $\int_a^b \log \frac{x}{b-x} \, dx = b \log \frac{b}{a} + (b-a) \log \frac{a}{b-a}$, we get

$$\int_{\ell(R)/2}^{|y-z|} \log \frac{2\ell}{|y-z| - \ell} \, d\ell \lesssim \ell(R).$$

Therefore,

$$\int_{\ell(R)/2}^{\ell(R)} |E_1(\ell)| d\ell \lesssim B\Theta(R) \iint_{\substack{|y-z|>\ell(R)/2 \\ y,z\in 5R}} d\mu(y) d\mu(z) \lesssim B\Theta(R) \mu(R)^2.$$

As a consequence, there exists some $\ell \in [\ell(R)/2, \ell(R)]$ such that $|E_1(\ell)| \lesssim B\Theta(R)^2 \mu(R)$. $\square$

Lemma 4.3. *Let us denote $S_\ell = \{(x,y,z) \in \mathbb{C}^3 : |x-y| \leq \ell,\ |x-z| \leq \ell,\ |y-z| \leq \ell\}$. The term E_2 in (4.5) can be written in the following way:*

$$\begin{aligned} E_2 = &\iiint_{S_\ell} \frac{|y-z|^2 \Phi(x)\Phi(y) + |x-z|^2 \Phi(y)\Phi(z) + |x-y|^2 \Phi(x)\Phi(z)}{D_\Phi(x,y,z)} \, d\mu_1(x) d\mu_2(y) d\mu_3(z) \\ &+ \iiint_{S_\ell} \frac{|y-z|^2 \Phi(z)\big(\Phi(x) - \Phi(y)\big)}{D_\Phi(x,y,z)} \, d\mu_1(x) d\mu_2(y) d\mu_3(z) \\ &+ \iiint_{S_\ell} \frac{|x-z|^2 \Phi(x)\big(\Phi(y) - \Phi(z)\big)}{D_\Phi(x,y,z)} \, d\mu_1(x) d\mu_2(y) d\mu_3(z) \\ &+ \iiint_{S_\ell} \frac{|x-y|^2 \Phi(y)\big(\Phi(z) - \Phi(x)\big)}{D_\Phi(x,y,z)} \, d\mu_1(x) d\mu_2(y) d\mu_3(z) \\ &=: E_2^+ + F_1 + F_2 + F_3. \end{aligned}$$

Observe that the term E_2^+ is positive, while F_1, F_2, F_3 may be negative. On the other hand, if Φ is constant (i.e. $\Phi(x) = \varepsilon$ for all x), then F_1, F_2, F_3 vanish, and so $E_2 = E_2^+ \geq 0$. We will see that if Φ is Lipschitz with small enough Lipschitz constant, under appropriate conditions $E_2 \geq 0$ too. This will be one the basic ingredients of the proof of Main Lemma 3.1.

Proof. By the cosinus theorem, for $x, y, z \in \mathbb{C}$ we have

$$\mathrm{Re}\big[(x-y)(\overline{x-z})\big] = |x-y||x-z|\cos\widehat{yxz} = \frac{1}{2}\big(|x-y|^2+|x-z|^2-|y-z|^2\big).$$

So the numerator inside the integral in the definition of E_2 in (4.5) equals

$$\begin{aligned}
&\sum_\sigma (x_{\sigma_1}-x_{\sigma_2})(\overline{x_{\sigma_1}-x_{\sigma_3}})\Phi(x_{\sigma_2})\Phi(x_{\sigma_3}) \\
&\quad = \frac{1}{2}\sum_\sigma \big(|x_{\sigma_1}-x_{\sigma_2}|^2+|x_{\sigma_1}-x_{\sigma_3}|^2-|x_{\sigma_2}-x_{\sigma_3}|^2\big)\Phi(x_{\sigma_2})\Phi(x_{\sigma_3}) \\
&\quad = |y-z|^2\big(\Phi(x)\Phi(y)+\Phi(x)\Phi(z)-\Phi(y)\Phi(z)\big) \\
&\qquad + |x-z|^2\big(\Phi(x)\Phi(y)+\Phi(y)\Phi(z)-\Phi(x)\Phi(z)\big) \\
&\qquad + |x-y|^2\big(\Phi(x)\Phi(z)+\Phi(y)\Phi(z)-\Phi(x)\Phi(y)\big),
\end{aligned}$$

where we set $x = x_1$, $y = x_2$, and $z = x_3$. If we plug this identity into (4.5), the lemma follows. $\square$

5. Flatness of μ when $\widetilde{\mathcal{C}}_\varepsilon\mu$ is small

The main result of this section is the following.

Lemma 5.1. *Let $x_0 \in \mathbb{C}$ and $\varepsilon > r > 0$. There exists a line L through x_0 such that if $x \in B(x_0, r)$ satisfies*

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x) - \widetilde{\mathcal{C}}_\varepsilon\mu(x_0)| \leq \frac{\delta\, P(x_0,\varepsilon)}{\varepsilon}\,|x-x_0|, \tag{5.1}$$

then

$$\mathrm{dist}(x, L) \lesssim \Big(\delta + \frac{r}{\varepsilon}\Big)\frac{P(x_0,\varepsilon)}{P_2(x_0,\varepsilon)}\,|x-x_0|. \tag{5.2}$$

This lemma will be used below to show if $\mathcal{C}_*\mu(z)$ is small at some points z with respect to density, then these points must be close to some common line. This fact will be crucial for some future estimates. The proof is inspired by ideas from [MaP] and [Ma1, Theorem 20.9]. It requires the following auxiliary result:

Lemma 5.2. *For all $\varepsilon > 0$ and $x \in \mathbb{C}$, we have*

$$\widetilde{\mathcal{C}}_\varepsilon\mu(x) = \widetilde{\mathcal{C}}_\varepsilon\mu(0) + P(0,\varepsilon)\,\frac{\overline{x}}{\varepsilon} - 2\int \frac{\overline{y}}{\big(|y|^2+\varepsilon^2\big)^2}\,\langle y, x\rangle\,d\mu(y) + E, \tag{5.3}$$

with

(5.4)
$$E = -\int \frac{2\langle y,x\rangle x + |x|^2(\overline{x-y})}{\big(|y|^2+\varepsilon^2\big)^2}\,d\mu(y) + \int \frac{\big(|x|^2 - 2\langle y,x\rangle\big)^2(\overline{x-y})}{\big(\xi_{x,y}+\varepsilon^2\big)^3}\,d\mu(y),$$

where $\xi_{x,y} \in [|y|^2, |y-x|^2]$.

Proof. The Taylor expansion of the function $1/(s+\varepsilon^2)$ at s_0 is

$$\frac{1}{s+\varepsilon^2} = \frac{1}{s_0+\varepsilon^2} - \frac{1}{(s_0+\varepsilon^2)^2}\,(s-s_0) + \frac{1}{(\xi+\varepsilon^2)^3}\,(s-s_0)^2,$$

where $\xi \in [s_0, s]$. If we set $s_0 = |y|^2$, $s = |y-x|^2$, and we multiply by $\overline{x-y}$, we get

$$\begin{aligned}\frac{\overline{x-y}}{|y-x|^2+\varepsilon^2} =& \frac{\overline{x-y}}{|y|^2+\varepsilon^2} - \frac{\overline{x-y}}{(|y|^2+\varepsilon^2)^2}\,(|x|^2 - 2\langle x,y\rangle)\\ &+ \frac{\overline{x-y}}{(\xi_{x,y}+\varepsilon^2)^3}\,(|x|^2-2\langle x,y\rangle)^2,\end{aligned}$$

where $\xi_{x,y} \in [|y|^2, |y-x|^2]$. The lemma follows if we integrate with respect to μ. □

Proof of Lemma 5.1. Without loss of generality, we assume that $x_0 = 0$. The lemma follows from a careful study of the following $\mathbb{R}$-linear map:

$$T(x) = \int \left(\frac{x}{|y|^2+\varepsilon^2} - \frac{2\langle y,x\rangle y}{\big(|y|^2+\varepsilon^2\big)^2} \right) d\mu(y). \tag{5.5}$$

By the identity (5.3) we have

$$T(x) = \overline{\widetilde{\mathcal{C}}_\varepsilon\mu(x)} - \overline{\widetilde{\mathcal{C}}_\varepsilon\mu(0)} - \overline{E},$$

with E given by (5.4). If x, x_0 satisfy the assumptions of the lemma we deduce

$$|T(x)| \le \frac{\delta\, P(0,\varepsilon)}{\varepsilon}\,|x| + |E|.$$

To estimate E we set

$$|E| \le \int \frac{|x|^2(3|y|+|x|)}{(|y|^2+\varepsilon^2)^2}\,d\mu(y) + \int \frac{(|x|^2+2|y|\,|x|)^2(|x|+|y|)}{\big(\xi_{x,y}+\varepsilon^2\big)^3}\,d\mu(y) = E_1 + E_2.$$

To deal with E_1 we use $\varepsilon(3|y|+|x|) \lesssim |y|^2+\varepsilon^2$ (recall that $|x| \le r < \varepsilon$), and then we get

$$E_1 \lesssim \frac{|x|^2}{\varepsilon} \int \frac{1}{|y|^2+\varepsilon^2}\,d\mu(y) \le \frac{P(0,\varepsilon) r}{\varepsilon^2}\,|x|.$$

For E_2 we take into account that $\xi_{x,y}+\varepsilon^2\approx|y|^2+\varepsilon^2$ and $|x|\le\varepsilon$. We obtain

$$\begin{aligned}E_2&\lesssim|x|^2\int\frac{(|x|+|y|)^3}{\left(|y|^2+\varepsilon^2\right)^3}\,d\mu(y)\lesssim|x|^2\int\frac{|x|+|y|}{(|y|^2+\varepsilon^2)^2}\,d\mu(y)\\&\lesssim\frac{|x|^2}{\varepsilon}\int\frac{1}{|y|^2+\varepsilon^2}\,d\mu(y)\le\frac{P(0,\varepsilon)r}{\varepsilon^2}\,|x|.\end{aligned}$$

We infer that

$$|T(x)|\lesssim\Big(\delta+\frac{r}{\varepsilon}\Big)\frac{P(0,\varepsilon)}{\varepsilon}\,|x|. \tag{5.6}$$

Let $x_1,x_2\in B(0,r)$ be two points different from $x_0=0$ satisfying (5.1). Suppose that $s:=|\sin\widehat{x_1 0\,x_2}|\neq0$. By linearity, inequality (5.6) also holds for $\lambda_i x_i$, with $\lambda_i\in\mathbb{R}$. On the other hand, since $|\lambda_1x_1|^2+|\lambda_2x_2|^2\le s^{-2}\,|\lambda_1x_1+\lambda_2x_2|^2$, we have

$$\begin{aligned}|T(\lambda_1x_1+\lambda_2x_2)|^2&\le2|T(\lambda_1x_1))|^2+2|T(\lambda_2x_2)|^2\\&\lesssim\Big(\delta+\frac{r}{\varepsilon}\Big)^2\frac{P(0,\varepsilon)^2}{s^2\varepsilon^2}\,|\lambda_1x_1+\lambda_2x_2|^2.\end{aligned}$$

As a consequence, the trace of T fulfills

$$|\mathrm{tr}(T)|=|\langle Te_1,e_1\rangle+\langle Te_2,e_2\rangle|\lesssim\Big(\delta+\frac{r}{\varepsilon}\Big)\frac{P(0,\varepsilon)}{s\varepsilon}, \tag{5.7}$$

where e_1,e_2 is the standard orthonormal basis of $\mathbb{R}^2$.

On the other hand, by the definition of T in (5.5), the precise value of $\mathrm{tr}(T)$ is the following:

$$\begin{aligned}\mathrm{tr}(T)&=\int\bigg(\frac{2}{|y|^2+\varepsilon^2}-\frac{2|y|^2}{(|y|^2+\varepsilon^2)^2}\bigg)\,d\mu(y)\\&=2\int\frac{\varepsilon^2}{(|y|^2+\varepsilon^2)^2}\,d\mu(y)=\frac{2}{\varepsilon}\,P_2(0,\varepsilon).\end{aligned}$$

From (5.7) and the preceding calculation we infer that

$$|\sin\widehat{x_1 0\,x_2}|=s\lesssim\Big(\delta+\frac{r}{\varepsilon}\Big)\frac{P(0,\varepsilon)}{P_2(0,\varepsilon)},$$

which implies the existence of the line L in (5.2) because this holds for all points $x_1,x_2\in B(0,r)$ satisfying (5.1). $\square$

6. Proof of Main Lemma 3.1

6.1. **The strategy.** The main tool for the proof is the identity (4.3) applied to $\mu_1=\mu_2=\mu$ and $\mu_3=\mu_{|R_0}$. In this case we have

$$2\int_{R_0}|\mathcal{C}_\Phi^\ell\mu|^2d\mu+4\mathrm{Re}\int(\mathcal{C}_\Phi^\ell\mu)\,\overline{\mathcal{C}_\Phi^\ell\mu_{|R_0}}\,d\mu=c_\Phi^{2,\ell}(\mu,\mu,\mu_{|R_0})+E_1+E_2, \tag{6.1}$$

where E_1 and E_2 are given by (4.4) and (4.5). Roughly speaking, we will see that the integrals on the left side are small compared to $\Theta(R_0)^2$ because $\mathcal{C}_*\mu(z)\ll\Theta(R_0)$ for many $z\in R_0$. On the other hand, we will also show that if the density $\Theta(Q)$ is too big for many squares $Q \subset R_0$ and $\mu_{at}(9R_0) \leq \eta\mu(R_0)$, then the term E_2 on the right side is positive and much bigger than E_1 and than the left side of (6.1), and then we will get a contradiction.

Recall that $E_2 = E_2^+ + F_1 + F_2 + F_3$, with $E_2^+ \geq 0$, while F_1, F_2, F_3 may be positive or negative. For $i = 1, 2, 3$ we split $F_i = F_i^+ - F_i^-$, according to the sign of the integrand in F_i. That is,

$$\begin{aligned} F_1 &= \iiint_{S_\ell} \frac{|y-z|^2 \Phi(z)\big(\Phi(x)-\Phi(y)\big)}{D_\Phi(x,y,z)}\, d\mu(x)d\mu(y)d\mu_{|R_0}(z) \\ &= \iiint_{\substack{(x,y,z)\in S_\ell \\ \Phi(x)>\Phi(y)}} \cdots + \iiint_{\substack{(x,y,z)\in S_\ell \\ \Phi(x)<\Phi(y)}} \cdots =: F_1^+ - F_1^-, \end{aligned} \tag{6.2}$$

where $S_\ell = \{(x,y,z) \in \mathbb{C}^3 : |x-y| \leq \ell,\ |x-z| \leq \ell,\ |y-z| \leq \ell\}$. Also,

$$\begin{aligned} F_2 &= \iiint_{S_\ell} \frac{|x-z|^2 \Phi(x)\big(\Phi(y)-\Phi(z)\big)}{D_\Phi(x,y,z)}\, d\mu(x)d\mu(y)d\mu_{|R_0}(z) \\ &= \iiint_{\substack{(x,y,z)\in S_\ell \\ \Phi(y)>\Phi(z)}} \cdots + \iiint_{\substack{(x,y,z)\in S_\ell \\ \Phi(y)<\Phi(z)}} \cdots =: F_2^+ - F_2^-, \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} F_3 &= \iiint_{S_\ell} \frac{|x-y|^2 \Phi(y)\big(\Phi(z)-\Phi(x)\big)}{D_\Phi(x,y,z)}\, \mu(x)d\mu(y)d\mu_{|R_0}(z) \\ &= \iiint_{\substack{(x,y,z)\in S_\ell \\ \Phi(z)>\Phi(x)}} \cdots + \iiint_{\substack{(x,y,z)\in S_\ell \\ \Phi(z)<\Phi(x)}} \cdots =: F_3^+ - F_3^-. \end{aligned} \tag{6.4}$$

Then we have

$$\begin{aligned} &c_\Phi^2(\mu_{|R_0}) + E_2^+ \\ &\qquad \leq 2\int_{R_0} |\mathcal{C}_\Phi^\ell \mu|^2 d\mu + 4\int \big|(\mathcal{C}_\Phi^\ell \mu)\, \mathcal{C}_\Phi^\ell \mu_{|R_0}\big|\, d\mu + |E_1| + F_1^- + F_2^- + F_3^-. \end{aligned}$$

6.2. **Construction of Φ.** Let R_0 be a square satisfying the assumptions of Main Lemma 3.1. We denote

$$G_0 := \{x \in \operatorname{supp}(\mu) \cap 10R_0 : \mathcal{C}_*\mu(x) \leq \varepsilon_0 \Theta(R_0)\}.$$

Roughly speaking, a square Q with center in $7R_0 \cap \operatorname{supp}(\mu)$ and $\ell(Q) \leq \ell(R_0)/4$ is not good if $Q \backslash G_0$ is too big. The precise definition is the following. If Q is a square centered at some point $x \in 7R_0 \cap \operatorname{supp}(\mu)$, and

$$\mu(Q \setminus G_0) > \tau^{1/2}\, \mu(Q),$$

we set $Q \in BC$. If $x \in 7R_0$ is the center of some square from BC, we write $x \in B_0$. Moreover, for $x \in B_0$ we set

$$\ell(x) := \sup\{\ell(Q) : Q \in BC \text{ centered at } x\}.$$

Given $x \in B_0$, take a square $Q_x \in BC$ centered at x such that $\ell(Q_x) \geq \ell(x)/2$. By Besicovitch covering theorem, there exists a subfamily $BC_0 \subset \{Q_x\}_{x\in B_0}$ with finite overlap which covers B_0.

Abusing language, sometimes we will write BC_0 instead of $\bigcup_{Q\in BC_0} Q$.

Let $0<\kappa<1/10$ be some constant to be fixed below. We denote

$$\Phi_{BC}(x) = \inf_{Q\in BC_0}\Big\{\kappa \operatorname{dist}(x,Q) + \ell(Q),\ \kappa \operatorname{dist}\Big(x, \mathbb{C}\setminus \bigcup_{Q\in BC_0} Q\Big)\Big\}. \tag{6.5}$$

Notice that Φ_{BC} is κ-Lipschitz.

Let $A\geq 100$ be another constant which will be fixed below. Following [NTV1], given $x\in 7R_0$, $0<r\leq \ell(R_0)/4$, we say that $B(x,r)$ is a *non Ahlfors disk* if $\mu(B(x,r)) > A\Theta(R_0)r$ and moreover $B(x,r)\cap \operatorname{supp}(\mu)\not\subset BC_0\cap\operatorname{supp}(\mu)$. We say that $x\in 7R_0$ is a *non Ahlfors point* if there exists some $r>0$ such that $B(x,r)$ is a non Ahlfors disk. For any $x\in F$, we denote

$$\mathcal{R}(x) = \sup\{r>0:\ B(x,r) \text{ is a non Ahlfors disk}\}.$$

If $x\in 7R_0$ is an Ahlfors point, we set $\mathcal{R}(x)=0$.

We denote

$$H = \bigcup_{x\in F,\,\mathcal{R}(x)>0} B(x,\mathcal{R}(x)).$$

By Vitali's $5r$-Covering Theorem there is a disjoint family $\{B(x_h, 2\mathcal{R}(x_h))\}_h$ such that $H\subset \bigcup_h B(x_h, 10\mathcal{R}(x_h))$. We set

$$H_0 = \bigcup_h B(x_h, 10\mathcal{R}(x_h)). \tag{6.6}$$

Since $H\subset H_0$, all non Ahlfors disks are contained in H_0 and then,

$$\operatorname{dist}(x, F\setminus H_0)\geq \mathcal{R}(x) \tag{6.7}$$

for all $x\in 7R_0$.

We set

$$\Phi(x) = \max\Big(\Phi_{BC}(x),\ \kappa\operatorname{dist}(x,\mathbb{C}\setminus H_0)\Big), \tag{6.8}$$

and given $\varepsilon>0$,

$$\Phi_\varepsilon(x) = \max(\Phi(x),\varepsilon).$$

It is clear that Φ and Φ_ε are κ-Lipschitz.

Notice that the balls $B(x,\mathcal{R}(x))$, $\mathcal{R}(x)>0$, are (a,b)-doubling for all $a>0$ and $b=b(a)$.

Remark 6.1. The constants that we denote by C (with or without subindex) do not depend on A, κ, τ or ε_0, unless stated otherwise.

Below we will take A, κ, τ and ε_0 so that $\varepsilon_0,\tau\ll\kappa\leq 1/10<100\leq A$. First we choose A and κ in Subsection 6.4, and then ε_0,τ are chosen in the rest of the proof of the Main Lemma.

Remark 6.2. It is easy to check that $\mu(BC_0)$ is small:

$$\begin{aligned}\mu(BC_0) &\leq \sum_{Q\in BC_0}\mu(Q)\leq \sum_{Q\in BC_0}\tau^{-1/2}\mu(Q\setminus G_0)\\ &\lesssim \tau^{-1/2}\mu(10R_0\setminus G_0)\lesssim C\tau^{1/2}\mu(R_0).\end{aligned}$$

Observe that $\mu_{|R_0\setminus H_0}$ has linear growth. Estimating $\mu(H_0)$ from above is the main objective of the rest of the proof of the Main Lemma.

In next lemma we collect a couple of basic properties dealing with Φ.

Lemma 6.3. *Let $x \in 7R_0 \cap \operatorname{supp}(\mu)$. We have:*

(a) *For all $\delta > 0$ there exists some square $P \in BC_0$ or point $P \in \mathbb{C} \setminus \bigcup_{Q\in BC_0} Q$ such that $\operatorname{dist}(x,P) \leq \kappa^{-1}\Phi_{BC}(x) + \delta$ and $\ell(P) \leq \Phi_{BC}(x) + \delta$.*

(b) *If Q is centered at x and $4\kappa^{-1}\Phi(x) \leq \ell(Q) \leq \ell(R_0)/10$, then $\Theta(Q) \leq 2A\Theta(R_0)$ and $\mu(Q \setminus G_0) \leq \tau^{1/2}\,\mu(2Q)$.*

Proof. The statement (a) is a direct consequence of the definition of Φ.

Let us turn our attention to (b). First we show $\Theta(Q) \leq 2A\Theta(R_0)$. We only have to consider the case $\mathcal{R}(x) > 0$. Then we have $B(x,\mathcal{R}(x)) \subset H_0$ by construction. Therefore, $\Phi(x) \geq \kappa \operatorname{dist}(x, \mathbb{C}\setminus H_0) \geq \kappa\mathcal{R}(x)$. So $\ell(Q) \geq 4\mathcal{R}(x)$ and then $\Theta(Q) \leq 2\Theta(B(x,\ell(Q)) \leq 2A\Theta(R_0)$.

Let us see that $\mu(Q \setminus G_0) \leq \tau^{1/2}\,\mu(2Q)$ now. Let $\delta > 0$ be a fixed small constant ($\delta \ll \ell(Q)$). Let $P \in BC_0$ be such that $\operatorname{dist}(x,P) \leq \kappa^{-1}\Phi_{BC}(x)+\delta$ and $\ell(P) \leq \Phi_{BC}(x) + \delta$. Let $\widetilde{P}$ be the smallest square concentric with P which contains Q. It is easy to check that $\ell(\widetilde{P}) \leq 2|x - x_P| + \ell(Q)$, where x_P is the center of P. Moreover, $\ell(\widetilde{P}) \leq \ell(R_0)/4$ because $\ell(Q) \leq \ell(R_0)/10$. Thus,

$$\ell(\widetilde{P}) \leq 2\kappa^{-1}\Phi_{BC}(x) + 2\delta + \ell(Q) \leq 2\ell(Q),$$

assuming δ small enough and taking into account that $\Phi_{BC}(x) \leq \Phi(x)$. We derive

$$\mu(Q \setminus G_0) \leq \mu(\widetilde{P} \setminus G_0) \leq \tau^{1/2}\mu(\widetilde{P}) \leq \tau^{1/2}\mu(2Q),$$

since $\widetilde{P} \subset 2Q$ (because $Q \subset \widetilde{P}$ and $\ell(Q) \geq \ell(\widetilde{P})/2$). $\square$

From (b) in the preceding lemma one infers that if Q is centered at $x \in \operatorname{supp}(\mu) \cap 7R_0$ and $\ell(Q) \geq \Phi(x)$, then $\Theta(Q) \lesssim \kappa^{-1}A\Theta(R_0)$. For technical reasons, we would like to obtain an analogous estimate not involving the constant κ. Below, in Lemma 6.10, we will get a result of this type, under the appropriate assumptions.

By Lemma 4.2 and the preceding comments, we deduce that there exists some $\ell \in [\ell(R_0)/2, \ell(R_0)]$ such that the term E_1 in (6.1) verifies $|E_1| \lesssim \kappa^{-1}A\Theta(R_0)^2\mu(R_0)$. We choose one of these precise values of ℓ.

6.3. **Estimates for the terms in** (6.1)**.** In this subsection we will state some estimates involving the terms in (6.1). They will be proved below.

Lemma 6.4. *For all $\varepsilon > 0$ we have*

$$\int_{3R_0} |\mathcal{C}^{\ell}_{\Phi_\varepsilon}\mu|^2 d\mu \lesssim A^2\Theta(R_0)^2\Big[\big(\varepsilon_0^2+\kappa^2+\tau^{1/2}\kappa^{-4}\big)\mu(R_0)+\kappa^{-1}\mu_{at}(9R_0)\Big]. \tag{6.9}$$

and

$$\int |\mathcal{C}^{\ell}_{\Phi_\varepsilon}\mu_{|R_0}|^2\, d\mu \lesssim A^2\Theta(R_0)^2\big(\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big). \tag{6.10}$$

Lemma 6.5. *For all $\varepsilon > 0$ we have*

$$F_1^- + F_2^- + F_3^- \lesssim A^2\Theta(R_0)^2\big[\big(\kappa^{1/2} + C(\kappa)\tau^{1/4}\big)\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big].$$

Lemma 6.6. *If $\tau, \varepsilon_0, \varepsilon > 0$ are small enough (in particular $\varepsilon_0, \tau \ll \kappa$), then*

$$E_2^+ \gtrsim A^2\Theta(R_0)^2\big[\mu(\tfrac12 R_0 \cap H_0 \setminus BC_0) - C\kappa^{-1}\mu_{at}(R_0)\big].$$

6.4. **Proof of Main Lemma 3.1 using Lemmas 6.4, 6.5 and 6.6.** Recall that

$$c^2_{\Phi_\varepsilon}(\mu_{|R_0}) + E_2^+ \lesssim \int_{R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu|^2 d\mu + \int |(\mathcal{C}^\ell_{\Phi_\varepsilon}\mu)\,\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}| d\mu + |E_1| + F_1^- + F_2^- + F_3^-. \tag{6.11}$$

Lemmas 6.4 and 6.5 give estimates for the first integral on the right side and for the terms F_i^- respectively. By Lemma 4.2 and our choice of ℓ we have $|E_1| \lesssim \kappa^{-1}A\Theta(R_0)^2\mu(R_0)$.

To deal with the integral $\int |(\mathcal{C}^\ell_{\Phi_\varepsilon}\mu)\,\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|\,d\mu$, notice that $\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}$ vanishes on $\mathbb{C}\setminus 3R_0$. Thus by Lemma 6.4 we have

$$\begin{aligned}
\int |(\mathcal{C}^\ell_{\Phi_\varepsilon}\mu)\,\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|\,d\mu &\le \left(\int_{3R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu|^2\,d\mu\right)^{1/2}\left(\int_{3R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2\,d\mu\right)^{1/2}\\
&\lesssim A^2\Theta(R_0)^2\Big[\big(\varepsilon_0^2 + \kappa^2 + \tau^{1/2}\kappa^{-4}\big)\mu(R_0) + \kappa^{-1}\mu_{at}(9R_0)\Big]^{1/2}\\
&\quad\times\Big[\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\Big]^{1/2}\\
&\lesssim A^2\Theta(R_0)^2\big[\big(\varepsilon_0^2 + \kappa^2 + \tau^{1/2}\kappa^{-4}\big)^{1/2}\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big]\\
&\lesssim A^2\Theta(R_0)^2\big[(\varepsilon_0 + \kappa + \tau^{1/4}\kappa^{-2})\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big].
\end{aligned}$$

If we gather all the estimates obtained for all the terms on the right side of (6.11) we obtain

$$\begin{aligned}
c^2_{\Phi_\varepsilon}(\mu_{|R_0}) + E_2^+ &\lesssim A^2\Theta(R_0)^2\big[(\varepsilon_0 + \kappa^{1/2} + C(\kappa)\tau^{1/4})\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big]\\
&\quad + \kappa^{-1}A\Theta(R_0)^2\mu(R_0),
\end{aligned} \tag{6.12}$$

where we took into account that $\varepsilon_0^2 \le \varepsilon_0$, and analogously for τ and κ. From Lemma 6.6 and the preceding estimate we infer that if $\varepsilon > 0$ is small enough, then

$$\mu(\tfrac12 R_0 \cap H_0 \setminus BC_0) \lesssim \big(\varepsilon_0 + \kappa^{1/2} + C(\kappa)\tau^{1/4} + A^{-1}\kappa^{-1}\big)\,\mu(R_0) + C(\kappa)\mu_{at}(9R_0).$$

If $\varepsilon_0, \tau, \kappa$ are small enough and A big enough (in particular, $A > \kappa^{-1}$), taking into account that $\mu(R_0) \approx \mu(\frac12 R_0)$, we get

$$\mu(\tfrac12 R_0 \cap H_0 \setminus BC_0) \le \frac14\,\mu(\tfrac12 R_0) + C(\kappa)\mu_{at}(9R_0).$$

Therefore, recalling Remark 6.2, we have

$$\mu(\tfrac12 R_0 \setminus (H_0 \cup BC_0)) + C(\kappa)\mu_{at}(9R_0) \ge \Big(1 - \frac14 - C\tau^{1/2}\Big)\mu(\tfrac12 R_0) \ge \frac12\,\mu(\tfrac12 R_0).$$

Thus either $\mu(\frac{1}{2}R_0 \setminus (H_0 \cup BC_0)) \geq \mu(\frac{1}{2}R_0)/4$ or $\mu_{at}(9R_0) \geq C(\kappa)\mu(R_0)$.

On the other hand, since (6.12) holds for $\varepsilon > 0$ we have

$$c^2(\mu_{|\frac{1}{2}R_0\setminus(H_0\cup BC_0)}) \leq c^2_\Phi(\mu_{|R_0}) \leq C(\varepsilon_0, A, \tau, \kappa)\mu(R_0) < \infty.$$

$\square$

6.5. **Auxiliary lemmas.** In order to prove Lemmas 6.4, 6.5 and 6.6 we will need some auxiliary results. The first one is the following.

Lemma 6.7. *Let Q be a square centered at some point in $G_0 \cap 7R_0$, with $\ell(Q) \leq \ell(R_0)/4$, such that $\mu(Q \setminus G_0) \lesssim \tau^{1/2}\mu(2Q)$, $\Theta(Q) \gtrsim \Theta(R_0)$, and $\Theta(2Q) \lesssim A\Theta(R_0)$. If ε_0 and τ are small enough, one of the following happens:*

(a) *either there exists some atom z_0 such that $Q \cap G_0 = \{z_0\}$, or*
(b) diam$(G_0 \cap Q) \geq \ell(Q)/6$, *and moreover there are two squares $Q_1, Q_2 \subset 5Q$ with* dist$(Q_1, Q_2) \gtrsim \ell(Q)$ *such that $\mu(Q_i) \gtrsim \mu(Q)$, for $i = 1, 2$.*

Proof. Suppose that (a) does not hold. Then diam$(Q \cap G_0) > 0$.

First we will prove that diam$(G_0 \cap Q) \geq \ell(Q)/6$. Suppose that this fails and let us argue by contradiction. Let $x_1, x_2 \in \overline{Q \cap G_0}$ be such that $|x_1 - x_2| = \text{diam}(Q \cap G_0)$. Consider the half planes

$$\Pi_1 := \{z : |z - x_1| \leq |z - x_2|\}, \qquad \Pi_2 := \{z : |z - x_1| \geq |z - x_2|\}.$$

Suppose that $\mu(\Pi_2 \cap G_0 \cap Q) \geq \mu(\Pi_1 \cap G_0 \cap Q)$. See Figure 1. We set

$$I := \int_{|x_2 - x_1|/2 \leq |z - x_1| \leq |x_2 - x_1|} \frac{1}{z - x_1}\, d\mu(z).$$

Since $x_1 \in \overline{G_0}$, we have

$$|I| \leq 2\mathcal{C}_*\mu(x_1) \leq 2\varepsilon_0\Theta(R_0). \tag{6.13}$$

Now we will estimate I from below. First we check that the annulus $A_0 := \{z : |x_2 - x_1|/2 \leq |z - x_1| \leq |x_2 - x_1|\}$ is contained Q: if x_Q stands for the center of Q, we have $|x_1 - x_Q| \leq \text{diam}(Q \cap G_0)$ since $x_Q \in G_0$. Thus for $z \in A_0$, we get

$$|z - x_Q| \leq |z - x_1| + |x_1 - x_Q| \leq \frac{\ell(Q)}{6} + \frac{\ell(Q)}{6} = \frac{\ell(Q)}{3},$$

and so $z \in Q$. We have

$$|I| \geq \left| \int_{z \in A_0 \cap G_0} \frac{1}{z - x_1}\, d\mu(z) \right| - \int_{z \in Q \setminus G_0} \frac{1}{|z - x_1|}\, d\mu(z). \tag{6.14}$$

Consider now the half plane

$$\Pi_3 := \{z : \langle z - x_1,\, x_1 - x_2 \rangle > 0\}.$$

From the definition of x_1, x_2 and the fact that $A_0 \subset Q$ we infer that $\mu(A_0 \cap G_0 \setminus \Pi_3) = 0$. Observe also that $\Pi_2 \cap G_0 \cap Q \subset A_0 \cap \Pi_2 \subset A_0 \setminus \Pi_3$ (see Figure 1). From this fact and some elementary geometry we deduce

$$\left| \int_{z \in A_0 \cap G_0} \frac{1}{z - x_1} \, d\mu(z) \right| \gtrsim \frac{\mu(\Pi_2 \cap G_0 \cap Q)}{|x_1 - x_2|} \gtrsim \frac{\mu(Q)}{|x_1 - x_2|}.$$

For the last integral in (6.14) we use the following rough estimate

$$\int_{z \in Q \setminus G_0} \frac{1}{|z - x_1|} \, d\mu(z) \lesssim \frac{\mu(Q \setminus G_0)}{|x_1 - x_2|} \leq \tau^{1/2} \frac{\mu(2Q)}{|x_1 - x_2|}.$$

Since $\mu(2Q) \lesssim A\mu(Q)$ and $\tau \ll A^{-2}$, we get

$$|I| \geq \frac{(C^{-1} - A\tau^{1/2})\, \mu(Q)}{|x_1 - x_2|} \gtrsim \frac{\mu(Q)}{|x_1 - x_2|} \gtrsim \frac{\ell(Q)}{|x_1 - x_2|} \Theta(R_0).$$

From this estimate and (6.13) we deduce that $|x_1 - x_2| \gtrsim \ell(Q)/\varepsilon_0 > \ell(Q)/6$, which is a contradiction.

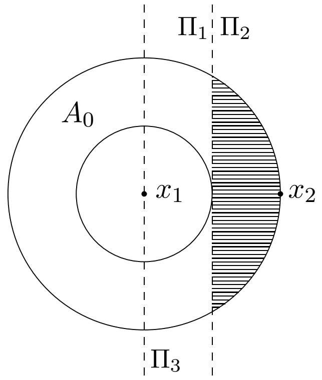

FIGURE 1. The dashed vertical lines are the borders of the half planes Π_1, Π_2, Π_3. The annulus A_0 is the region between both circles, $G_0 \cap A_0$ is contained in $\Pi_3 \cap A_0$, and $\Pi_2 \cap G_0 \cap A_0$ is contained in the region marked with horizontal lines.

It remains to prove the existence of the squares $Q_1, Q_2 \subset 5Q$ described in (b). Divide Q into 900 squares of side length $\ell(Q)/30$ with disjoint interiors. Let Q_1 be the square with maximal measure. Since $\operatorname{diam}(G_0 \cap Q) \geq \ell(Q)/6$, there exists some point $x_0 \in G_0 \cap Q$ such that $\operatorname{dist}(x_0, Q_1) > \ell(Q)/10$. Consider the annulus $A_1 := \{z : \operatorname{dist}(x_0, Q_1) - \ell(Q_1)/100 < |z - x_0| \leq \operatorname{dist}(x_0, Q_1) + \operatorname{diam}(Q_1)\}$, which contains Q_1. From the fact that $\mathcal{C}_*\mu(x_0) \lesssim \varepsilon_0 \Theta(Q)$, one easily infers the existence of another square Q_2 intersecting A_1, with $\ell(Q_2) = \ell(Q_1)$, $Q_2 \subset 5Q$, $\operatorname{dist}(Q_1, Q_2) \gtrsim \ell(Q)$ and $\mu(Q_2) \gtrsim \mu(Q)$. The details are left for the reader. $\square$

Remark 6.8. The conclusion of the preceding lemma also holds if one asks $\mu(Q \setminus G_0) \lesssim C(A, \kappa)\tau^{1/2}\mu(2Q)$, $\Theta(Q) \gtrsim C(A, \kappa)^{-1}\Theta(R_0)$ and $\Theta(2Q) \lesssim$

$C(A,\kappa)\Theta(R_0)$ instead of $\mu(Q\setminus G_0)\lesssim \tau^{1/2}\mu(2Q)$, $\Theta(Q)\gtrsim\Theta(R_0)$ and $\Theta(2Q)\lesssim A\Theta(R_0)$. The same proof works because $\varepsilon_0,\tau\ll A^{-1},\kappa$.

Next lemma is a direct consequence of Lemmas 5.1 and 6.7. The main difficulty consists in the choice of the appropriate parameter ε in Lemma 5.1.

Lemma 6.9. *Let Q be a square centered at some point in $G_0\cap 7R_0$, with $\ell(Q)\leq \ell(R_0)/4$, such that $\mu(Q\setminus G_0)\lesssim\tau^{1/2}\mu(2Q)$, $\Theta(Q)\gtrsim\Theta(R_0)$, and $\Theta(\lambda Q)\lesssim A\Theta(R_0)$ for all $\lambda>1$ with $\lambda Q\subset 2R_0$. If ε_0 and τ are small enough, then one of the following happens:*

(a) *either there exists some atom z_0 such that $Q\cap G_0=\{z_0\}$, or*
(b) *there exists a line L_Q such that any $z\in G_0\cap Q$ fulfills*

$$\operatorname{dist}(z,L_Q)\leq \varepsilon_0^{1/12}\ell(Q). \tag{6.15}$$

Proof. Suppose that (a) does not hold. By the preceding lemma there are two points $x_1,x_2\in Q\cap G_0$ such that $|x_1-x_2|\geq \ell(Q)/6$. Let L_Q be the line through x_1 and x_2. Consider $z\in Q\cap G_0$. One of the points x_i, say x_1, satisfies $|z-x_i|\geq\ell(Q)/12$. We wish to apply Lemma 5.1 with $r=2\ell(Q)$ and x_1 instead of x_0. To choose the appropriate value of ε in Lemma 5.1 we argue as follows. Set $\widetilde{\varepsilon}=\varepsilon_0^{-1/5}\ell(Q)$. We distinguish two cases:

(i) If for all $r\in[\widetilde{\varepsilon},\widetilde{\varepsilon}^2/\ell(Q)]$ we have

$$\Theta(B(x_1,r))\leq \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\Theta(B(x_1,\widetilde{\varepsilon})),$$

then we set $\varepsilon=\widetilde{\varepsilon}$.

(ii) Otherwise we choose $\varepsilon\in[\widetilde{\varepsilon},\widetilde{\varepsilon}^2/\ell(Q)]$ such that

$$\Theta(B(x_1,\varepsilon))\geq \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\Theta(B(x_1,\widetilde{\varepsilon})).$$

Observe that in both cases we have

$$\Theta(B(x_1,\varepsilon))\geq\Theta(B(x_1,\widetilde{\varepsilon}))\gtrsim\frac{\ell(Q)}{\widetilde{\varepsilon}}\,\Theta(Q). \tag{6.16}$$

Let L_{x_1} be the line through x_1 given by Lemma 5.1. Indeed, notice that x_1 satisfies the assumptions of the lemma, since every point $z\in Q\cap G_0$ (in particular x_1) satisfies $\mathcal{C}_*\mu(z)\leq\varepsilon_0\Theta(R_0)$, and so

$$\begin{aligned}|\widetilde{\mathcal{C}}_\varepsilon\mu(z)-\widetilde{\mathcal{C}}_\varepsilon\mu(x_1)| &\leq \mathcal{C}_*\mu(z)+\mathcal{C}_*\mu(x_1)\leq 2\varepsilon_0\Theta(R_0)\\ &=:\delta\,P(x_1,\varepsilon)\frac{\ell(Q)}{12\varepsilon}\leq\delta\,P(x_1,\varepsilon)\frac{|z-x_1|}{\varepsilon},\end{aligned}$$

where we set $\delta=24\varepsilon_0\varepsilon\Theta(R_0)/[P(x_1,\varepsilon)\ell(Q)]$. By (6.16) in any case we have

$$\delta\lesssim\varepsilon_0\frac{\varepsilon\widetilde{\varepsilon}}{\ell(Q)^2}\leq\varepsilon_0\frac{\widetilde{\varepsilon}^3}{\ell(Q)^3}.$$

Then Lemma 5.1 tells us that

$$\begin{aligned}\text{dist}(z, L_{x_1}) &\lesssim \Big(\delta + \frac{\ell(Q)}{\varepsilon}\Big)\frac{P(x_1,\varepsilon)}{P_2(x_1,\varepsilon)}\,|z-x_1| \\ &\lesssim \Big(\varepsilon_0\frac{\widetilde{\varepsilon}^3}{\ell(Q)^3} + \frac{\ell(Q)}{\varepsilon}\Big)\frac{P(x_1,\varepsilon)}{\Theta(B(x_1,\varepsilon))}\,\ell(Q).\end{aligned} \tag{6.17}$$

In case (i), from the fact that $\varepsilon = \widetilde{\varepsilon}$, the definition of $P(x_1,\varepsilon)$, and (6.16), we deduce

$$\begin{aligned}P(x_1,\varepsilon) = P(x_1,\widetilde{\varepsilon}) &\lesssim \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\Theta(B(x_1,\widetilde{\varepsilon})) + \frac{\ell(Q)}{\widetilde{\varepsilon}}A\Theta(R_0) \\ &\lesssim \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\Theta(B(x_1,\widetilde{\varepsilon})) + A\Theta(B(x_1,\widetilde{\varepsilon})) \lesssim \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\Theta(B(x_1,\widetilde{\varepsilon})),\end{aligned}$$

where we took into account that $\varepsilon_0 \ll A^{-1}$. Thus

$$\frac{P(x_1,\varepsilon)}{\Theta(B(x_1,\varepsilon))} \lesssim \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}.$$

On the other hand, in case (ii) we have

$$\Theta(B(x_1,\varepsilon)) \geq \frac{\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\Theta(B(x_1,\widetilde{\varepsilon})) \gtrsim \frac{\ell(Q)^{1/2}}{\widetilde{\varepsilon}^{1/2}}\,\Theta(Q),$$

and $P(x_1,\varepsilon) \lesssim A\Theta(R_0) \lesssim A\Theta(Q)$. So,

$$\frac{P(x_1,\varepsilon)}{\Theta(B(x_1,\varepsilon))} \lesssim \frac{A\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}.$$

Therefore, from (6.17) in both cases we obtain

$$\text{dist}(z, L_{x_1}) \lesssim \Big(\frac{\varepsilon_0\,\widetilde{\varepsilon}^3}{\ell(Q)^3} + \frac{\ell(Q)}{\widetilde{\varepsilon}}\Big)\frac{A\widetilde{\varepsilon}^{1/2}}{\ell(Q)^{1/2}}\,\ell(Q) = A\Big(\frac{\varepsilon_0\,\widetilde{\varepsilon}^{7/2}}{\ell(Q)^{7/2}} + \frac{\ell(Q)^{1/2}}{\widetilde{\varepsilon}^{1/2}}\Big)\ell(Q).$$

Recalling that $\widetilde{\varepsilon} = \varepsilon_0^{-1/5}$ and that $\varepsilon_0 \ll A^{-1}$, we get

$$\text{dist}(z, L_{x_1}) \lesssim \varepsilon_0^{1/11}\ell(Q).$$

This estimate also applies to the case $z = x_2$. Since $|x_1 - x_2| \gtrsim \ell(Q)$, we infer that the Hausdorff distance between $L_Q \cap 2Q$ and $L_{x_1} \cap 2Q$ is $\lesssim \varepsilon_0^{1/11}\ell(Q)$. Then it easily follows that $\text{dist}(z, L_Q) \leq \varepsilon_0^{1/12}\ell(Q)$. $\square$

Lemma 6.10. *Let Q be a square centered at some point in $G_0 \cap 7R_0$, with $\ell(Q) \leq \ell(R_0)/4$, such that $\mu(Q \setminus G_0) \lesssim \tau^{1/2}\mu(2Q)$, $\Theta(Q) \gtrsim \Theta(R_0)$, and $\Theta(\lambda Q) \lesssim A\Theta(R_0)$ for all $\lambda > 1$ with $\lambda Q \subset 2R_0$. If ε_0 and τ are small enough, then one of the following happens:*

(a) *either there exists some atom $z_0 \in Q$ such that $\mu(z_0) \geq C^{-1}\kappa\mu(Q)$, or*
(b) *any square $P \subset \frac{1}{2}Q$ with $\ell(P) \geq \kappa\ell(Q)$ satisfies*

$$\Theta(P) \leq C_2\Theta(Q).$$

The important fact about this lemma is that the constant C_2 does not depend on κ. That is, for a fixed $\kappa > 0$ one can choose $\varepsilon_0, \tau \ll \kappa$ so that in case (b), $\Theta(P) \leq C_2\Theta(Q)$, where C_2 is some absolute constant. An analogous result would hold with any constant different from κ. We have stated the lemma in terms of κ because this is the precise result that will be needed below.

Proof. Suppose that there is not any atom $z_0 \in Q$ with $\mu(z_0) \geq C_3^{-1}\kappa\mu(Q)$, where C_3 will be fixed below. By Lemma 6.9 there exists a line L_Q such that any $z \in G_0 \cap Q$ satisfies $\operatorname{dist}(z, L_Q) \lesssim \varepsilon_0^{1/12}\ell(Q)$.

We will prove the following:

Claim. *Let B be a closed ball centered at some point in $L_Q \cap \frac{3}{4}Q$ with $\Theta(B) \geq \Theta(Q)$ and $\frac{\kappa}{2}\ell(Q) \leq \operatorname{diam}(B) \leq \ell(Q)/100$. Then there are two closed balls B_a, B_b with centers on L_Q such that for $i = a, b$*

$$C_4^{-1}\operatorname{diam}(B) \leq \operatorname{dist}(B_i, B) \leq C_4\operatorname{diam}(B),$$

$$\operatorname{diam}(B) \leq \operatorname{diam}(B_i) \leq \operatorname{diam}(B)(1 + \varepsilon_1),$$

and

$$\mu(B_i) \geq \mu(B)(1 - \varepsilon_1),$$

where ε_1 is some constant such that $\varepsilon_1 \to 0$ as $\varepsilon_0 \to 0$ and $\tau \to 0$ (for a fixed κ). Moreover, the center of B is located between the centers of B_a and B_b (that is to say, B_a and B_b are "at opposite sides of B").

Let us see how the statement (b) of the lemma follows from our claim. Let P be a square such that $\kappa\ell(Q) \leq \ell(P)$ and $P \subset \frac{1}{2}Q$. To prove (b) we may assume $\ell(P) \leq \ell(Q)/200$. Suppose also that $\Theta(P) \geq 10\Theta(Q)$. Since $\tau \ll \kappa$, we have $G_0 \cap P \neq \varnothing$ and so $\operatorname{dist}(L_Q, P) \ll \ell(P)$. Let B be a ball with center in L_Q and radius $1.1\operatorname{diam}(P)$ such that $P \subset B$. It is easy to check that B satisfies the assumptions of the claim. As a consequence, we derive the existence of two other balls B_1, B_2 as above. Further, if ε_1 is small enough, B_a and B_b also satisfy the assumptions, and the claim can be applied again.

Proceeding inductively one proves the existence of a sequence of pairwise disjoint balls $B_1, B_2, \ldots, B_N \subset Q$ with $N \approx \ell(Q)/\operatorname{diam}(B) \lesssim \kappa^{-1}$. Indeed, recall that the starting ball B intersects $\frac{1}{2}Q$ and at each step j one will get $\operatorname{diam}(B) \leq \operatorname{diam}(B_j) \leq \operatorname{diam}(B)(1+\varepsilon_1)^j$, and $\mu(B_i) \geq \mu(B)(1-\varepsilon_1)^j$, with $j \lesssim \kappa^{-1}$, and so each ball B_j will satisfy the assumptions in the claim for ε_1 small enough, etc.

Then we deduce

$$\mu(Q) \geq \sum_{j=1}^{N} \mu(B_j) \geq N\mu(B)(1-\varepsilon_1)^N \gtrsim \frac{\ell(Q)}{\operatorname{diam}(B)}\,\mu(B) \gtrsim \frac{\ell(Q)}{\operatorname{diam}(P)}\,\mu(P),$$

where used the fact that if ε_1 is small enough, then $(1-\varepsilon_1)^N \approx 1$. So we get $\Theta(Q) \geq C_5^{-1}\Theta(P)$, and the lemma follows.

It remains to prove the claim above. If τ is small enough it follows easily that $G_0 \cap B \neq \varnothing$. Let $z \in G_0 \cap B$ and take a square S centered at z with $\ell(S) = 12\mathrm{diam}(B)$. Since $S \subset Q$ we have $\mu(S \setminus G_0) \leq \mu(Q \setminus G_0) \leq \tau^{1/2}\mu(2Q) \leq \tau^{1/2}C(A,\kappa)\mu(S)$. Then, by Lemma 6.7 and Remark 6.8,

$$\mathrm{diam}(S \cap G_0) \geq \frac{1}{6}\,\ell(S) = 2\mathrm{diam}(B).$$

So there exists some point $z_0 \in G_0 \cap S$ with $\mathrm{dist}(z_0, B) \geq \mathrm{diam}(B)$.

Now we will use the fact that $\mathcal{C}_*\mu(z_0) \leq \varepsilon_0\Theta(R_0)$ to prove the existence of a ball B_1 like the one described in the claim. To this end, consider the annulus

$$R := \{x \in \mathbb{C} :\, \mathrm{dist}(z_0, B) \leq |x - z_0| \leq \mathrm{dist}(z_0, B) + \mathrm{diam}(B)\},$$

which contains B. Suppose for simplicity that L_Q is parallel to the horizontal axis. For $\varepsilon_0 \leq \kappa$, the strip $T := \{x \in \mathbb{C} :\, \mathrm{dist}(x, L_Q) \leq \varepsilon_0^{1/12}\ell(Q)\}$ which contains $G_0 \cap Q$ is much thinner than R (i.e. $\varepsilon_0^{1/12}\ell(Q) \ll \mathrm{diam}(B)$). Thus $R \cap T$ is the union of two disjoint connected regions U_1, U_2. One of them, say U_2, contains $B \cap G_0$. Let B_a be the ball with center in $U_1 \cap L_Q$ with minimal radius such that $U_1 \subset B_a$. It is clear that when the width of T tends to zero, $\mathrm{diam}(B_a) \to \mathrm{diam}(B)$. One can check easily that $\mathcal{C}_*\mu(z_0) \leq \varepsilon_0\Theta(R_0)$ implies that $\mu(U_1) \approx \mu(U_2)$. However we need a better estimate (recall that we need $\mu(B_a) \geq \mu(B)(1 - \varepsilon_1)$). To this end, we split R into n disjoint concentric annuli R^j of width $\mathrm{diam}(B)/n$. Then we have

$$\left|\int_{y \in R^j} \frac{1}{y - z_0}\, d\mu(y)\right| \leq 2\mathcal{C}_*\mu(z_0) \leq 2\varepsilon_0\Theta(R_0).$$

Thus,

$$\begin{aligned}\left|\int_{y \in R^j \cap G_0} \mathrm{Re}\frac{1}{y - z_0}\, d\mu(y)\right| &\leq 2\varepsilon_0\Theta(R_0) + C\frac{\mu(R^j \setminus G_0)}{\mathrm{diam}(R^j)} \\ &\leq 2\varepsilon_0\Theta(R_0) + C\tau^{1/2}A\kappa^{-1}\Theta(R_0).\end{aligned}$$

On the other hand, if we denote by r_j the inner radius of R^j, we have

$$\left|\int_{y \in R^j \cap G_0} \mathrm{Re}\frac{1}{y - z_0}\, d\mu(y)\right| \geq \frac{\mu(R^j \cap U_1 \cap G_0)}{r_j + \frac{2\mathrm{diam}(B)}{n}} - \frac{\mu(R^j \cap U_2 \cap G_0)}{r_j - \frac{2\mathrm{diam}(B)}{n}}.$$

We derive

$$\begin{aligned}\mu(R^j \cap U_2 \cap G_0) \geq{}& \frac{r_j - \frac{2\mathrm{diam}(B)}{n}}{r_j + \frac{2\mathrm{diam}(B)}{n}}\mu(R^j \cap U_1 \cap G_0) \\ &- C(\varepsilon_0 + \tau^{1/2}A\kappa^{-1})\mathrm{diam}(B)\Theta(R_0).\end{aligned} \tag{6.18}$$

If we take $n = [\ell(Q)\min(\varepsilon_0^{-1/12}, \tau^{-1/4}]$ (where $[\cdot]$ stands for the integer part), we will have $(r_j - \frac{2\mathrm{diam}(B)}{n})/(r_j + \frac{2\mathrm{diam}(B)}{n}) \geq 1 - \frac{1}{2}\varepsilon_1$, with $\varepsilon_1 \to 0$ as

$\varepsilon_0, \tau \to 0$. Summing (6.18) on j, we get

$$\begin{aligned}\mu(U_2 \cap G_0) &\geq (1 - \tfrac{1}{2}\varepsilon_1)\mu(U_1 \cap G_0) - Cn(\varepsilon_0 + \tau^{1/2} a\kappa^{-1})\mu(B) \\ &\geq (1 - \tfrac{1}{2}\varepsilon_1)\mu(U_1 \cap G_0) - C(\varepsilon_0^{11/12} + \tau^{1/4} A\kappa^{-1})\mu(B).\end{aligned}$$

Since

$$\mu(U_1 \cap G_0) \geq \mu(B \cap G_0) \geq \mu(B) - \mu(Q \setminus G_0) \geq \mu(B) - C\tau^{1/2} A\kappa^{-1}\mu(B),$$

for τ small enough we get $\mu(B_a) \geq \mu(U_2 \cap G_0) \geq (1 - \varepsilon_1)\mu(B)$. So B_a is as needed.

To show the existence of the ball B_b, let z_0' be the point in $B \cap G_0$ which is at maximal distance from B_a. Arguing as above, with z_0' instead of z_0, and with B_a instead of B, one can show that the fact that $\mathcal{C}_*\mu(z_0') \leq \varepsilon_0 \Theta(R_0)$ and the existence of the ball B_a which is far from z_0' implies the existence of another ball B_b satisfying the requirements of the claim, whose center is very close to the point symmetric to the center of B_a with respect to z_0 (with a slightly bigger radius). □

Given a positive measure σ and $x \in 7R_0$, we denote

$$M_\mu \sigma(x) = \sup_{r>0} \frac{\sigma(B(x,r))}{\mu(B(x,r))},$$

and

$$M_\Phi \sigma(x) = \sup_{r:\Phi(x) \leq r \leq \ell(R_0)} \frac{\sigma(B(x,r))}{r}.$$

Using Lemmas 6.3 and 6.10 we obtain the following result.

Lemma 6.11. *For all $x \in 7R_0$ we have*

$$M_\Phi \mu(x) \lesssim \kappa^{-1} A\, \Theta(R_0).$$

Also, there exists some absolute constant C_6 such that if $M_\mu(\chi_{9R_0 \setminus G_0}\mu)(x) \leq \tau^{1/2}$ and $M_\mu(\mu_{at|9R_0})(x) \leq C_6 \kappa$, we have

$$M_\Phi \mu(x) \lesssim A\, \Theta(R_0).$$

Remark 6.12. Let

$$G_1 = \big\{x \in 7R_0 : M_\mu(\chi_{9R_0 \setminus G_0}\mu)(x) \leq \tau^{1/2} \text{ and } M_\mu(\mu_{at|9R_0})(x) \leq C_6\kappa\big\}.$$

The preceding lemma asserts that if $x \in G_1$ we have $M_\Phi \mu(x) \lesssim A\,\Theta(R_0)$, while for $x \notin G_1$ the estimate is worse: $M_\Phi\mu(x) \lesssim \kappa^{-1} A\,\Theta(R_0)$. However, one should keep in mind that $7R_0 \setminus G_1$ is a small set, because

$$\begin{aligned}\mu(7R_0 \setminus G_1) &\leq \mu\Big\{x \in 7R_0 : M_\mu(\chi_{9R_0 \setminus G_0}\mu)(x) > \tau^{1/2}\Big\} \\ &\quad + \mu\Big\{x \in 7R_0 : M_\mu(\mu_{at|9R_0})(x) > C_6\kappa\Big\} \\ &\lesssim \tau^{-1/2}\mu(9R_0 \setminus G_0) + \kappa^{-1}\mu_{at}(9R_0) \lesssim \tau^{1/2}\mu(R_0) + \kappa^{-1}\mu_{at}(9R_0), \qquad (6.19)\end{aligned}$$

since M_μ is bounded from $M(\mathbb{C})$ (the space of finite Radon measures) into $L^{1,\infty}(\mu)$.

The following well known estimate will be used often in next subsection. It can be proved by splitting the domain of integration into annuli, for example.

Lemma 6.13. *For all $s>0$, $f\in L^1_{loc}(\mu)$ and $x\in 7R_0$ we have*
$$\int_{|x-y|\leq\ell(R_0)}\frac{\Phi_\varepsilon(x)^s}{|y-x|^{1+s}+\Phi_\varepsilon(x)^{1+s}}\,f(y)\,d\mu(y)\leq C\,M_\Phi(f\,d\mu)(x),$$
where C depends only on s; and if $0\leq r_1<r_2\leq\ell(R_0)$,
$$\int_{r_1\leq|x-y|\leq r_2}\frac{1}{|y-x|+\Phi_\varepsilon(x)}\,d\mu(y)\lesssim\log\biggl(2+\frac{r_2}{r_1+\Phi(x)}\biggr)\,M_\Phi\mu(x).$$

6.6. **Proof of Lemma 6.4.**

6.6.1. **Proof of** (6.9)**.** We have to estimate the integral $\int_{3R_0}|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu|^2d\mu$. Since Φ_ε is κ-Lipschitz and by Lemma 6.13, for all $x\in 3R_0$ we have
$$\begin{aligned}|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x)\;-\;\widetilde{\mathcal{C}}^\ell_{\Phi_\varepsilon}\mu(x)|&\\
&\leq\int_{|x-y|\leq\ell}\left|\frac{\overline{x-y}}{|x-y|^2+\Phi_\varepsilon(x)\Phi_\varepsilon(y)}-\frac{\overline{x-y}}{|x-y|^2+\Phi_\varepsilon(x)^2}\right|d\mu(y)\\
&\lesssim\int_{|x-y|\leq\ell}\frac{|x-y|\,|\Phi_\varepsilon(x)^2-\Phi_\varepsilon(x)\Phi_\varepsilon(y)|}{\bigl(|x-y|^2+\Phi_\varepsilon(x)^2\bigr)^2}\,d\mu(y)\\
&\lesssim\kappa\int_{|x-y|\leq\ell}\frac{\Phi_\varepsilon(x)}{|x-y|^2+\Phi_\varepsilon(x)^2}\,d\mu(y)\lesssim\kappa\,M_\Phi\mu(x).\end{aligned}$$
Therefore, for $x\in G_0$, since $|\widetilde{\mathcal{C}}^\ell_{\Phi_\varepsilon}\mu(x)|\lesssim\mathcal{C}_*\mu(x)\leq\varepsilon_0\Theta(R_0)$, we get
$$|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x)|\lesssim\varepsilon_0\Theta(R_0)+\kappa\,M_\Phi\mu(x).\tag{6.20}$$
Suppose that $x\notin G_0$. Then, by (b) in Lemma 6.3, there exists some $x'\in G_0$ such that $|x-x'|\lesssim\kappa^{-1}\Phi(x)$. So we have
$$|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x)|\leq|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x)-\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x')|+|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x')|.$$
Since $k_{\Phi_\varepsilon}(x,y)$ is a Calderón-Zygmund kernel satisfying (4.1), it is easy to check that
$$\begin{aligned}|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x)-\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x')|&\leq|\mathcal{C}_{\Phi_\varepsilon}\mu_{|5R_0}(x)-\mathcal{C}_{\Phi_\varepsilon}\mu_{|5R_0}(x')|+C\,\frac{\mu(5R_0)}{\ell(R_0)}\\
&\lesssim\int_{|x-y|\leq10\ell(R_0)}\frac{|x-x'|}{|x-y|^2+\Phi_\varepsilon(x)^2}\,d\mu(y)+\Theta(R_0)\\
&\lesssim\kappa^{-1}M_\Phi(x)+\Theta(R_0)\lesssim\kappa^{-2}A\Theta(R_0).\end{aligned}$$
Also, since x' satisfies (6.20),
$$|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x')|\lesssim\varepsilon_0\Theta(R_0)+A\Theta(R_0)\lesssim A\Theta(R_0).$$

So we get $|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu(x)| \lesssim \kappa^{-2}A\Theta(R_0)$. Therefore,

$$\begin{aligned}\int_{3R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu|^2 d\mu &\lesssim \varepsilon_0^2\,\Theta(R_0)^2\mu(R_0)^2 + \int_{3R_0} \kappa^2 M_\Phi\mu(x)^2\,d\mu(x)\\ &\qquad + \kappa^{-4}A^2\Theta(R_0)^2\mu(3R_0\setminus G_0)\\ &\lesssim (\varepsilon_0^2 + \kappa^{-4}A^2\tau)\Theta(R_0)^2\mu(R_0)^2 + \int_{3R_0}\kappa^2 M_\Phi\mu(x)^2\,d\mu(x). \end{aligned}\tag{6.21}$$

To estimate the integral on the right side we will use Lemma 6.11 and the subsequent remark (recall the definition of G_1 in the remark):

$$\begin{aligned}\int_{3R_0}\kappa^2|M_\Phi\mu|^2 d\mu &= \int_{3R_0\cap G_1} + \int_{3R_0\setminus G_1}\\ &\lesssim A^2\Theta(R_0)^2\big(\kappa^2\mu(R_0) + \tau^{1/2}\mu(R_0) + \kappa^{-1}\mu_{at}(9R_0)\big).\end{aligned}\tag{6.22}$$

Inequality (6.9) follows from (6.21) and (6.22). □

6.6.2. **Proof of** (6.10)**.** Now we have to deal with the integral $\int |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2\,d\mu$. We set

$$\begin{aligned}\int |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2\,d\mu &\le \int_{\mathbb{C}\setminus R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2\,d\mu + \int_{R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2\,d\mu\\ &\le \int_{\mathbb{C}\setminus R_0} |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2 d\mu + 2\int_{R_0}|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu|^2 d\mu + 2\int_{R_0}|\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|\mathbb{C}\setminus R_0}|^2 d\mu\\ &= I_1 + 2I_2 + 2I_3.\end{aligned}$$

Inequality (6.9) yields a good estimate for I_2. Since $\varepsilon_0 \le 1$ and $\tau \ll \kappa$, clearly we get

$$I_2 \le A^2\Theta(R_0)^2\mu(R_0) + C(\kappa)A^2\Theta(R_0)^2\mu_{at}(9R_0).$$

This rough estimate is enough for our purposes.

So we only have to consider I_1 and I_3. To deal with them we will use the assumption of thin boundaries for R_0. First we will estimate I_1. Since $\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}$ vanishes in $\mathbb{C}\setminus 3R_0$, we have

$$I_1 \lesssim \int_{x\in 3R_0\setminus R_0}\left|\int_{y\in R_0}\frac{1}{|y-x| + \Phi_\varepsilon(x)}\,d\mu(x)\right|^2 d\mu(y).$$

By Lemma 6.13, for $x \in 3R_0\setminus R_0$ we obtain

$$\begin{aligned}\int_{y\in R_0}\frac{1}{|y-x|+\Phi_\varepsilon(x)}\,d\mu(y) &\lesssim \log\left(2 + \frac{\ell(6R_0)}{\operatorname{dist}(x,\partial R_0) + \Phi(x)}\right) M_\Phi\mu(x)\\ &\lesssim \log\left(\frac{\ell(6R_0)}{\operatorname{dist}(x,\partial R_0)}\right) M_\Phi\mu(x).\end{aligned}$$

Thus,

$$I_1 \lesssim \int_{x\in 3R_0\setminus R_0}\log^2\left(\frac{\ell(6R_0)}{\operatorname{dist}(x,\partial R_0)}\right) M_\Phi\mu(x)^2\,d\mu(x).$$

We split the last integral as follows:

$$I_1 \lesssim \int_{x\in 3R_0\cap G_1\setminus R_0} \cdots + \int_{x\in 3R_0\setminus(G_1\cup R_0)} \cdots = I_{1,1} + I_{1,2}.$$

According to Lemma 6.11,

$$I_{1,1} \lesssim A^2\Theta(R_0)^2 \int_{x\in 3R_0\setminus R_0} \log^2\left(\frac{\ell(6R_0)}{\operatorname{dist}(x,\partial R_0)}\right) d\mu(x).$$

From the fact that R_0 has thin boundaries and standard estimates, it follows easily that

$$I_{1,1} \lesssim A^2\Theta(R_0)^2\mu(3R_0) \lesssim A^2\Theta(R_0)^2\mu(R_0).$$

To deal with $I_{1,2}$ we will use Lemma 6.11 and Cauchy-Schwarz:

$$\begin{aligned} I_{1,2} &\lesssim \kappa^{-2}A^2\Theta(R_0)^2 \int_{x\in 3R_0\setminus(G_1\cup R_0)} \log^2\left(\frac{\ell(6R_0)}{\operatorname{dist}(x,\partial R_0)}\right) d\mu(x) \\ &\le \kappa^{-2}A^2\Theta(R_0)^2\mu(3R_0\setminus G_1)^{1/2}\left(\int_{x\in 3R_0\setminus R_0} \log^4\left(\frac{\ell(6R_0)}{\operatorname{dist}(x,\partial R_0)}\right) d\mu(x)\right)^{1/2}. \end{aligned}$$

From the fact that R_0 has thin boundaries again, it is easy to check that the last integral is $\lesssim \mu(3R_0) \lesssim \mu(R_0)$. Using also the estimate for $\mu(3R_0\setminus G_1)$ in Remark 6.12 we get

$$\begin{aligned} I_{1,2} &\lesssim \kappa^{-2}A^2\Theta(R_0)^2\mu(R_0)^{1/2}\big(\tau^{1/2}\mu(R_0) + \kappa^{-1}\mu_{at}(9R_0)\big)^{1/2} \\ &\le C(\kappa)A^2\Theta(R_0)^2\big(\tau^{1/4}\mu(R_0) + \mu(R_0)^{1/2}\mu_{at}(9R_0)^{1/2}\big). \end{aligned}$$

So if $\tau \ll \kappa$ we obtain

$$I_1 \lesssim A^2\Theta(R_0)^2\big(\mu(R_0) + C(\kappa)\mu(R_0)^{1/2}\mu_{at}(9R_0)^{1/2}\big). \tag{6.23}$$

Almost the same arguments used for I_1 work for I_3. The estimate obtained for I_1 in (6.23) is also valid for I_3. We leave the details for the reader. So we have

$$\begin{aligned} \int |\mathcal{C}^\ell_{\Phi_\varepsilon}\mu_{|R_0}|^2\,d\mu &\le I_1 + 2I_2 + 2I_3 \\ &\lesssim A^2\Theta(R_0)^2\big(\mu(R_0) + C(\kappa)\mu_{at}(9R_0) + C(\kappa)\mu(R_0)^{1/2}\mu_{at}(9R_0)^{1/2}\big) \\ &\le A^2\Theta(R_0)^2\big(\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big). \end{aligned}$$

□

6.7. **Proof of Lemma 6.5.** Now we have to deal with the terms F_1^-, F_2^-, F_3^- defined in (6.2), (6.3) and (6.4), with Φ_ε instead of Φ. We will estimate F_1^- and, by symmetry, the same arguments used for F_1^- will also work for F_2^- and F_3^-.

Claim. *We have*

$$F_1^- \lesssim \kappa^{1/2}\int_{5R_0} M_\Phi\big((M_\Phi\mu)d\mu\big)\,d\mu. \tag{6.24}$$

Proof. Since $\ell \leq \ell(R_0)$, F_1^- is bounded above by

$$\iiint_{(5R_0)^3\cap S_\ell \cap\{\Phi_\varepsilon(y)>\Phi_\varepsilon(x)\}} \frac{|y-z|^2\Phi_\varepsilon(z)\big(\Phi_\varepsilon(y)-\Phi_\varepsilon(x)\big)}{D_{\Phi_\varepsilon}(x,y,z)} d\mu(x)d\mu(y)d\mu(z).$$

Recall that $S_\ell = \{(x,y,z)\in\mathbb{C}^3 : |x-y|\leq \ell,\ |x-z|\leq\ell,\ |y-z|\leq\ell\}$. From $|y-z|\leq|x-y|+|x-z|$, one gets

$$|y-z|^2 \lesssim |y-z|^{1/2}|x-y|^{3/2}+|x-z|^2.$$

Since Φ_ε is κ-Lipschitz we have

$$\Phi_\varepsilon(y)-\Phi_\varepsilon(x)\leq \kappa^{1/2}\big(\Phi_\varepsilon(y)-\Phi_\varepsilon(x)\big)^{1/2}|y-x|^{1/2}\leq \kappa^{1/2}\Phi_\varepsilon(y)^{1/2}|y-x|^{1/2}.$$

Recall that we are assuming $\Phi_\varepsilon(y)>\Phi_\varepsilon(x)$. So we obtain

$$\begin{aligned}\frac{|y-z|^2\Phi_\varepsilon(z)\big(\Phi_\varepsilon(y)-\Phi_\varepsilon(x)\big)}{D_{\Phi_\varepsilon}(x,y,z)} &\lesssim \frac{\kappa^{1/2}|x-y|^2|y-z|^{1/2}\Phi_\varepsilon(y)^{1/2}\Phi_\varepsilon(z)}{D_{\Phi_\varepsilon}(x,y,z)}\\ &\quad+\frac{\kappa^{1/2}|x-y|^{1/2}|x-z|^2\Phi_\varepsilon(y)^{1/2}\Phi_\varepsilon(z)}{D_{\Phi_\varepsilon}(x,y,z)}.\end{aligned}$$

By the definition of D_{Φ_ε} in (4.2) it is clear that

$$\begin{aligned}D_{\Phi_\varepsilon}(x,y,z)\approx&\big(|x-y|^2+\Phi_\varepsilon(x)^2+\Phi_\varepsilon(y)^2\big)\big(|x-z|^2+\Phi_\varepsilon(x)^2+\Phi_\varepsilon(z)^2\big)\\ &\times\big(|y-z|^2+\Phi_\varepsilon(y)^2+\Phi_\varepsilon(z)^2\big).\end{aligned}$$

Thus

$$\begin{aligned}\frac{|y-z|^2\Phi_\varepsilon(z)\big(\Phi_\varepsilon(y)-\Phi_\varepsilon(x)\big)}{D_{\Phi_\varepsilon}(x,y,z)} &\lesssim \frac{\kappa^{1/2}|y-z|^{1/2}\Phi_\varepsilon(y)^{1/2}\Phi_\varepsilon(z)}{\big(|x-z|^2+\Phi_\varepsilon(z)^2\big)\big(|y-z|^2+\Phi_\varepsilon(y)^2\big)}\\ &\quad+\frac{\kappa^{1/2}|x-y|^{1/2}\Phi_\varepsilon(y)^{1/2}\Phi_\varepsilon(z)}{\big(|x-y|^2+\Phi_\varepsilon(y)^2\big)\big(|y-z|^2+\Phi_\varepsilon(z)^2\big)}.\end{aligned}$$

Therefore,

$$\begin{aligned}F_1^-\lesssim&\iiint_{(5R_0)^3\cap S_\ell}\frac{\kappa^{1/2}\Phi_\varepsilon(y)^{1/2}\Phi_\varepsilon(z)}{\big(|x-z|^2+\Phi_\varepsilon(z)^2\big)\big(|y-z|^{3/2}+\Phi_\varepsilon(y)^{3/2}\big)}d\mu(x)d\mu(y)d\mu(z)\\ &+\iiint_{(5R_0)^3\cap S_\ell}\frac{\kappa^{1/2}\Phi_\varepsilon(y)^{1/2}\Phi_\varepsilon(z)}{\big(|x-y|^{3/2}+\Phi_\varepsilon(y)^{3/2}\big)\big(|y-z|^2+\Phi_\varepsilon(z)^2\big)}d\mu(x)d\mu(y)d\mu(z)\\ &=: J_1+J_2.\end{aligned}$$

First we deal with J_1. Applying Lemma 6.13 twice, we obtain

$$\begin{aligned}J_1&=\iint_{\substack{y,z\in 5R_0\\ |y-z|\leq\ell}}\frac{\kappa^{1/2}\Phi_\varepsilon(y)^{1/2}}{|y-z|^{3/2}+\Phi_\varepsilon(y)^{3/2}}\int_{\substack{x\in 5R_0\\ |x-z|\leq\ell}}\frac{\Phi_\varepsilon(z)}{|x-z|^2+\Phi_\varepsilon(z)^2}d\mu(x)d\mu(y)d\mu(z)\\ &\lesssim\iint_{\substack{y,z\in 5R_0\\ |y-z|\leq\ell}}\frac{\kappa^{1/2}\Phi_\varepsilon(y)^{1/2}}{|y-z|^{3/2}+\Phi_\varepsilon(y)^{3/2}}\,M_\Phi\mu(z)\,d\mu(z)d\mu(y)\\ &\lesssim\kappa^{1/2}\int_{5R_0}M_\Phi\big((M_\Phi\mu)d\mu\big)(y)\,d\mu(y).\end{aligned}$$

The arguments for J_2 are analogous:

$$\begin{aligned}
J_2 &= \iint_{\substack{y,z\in 5R_0\\ |y-z|\leq\ell}} \frac{\kappa^{1/2}\Phi_\varepsilon(z)}{|y-z|^2+\Phi_\varepsilon(z)^2} \int_{\substack{x\in 5R_0\\ |x-y|\leq\ell}} \frac{\Phi_\varepsilon(y)^{1/2}}{|x-y|^{3/2}+\Phi_\varepsilon(y)^{3/2}} d\mu(x)d\mu(y)d\mu(z)\\
&\lesssim \iint_{\substack{y,z\in 5R_0\\ |y-z|\leq\ell}} \frac{\kappa^{1/2}\Phi_\varepsilon(z)}{|y-z|^2+\Phi_\varepsilon(z)^2}\, M_\Phi\mu(y)\, d\mu(y)d\mu(z)\\
&\lesssim \kappa^{1/2}\int_{5R_0} M_\Phi\big((M_\Phi\mu)d\mu\big)(z)\, d\mu(z).
\end{aligned}$$

The claim is proved. □

Analogous estimates hold for F_2^- and F_3^-, and then we get

$$F_1^- + F_2^- + F_3^- \lesssim \kappa^{1/2}\int_{5R_0} M_\Phi\big((M_\Phi\mu)d\mu\big)\, d\mu.$$

To deal with the last integral, for $x\in 5R_0$ we set

$$\begin{aligned}
M_\Phi\big((M_\Phi\mu)d\mu\big)(x) &\leq M_\Phi\big(\chi_{7R_0\cap G_1}(M_\Phi\mu)d\mu\big)(x) + M_\Phi\big(\chi_{7R_0\setminus G_1}(M_\Phi\mu)d\mu\big)(x)\\
&\lesssim A\Theta(R_0)M_\Phi\mu(x) + \kappa^{-1}A\Theta(R_0)\, M_\Phi(\chi_{\mathbb{C}\setminus G_1}d\mu)(x).
\end{aligned}$$

Thus,

$$F_1^- + F_2^- + F_3^- \lesssim \kappa^{1/2}A\Theta(R_0)\int_{5R_0}\big(M_\Phi\mu + \kappa^{-1}M_\Phi(\chi_{\mathbb{C}\setminus G_1}d\mu)\big)\, d\mu.$$

We have

$$\begin{aligned}
\int_{5R_0} M_\Phi\mu\, d\mu &= \int_{5R_0\cap G_1}\cdots + \int_{5R_0\setminus G_1}\cdots\\
&\lesssim A\Theta(R_0)\mu(5R_0) + A\kappa^{-1}\Theta(R_0)\mu(5R_0\setminus G_1).
\end{aligned}$$

To estimate $\int_{5R_0} M_\Phi(\chi_{\mathbb{C}\setminus G_1}d\mu)\, d\mu$, notice that for $x\in 5R_0$ we have

$$\begin{aligned}
M_\Phi(\chi_{\mathbb{C}\setminus G_1}d\mu)(x) &\leq M_\Phi\mu(x) \sup_{r:\Phi(x)\leq r\leq \ell(R_0)} \frac{\mu(B(x,r)\setminus G_1)}{\mu(B(x,r))}\\
&\leq \kappa^{-1}A\Theta(R_0)\, M_\mu(\chi_{7R_0\setminus G_1}d\mu)(x).
\end{aligned}$$

Therefore, since M_μ is bounded in $L^2(\mu)$,

$$\begin{aligned}
\int_{5R_0} M_\Phi(\chi_{\mathbb{C}\setminus G_1}d\mu)\, d\mu &\lesssim \kappa^{-1}A\Theta(R_0)\int_{5R_0} M_\mu(\chi_{7R_0\setminus G_1}d\mu)\, d\mu\\
&\leq \kappa^{-1}A\Theta(R_0)\mu(5R_0)^{1/2}\left(\int_{5R_0} M_\mu(\chi_{7R_0\setminus G_1}d\mu)^2 d\mu\right)^{1/2}\\
&\lesssim \kappa^{-1}A\Theta(R_0)\mu(R_0)^{1/2}\mu(7R_0\setminus G_1)^{1/2}.
\end{aligned}$$

So we obtain

$$\begin{aligned}
F_1^- + F_2^- + F_3^- &\lesssim A^2\Theta(R_0)^2\big(\kappa^{1/2}\mu(R_0) + C(\kappa)\mu(7R_0\setminus G_1)\big)\\
&\lesssim A^2\Theta(R_0)^2\big[\big(\kappa^{1/2} + C(\kappa)\tau^{1/2}\big)\mu(R_0) + C(\kappa)\mu_{at}(9R_0)\big].
\end{aligned}$$

□

6.8. **Proof of Lemma 6.6.** Recall that

$$\begin{aligned} E_2^+ = & \iiint_{S_\ell} \frac{|y-z|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu_{|R_0}(z) \\ & + \iiint_{S_\ell} \frac{|x-z|^2 \Phi_\varepsilon(y)\Phi_\varepsilon(z)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu_{|R_0}(z) \\ & + \iiint_{S_\ell} \frac{|x-y|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu_{|R_0}(z), \end{aligned}$$

with $S_\ell = \{(x,y,z) \in \mathbb{C}^3 : |x-y| \leq \ell,\ |x-z| \leq \ell,\ |y-z| \leq \ell\}$ and

$$D_{\Phi_\varepsilon}(x,y,z) = \big(|x-y|^2 + \Phi_\varepsilon(x)\Phi_\varepsilon(y)\big)\big(|x-z|^2 + \Phi_\varepsilon(x)\Phi(z)\big)\big(|y-z|^2 + \Phi_\varepsilon(y)\Phi_\varepsilon(z)\big).$$

So,

$$E_2^+ \geq \iiint_{S_\ell \cap R_0^3} \frac{|y-z|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu(z).$$

Recall also that $H_0 = \bigcup_{h \in I_{H_0}} B(x_h, 10\mathcal{R}(x_h))$, where the balls $B(x_h, 2\mathcal{R}(x_h))$ are pairwise disjoint. To simplify notation we will write $\mathcal{R}_h := \mathcal{R}(x_h)$. Among the balls $B(x_h, 10\mathcal{R}_h)$ which intersect $\frac{1}{2}R_0$, take a maximal subfamily $\{B(x_h, 10\mathcal{R}_h)\}_{h \in I_M} \subset \{B(x_h, 10\mathcal{R}_h)\}_{h \in I_{H_0}}$ with the property that, for $h \in I_M$, $B(x_h, 20\mathcal{R}_h)$ is not contained in any ball $B(x_k, 20\mathcal{R}_k)$, $k \in I_M$, $k \neq h$. It follows easily that

$$H_0 \cap \frac{1}{2}R_0 \subset \bigcup_{h \in I_M} B(x_h, 20\mathcal{R}_h).$$

If A is chosen big enough, $\mathcal{R}_h \leq \ell(R_0)/10$, and so $B(x_h, 2\mathcal{R}_h) \subset R_0$ for $h \in I_M$. Therefore,

$$E_2^+ \geq \sum_{h \in I_M} \iiint_{B(x_h, 2\mathcal{R}_h)^3} \frac{|y-z|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu(z),$$

For each $h \in I_M$ we will show below that if $B(x_h, 2\mathcal{R}_h)$ does not contain any atom z_h such that $\mu(z_h) \gtrsim \kappa\mu(B(x_h, \mathcal{R}_h))$ (in this case we set $h \in \widetilde{I}_M$), and $\varepsilon \leq \kappa\mathcal{R}_h$, then

$$\begin{aligned} \text{(6.25)} \quad & \iiint_{B(x_h, 2\mathcal{R}_h)^3} \frac{|y-z|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu(z) \\ & \qquad \gtrsim A^2 \Theta(R_0)^2 \mu\big(B(x_h, 20\mathcal{R}_h) \setminus BC_0\big). \end{aligned}$$

This implies

$$\text{(6.26)} \qquad E_2^+ \gtrsim A^2 \Theta(R_0)^2 \Big(\mu(H_0 \cap \tfrac{1}{2}R_0 \setminus BC_0) - C\kappa^{-1}\mu_{at}(R_0)\Big),$$

for ε small enough. Indeed, from (6.25) we deduce

$$\begin{aligned} E_2^+ &\gtrsim A^2\Theta(R_0)^2 \sum_{h\in\widetilde{I}_M} \mu\big(B(x_h, 20\mathcal{R}_h)\setminus BC_0\big) \\ &\geq A^2\Theta(R_0)^2 \Big(\mu(H_0\cap \tfrac{1}{2}R_0\setminus BC_0) - C\sum_{h\in I_M\setminus\widetilde{I}_M} \mu\big(B(x_h,\mathcal{R}_h)\big)\Big) \\ &\gtrsim A^2\Theta(R_0)^2 \Big(\mu(H_0\cap \tfrac{1}{2}R_0\setminus BC_0) - C\kappa^{-1}\sum_{h\in I_M\setminus\widetilde{I}_M} \mu_{at}\big(B(x_h,2\mathcal{R}_h)\big)\Big). \end{aligned}$$

which gives (6.26), taking into account that the balls $B(x_h, 2\mathcal{R}_h)$ are pairwise disjoint.

To prove (6.25) we will need the following auxiliary result.

Lemma 6.14. *For $h\in\widetilde{I}_M$, suppose that $B(x_h, 20\mathcal{R}_h)\cap \operatorname{supp}(\mu)\not\subset BC_0$. If $z\in B(x_h,\mathcal{R}_h)$, then $\Phi(z)\approx\kappa\mathcal{R}_h$.*

Notice that $h\in\widetilde{I}_M$ implies that $B(x_h, 2\mathcal{R}_h)$ does not contain any atom z_h such that $\mu(z_h)\gtrsim\kappa\mu(B(x_h,\mathcal{R}_h))$. Recall also that the constants involved in the relationship $\approx$ are absolute constants, independent of A, ε_0, etc.

Proof of Lemma 6.14. Recall that $\Phi(z) = \max\Big(\Phi_{BC}(z),\, \kappa\operatorname{dist}(z,\mathbb{C}\setminus H_0)\Big)$. Thus, if $z\in B(x_h,\mathcal{R}_h)$, we have

$$\Phi(z)\geq\kappa\operatorname{dist}(z,\mathbb{C}\setminus H_0)\geq\kappa\operatorname{dist}\big(z,\mathbb{C}\setminus B(x_h,10\mathcal{R}_h)\big)\geq\kappa\mathcal{R}_h.$$

Let us see that $\Phi(z)\lesssim\kappa\mathcal{R}_h$ now. Take $z_1\in B(x_h,20\mathcal{R}_h)\cap\operatorname{supp}(\mu)\setminus BC_0$. For $z\in B(x_h,\mathcal{R}_h)$, from the definition of Φ_{BC} in (6.5), we deduce that $\Phi_{BC}(z)\leq\kappa|z-z_1|\leq 40\kappa\mathcal{R}_h$.

So we only have to check that

$$\operatorname{dist}(z,\mathbb{C}\setminus H_0)\lesssim\mathcal{R}_h. \tag{6.27}$$

Denote $B_h := B(x_h,\mathcal{R}_h)$, and consider the square $Q_1 := Q(x_h, 200\mathcal{R}_h)$. From Lemma 6.9 we deduce that there exists a line L_{Q_1} such that any $z\in G_0\cap Q_1$ fulfills

$$\operatorname{dist}(z, L_{Q_1})\lesssim\varepsilon_0^{1/12}\ell(Q_1).$$

That is, $G_0\cap Q_1$ is very close to L_{Q_1}. Notice that the assumptions in Lemma 6.9 hold because of the existence of $z_1\in B(x_h,10\mathcal{R}_h)\cap\operatorname{supp}(\mu)\setminus BC_0$, $\Theta(Q)\gtrsim\Theta(R_0)$, and $h\in\widetilde{I}_M$. Let $B\subset\frac{1}{2}Q_1$ be a ball with radius $\mathcal{R}_h$ such that $\operatorname{dist}(B,L_{Q_1})\geq 50\mathcal{R}_h$ and $\operatorname{dist}(B,B_h)\geq 50\mathcal{R}_h$. We will show that $B\not\subset H_0$. Clearly this implies (6.27).

Let $B_j := B(x_j,\mathcal{R}_j)$, $j\in I_{H_0}$, be such that $10B_j\cap B\neq\varnothing$. Let us check that $\mathcal{R}_j\leq 17\mathcal{R}_h$. Suppose that this fails. This implies $20B_h\subset 20B_j$ (which is a contradiction because of the definition of I_M) since if $z\in 20B_h$ and we denote by x_B the center of B, we have

$$\begin{aligned}|z - x_j| &\leq |x - x_h| + |x_h - x_B| + |x_B - x_j| \\ &\leq 20\mathcal{R}_h + 100 \cdot 2^{1/2}\mathcal{R}_h + \mathcal{R}_h + 10\mathcal{R}_j \\ &\leq 170\mathcal{R}_h + 10\mathcal{R}_j \leq 20\mathcal{R}_j.\end{aligned}$$

Since $\mathcal{R}_j \leq 17\mathcal{R}_h$, $B_j \cap B \neq \varnothing$, and $\operatorname{dist}(B, L_{Q_1}) \geq 50\mathcal{R}_h$, we infer

$$\operatorname{dist}(B_j, L_{Q_1}) \geq \operatorname{dist}(B, L_{Q_1}) - 2\mathcal{R}_j \geq 50\mathcal{R}_h - 34\mathcal{R}_h = 16\mathcal{R}_h.$$

That is, B_j is far from L_{Q_1}. Moreover, it is straightforward to check that $B_j \subset Q_1$ (recall that $B \subset \frac{1}{2}Q_1$). Thus,

$$B_j \subset Q_1 \setminus G_0 \subset Q(z_1, 300\mathcal{R}_h) \setminus G_0,$$

where $z_1 \in B(x_h, 10\mathcal{R}_h) \setminus BC_0$ was introduced above. Therefore,

$$\mu(B_j) \leq \mu\big(Q(z_1, 300\mathcal{R}_h) \setminus G_0\big) \lesssim \tau^{1/2}\mu(B_h). \tag{6.28}$$

Since $\Theta(B_j) \approx \Theta(B_h) \approx A\Theta(R_0)$, we derive $\mathcal{R}_j \lesssim \tau^{1/2}\mathcal{R}_h$.

We have proved that any ball B_j, $j \in I_{H_0}$, such that $10B_j \cap B \neq \varnothing$ must be very small. We will deduce now that if $B \subset H_0 = \bigcup_{j\in I_{H_0}} 10B_j$, then $\mu(B)$ must be large, and then we will obtain a contradiction. The detailed arguments are the following. Since the balls $2B_j$ are disjoint and $\mu(B_j) \approx \mu(10B_j)$, we get

$$\begin{aligned}\mu(B) &\gtrsim \sum_{j\in I_{H_0}: B_j\cap\frac{1}{2}B\neq\varnothing} \mu(B_j) \gtrsim A\Theta(R_0) \sum_{j\in I_{H_0}: B_j\cap\frac{1}{2}B\neq\varnothing} \mathcal{R}_j \\ &\gtrsim A\Theta(R_0) \sum_{j\in I_{H_0}: B_j\cap\frac{1}{2}B\neq\varnothing} \frac{\mathcal{R}_j^2}{\tau^{1/2}\mathcal{R}_h} \gtrsim A\Theta(R_0)\frac{\mathcal{R}_h^2}{\tau^{1/2}\mathcal{R}_h},\end{aligned}$$

because $\mathcal{L}^2(\frac{1}{2}B_h) \leq \mathcal{L}^2\big(\bigcup_{j\in I_{H_0}: B_j\cap\frac{1}{2}B\neq\varnothing} B_j\big)$, where $\mathcal{L}^2$ stands for the Lebesgue measure on $\mathbb{R}^2$. Thus,

$$\mu(B) \gtrsim A\Theta(R_0)\tau^{-1/2}\mathcal{R}_h \approx \tau^{-1/2}\mu(B_h).$$

That is, $\mu(B)$ is very large, as claimed above. However, in (6.28) we saw that $\mu\big(Q(z_1, 300\mathcal{R}_h) \setminus G_0\big) \lesssim \tau^{1/2}\mu(B_h) \ll \mu(B_h)$. This is a contradiction because $B \subset Q(z_1, 300\mathcal{R}_h) \setminus G_0$. □

6.8.1. **Proof of** (6.25)**.** Given $\varepsilon \leq \kappa\mathcal{R}_h$, we only have to prove (6.25) for the indices $h \in I_M$ such that $B(x_h, 20\mathcal{R}_h) \cap \operatorname{supp}(\mu) \setminus BC_0 \neq \varnothing$. Consider such an h. We denote $B_h := B(x_h, \mathcal{R}_h)$. By the preceding lemma, $\Phi(z) \approx \kappa\mathcal{R}_h$ for all $z \in B_h$. By the Besicovitch covering theorem, $G_0 \cap B_h$ can be covered by a family of squares Q_j^h centered at points in $G_0 \cap B_h$ with side length $\kappa\mathcal{R}_h$ so that this family can be split into N_1 (where N_1 is an absolute constant) subfamilies of pairwise disjoint squares. We infer that there exists a pairwise disjoint subfamily of squares $\{Q_j^h\}_{j\in I_h}$ such that

$$\mu(B_h) \leq N_1 \mu\Big(\bigcup_{j\in I_h} Q_j^h \cap B_h\Big).$$

By Lemma 6.10, each Q_j^h, $j \in I_h$, satisfies $\Theta(Q_j^h) \lesssim \Theta(B(x_h, \mathcal{R}_h))$. Let us see that the opposite inequality is also true for "many" of the squares Q_j^h, $j \in I_h$, too. To this end, take $z_1 \in B(x_h, 20\mathcal{R}_h) \cap \operatorname{supp}(\mu) \setminus BC_0$. By definition the square centered at z_1 and side length $40\mathcal{R}_h$ [which contains $B(x_h, 20\mathcal{R}_h)$] does not belong to BC. Then, by Lemma 6.9 there exists some line L_h such that any $z \in B_h$ satisfies $\operatorname{dist}(z, L_h) \lesssim \varepsilon_0^{1/12}\mathcal{R}_h \ll \kappa\mathcal{R}_h$. This implies that all the squares Q_j^h, $j \in I_h$, intersect L_h, and then one easily deduces that $\sum_{j\in I_h} \ell(Q_j^h) \leq 10\mathcal{R}_h$, say. Let us denote

$$I_L = \{h \in I_h : \Theta(Q_j^h) \leq C_7\Theta(B_h)\},$$

where C_7 is some small positive absolute to be fixed below. Then we have

$$\begin{aligned}
\mu\Big(\bigcup_{j\in I_L} Q_j^h \cap B_h\Big) &\leq \sum_{h\in I_L} \mu(Q_j^h) \leq C_7\,\Theta(B_h) \sum_{h\in I_L} \ell(Q_j^h) \\
&\leq C_7\,\Theta(B_h)\, 10\mathcal{R}_h = \frac{3}{2} C_7\, \mu(B_h) \\
&\leq \frac{1}{2}\mu\Big(\bigcup_{j\in I_h} Q_j^h \cap B_h\Big),
\end{aligned}$$

if C_7 is small enough. So,

$$\mu\Big(\bigcup_{j\in I_L\setminus I_h} Q_j^h \cap B_h\Big) \geq \frac{1}{2N_1}\,\mu(B_h). \tag{6.29}$$

In other words, the family of squares Q_j^h, $j \in I_h \setminus I_L$, contains a big part of $\mu(B_h)$. These squares satisfy $C_7\Theta(B_h) \leq \Theta(Q_j^h) \leq C\Theta(B_h)$. Observe also that $5Q_j^h \subset 2B_h$ (assuming κ small enough).

We have

$$\iiint_{(2B_h)^3} \frac{|y-z|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)}\, d\mu(x)d\mu(y)d\mu(z) \geq \sum_{j\in I_h\setminus I_L} \iiint_{\substack{x\in Q_j^h\\ y,z\in 5Q_j^h}} \cdots .$$

Since $\varepsilon \leq \kappa\mathcal{R}_h$ and $\Phi(x) \approx \Phi(y) \approx \kappa\mathcal{R}_h = \ell(Q_j^h)$, for each $j \in I_h \setminus I_L$ the latter integral can be estimated as follows:

$$\iiint_{\substack{x\in Q_j^h\\ y,z\in 5Q_j^h}} \cdots \gtrsim \iiint_{\substack{x\in Q_j^h\\ y,z\in 5Q_j^h}} \frac{|y-z|^2 \ell(Q_j^h)^2}{\ell(Q_j^h)^6}\, d\mu(x)d\mu(y)d\mu(z).$$

Recall now that, by Lemma 6.7 there exist two squares $P_1, P_2 \subset 5Q_j^h$ with $\operatorname{dist}(P_1, P_2) \gtrsim \ell(Q_j^h)$ such that $\mu(P_i) \gtrsim \mu(Q_j^h)$, for $i = 1, 2$. Therefore,

$$\iiint_{\substack{x\in Q_j^h\\ y,z\in 5Q_j^h}} \frac{|y-z|^2 \ell(Q_j^h)^2}{\ell(Q_j^h)^6}\, d\mu(x)d\mu(y)d\mu(z) \gtrsim \frac{\mu(Q_j^h)^3}{\ell(Q_j^h)^2}.$$

Since $\Theta(Q_j^h) \approx \Theta(B_h) \approx A\Theta(R_0)$ for $j \in I_h \setminus I_L$, by (6.29) we get

$$\begin{aligned}\iiint_{(2B_h)^3} \frac{|y-z|^2 \Phi_\varepsilon(x)\Phi_\varepsilon(y)}{D_{\Phi_\varepsilon}(x,y,z)} d\mu(x)d\mu(y)d\mu(z) &\gtrsim A^2\Theta(R_0)^2 \sum_{j\in I_h\setminus I_L} \mu(Q_j^h)\\ &\gtrsim A^2\Theta(R_0)^2\mu(B_h) \gtrsim A^2\Theta(R_0)^2\mu(20B_h),\end{aligned}$$

which yields (6.25). □

7. Proof of Theorem 1.1

Recall that $E = \{x \in \operatorname{supp}(\mu) : \mathcal{C}_*\mu(x) < \infty\}$. Let $E_1 = \{x \in E : \Theta_\mu^*(x) < \infty\}$, and $F = E \setminus E_1$. Arguing as in [To2] one can easily show that $\mu_{|E_1}$ can be written as in the statement of the theorem. So we only have to show that $\mu_{|F} = \mu_{at}$.

Let

$$F_m = \{x \in E : \Theta_\mu^*(x) = \infty,\, \mathcal{C}_*\mu(x) \le m,\, \mu(\{x\}) = 0\}.$$

So F equals the union of the sets F_m plus a countable collection of atoms and a set of μ-measure 0. We have to show that $\mu(F_m) = 0$ for all m.

Suppose that $\mu(F_m) > 0$ and let $x \in F_m$ be a density point of F_m. Since $\Theta_\mu^*(x) = \infty$, there exists a sequence of radii $\{r_n\}_n$ such that $\{\Theta_\mu(B(x, r_n))\}_n$ is an increasing sequence with

$$\Theta_\mu(B(x,r)) \le 2\,\Theta_\mu(B(x, \tfrac{1}{2}r_n)) \quad \text{if } r > r_n. \tag{7.1}$$

By Lemma 2.2 there exists some $r_n' \in [r_n, 1.1r_n]$ such that the ball $B(x, r_n')$ has thin boundaries. It is straightforward to check that for n big enough, $B(x, r_n')$ satisfies the assumptions of Main Lemma 3.1 (of course, the lemma also holds for balls instead of squares). Moreover, for n big enough the option (a) of the Main Lemma cannot hold because

$$\frac{\mu_{at}(B(x, 9r_n'))}{\mu(B(x, r_n'))} \le C\,\frac{\mu(B(x, 9r_n') \setminus F_m)}{\mu(B(x, 9r_n'))} \to 0 \qquad \text{as } n \to \infty$$

(recall that $B(x, r_n)$ is doubling because of (7.1), and F_m contains no atoms). As a consequence, for n big enough, in each ball $B(x, r_n')$, there is a subset $G \subset B(x, r_n')$ with $\mu(G) \ge \eta\,\mu(B(x, r_n'))$ where $\mu_{|G}$ has linear growth. In particular, $\Theta_\mu^*(y) < \infty$ at μ-a.e. $y \in G$, and so $\mu(B(x, r_n') \setminus F_m) \ge \eta\,\mu(B(x, r_n'))$, which contradicts the fact that x is a density point of F_m.

8. Proof of Theorem 1.3

Inspecting the proof of Main Lemma 3.1 it is easy to check that the assumption

$$\mu(10R_0 \setminus G_0) = \mu\{x \in 10R_0 : \mathcal{C}_*\mu(x) > \varepsilon_0\Theta_\mu(R_0)\} \leq \tau\mu(R_0)$$

can be relaxed. So it is not necessary to ask $|\mathcal{C}_\varepsilon\mu(x)| \leq \varepsilon_0\Theta(R_0)$ to hold for *all* $\varepsilon > 0$, for $x \in G_0$. Instead, it is enough to ask

$$|\mathcal{C}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon^\ell\mu(x)| \leq \varepsilon_0\Theta_\mu(R_0)$$

for all ε, ℓ such that $0 < \varepsilon \leq \ell \leq \varepsilon_0^{-1}\ell(R_0)$ with

$$\sup_{\eta\geq\varepsilon} \Theta_\mu(B(x,\eta)) \lesssim \kappa^{-1}A\Theta_\mu(R_0).$$

Let us remark that the estimates on $\widetilde{\mathcal{C}}_\varepsilon\mu$ were used in the proof on Lemma 6.9, while the ones for $\widetilde{\mathcal{C}}_\varepsilon^\ell\mu$ were necessary in the proof of Lemma 6.4 (with $\Phi_\varepsilon(x)$ instead of ε).

As a consequence, the following variant of the Main Lemma holds. We leave for the reader to check the precise details.

Lemma 8.1. *Let μ be a finite Radon measure on $\mathbb{C}$. Let R_0 be a square with thin boundaries such that $\Theta_\mu(\lambda R_0) \leq 100\Theta_\mu(\frac{1}{2}R_0)$ for all $\lambda > 1$. Let ε_0 and τ be two (small) positive constants and $A' > 1$ big enough. Let G_0 be the set of points in $10R_0$ satisfying $|\mathcal{C}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon^\ell\mu(x)| \leq \varepsilon_0\Theta_\mu(R_0)$ for all ε, ℓ such that $0 < \varepsilon \leq \ell \leq \varepsilon_0^{-1}\ell(R_0)$ with $\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta)) \leq A'\Theta_\mu(R_0)$. Suppose that $\mu(10R_0 \setminus G_0) \leq \tau\mu(R_0)$ and that ε_0 and τ are small enough. Then there exists some positive constant η such that one of the following holds:*

(a) *either $\mu_{at}(9R_0) \geq \eta\,\mu(R_0)$, or*
(b) *there exists $E \subset R_0$ with $\mu(E) \geq \eta\,\mu(R_0)$ such that $\mu_{|E}$ has linear growth and $c^2(\mu_{|E}) < \infty$.*

To see how Theorem 1.3 follows from this lemma, we need the following auxiliary result.

Lemma 8.2. *Let $x \in \mathbb{C}$ be such that $\mathcal{C}_\varepsilon\mu(x) = o(\Theta_\mu(B(x,\varepsilon)))$ as $\varepsilon \to 0$. Then we have*

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon^\ell\mu(x)| = o\Big(\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))\Big) \quad \text{as } \varepsilon, \ell \to 0.$$

Proof. Let us denote

$$M = \sup_{\varepsilon>0} \frac{|\mathcal{C}_\varepsilon\mu(x)|}{\Theta_\mu(B(x,\varepsilon))}.$$

Let us check that $|\widetilde{\mathcal{C}}_\varepsilon\mu(x)| = o\big(\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))\big)$ as $\varepsilon \to 0$. It is easy to check that

$$\widetilde{\mathcal{C}}_\varepsilon\mu(x) = \int_0^\infty \mathcal{C}_t\mu(x)\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt.$$

Therefore,

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x)| \leq \int_0^\infty |\mathcal{C}_t\mu(x)|\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt = \int_0^\varepsilon + \int_\varepsilon^{\varepsilon^{1/2}} + \int_{\varepsilon^{1/2}}^\infty = I_1 + I_2 + I_3.$$

First we deal with I_2:

$$\begin{aligned}\frac{I_2}{\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))} &\leq \int_\varepsilon^{\varepsilon^{1/2}} \frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,t))}\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt\\ &\leq \sup_{t\in[\varepsilon,\varepsilon^{1/2}]} \frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,t))} \to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

Now we consider I_3:

$$\begin{aligned}\frac{I_3}{\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))} &\leq \int_{\varepsilon^{1/2}}^\infty \frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,t))}\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt \leq M\int_{\varepsilon^{1/2}}^\infty \frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt\\ &= M\frac{\varepsilon^2}{\varepsilon+\varepsilon^2} \to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

For I_1 we take into account that if $t\leq\varepsilon$, we have $\Theta_\mu(B(x,t))\leq \varepsilon t^{-1}\Theta_\mu(B(x,\varepsilon))$. Then,

$$\begin{aligned}\frac{I_1}{\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))} &\leq \int_0^\varepsilon \frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,\varepsilon))}\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt\\ &\leq \int_0^\varepsilon \frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,t))}\,\frac{2\varepsilon^3}{(t^2+\varepsilon^2)^2}\,dt\\ &\leq \int_0^\varepsilon \frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,t))}\,\frac{2}{\varepsilon}\,dt \leq 2\sup_{t\in[0,\varepsilon]}\frac{|\mathcal{C}_t\mu(x)|}{\Theta_\mu(B(x,t))} \to 0,\end{aligned}$$

as $\varepsilon\to 0$. Therefore, $\widetilde{\mathcal{C}}_\varepsilon\mu(x) = o\big(\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))\big)$.

For $|\widetilde{\mathcal{C}}^\ell_\varepsilon\mu(x)|$ we use similar arguments. We have

$$\widetilde{\mathcal{C}}^\ell_\varepsilon\mu(x) = \int_0^\ell \mathcal{C}^\ell_t\mu(x)\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt,$$

where $\mathcal{C}^\ell_t\mu(x) = \int_{t<|x-y|\leq\ell}(x-y)^{-1}d\mu(y)$. So

$$|\widetilde{\mathcal{C}}_\varepsilon\mu(x)| \leq \int_0^\ell |\mathcal{C}^\ell_t\mu(x)|\,\frac{2\varepsilon^2 t}{(t^2+\varepsilon^2)^2}\,dt = \int_0^\varepsilon + \int_\varepsilon^\ell = J_1 + J_2.$$

Arguing as in the case of I_1, I_2, one can check that

$$\frac{J_1+J_2}{\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))} \to 0 \text{ as } \varepsilon,\ell\to 0.$$

The details are left for the reader. □

Now we are ready to prove Theorem 1.3. The arguments are very similar to the ones in Section 7. Set $E = \{x\in\operatorname{supp}(\mu) : \mathcal{C}_\varepsilon\mu(x) = o(\Theta_\mu(B(x,\varepsilon))\}$, $E_1 = \{x\in E : \Theta^*_\mu(x) < \infty\}$, and $F = E\setminus E_1$. Arguing as in [To2] it follows that $\mu_{|E_1}$ can be written as in the statement of the theorem, because $\mathcal{C}_*\mu(x) + \Theta^*_\mu(x) < \infty$ on E_1.

It only remains to show that $\mu_{|F} = \mu_{at}$. Given an arbitrary $0 < \delta < 1$, let $F_\delta \subset F$ be such that $\mu(F \setminus F_\delta) \leq \delta$ and

$$\frac{|\mathcal{C}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}^\ell_\varepsilon\mu(x)|}{\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta))} \to 0 \quad \text{as } \varepsilon, \ell \to 0$$

uniformly for $x \in F_\delta$. It is enough to show that $\mu_{|F_\delta}$ is atomic (for any $\delta > 0$). To this end we set

$$F_0 = \{x \in F_\delta :\ \mu(\{x\}) = 0\}.$$

Suppose that $\mu(F_0) > 0$ and let $x \in F_0$ be a density point of F_0. Since $\Theta^*_\mu(x) = \infty$, there exists a sequence of radii $\{r_n\}_n$ such that $\{\Theta_\mu(B(x, r_n))\}_n$ is an increasing sequence with

$$\Theta_\mu(B(x,r)) \leq 2\,\Theta_\mu(B(x, \tfrac{1}{2}r_n)) \quad \text{if } r > r_n. \tag{8.1}$$

By Lemma 2.2 there exists some $r'_n \in [r_n, 1.1r_n]$ such that the ball $B(x, r'_n)$ has thin boundaries.

Let us check that $R_0 := B(x, r'_n)$ satisfies the assumptions of Lemma 8.1 for n big enough. Suppose that r'_n is small enough so that

$$|\mathcal{C}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}_\varepsilon\mu(x)| + |\widetilde{\mathcal{C}}^\ell_\varepsilon\mu(x)| \leq \varepsilon_0 A'^{-1} \sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta)) \tag{8.2}$$

for all $x \in F_\delta$ and $\varepsilon \leq \ell \leq \varepsilon_0^{-1} r'_n$. Notice that this condition implies that if $x \in F_\delta$ satisfies $\sup_{\eta\geq\varepsilon}\Theta_\mu(B(x,\eta)) \leq A'\Theta_\mu(R_0)$, then the left hand side of (8.2) is bounded above by $\varepsilon_0\Theta_\mu(R_0)$. Thus, $F_\delta \cap 10R_0 \subset G_0$.

On the other hand, since x is a density point of F_δ, if r'_n is small enough we will have $\mu(10R_0 \setminus F_\delta) \leq \tau\mu(R_0)$. Therefore, the assumptions in Lemma 8.1 hold.

Moreover, for n big enough the option (a) of the lemma cannot hold because

$$\frac{\mu_{at}(B(x, 9r'_n))}{\mu(B(x, r'_n))} \leq C\,\frac{\mu(B(x, 9r'_n) \setminus F_0)}{\mu(B(x, 9r'_n))} \to 0 \quad \text{as } n \to \infty$$

(recall that $B(x, r'_n)$ is doubling and F_0 contains no atoms). As a consequence, for n big enough, in each ball $B(x, r'_n)$, there is a subset $G \subset B(x, r'_n)$ with $\mu(G) \geq \eta\,\mu(B(x, r'_n))$ where $\mu_{|G}$ has linear growth. In particular, $\Theta^*_\mu(y) < \infty$ at μ-a.e. $y \in G$, and so $\mu(B(x, r'_n) \setminus F_0) \geq \eta\,\mu(B(x, r'_n))$, which contradicts the fact that x is a density point of F_0. Thus $\mu(F_0) = 0$ and $\mu_{|F_\delta}$ is atomic.

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