RIESZ TRANSFORMS AND HARMONIC LIP$_1$−CAPACITY IN CANTOR SETS

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Abstract. We estimate the L^2 norm of the $s-$dimensional Riesz transforms on some Cantor sets in $\mathbb{R}^d$. To this end, we show that the Riesz transforms truncated at different scales behave in quasiorthogonal way. As an application, we obtain some precise numerical estimates for the Lipschitz harmonic capacity of these sets.

1. Introduction

In this paper we estimate the L^2 norm of the $s-$dimensional Riesz transforms on some Cantor sets in $\mathbb{R}^d$ with respect to the natural probability measure associated to these Cantor sets. As an application, we obtain precise estimates of their Lipschitz harmonic capacity κ. We also show that the capacity κ of these Cantor type sets is comparable to their capacity κ_+, which is originated by $(d-1)-$dimensional Riesz transforms of positive measures. These results are the natural generalization to higher dimensions of the estimates obtained for analytic capacity of Cantor sets in [MTV]. The main new difficulty to overcome in the present situation is the absence of a notion similar to curvature of measures, which plays an important role in [MTV].

For general compact sets, in [To] it has been proved that analytic capacity and positive analytic capacity are comparable. Recently, Volberg [Vo] has shown that the capacities κ and κ_+ are also comparable. So the comparability of κ and κ_+ for Cantor sets mentioned above is only a particular case of Volberg's result. However, the estimates of Lipschitz harmonic capacity for Cantor sets that we obtain in the present paper do not follow from Volberg's theorem. Let us also mention that in [MPV], for $0 < s < 1$, capacities associated to $s-$dimensional Riesz transforms are studied and it is proved that they are comparable to their corresponding positive versions.

To state our results in detail we need some definitions. For $0 < s < d$, the $s-$dimensional Riesz kernel is the vectorial kernel

$$K^s(x) = \frac{x}{|x|^{s+1}}, \quad x \in \mathbb{R}^d,\, x \neq 0,$$

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and the $s-$dimensional Riesz transform (or $s-$Riesz transform) of a real Radon measure ν with compact support is

$$R^s\nu(x) = \int K^s(y-x)\,d\nu(y), \qquad x \notin \operatorname{supp}(\nu).$$

For $x \in \operatorname{supp}(\nu)$, the integral above is not absolutely convergent in general. This is the reason why one considers the truncated $s-$Riesz transform of ν, which is defined as

$$R^s_\varepsilon\nu(x) = \int_{|y-x|>\varepsilon} K^s(y-x)\,d\nu(y), \qquad x \in \mathbb{R}^d,\ \varepsilon > 0.$$

These definitions also make sense if we consider distributions instead of measures. Given a compactly supported distribution T, we set

$$R^s(T) = K^s * T$$

(in the principal value sense for $s = d$), and analogously

$$R^s_\varepsilon(T) = K^s_\varepsilon * T,$$

where $K^s_\varepsilon(x) = \chi_{|x|>\varepsilon}\, x/|x|^{s+1}$. Given a positive Radon measure with compact support and a function $f \in L^1(\mu)$, we consider the operators $R^s_\mu(f) := R^s(f\,d\mu)$ and $R^s_{\mu,\varepsilon}(f) := R^s_\varepsilon(f\,d\mu)$. We say that R^s_μ is bounded on $L^2(\mu)$ if $R^s_{\mu,\varepsilon}$ is bounded on $L^2(\mu)$ uniformly in $\varepsilon > 0$, and we set

$$\|R^s_\mu\|_{L^2(\mu)} = \sup_{\varepsilon>0} \|R^s_{\mu,\varepsilon}\|_{L^2(\mu)}.$$

If the measure μ is fixed, to simplify notation, we will write also $R^s f$ instead of $R^s_\mu f$, and $\|R^s\|_{L^2(\mu)}$ instead of $\|R^s_\mu\|_{L^2(\mu)}$.

Given a compact set $E \subset \mathbb{R}^d$, the Lipschitz harmonic capacity of E is

$$\kappa(E) = \sup |\langle T, 1\rangle|, \tag{1.1}$$

where the supremum is taken over all distributions T supported on E such that $\|R^{d-1}(T)\|_{L^\infty(\mathbb{R}^d)} \leq 1$. The capacity κ was introduced by Paramonov [Pa] in order to study problems of $\mathcal{C}^1$ approximation by harmonic functions in $\mathbb{R}^d$ (the reader should notice that κ is called κ' in [Pa]). For $d = 2$, κ coincides with the real version of analytic capacity, γ_{Re}, which is originated by real distributions. Some properties of κ have been studied in [MP1] and [MP2].

If in the supremum in (1.1) we only consider distributions T given by positive Radon measures supported on E, we obtain the capacity κ_+. For $d = 2$, κ_+ coincides with the positive analytic capacity γ_+.

Now we turn our attention to Cantor sets. Given a sequence $\lambda = (\lambda_n)_{n=1}^\infty$, $0 \leq \lambda_n \leq 1/2$, we construct a Cantor set by the following algorithm. Consider the unit cube $Q^0 = [0,1]^d$. At the first step we take 2^d closed cubes inside Q^0, of side-length λ_1, with sides parallel to the coordinate axes, such that each cube contains a vertex of Q^0. At step 2 we apply the preceding procedure to each of the 2^d cubes produced at step 1, but now using the proportion factor λ_2. Then we obtain 2^{2d} cubes of side length $\sigma_2 = \lambda_1\lambda_2$.

Proceeding inductively, we have at the $n-$th step 2nd cubes Q_j^n, $1 \leq j \leq 2^{nd}$, of side length $\sigma_n = \prod_{j=1}^n \lambda_j$. We consider

$$E_n = E(\lambda_1, \dots, \lambda_n) = \bigcup_{j=1}^{2^{nd}} Q_j^n,$$

and we define the Cantor set associated to $\lambda = (\lambda_n)_{n=1}^\infty$ as

$$E = E(\lambda) = \bigcap_{n=1}^\infty E_n.$$

For example, if $\lim_{n\to\infty} \sigma_n / 2^{-nd/s} = 1$, then the Hausdorff dimension of $E(\lambda)$ is s. If moreover $\sigma_n = 2^{-nd/s}$ for each n, then $0 < H^s(E(\lambda)) < \infty$, where H^s stands for the $s-$dimensional Hausdorff measure.

Given a fixed Cantor set $E(\lambda)$, we denote by p the uniform probability measure on $E(\lambda)$, so that $p(Q_j^n) = 2^{-nd}$ for $n \geq 0$ and $1 \leq j \leq 2^{nd}$. Also, given the $N-$th generation E_N of $E(\lambda)$, we denote by p_N the probability measure on E_N given by

$$p_N = \frac{\mathcal{L}^d | E_N}{\mathcal{L}^d(E_N)},$$

where $\mathcal{L}^d$ stands for the Lebesgue measure in $\mathbb{R}^d$.

Our first result reads as follows.

Theorem 1.1. *Let $E(\lambda)$, with $2^{-d/s} \leq \lambda_n \leq \lambda_0 < \frac{1}{2}$, be a Cantor set and $0 < s < d$. For all $N = 1, 2, \dots$ we have*

$$C^{-1} \left(\sum_{n=1}^N \frac{1}{(2^{nd}\sigma_n^s)^2} \right)^{1/2} \leq \|R^s\|_{L^2(p_N)} \leq C \left(\sum_{n=1}^N \frac{1}{(2^{nd}\sigma_n^s)^2} \right)^{1/2},$$

where the constant C depends on λ_0, s and d, but not on N.

In the theorem, R^s is the operator $R^s_{p_N}$. Notice that the number $2^{-d/s}$ appearing in the theorem is a critical value, so that if, for some positive constant $\eta < 1$, we have $\lambda_n \leq \eta 2^{-d/s}$ for all n, then $H^s(E(\lambda)) = 0$ and thus $\|R^s\|_{L^2(p)} = 0$.

For $s = 1$ one can use the relationship between curvature of measures and the Cauchy kernel (see [Me] and [MeV]) to obtain the estimates above. Analogous arguments would also work for $0 < s < 1$ (see [MPV]). However, for $s > 1$ we don't have at our disposal a notion such as curvature of measures (see [Fa]). Nevertheless, we will show in Section 3 that, on $E(\lambda)$, the Riesz transforms truncated at different scales behave in a quasiorthogonal way because of the antisymmetry of the kernel and the special structure of $E(\lambda)$. More precisely, we show that if R^s is bounded on $L^2(p_N)$, then $\sum_{k\in\mathbb{Z}} \|R_k 1\|^2_{L^2(p_N)} < \infty$, where R_k is an appropriate truncation of the operator R^s (see (3.2)). So one can consider $\left(\sum_{k\in\mathbb{Z}} |R_k 1(x)|^2\right)^{1/2}$ as a suitable square function that relates the $L^2(p_N)$ boundedness of R^s with the

geometry of $E(\lambda)$ (for $s = 1$ another suitable square function is given by curvature).

We will apply Theorem 1.1 to prove the following estimates for the Lipschitz harmonic capacity of $E_N = E(\lambda_1, \dots, \lambda_N)$.

Theorem 1.2. *Assume that for all n, $2^{\frac{-d}{d-1}} \leq \lambda_n \leq \lambda_0 < \frac{1}{2}$. For any $N = 1, 2, \dots$ we have*

$$C^{-1} \left(\sum_{n=1}^{N} \frac{1}{(2^{nd} \sigma_n^{d-1})^2} \right)^{-1/2} \leq \kappa_+(E_N) \leq \kappa(E_N) \leq C \left(\sum_{n=1}^{N} \frac{1}{(2^{nd} \sigma_n^{d-1})^2} \right)^{-1/2},$$

where the constant C depends on d and λ_0, but not on N.

The proof of this result is inspired by the arguments in [MTV] for analytic capacity. We will argue by induction on N, and a basic tool for the proof will be the local $T(b)-$theorem of M. Christ [Ch1], as in [MTV].

The plan of the paper is the following. In next section we state some preliminary results which will be used to obtain the theorems above. Theorem 1.1 is proved in Section 3 and Theorem 1.2 in Section 4. Finally, in Section 5 we will show another consequence of Theorem 1.1. We will prove that if the $s-$Riesz transforms are bounded on $E(\lambda)$ (in $L^2(p)$), then all $s-$dimensional Calderón-Zygmund operators with antisymmetric kernel (smooth enough) are also bounded on $L^2(p)$. In the case $s = 1$, this result is already known for the Cauchy transform with respect to the length on AD-regular $1-$dimensional sets, by the results of [MMV] and [DS], and also for homogeneous kernels and all measures μ on $\mathbb{C}$ satisfying $\mu(B(x, r)) \leq Cr$ for all $x \in \mathbb{C}$, $r > 0$ [Ve]. For other measures and $s-$Riesz transforms it is an open question.

Throughout all the paper, the letter C will stand for an absolute constant (which may depend on d and s) that may change at different occurrences. Constants with subscripts, such as C_1, will retain its value, in general. The statement $A \approx B$ means that there exists some positive constant C such that $C^{-1}A \leq B \leq CA$.

2. Preliminaries

As mentioned above, one of the essential tools for estimating $\kappa(E_N)$ in Theorem 1.2 will the local $T(b)-$Theorem of M. Christ [Ch1]. We state below a very particular version of this result, which is adapted to the case of $(d-1)-$Riesz transforms on E_N.

Theorem 2.1 (Christ)**.** *Let μ be a positive Borel measure supported on ∂E_N satisfying, for some absolute constant C, the following conditions:*

(i) $\mu(B(x, r)) \leq Cr^{d-1}$, *for* $x \in \partial E_N$, $r > 0$.
(ii) $\mu(B(x, 2r)) \leq C\mu(B(x, r))$, *for* $x \in \partial E_N$, $r > 0$.

(iii) *For each ball B centered at a point in ∂E_N there exists a function b_B in $L^\infty(\mu)$, supported on B, satisfying $|b_B| \leq 1$ and $|R_\varepsilon^{d-1}(b_B\, d\mu)| \leq 1$ $\mu-$almost everywhere on ∂E_N, and $\mu(B) \leq C\Big|\int b_B d\mu\Big|$.*

Then

$$\int |R_\varepsilon^{d-1}(f\, d\mu)|^2 d\mu \leqslant C' \int |f|^2 d\mu, \quad f \in L^2(\mu), \tag{2.1}$$

for some absolute constant C' (depending only on C and d).

Now we recall a well known dualization method for the weak $(1,1)$ inequality of $(d-1)-$Riesz transforms.

Theorem 2.2. *Suppose that $\mu(B(x,r)) \leq Cr^{d-1}$ for all $x \in \mathbb{R}^d$, $r > 0$, and that R^{d-1} is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. For all $F \subset \mathbb{R}^d$ there exists some function b supported on F, such that $0 \leq b \leq 1$, $\int b\, d\mu \geq \mu(F)/2$, and $\|R_\varepsilon^{d-1}(b\, d\mu)\|_{L^\infty(\mathbb{R}^d)} \leq C_1$ for all $\varepsilon > 0$.*

The original argument for the proof is in [DØ] (see also [Uy], and [Ch2, p.107-108]). Regarding the fact that b can be chosen independently of ε, see [NTV, Theorem 2.1] for example.

Let φ be a fixed $\mathcal{C}^\infty$ radial function supported on $B(0,1)$ such that $0 \leq \varphi \leq 1$, $\|\nabla\varphi\|_\infty \leq C$, and $\int \varphi d\mathcal{L}^d = 1$. For $\varepsilon > 0$ we denote $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. The regularized operators $\widetilde{R}_\varepsilon^{d-1}$, $\varepsilon > 0$, are defined as

$$\widetilde{R}_\varepsilon^{d-1}\nu := \varphi_\varepsilon * R^{d-1}\nu = \varphi_\varepsilon * K^{d-1} * \nu.$$

Let $\widetilde{K}_\varepsilon^{d-1} = \varphi_\varepsilon * K^{d-1}$. It is easily seen that $\widetilde{K}_\varepsilon^{d-1}(x) = K^{d-1}(x)$ if $|x| > \varepsilon$, $\|\widetilde{K}_\varepsilon^{d-1}\|_\infty \leq C/\varepsilon^{d-1}$, and $|\nabla\widetilde{K}_\varepsilon^{d-1}(x)| \leq C/|x|^d$. Further, since $\widetilde{K}_\varepsilon^{d-1}$ is a uniformly continuous kernel, $\widetilde{R}_\varepsilon^{d-1}\nu$ is continuous on $\mathbb{R}^d$. Notice also that if $|R^{d-1}\nu(x)| \leq B$ a.e. $x \in \mathbb{R}^d$ with respect to Lebesgue measure, then $|\widetilde{R}_\varepsilon^{d-1}\nu(x)| \leq B$ for all $x \in \mathbb{R}^d$. Moreover, we have

$$\left|R_\varepsilon^{d-1}\nu(x) - \widetilde{R}_\varepsilon^{d-1}\nu(x)\right| = \left|\int_{|y-x|\leq\varepsilon} \widetilde{K}_\varepsilon^{d-1}(y-x)d\nu(y)\right| \leq C\,\frac{|\nu|(B(x,\varepsilon))}{\varepsilon^{d-1}}. \tag{2.2}$$

3. Riesz transforms on Cantor sets

This section is devoted to the proof of Theorem 1.1. Given a Cantor set $E(\lambda)$, for any $m \geq 0$ the family of cubes of the m-th generation of $E_N(\lambda)$ is denoted Δ_m. Given a cube Q we set

$$\theta(Q) := \frac{p_N(Q)}{\ell(Q)^s}.$$

So $\theta(Q)$ is the average s-dimensional density of p_N over Q. We also set $\theta_m := \theta(Q)$, with $Q \in \Delta_m$.

To simplify notation, in this section we will write $Rf(x)$ instead of $R^s f(x)$, $K(x)$ instead of $K^s(x)$, and μ instead of p_N. Also, $\|\cdot\|$ stands for the $L^2(\mu)$ norm.

For the sake of clarity, we will split the proof of Theorem 1.1 in two parts.

3.1. **Proof of $\|R\|_{L^2(\mu),L^2(\mu)} \leq C\left(\sum_{n=1}^N \theta_n^2\right)^{1/2}$.** By the $T(1)-$Theorem of David and Journé (for homogeneous spaces), we have

$$\|R\|_{L^2(\mu),L^2(\mu)} \leq C \sup_{j,m} \frac{\mu(Q_j^m)}{\ell(Q_j^m)^s} + C \sup_{j,m} \frac{\|R\chi_{Q_j^m}\|_{L^2(\mu|Q_j^m)}}{\mu(Q_j^m)^{1/2}}.$$

So the inequality $\|R\|_{L^2(\mu),L^2(\mu)} \leq C\left(\sum_{n=1}^N \theta_n^2\right)^{1/2}$ follows from next result.

Proposition 3.1. *Given m with $0 \leq m \leq N$, for any cube Q^m of the $m-$th generation of E_N, we have*

$$\|R\chi_{Q^m}\|^2_{L^2(\mu|Q^m)} \leq C \sum_{n=m}^N \theta_n^2 \mu(Q^m). \tag{3.1}$$

Proof. For $k \in \mathbb{Z}$, we set

$$R_k f(x) = \int_{Q_x^k \setminus Q_x^{k+1}} K(y-x) f(y)\, d\mu(y), \tag{3.2}$$

where Q_x^k stands for the cube of generation k that contains x if $m \leq k \leq N$, and $Q_x^k = \varnothing$ if $k < m$ or $k > N$. Notice that $R_k = 0$ if either $k < m$ or $k > N$. Moreover, for $x \in Q^m$ we have

$$R\chi_{Q^m}(x) = \sum_{k\in\mathbb{Z}} R_k \chi_{Q^m}(x).$$

Notice also that $\|R_k\chi_{Q^m}\|^2_{L^2(\mu|Q^m)} = \|R_k\chi_{Q^m}\|^2_{L^2(\mu)}$ for all $k \in \mathbb{Z}$, by the definition of R_k. We will prove that

$$\|R\chi_{Q^m}\|^2_{L^2(\mu|Q^m)} \leq C \sum_{k\in\mathbb{Z}} \|R_k\chi_{Q^m}\|^2. \tag{3.3}$$

Because of the special geometric properties of the Cantor set $E(\lambda)$, it is easy to check that

$$\|R_k\chi_{Q^m}\| \approx \theta_k \mu(Q^m)^{1/2}, \tag{3.4}$$

for k with $m \leq k \leq N$. Therefore, (3.1) follows from (3.3).

We write

$$\begin{aligned} \|R\chi_{Q^m}\|^2_{L^2(\mu|Q^m)} &= \|\sum_{k\in\mathbb{Z}} R_k\chi_{Q^m}\|^2 \\ &= \sum_{k\in\mathbb{Z}} \|R_k\chi_{Q^m}\|^2 + \sum_{j,k:\, j\neq k} \langle R_j\chi_{Q^m}, R_k\chi_{Q^m}\rangle. \end{aligned} \tag{3.5}$$

We claim that the following estimate holds:

$$|\langle R_j\chi_{Q^m},\, R_k\chi_{Q^m}\rangle| \leq C\, 2^{-|j-k|} \|R_j\chi_{Q^m}\|\, \|R_k\chi_{Q^m}\|. \tag{3.6}$$

Then we deduce

$$\sum_{j,k:j\neq k} |\langle R_j\chi_{Q^m},\, R_k\chi_{Q^m}\rangle| \leq C_2 \sum_j \|R_j\chi_{Q^m}\|^2,$$

since the matrix $\{2^{-|j-k|}\}_{j,k}$ defines a bounded operator on ℓ^2. Now (3.3) follows from the decomposition (3.5) and the last estimate.

Let us turn our attention to (3.6). Assuming $k > j$, let Q_i^k be some cube of the $k-$th generation, and x_i^k its center. Notice that

$$\int_{Q_i^k} R_k\chi_{Q^m}(x)\, d\mu(x) = 0,$$

due to the antisymmetry of $K(x)$. Then we have

$$\begin{aligned} I_i^k &:= \int_{Q_i^k} R_j\chi_{Q^m}(x)\, R_k\chi_{Q^m}(x)\, d\mu(x) \\ &= \int_{Q_i^k} \big(R_j\chi_{Q^m}(x) - R_j\chi_{Q^m}(x_i^k)\big)\, R_k\chi_{Q^m}(x)\, d\mu(x). \end{aligned}$$

Thus,

$$|I_i^k| \leq \|R_k\chi_{Q^m}\|_{L^1(\mu|Q_i^k)} \sup_{x\in Q_i^k} \big|R_j\chi_{Q^m}(x) - R_j\chi_{Q^m}(x_i^k)\big|.$$

Moreover, for each $x \in Q_i^k$ we have

$$\begin{aligned} \big|R_j\chi_{Q^m}(x) - R_j\chi_{Q^m}(x_i^k)\big| &\leq \frac{C\ell(Q_i^k)\,\mu(Q_x^j)}{\operatorname{dist}(Q_x^{j+1}, Q_x^j \setminus Q_x^{j+1})^{s+1}} \\ &\leq \frac{C\ell(Q_i^k)\,\mu(Q_x^j)}{\ell(Q_x^j)^{s+1}} \\ &= C\,\theta_j\,\frac{\ell(Q_i^k)}{\ell(Q_x^j)} \leq C\,\theta_j\,2^{-|j-k|}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} |\langle R_j\chi_{Q^m},\, R_k\chi_{Q^m}\rangle| &\leq \sum_{i:Q_i^k\subset Q^m} |I_i^k| \\ &\leq C\,\theta_j\,2^{-|j-k|} \sum_{i:Q_i^k\subset Q^m} \|R_k\chi_{Q^m}\|_{L^1(\mu|Q_i^k)} \\ &= C\,\theta_j\,2^{-|j-k|}\|R_k\chi_{Q^m}\|_{L^1(\mu)} \\ &\leq C\,\theta_j\,2^{-|j-k|}\mu(Q^m)^{1/2}\|R_k\chi_{Q^m}\|. \end{aligned}$$

From this estimate and (3.4), inequality (3.6) follows. □

3.2. **Proof of $\|R\|_{L^2(\mu),L^2(\mu)} \geq C^{-1}\left(\sum_{n=1}^N \theta_n^2\right)^{1/2}$.** Let $R_N 1$ be the following truncated version of $R1$:

$$R_N 1(x) = \int_{y\notin Q_x^N} K(y-x)\,d\mu(y) \tag{3.7}$$

The preceding estimate follows from next result.

Proposition 3.2. *We have*

$$\|R_N 1\|^2 \geq C^{-1}\sum_{n=0}^N \theta_n^2.$$

Given $P\in\Delta_m$ and $k>m$, we define

$$R_k^P 1(x) = \chi_P(x)\int_{P\setminus Q^k(x)} K(y-x)\,d\mu(y),$$

where $Q^k(x)$ stands for the cube of the k-th generation that contains x.

Next lemma concentrates the main difficulties of the proof of Proposition 3.2.

Lemma 3.3. *Let P_0 be a cube from Δ_{m_0}, $0<m_0\leq N$, and take $0<\delta<1$ and M with $m_0\leq M\leq N$. Assume that $\theta(P_0)\geq\theta(Q)\geq\delta\theta(P_0)$ for any $Q\in\Delta_m$ with $m_0\leq m\leq M$. Then*

$$\|R_M^{P_0}1\|^2 \geq C(\delta)\sum_{m=m_0}^M \sum_{\substack{Q\in\Delta_m\\ Q\subset P_0}} \theta(Q)^2\mu(Q). \tag{3.8}$$

Proof. Unless stated otherwise, all the cubes that we will consider in the proof are contained in P_0 and belong to $\bigcup_{m=m_0}^M \Delta_m$.

Let A be some big constant which will be fixed below. Given any cube P, we say that Q belongs to $\mathrm{Stop}(P)$ if Q is a cube contained in P such that

$$\left|\int_{P\setminus Q} K(y-x)\,d\mu(y)\right| \geq A\theta(P_0) \quad \text{for all } x\in Q$$

and for all $\widetilde{Q}$ with $Q\subsetneq\widetilde{Q}\subset P$, there exists $\tilde{x}\in\widetilde{Q}$ such that

$$\left|\int_{P\setminus\widetilde{Q}} K(y-\tilde{x})\,d\mu(y)\right| < A\theta(P_0).$$

Now we define $\mathrm{Stop}^k(P)$ by induction: we set $\mathrm{Stop}^1(P):=\mathrm{Stop}(P)$, and for $k\geq 2$,

$$\mathrm{Stop}^k(P) = \{Q:\ \exists\widetilde{P}\in\mathrm{Stop}^{k-1}(P) \text{ such that } Q\in\mathrm{Stop}(\widetilde{P})\}.$$

That is, $\mathrm{Stop}^k(P)=\mathrm{Stop}(\mathrm{Stop}^{k-1}(P))$. Finally, we denote

$$\mathrm{Top} := \{P_0\}\cup\bigcup_{k\geq 1}\mathrm{Stop}^k(P_0).$$

Given $P \in \mathrm{Top}$, observe that the cubes from $\mathrm{Stop}(P)$ are pairwise disjoint. The family $\mathrm{Stop}(P)$ may or may not cover P (remember that we are only considering a finite number of generations). We denote $P^{stp} = \bigcup_{Q\in \mathrm{Stop}(P)} Q$ (this is the part of P covered by the cubes in $\mathrm{Stop}(P)$).

Now we define

$$K_P 1(x) = \sum_{Q\in \mathrm{Stop}(P)} \chi_Q(x) \int_{P\setminus Q} K(y-x)d\mu(y) + \chi_{P\setminus P^{stp}}(x) \int_{P\setminus Q^M(x)} K(y-x)d\mu(y).$$

Then, by construction, we have

$$R_M^{P_0} 1 = \sum_{P\in \mathrm{Top}} K_P 1.$$

Thus,

$$\|R_M^{P_0} 1\|^2 = \sum_{P\in \mathrm{Top}} \|K_P 1\|^2 + \sum_{P,Q\in \mathrm{Top}:\, P\neq Q} \langle K_P 1, K_Q 1\rangle. \tag{3.9}$$

We claim that

$$\|K_P 1\|^2 \geq C^{-1}\theta(P)^2\mu(P), \tag{3.10}$$

and also that

$$\sum_{P,Q\in \mathrm{Top}:\, P\neq Q} |\langle K_P 1, K_Q 1\rangle| \leq \frac{C}{A} \sum_{P\in \mathrm{Top}} \|K_P 1\|^2. \tag{3.11}$$

Assume the estimates (3.10) and (3.11) for the moment. Notice that if A has been chosen big enough, from (3.9) and (3.11) we get

$$\|R_M^{P_0} 1\|^2 \approx \sum_{P\in \mathrm{Top}} \|K_P 1\|^2,$$

and then by (3.10),

$$\|R_M^{P_0} 1\|^2 \geq C^{-1} \sum_{P\in \mathrm{Top}} \theta(P)^2\mu(P) \geq C^{-1}\delta^2\theta(P_0)^2 \sum_{P\in \mathrm{Top}} \mu(P). \tag{3.12}$$

We will show now that the last sum in the inequality above is comparable to the sum on the right hand side of (3.8). Indeed, for each $Q \in \Delta_m$, with $m_0 \leq m \leq M$ and m far enough from M, it is easy to see that there exists $P_Q \in \mathrm{Top}$, with $P_Q \subset Q$, with comparable size (i.e. $P_Q \in \Delta_{m'}$, with $m \leq m' \leq m+n_0 \leq M$ for some fixed n_0 *which may depend on* A). Assume $Q \notin \mathrm{Top}$ and take the smallest cube $P \in \mathrm{Top}$ containing Q. Because of the definition of $\mathrm{Stop}(P)$, there exists some $\tilde{x} \in Q$ such that

$$\left| \int_{P\setminus Q} K(y-\tilde{x})\, d\mu(y) \right| < A\theta(P_0).$$

It is easily checked that this implies that

$$\left|\int_{P\setminus Q} K(y-x)\,d\mu(y)\right| \leq (A+C)\theta(P_0) \leq C_3 A\theta(P_0) \quad \text{for all } x\in Q. \tag{3.13}$$

Let $\widetilde{P}_Q \in \Delta_{m+n_0}$ be a cube which contains one of four corners of Q (for instance, if $Q=[0,\sigma_m]^d$, then we can take $\widetilde{P}_Q=[0,\sigma_{m+n_0}]^d$). Because of the geometry of $E_N(\lambda)$ we have

$$\left|\int_{Q\setminus \widetilde{P}_Q} K(y-x)\,d\mu(y)\right| \geq C^{-1} n_0\delta\,\theta(P_0) \geq (C_3+1)A\theta(P_0),$$

assuming that $n_0=n_0(A,\delta)$ has been chosen big enough. As a consequence, by (3.13),

$$\left|\int_{P\setminus \widetilde{P}_Q} K(y-x)\,d\mu(y)\right| \geq A\theta(P_0) \quad \text{for all } x\in Q.$$

Thus there exists some $P_Q\in \mathrm{Stop}(P)\subset \mathrm{Top}$ (which may coincide with $\widetilde{P}_Q$) such that $\widetilde{P}_Q\subset P_Q\subset Q$, assuming $m+n_0\leq M$. From the fact that Q and P_Q have comparable sizes, it easily follows that

$$\sum_Q \mu(Q) \leq C\sum_Q \mu(P_Q) + n_0\mu(P_0) \leq C\sum_{P\in \mathrm{Top}} \mu(P).$$

So (3.8) follows from (3.12).

To conclude the proof of the lemma it only remains to prove (3.10) and (3.11). This task is carried out in the Lemmas 3.4 and 3.5 below. □

Lemma 3.4. *Using the notation above, for $P\in$ Top we have*

$$\|K_P 1\|^2 \geq C^{-1}\theta(P)^2\mu(P) \tag{3.14}$$

and also

$$\|K_P 1\|^2 \geq A\theta(P)^2\mu(P^{stp}). \tag{3.15}$$

Proof. Let us prove the first estimate. Suppose that $P\in\Delta_m$ and let $P_1,\dots,P_{2^d}$ be the siblings of P (i.e. the cubes in Δ_{m+1} which are contained in P). We have

$$\chi_{P_1} K_P 1 = \chi_{P_1} R^P_{m+1} 1 + {\sum_{S\subset P_1}}' R^S_{m_S} 1,$$

where the $'$ in the last sum means that the sum is only over an appropriate collection of cubes $S\subset P_1$. By antisymmetry, we have $\int_S R^S_{m_S} 1\,d\mu = 0$. Thus,

$$\left|\int_{P_1} K_P 1\,d\mu\right| = \left|\int_{P_1} R^P_{m+1} 1\,d\mu\right|.$$

From the geometric properties of the Cantor sets E_N it is easily seen that

$$\left|\int_{P_1} R^P_{m+1} 1\,d\mu\right| \geq C^{-1}\,\frac{\mu(P)}{\mathrm{dist}(P_1, E_N\setminus P_1)^s}\,\mu(P_1) \geq C^{-1}\,\frac{\mu(P)^2}{\ell(P)^s}.$$

Therefore,

$$\theta(P)\,\mu(P) \leq C\int_P |K_P1|\,d\mu \leq C\|K_P1\|\mu(P)^{1/2},$$

and (3.14) follows.

The inequality (3.15) holds by construction, because of the choice of the cubes in Stop(P). □

Lemma 3.5. *There exists a constant C such that*

$$\sum_{P,Q\in\mathrm{Top}:\,P\neq Q} |\langle K_P1, K_Q1\rangle| \leq \frac{C}{A}\sum_{P\in\mathrm{Top}} \|K_P1\|^2. \tag{3.16}$$

Proof. Let $P, Q \in$ Top, with $Q \subsetneq P$. Denote by P_Q the cube in Stop(P) which contains Q. We are going to show first that the following estimate holds:

$$|\langle K_P1, K_Q1\rangle| \leq \frac{C}{A}\left(\frac{\ell(Q)}{\ell(P_Q)}\right)\left(\frac{\mu(Q)}{\mu(P^{stp})}\right)^{1/2}\|K_Q1\|\,\|K_P1\|. \tag{3.17}$$

Because of the antisymmetry of $K(y-x)$, for any cube $S\in\Delta_k$, we have

$$\int_P R^S_{k+1}1\,d\mu = 0.$$

Then we infer that

$$\int_P K_Q1\,d\mu = 0,$$

because K_Q is the addition of a finite number of functions R^S_{k+1} (for appropriate S's). So we have

$$\begin{aligned}\left|\int_Q K_Q1(x)K_P1(x)d\mu(x)\right| &= \left|\int_Q K_Q1(x)(K_P1(x)-K_P1(x_Q))d\mu(x)\right| \\ &\leq \|K_Q1\|_{L^1(Q)}\sup_{x\in Q}|K_P1(x)-K_P1(x_Q)|,\end{aligned}$$

where x_Q is the center of Q. For any $x\in Q$,

$$|K_P1(x)-K_P1(x_Q)| = \int_{P\setminus P_Q}|K(y-x)-K(y-x_Q)|d\mu(y) \leq C\ell(Q)\int_{P\setminus P_Q}\frac{d\mu(y)}{|x-y|^{s+1}}.$$

By standard arguments, the integral on the right side is bounded above by

$$\frac{C}{\ell(P_Q)}\sup_{\eta\geq 1}\frac{\mu(P\cap\eta P_Q)}{\ell(\eta P_Q)^s} \leq C\,\frac{\theta(P_0)}{\ell(P_Q)}.$$

Therefore, using (3.15) we deduce

$$\begin{aligned}|\langle K_P1, K_Q1\rangle| &\leq C\theta(P_0)\,\frac{\ell(Q)}{\ell(P_Q)}\,\|K_Q1\|\,\mu(Q)^{1/2} \\ &\leq \frac{C}{A}\left(\frac{\mu(Q)}{\mu(P^{stp})}\right)^{1/2}\frac{\ell(Q)}{\ell(P_Q)}\|K_Q1\|\,\|K_P1\|.\end{aligned}$$

So (3.17) holds.

To estimate the sum on the left hand side of (3.16), notice that if $P, Q \in$ Top and $Q \in \mathrm{Stop}^k(P)$, then $Q \subset P^{stp}$ and $\ell(Q)/\ell(P_Q) \leq 2^{-k+1}$. Using (3.17) we obtain

$$\begin{aligned}
\sum_{P,Q\in \mathrm{Top}:\, P\neq Q} |\langle K_P 1, K_Q 1\rangle| \;&=\; 2 \sum_{\substack{P,Q\in \mathrm{Top}\\ Q\subsetneq P^{stp}}} |\langle K_P 1, K_Q 1\rangle| \\
&\leq \sum_{P\in \mathrm{Top}} \sum_{\substack{Q\in \mathrm{Top}\\ Q\subsetneq P^{stp}}} \frac{C}{A}\left(\frac{\ell(Q)}{\ell(P_Q)}\right)\left(\frac{\mu(Q)}{\mu(P^{stp})}\right)^{1/2} \|K_P 1\|\,\|K_Q 1\| \\
&\leq \frac{C}{A}\sum_{P\in \mathrm{Top}}\Bigg[\sum_k 2^{-k} \sum_{Q\in \mathrm{Stop}^k(P)} \left(\frac{\mu(Q)}{\mu(P^{stp})}\right)^{1/2} \|K_Q 1\|\Bigg]\|K_P 1\|.
\end{aligned}$$

By Cauchy-Schwartz we have

$$\begin{aligned}
&\sum_{Q\in \mathrm{Stop}^k(P)} \left(\frac{\mu(Q)}{\mu(P^{stp})}\right)^{1/2} \|K_Q 1\| \\
&\qquad\qquad \leq \Bigg(\sum_{Q\in \mathrm{Stop}^k(P)} \frac{\mu(Q)}{\mu(P^{stp})}\Bigg)^{1/2} \Bigg(\sum_{Q\in \mathrm{Stop}^k(P)} \|K_Q 1\|^2\Bigg)^{1/2} \\
&\qquad\qquad \leq \Bigg(\sum_{Q\in \mathrm{Stop}^k(P)} \|K_Q 1\|^2\Bigg)^{1/2}.
\end{aligned}$$

since the cubes in $\mathrm{Stop}^k(P)$ are contained in P^{stp} and pairwise disjoint. Consequently, by Cauchy-Schwartz again,

$$\begin{aligned}
\sum_{P,Q\in \mathrm{Top}:\, P\neq Q} |\langle K_Q 1, K_P 1\rangle| &\leq \frac{C}{A}\sum_k 2^{-k} \sum_{P\in \mathrm{Top}} \|K_P 1\| \Bigg(\sum_{Q\in \mathrm{Stop}^k(P)} \|K_Q 1\|^2\Bigg)^{1/2} \\
&\leq \frac{C}{A}\sum_k 2^{-k} \Bigg(\sum_{P\in \mathrm{Top}} \|K_P 1\|^2\Bigg)^{1/2} \Bigg(\sum_{P\in \mathrm{Top}} \sum_{Q\in \mathrm{Stop}^k(P)} \|K_Q 1\|^2\Bigg)^{1/2} \\
&\leq \frac{C}{A}\sum_{P\in \mathrm{Top}} \|K_P 1\|^2.
\end{aligned}$$

□

Remember that we are assuming that the sequence of densities $\{\theta_i\}_{i\geq 0}$ is non-increasing, which is equivalent to saying that $\lambda_i \geq 2^{-d/s}$ for all i.

If all the densities θ_i are (uniformly) comparable to θ_0, then Proposition 3.2 follows immediately from the estimates obtained in Lemma 3.3. If the densities θ_j decrease too much, we need to introduce some additional stopping time arguments. To this end we define a finite increasing sequence of integers $\{d_i\}_{0\leq i\leq p}$, with $0 \leq d_i \leq N$, as follows. We set $d_0 := 0$. If d_i has

already been defined and $d_i < N$, then we let d_{i+1} be the least integer such that $d_{i+1} > d_i$ and $\theta_{d_{i+1}} \leq \delta\theta_{d_i}$. If d_{i+1} does not exist because $\theta_i > \delta\theta_{d_i}$ for $d_i < i \leq N$, then we set $d_{i+1} = N$ and the construction of $\{d_i\}$ is finished.

We denote

$$K_i 1(x) = \int_{Q^{d_i}(x)\setminus Q^{d_{i+1}}(x)} K(y-x)\, d\mu(y).$$

By convenience, if $d_i = N$, we set $K_i = 0$. Observe that

$$R_N 1 = \sum_{i=0}^{p-1} K_i 1.$$

In next lemma we show that the functions $K_i 1$, $i \geq 0$, behave in a quasiorthogonal way.

Lemma 3.6. *If δ is small enough, then*

$$\|R_N 1\|^2 \approx \sum_{i=0}^{p-1} \|K_i 1\|^2.$$

Proof. First we will show that the following estimate holds for $i < j$:

$$|\langle K_i 1,\, K_j 1\rangle| \leq C\delta^{\varepsilon}\theta_{d_i}\, \frac{\sigma_{d_j}}{\sigma_{d_{i+1}}} \|K_j 1\|, \tag{3.18}$$

where $\varepsilon > 0$ is some fixed constant depending only on d and s. Indeed we have

$$\left|\int K_i 1 \cdot K_j 1\, d\mu\right| \leq \sum_{Q\in\Delta_{d_j}} \left|\int_Q (K_i 1 - K_i 1(x_Q))\, K_j 1\, d\mu\right|. \tag{3.19}$$

For $x \in Q$ we have

$$|K_i 1(x) - K_i 1(x_Q)| \leq C \int_{R_Q^{d_i}\setminus R_Q^{d_{i+1}}} \frac{\ell(Q)}{|x-y|^{s+1}}\, d\mu(y),$$

where R_Q^m is the cube from Δ_m which contains Q. To estimate the integral on the right side, we consider an intermediate cube R_Q^k, with $d_i \leq k \leq d_{i+1}$ to be chosen below, and we write

$$\begin{aligned} \int_{R_Q^{d_i}\setminus R_Q^{d_{i+1}}} \frac{\ell(Q)}{|x-y|^{s+1}}\, d\mu(y) &= \int_{R_Q^{d_i}\setminus R_Q^{k}} + \int_{R_Q^{k}\setminus R_Q^{d_{i+1}}} \\ &\leq C\sigma_{d_j}\left(\sum_{m=d_i}^{k} \frac{\theta_m}{\sigma_m} + \sum_{m=k+1}^{d_{i+1}-1} \frac{\theta_m}{\sigma_m}\right). \end{aligned} \tag{3.20}$$

For the first sum, since the densities θ_m are non increasing, we have

$$\sum_{m=d_i}^{k} \frac{\theta_m}{\sigma_m} \leq \theta_{d_i} \sum_{m=d_i}^{k} \frac{1}{\sigma_m} \leq \frac{C\theta_{d_i}}{\sigma_k} \leq \frac{C\theta_{d_i}}{2^{|k-d_{i+1}|}\sigma_{d_{i+1}}}.$$

We consider now the last sum in (3.20). From the construction of $E(\lambda)$ it easily follows that $\theta_{m-1} \leq 2^{d-s}\theta_m$ for any m. Then we get

$$\sum_{m=k+1}^{d_{i+1}-1} \frac{\theta_m}{\sigma_m} \leq 2^{(d-s)|k-d_{i+1}|}\theta_{d_{i+1}} \sum_{m=k+1}^{d_{i+1}-1} \frac{1}{\sigma_m} \leq \frac{C2^{(d-s)|k-d_{i+1}|}\delta\,\theta_{d_i}}{\sigma_{d_{i+1}}}.$$

If we choose k such that $2^{|k-d_{i+1}|} \approx \delta^{-1/(1+d-s)}$ (we will fix $\delta \ll 1$ below), from the preceding estimates we deduce

$$\sum_{m=d_i}^{d_{i+1}-1} \frac{\theta_m}{\sigma_m} \leq \frac{C\delta^{1/(1+d-s)}\theta_{d_i}}{\sigma_{d_{i+1}}} =: \frac{C\delta^{\varepsilon}\theta_{d_i}}{\sigma_{d_{i+1}}}.$$

Thus we obtain

$$|K_i1(x) - K_i1(x_Q)| \leq C\,\frac{\sigma_{d_j}}{\sigma_{d_{i+1}}}\,\delta^{\varepsilon}\theta_{d_i}.$$

If we plug this estimate into (3.19) and we apply Hölder's inequality, (3.18) follows.

Now since $\|K_i1\| \geq C^{-1}\theta_{d_i}$, from (3.18) we get

$$|\langle K_i1,\, K_j1\rangle| \leq C\delta^{\varepsilon}\,\frac{\sigma_{d_j}}{\sigma_{d_{i+1}}}\,\|K_i1\|\|K_j1\| \leq C\delta^{\varepsilon}2^{-|i-j|}\|K_i1\|\|K_j1\|. \tag{3.21}$$

We have

$$\|R_N1\|^2 = \sum_i \|K_i1\|^2 + 2\sum_{i,j:i<j} \langle K_i1,\, K_j1\rangle. \tag{3.22}$$

The last term on the right side is estimated using (3.21). Since the matrix $\{2^{-|i-j|}\}_{i,j}$ is bounded on ℓ^2, we have

$$\begin{aligned}
\sum_{i,j:i<j} |\langle K_i1,\, K_j1\rangle| &\leq C\delta^{\varepsilon} \sum_{i,j:i<j} 2^{-|i-j|}\|K_i1\|\|K_j1\| \\
&\leq C\delta^{\varepsilon}\Big(\sum_i \|K_i1\|^2\Big)^{1/2}\Big(\sum_j \|K_j1\|^2\Big)^{1/2} \\
&= C\delta^{\varepsilon}\sum_i \|K_i1\|^2.
\end{aligned}$$

Hence if we choose δ so that $2C\delta^{\varepsilon} \leq 1/2$, by (3.22) the lemma follows. □

Proof of Proposition 3.2. By Lemma 3.3, for each i we have

$$\begin{aligned}
\|K_i1\|^2 &= \sum_{Q\in\Delta_{d_i}} \int_Q \left|\int_{Q\setminus Q_x^{d_{i+1}}} K(y-x)\,d\mu(y)\right|^2 d\mu(x) = \sum_{Q\in\Delta_{d_i}} \|R_{d_{i+1}}^Q 1\|^2 \\
&\geq C(\delta) \sum_{Q\in\Delta_{d_i}} \sum_{\substack{P\in\Delta_{d_i}\cup\cdots\cup\Delta_{d_{i+1}-1} \\ P\subset Q}} \theta(Q)^2\mu(Q) = C(\delta)\sum_{j=d_i}^{d_{i+1}-1} \theta_j^2.
\end{aligned}$$

Thus, by Lemma 3.6, if δ is small enough,

$$\|R_N 1\|^2 \geq C^{-1} \sum_i \|K_i 1\|^2 \geq C(\delta) \sum_i \sum_{j=d_i}^{d_{i+1}-1} \theta_j^2 = C(\delta) \sum_{j=0}^{N} \theta_j^2.$$

□

4. The harmonic Lip$_1$−capacity of Cantor sets

In this section we will prove Theorem 1.2. For simplicity we change slightly the definition of p_N. Now we set

$$p_N = \frac{H^{d-1}|\partial E_N}{H^{d-1}(\partial E_N)}.$$

Remember that in section 3 we used Lebesgue measure instead of H^{d-1} in the definition of p_N. It is easily checked that Theorem 1.1 holds for this new p_N too.

We need to introduce the following auxiliary capacity of the sets E_N:

$$\kappa_p(E_N) = \sup\{\alpha : \, 0 \leq \alpha \leq 1, \|R^{d-1}_{\alpha p_N}\|_{L^2(\alpha p_N)} \leq 1\}.$$

We have

$$\kappa(E_N) \geq \kappa_+(E_N) \geq C^{-1} \kappa_p(E_N). \tag{4.1}$$

The first inequality is a trivial consequence of the definitions of κ and κ_+, and the second one follows from Theorem 2.2.

The capacity $\kappa_p(E_N)$ is easy to estimate using Theorem 1.1:

Lemma 4.1. *We have*

$$\kappa_p(E_N) \approx \left(\sum_{n=1}^{N} \frac{1}{(2^{nd} \sigma_n^{d-1})^2} \right)^{-1/2}.$$

Remember that the constant involved in the relationship $\approx$ depends on d, but not on N.

Proof. By Theorem 1.1, we have

$$\|R^{d-1}_{\alpha p_N}\|_{L^2(\alpha p_N)} = \alpha \|R^{d-1}_{p_N}\|_{L^2(p_N)} \approx \alpha \left(\sum_{n=1}^{N} \frac{1}{\left(2^{nd} \sigma_n^{d-1}\right)^2} \right)^{1/2}.$$

Since the sum on the right hand side is $\geq 2^{-d}$, the lemma follows. □

We will prove the following.

Lemma 4.2. *There exists an absolute constant C_0 such that for all $N = 1, 2, \ldots$ we have*

$$\kappa(E_N) \leq C_0 \kappa_p(E_N). \tag{4.2}$$

Observe that Theorem 1.2 is a straightforward consequence of this result, Lemma 4.1 and (4.1).

Proof. The arguments are similar to the ones of [MTV, Theorem 2] for analytic capacity, where it is shown that

$$\gamma(E_N) \approx \gamma_p(E_N)$$

(although in [MTV, Theorem 2] only the estimate $\gamma(E_N) \approx \gamma_+(E_N)$ is stated explicitly).

We set

$$S_n = \theta_1^2 + \theta_2^2 + \cdots + \theta_n^2.$$

Remember that $\theta_n = 2^{-nd}/\sigma_n^{d-1}$ We can assume without loss of generality that, for each $N > 1$, there exists M, $1 \leq M < N$, such that

$$S_M \leq \frac{S_N}{2} < S_{M+1}. \tag{4.3}$$

Otherwise $\frac{S_N}{2} < S_1$ and thus, using Lemma 4.1, we get $\kappa_p(E_N) \geq C^{-1}\lambda_1^{d-1}$. By [Pa, Lemma 2.2] we have

$$\kappa(E_N) \leq \kappa(E_1) \leq CH^{d-1}(\partial E_1) \leq C\lambda_1^{d-1}.$$

Thus, the conclusion of the lemma is trivial in this case if C_0 is big enough.

Now we will proceed to prove (4.2) by induction on N, assuming that (4.3) holds. The case $N = 1$ is obviously true. The induction hypothesis is

$$\kappa(E_n) \leq C_0 \kappa_p(E_n) \qquad \text{if } 0 < n < N,$$

where the precise value of C_0 will be determined later.

We distinguish two cases.

Case 1: For some absolute constant A_0 to be determined below,

$$\kappa(E(\lambda_{M+1}, \cdots, \lambda_N)) \leq A_0 \theta_M \, \kappa(E_N). \tag{4.4}$$

Case 2: (4.4) does not hold.

We deal first with Case 2. By induction hypothesis applied to the sequence $\lambda_{M+1}, \cdots, \lambda_N$ and by Lemma 4.1, we have

$$\begin{aligned}
\kappa(E_N) &\leq A_0^{-1}\theta_M^{-1}\kappa(E(\lambda_{M+1}, \ldots, \lambda_N)) \\
&\leq A_0^{-1}C_0\theta_M^{-1}\kappa_p(E(\lambda_{M+1}, \ldots, \lambda_N)) \\
&\leq A_0^{-1}C_0 C \frac{\theta_M^{-1}}{\left(\sum_{n=M+1}^N \frac{1}{(2^d\lambda_{M+1}\cdots 2^d\lambda_n)^2}\right)^{1/2}} \\
&= A_0^{-1}C_0 C \frac{1}{\left(\sum_{n=M+1}^N \theta_n^2\right)^{1/2}}.
\end{aligned}$$

Clearly, the inequality $S_M \leq \frac{S_N}{2}$ is equivalent to

$$\frac{1}{\sum_{n=M+1}^N \theta_n^2} \leq \frac{2}{\sum_{n=1}^N \theta_n^2} \tag{4.5}$$

and so, again by Lemma 4.1,

$$\kappa(E_N) \leq A_0^{-1} C_0 C \kappa_p(E_N). \tag{4.6}$$

Taking $A_0 = C$, (4.2) follows in Case 2.

Let us consider Case 1. We distinguish again two cases, according to whether θ_{M+1}^2 is greater than S_M or not. If $\theta_{M+1}^2 > S_M$, then

$$S_{M+1} = S_M + \theta_{M+1}^2 \approx \theta_{M+1}^2.$$

Thus

$$\kappa_p(E_{M+1}) \approx \frac{1}{S_{M+1}^{1/2}} \approx \frac{1}{\theta_{M+1}} = CH^{d-1}(\partial E_{M+1}).$$

Hence, by (4.3),

$$\kappa(E_N) \leq \kappa(E_{M+1}) \leq CH^{d-1}(\partial E_{M+1}) \approx \kappa_p(E_{M+1}) \approx \kappa_p(E_N),$$

and so (4.2) holds for C_0 big enough.

Suppose now that $\theta_{M+1}^2 \leq S_M$. Then we have

$$S_{M+1} \approx S_M \approx S_N. \tag{4.7}$$

We consider the measure

$$\mu = \frac{\kappa(E_N)}{H^{d-1}(\partial E_M)} H^{d-1}_{|\partial E_M}.$$

Clearly $\|\mu\| = \kappa(E_N)$. We will show below that R^{d-1} is bounded on $L^2(\mu)$ (with absolute constants). Assuming this fact for the moment, from Lemma 4.1 and (4.7) we deduce

$$\kappa(E_N) = \|\mu\| \leq C\kappa_p(E_M) \leq \frac{C}{S_M^{1/2}} \approx \frac{C}{S_N^{1/2}} \leq C\kappa_p(E_N),$$

and (4.2) follows.

To prove that R^{d-1} is bounded on $L^2(\mu)$, we will show that μ satisfies the conditions of the local $T(b)-$Theorem. First we verify condition (i). Let B be a ball of radius r centered at $x \in \partial E_M$. If $r \leq \sigma_M$, we have

$$\mu(B) \leq \frac{\kappa(E_N)}{H^{d-1}(\partial E_M)} C r^{d-1} \leq C r^{d-1}, \tag{4.8}$$

because $\kappa(E_N) \leq \kappa(E_M) \leq CH^{d-1}(\partial E_M)$. For $r > \sigma_M$ we can replace arbitrary balls centered at points in ∂E_M by cubes Q_n^j, $0 \leq n \leq M$, $0 \leq j \leq 2^{nd}$. In other words, it suffices to prove

$$\mu(Q_j^n) \leq CH^{d-1}(\partial Q_j^n), \qquad 0 \leq n \leq M,\ 1 \leq j \leq 2^{nd}. \tag{4.9}$$

We have

$$\begin{aligned} \mu(Q_j^n) &= \kappa(E_N)\frac{1}{2^{nd}} = \frac{\kappa(E_N)}{H^{d-1}(\partial E_n)} H^{d-1}(\partial Q_j^n) \\ &\leq \frac{\kappa(E_n)}{H^{d-1}(\partial E_n)} H^{d-1}(\partial Q_j^n) \leq CH^{d-1}(\partial Q_j^n), \end{aligned}$$

and so (4.9) holds.

The same kind of ideas can be used to verify that condition (ii) is fulfilled. The details are left for the reader.

Summing up, we have reduced the proof to checking that the hypothesis (iii) of the local $T(b)-$Theorem is satisfied. As above we can replace balls centered at points in ∂E_M by cubes Q_j^n, $1 \leq j \leq 2^{nd}$, $0 \leq n \leq M$. That is, it is enough to show that given a cube Q_j^n, $1 \leq j \leq 2^{nd}$, $0 \leq n \leq M$, there exists a function b_j^n in $L^\infty(\mu)$, supported on ∂Q_j^n, satisfying $\|b_j^n\|_\infty \leq 1$ and $\|R_\varepsilon^{d-1} b_j^n\|_\infty \leq 1$ uniformly in $\varepsilon > 0$, and such that

$$\mu(Q_j^n) \leq C \left| \int b_j^n \, d\mu \right|. \tag{4.10}$$

By definition there exists a distribution T supported on E_N such that $\kappa(E_N) \leq 2|\langle T, 1\rangle|$ and $\|R^{d-1}T\|_{L^\infty(\mathbb{R}^d)} \leq 1$. By an approximation argument, T can be replaced by a real measure ν supported on ∂E_N such that $\kappa(E_N) \leq C|\nu(E_N)|$ and $\|R^{d-1}\nu\|_{L^\infty(\mathbb{R}^d)} \leq 1$. For a fixed generation n, $0 \leq n \leq M$, there exists some index k, $1 \leq k \leq 2^{nd}$, such that

$$\kappa(E_N) \leq C \sum_{j=1}^{2^{nd}} |\nu(Q_j^n)| \leq C 2^{nd} |\nu(Q_k^n)|.$$

Equivalently, $\mu(Q_k^n) \leq C|\nu(Q_k^n)|$.

In order to define the functions b_j^n we need to describe a simple preliminary construction. By standard arguments, one can construct a compactly supported $\mathcal{C}^\infty$ function φ in $\mathbb{R}^d$, with $0 \leq \varphi \leq 1$, $\int_{\partial Q_0} \varphi \, dH^{d-1} \geq 1$, and

$$\|R_\varepsilon^{d-1}(\varphi \, dH^{d-1}_{|\partial Q_0})\|_\infty \leq C \qquad \text{uniformly in } \varepsilon > 0.$$

Set

$$\varphi_j^M(x) = \varphi\left(\frac{x - v_j^M}{\sigma_M}\right) \chi_{\partial Q_j^M}(x),$$

where v_j^M is vertex of Q_j^M closest to the origin. So, by translation and dilation invariance one has $\|R_\varepsilon^{d-1}(\varphi_j^M \, dH^{d-1}_{|\partial Q_0})\|_\infty \leq C$ uniformly in $\varepsilon > 0$ and

$$\int \varphi_j^M \, d\mu = \frac{1}{H^{d-1}(\partial Q_j^M)} \int \varphi_j^M \, dH^{d-1} \, \mu(Q_j^M) \geq C^{-1} \mu(Q_j^M).$$

First we define b_k^n. We set

$$b_k^n = \sum_{i : Q_i^M \subset Q_k^n} \nu(Q_i^M) \frac{\varphi_i^M}{\int \varphi_i^M \, d\mu}.$$

For $i \neq k$, we construct b_i^n by translation of b_k^n. More precisely, we have $Q_i^n = w_i^n + Q_k^n$ for some w_i^n, and then we put

$$b_i^n(x) = b_k^n(x - w_i^n), \qquad x \in \mathbb{R}^d.$$

Now we will prove that b_k^n satisfies condition (iii). It is clear that

$$\left|\int b_k^n d\mu\right| = |\nu(Q_k^n)| \geq C^{-1}\mu(Q_k^n).$$

In order to prove that b_k^n is bounded it is enough to verify that

$$|\nu(Q_j^M)| \leq C\mu(Q_j^M), \quad 1 \leq j \leq 2^{Md}. \tag{4.11}$$

By a result of Paramonov [Pa, Lema 4.1 p.192] on localization of singularities of harmonic functions, we get $\|R^{d-1}(\chi_{Q_j^M}\, d\nu)\|_{L^\infty(\mathbb{R}^d)} \leq C$. So we conclude that

$$|\nu(Q_j^M)| \leq C\kappa(Q_j^M \cap E_N),$$

since $R^{d-1}(\chi_{Q_j^M} d\nu)(x)$ is the gradient of a harmonic function outside $Q_j^M \cap E_N$. By (4.4), since the set $Q_j^M \cap E_N$ can be obtained from $E(\lambda_{M+1} \cdots, E_N)$ by a translation and a dilation of factor σ_M, we obtain

$$\kappa(Q_j^M \cap E_N) = \sigma_M^{d-1}\kappa(E(\lambda_{M+1} \cdots, E_N)) \leq A_0 \frac{1}{2^{Md}}\kappa(E_N) = A_0\mu(Q_j^M),$$

and then (4.11) follows.

To complete the proof we only need to check that $R_\varepsilon^{d-1}(b_k^n\, d\mu)$ is a bounded function. Actually, we will prove it only for $\varepsilon \leq \sigma_M$. Then, using the local $T(b)-$Theorem of M. Christ, we will show that R_ε^{d-1} is bounded on $L^2(\mu)$ uniformly in $0 < \varepsilon \leq \sigma_M$. By Cotlar's inequality, it will follow easily that R_ε^{d-1} is bounded in $L^2(\mu)$ uniformly for all $\varepsilon > 0$ (independently of M).

From $\|R^{d-1}(\chi_{Q_k^n} d\nu)\|_{L^\infty(\mathbb{R}^d)} \leq\ C$ we derive

$$\|\widetilde{R}_\varepsilon^{d-1}(\chi_{Q_k^n} d\nu)\|_{L^\infty(\mathbb{R}^d)} \leq\ C \qquad \text{uniformly in } \varepsilon > 0.$$

By (2.2), $R_\varepsilon^{d-1}(b_k^n\, d\mu)$ is bounded if and only if $\widetilde{R}_\varepsilon^{d-1}(b_k^n\, d\mu)$ is bounded. Thus we only need to estimate the difference

$$\widetilde{R}_\varepsilon^{d-1}(b_k^n d\mu)(x) - \widetilde{R}_\varepsilon^{d-1}(\chi_{Q_k^n} d\nu)(x) = \sum_{j:Q_j^M \subset Q_k^n} \widetilde{R}_\varepsilon^{d-1}\alpha_j^M(x), \tag{4.12}$$

where $\alpha_j^M = \nu(Q_j^M)\frac{\varphi_j^M d\mu}{\int \varphi_j^M d\mu} - \chi_{Q_j^M} d\nu$. Clearly, we have $\int d\alpha_j^M = 0$ and $\|\widetilde{R}_\varepsilon^{d-1}\alpha_j^M\|_\infty \leq C$ for $1 \leq j \leq 2^{nd}$. If we denote by x_j^M the center of Q_j^M, by [Pa, Corollary 3.4, p.191] we have

$$|R^{d-1}\alpha_j^M(x)| \leq C\frac{\sigma_M^d}{\operatorname{dist}(x, Q_j^M)^d}, \qquad \text{if } |x - x_j^M| > 2\sigma_M.$$

Since $\widetilde{R}_\varepsilon^{d-1}\alpha_j^M = R^{d-1}\alpha_j^M * \varphi_\varepsilon$, for $\varepsilon \leq \sigma_M$ we get

$$|\widetilde{R}_\varepsilon^{d-1}\alpha_j^M(x)| \leq C\frac{\sigma_M^d}{\operatorname{dist}(x, Q_j^M)^d}, \qquad \text{if } |x - x_j^M| > 3\sigma_M.$$

Using this estimate and arguing as in [MTV], it easily follows that (4.12) is uniformly bounded for $\varepsilon \leq \sigma_M$. Thus the function b_k^n associated to the cube Q_k^n satisfies the hypothesis (iii) in the $T(b)-$Theorem.

Now, by translation invariance it is clear that b_j^n for $j \neq k$ also satisfies hypothesis (iii).

Consequently, the local $T(b)-$Theorem can be applied (for $\varepsilon \leq \sigma_M$) and then the proof is complete. □

Remark 4.3. In the proof above we have replaced the distribution T supported on E_N such that $\kappa(E_N) \leq 2|\langle T, 1\rangle|$ and $\|R^{d-1}T\|_{L^\infty(\mathbb{R}^d)} \leq 1$ by a real measure ν supported on E_N satisfying analogous estimates. Actually, this step is not really needed in the proof. The reader can verify quite easily that all the arguments and estimates for ν also work for the distribution T.

5. Calderón-Zygmund operators on Cantor sets

In this section we prove that if $E(\lambda)$ is a Cantor set in $\mathbb{R}^d$ such that $\sum \theta_n^2 = \sum \frac{1}{(2^{nd}\sigma_n^s)^2} < \infty$, where $0 < s \leq d$, then any $s-$Calderón-Zygmund operator with antisymmetric kernel is bounded on $L^2(p)$. In fact, our result answers the following question in a particular case.

Question. Let μ be a positive Radon measure satisfying $\mu(B(x,r)) \leq Cr^s$ for $r > 0$, such that the $s-$Riesz transform is bounded on $L^2(\mu)$. Then, are all $s-$Calderón-Zygmund operators with antisymmetric kernel bounded on $L^2(\mu)$?

A kernel $K(\cdot,\cdot)$ from $L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,y) : x = y\})$ is an $s-$Calderón-Zygmund kernel, $0 < s \leq d$, if

(i) $|K(x,y)| \leq \frac{C}{|x-y|^s}$,

(ii) there exists $0 < \delta \leq 1$ such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\frac{|x-x'|^\delta}{|x-y|^{s+\delta}}$$

if $|x - x'| \leq |x-y|/2$.

The Calderón-Zygmund operator T_μ associated to the kernel $K(\cdot,\cdot)$ and the measure μ is defined as

$$T_\mu f(x) = \int K(x,y) f(y)\, d\mu(y), \qquad x \notin \operatorname{supp}(\mu).$$

The above integral may be not convergent for $x \in \operatorname{supp}(\mu)$ because $K(x,y)$ has a singularity for $x = y$. For this reason one introduces the truncated operators $T_{\mu,\varepsilon}$, $\varepsilon > 0$:

$$T_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} K(x,y) f(y) d\mu(y).$$

One says that T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly in $\varepsilon > 0$. When μ is fixed we will write T instead of T_μ and T_ε instead of $T_{\mu,\varepsilon}$.

The result that we will prove in this section is the following.

Theorem 5.1. *Let $\lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence such that $0 < \lambda_n \leq \lambda_0 < 1/2$. If the Cantor set $E(\lambda)$ satisfies*

$$\sum \theta_n^2 = \sum \frac{1}{(2^{nd}\sigma_n^s)^2} < \infty,$$

then any $s-$Calderón-Zygmund operator with antisymmetric kernel is bounded in $L^2(p)$, where p is the uniform probability measure on $E(\lambda)$.

Proof. For any Calderón-Zygmund operator T with antisymmetric kernel, we will prove that

$$\|T\|^2_{L^2(p),L^2(p)} \leq C \sum \theta_n^2 < \infty. \tag{5.1}$$

By the $T(1)-$Theorem, we have

$$\|T\|_{L^2(p),L^2(p)} \leq C \sup_{j,m} \frac{p(Q_j^m)}{\ell(Q_j^m)^s} + C \sup_{j,m} \frac{\|T\chi_{Q_j^m}\|_{L^2(p|Q_j^m)} + \|T^*\chi_{Q_j^m}\|_{L^2(p|Q_j^m)}}{p(Q_j^m)^{1/2}}.$$

So it is enough to show that

$$\|T\chi_{Q_j^m}\|_{L^2(p|Q_j^m)} \leq C\left(\sum_{k=m}^{\infty} \theta_k^2\right)^{1/2} p(Q_j^m)^{1/2}, \tag{5.2}$$

and analogously for T^*. In order to prove (5.2), we take a suitable decomposition of T. For any fixed $m > 0$ and $k \geq m$, let Q^m be a cube of the $m-$th generation of the Cantor set and $x \in Q^m$. We define

$$T_k f(x) = \int_{Q_x^k \setminus Q_x^{k+1}} K(x,y) f(y) dp(y),$$

where Q_x^k is the cube of generation k containing x. Then, we have

$$T\chi_{Q^m}(x) = \sum_{k=m}^{\infty} T_k \chi_{Q^m}(x).$$

We write

$$\|T\chi_{Q^m}\|^2_{L^2(p|Q^m)} = \sum_{k=m}^{\infty} \|T_k\chi_{Q^m}\|^2_{L^2(p|Q^m)} + \sum_{j,k\geq m,\, j\neq k} \langle T_j\chi_{Q^m}, T_k\chi_{Q^m}\rangle. \tag{5.3}$$

To estimate the first term in (5.3) we use condition (i) from the definition of Calderón-Zygmund kernel. So,

$$|T_k\chi_{Q^m}(x)| = \left|\int_{Q^m\cap(Q_x^k\setminus Q_x^{k+1})} K(x,y)dp(y)\right| \leq C\frac{p(Q_x^k)}{l(Q_x^{k+1})^s} = C\theta_k,$$

and consequently,

$$\|T_k\chi_{Q^m}\|^2_{L^2(p|Q^m)} \leq C\theta_k^2 p(Q^m). \tag{5.4}$$

Now we consider the second term in (5.3). Using the antisymmetry of the kernel K and arguing as in the proof of inequality (3.6), we have

$$\begin{aligned}\sum_{j,k\geq m,\, j\neq k} & |\langle T_j\chi_{Q^m}, T_k\chi_{Q^m}\rangle| \\ & \leq C \sum_{j,k\geq m,\, j\neq k} 2^{-|j-k|}\theta_j p(Q^m)^{1/2}\|T_k\chi_{Q^m}\|_{L^2(p|Q^m)} \\ & \leq C \sum_{j,k\geq m,\, j\neq k} 2^{-|j-k|}\theta_j\theta_k p(Q^m).\end{aligned}$$

Therefore, by (5.3),

$$\begin{aligned}\|T\chi_{Q^m}\|^2_{L^2(p|Q^m)} &\leq C\sum_{j\geq m}\|T_j\chi_{Q^m}\|^2_{L^2(p|Q^m)} + C\sum_{j,k\geq m,\, j\neq k} 2^{-|j-k|}\theta_j\theta_k p(Q^m) \\ &\leq C\sum_{j\geq m}\Big(\|T_j\chi_{Q^m}\|^2_{L^2(p|Q^m)} + \theta_j^2 p(Q^m)\Big) \\ &\leq C\sum_{j\geq m}\theta_j^2 p(Q^m),\end{aligned}$$

where the second inequality follows from the boundedness in ℓ^2 of the matrix $\{2^{-|j-k|}\}_{j,k}$, and the last one from (5.4).

So, (5.2) holds and the proof is complete. □

Final remarks

We don't know how to extend the arguments in Section 3 to Cantor sets $E(\lambda)$ associated to a general sequence $\lambda = \{\lambda_n\}$ with $0 < \lambda_n \leq \lambda_0 < 1/2$. However, it can be shown that a variation of the arguments in the proof of Theorem 1.1 can be adapted to the case where $\lambda_n \in (0, 2^{-d/s}]$ for all n (recall that we assumed $\lambda_n \in [2^{-d/s}, \lambda_0]$ in the paper).

The estimates for the L^2 norm of Riesz transforms in Theorem 1.1 can be extended to more general antisymmetric kernels.

By arguments analogous to the ones in Section 4 and some ideas from [MPV], one can estimate the s-dimensional signed Riesz capacity, γ_s, of the sets $E(\lambda)$ considered in this paper. More precisely, if $\lambda_n \in [2^{-d/s}, \lambda_0]$, one obtains

$$\gamma_s(E_N) \approx \left(\sum_{n=1}^N \frac{1}{(2^{nd}\sigma_n^s)^2}\right)^{-1/2}.$$

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