NON EXISTENCE OF PRINCIPAL VALUES OF SIGNED RIESZ TRANSFORMS OF NON INTEGER DIMENSION

ALEIX RUIZ DE VILLA AND XAVIER TOLSA

Abstract. In this paper we prove that, given $s \geq 0$, and a Borel non zero measure μ in $\mathbb{R}^m$, if for μ-almost every $x \in \mathbb{R}^m$ the limit

$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}} d\mu(y)$$

exists and $0 < \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} < \infty$, then s in an integer. In particular, if $E \subset \mathbb{R}^m$ is a set with positive and bounded s-dimensional Hausdorff measure H^s and for H^s-almost every $x \in E$ the limit

$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}} dH^s_{|E}(y)$$

exists, then s is integer.

1. Introduction

Given a Borel measure μ in $\mathbb{R}^m$ and $0 < s \leq m$, the s-Riesz transform of μ is

$$R^s\mu(x) = \int \frac{x-y}{|x-y|^{s+1}}\, d\mu(y), \qquad x \notin \operatorname{supp}(\mu).$$

Since for x in the support of μ the integral may not be convergent, for $\varepsilon > 0$ one considers the truncated Riesz transform

$$R^s_\varepsilon\mu(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}}\, d\mu(y), \qquad x \in \mathbb{R}^m.$$

The lower and upper s-dimensional densities of μ at x are defined by

$$\theta^s_{\mu,*}(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s}, \qquad \theta^{s,*}_\mu(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s}.$$

In the case where $\theta^s_{\mu,*}(x) = \theta^{s,*}_\mu(x)$ one calls this quantity the (s-dimensional) density of the measure μ at x, denoted by $\theta^s_\mu(x)$.

The main result of this paper is the following.

Theorem 1. *For $0 \leq s \leq m$, let μ be a finite Radon measure in $\mathbb{R}^m$ such that $0 < \theta^{s,*}_\mu(x) < \infty$ and $\lim_{\varepsilon\to 0} R^s_\varepsilon\mu(x)$ exists for all x in a set of positive μ-measure. Then $s \in \mathbb{Z}$.*

A. Ruiz de Villa was supported by grant AP-2004-5141. Also, both authors were partially supported by grants MTM2007-62817 (Spain) and 2005-SGR-00774 (Generalitat de Catalunya).

Consider now the case the case where μ coincides with the s-dimensional Hausdorff measure H^s on a set E with $0 < H^s(E) < \infty$. Recall that for H^s-almost every $x \in E$ we have $0 < \theta^{s,*}_{H^s_{|E}}(x) < \infty$. So we deduce the following corollary.

Corollary 1. *For $0 \leq s \leq m$, let $E \subset \mathbb{R}^m$ be a set satisfying $0 < H^s(E) < \infty$ such that for $\mathcal{H}^s$-almost every $x \in E$ the limit*

$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}} dH^s_{|E}(y)$$

exists. Then $s \in \mathbb{Z}$.

Let us remark that Mattila and Preiss [MP] already proved that if one assumes

$$\theta^s_{\mu,*}(x) > 0 \qquad \mu\text{−a.e. } x \in \mathbb{R}^n \tag{1}$$

(instead of $\theta^{s,*}_\mu(x) > 0$ μ-a.e.), then the μ-a.e. existence of the principal value $\lim_{\varepsilon \to 0} R^s_\varepsilon \mu(x)$ forces s to be an integer. Later on, Vihtilä [Vi] showed that this also holds if one assumes (1) and

$$\sup_{\varepsilon > 0} |R^s_\varepsilon \mu(x)| < \infty \qquad \mu\text{−a.e. } x \in \mathbb{R}^n \tag{2}$$

(instead of the existence of the principal value $\lim_{\varepsilon \to 0} R^s_\varepsilon \mu(x)$ μ-a.e.). The proofs in [MP] and [Vi] rely on the use of tangent measures, and for these arguments, and for all usual arguments involving tangent measures, the assumption (1) on the lower density is essential. So to prove theorem 1 we have followed a quite different approach, inspired in part by some of the techniques used in [To2] and [To3]. However, we have not been able to use the weaker assumption (2) instead of the one concerning the existence of principal values.

On the other hand, the case $0 \leq s \leq 1$ of theorem 1 follows from Prat's results [Pr1], [Pr2]. In this case, the so called curvature method works, and one can even assume (2) instead of the fact that principal value $\lim_{\varepsilon \to 0} R^s_\varepsilon \mu(x)$ exists μ-a.e.

If one combines corollary 1 with the results in [MM] and [To2] one gets:

Theorem. *For $0 < s \leq m$, let $E \subset \mathbb{R}^m$ be a set satisfying $0 < H^s(E) < \infty$. The principal value*

$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}} dH^s_{|E}(y)$$

exists for H^s-almost every $x \in E$ if and only if s is integer and E is s-rectifiable.

Recall that $E \subset \mathbb{R}^m$ is called s-rectifiable if it is contained H^s-a.e. in a countable union of s-dimensional C^1-submanifolds of $\mathbb{R}^m$. See also [MP] for other previous results concerning the case s integer, and [Mat], [To1], for the case $s = 1$.

It is interesting to compare the last theorem with well known results in geometric measure theory due essentially to Marstrand [Mar] and Preiss [Pre]:
For $0 < s \leq m$, let $E \subset \mathbb{R}^m$ be a set satisfying $0 < H^s(E) < \infty$. The density $\theta^s_{H^s|E}(x)$ exists for H^s-almost every $x \in E$ if and only if s is integer and E is s-rectifiable.
Notice the analogies between this statement and the previous theorem.

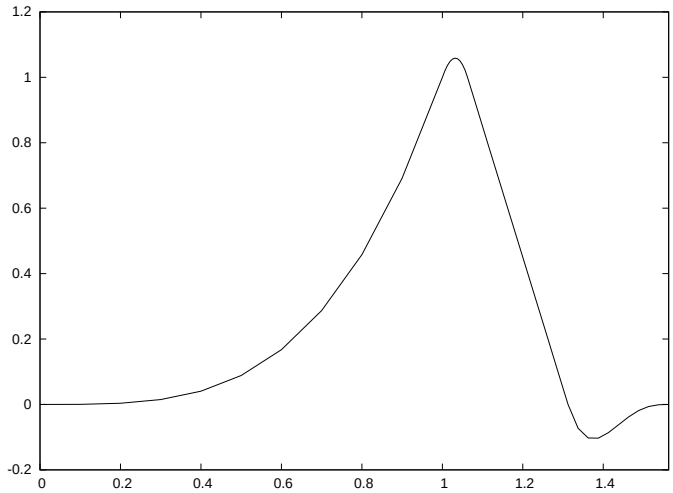

FIGURE 1. Example of the function φ for the values $\rho = 1/4$ and $s = 6$.

2. Main tools

Given two different quantities a, b we use the notation $a \lesssim b$ if there exists a fixed constant $C > 0$ satisfying $a \leq Cb$, with C depending at most on m and s. If also $b \lesssim a$, then we write $a \approx b$. Given $x \in \mathbb{R}^m$ and $r > 0$, $B(x, r)$ stands for the open ball of center x and radius r, and $\theta^s(x, r) := \mu(B(x, r))/r^s$ stands for the (average) s-dimensional density of the ball $B(x, r)$. In the case $x = 0$ we write $\theta^s(r) = \theta^s(0, r)$. Throughout the paper n will denote an integer satisfying $n < s \leq n + 1 \leq m$.

Given $0 < \rho < 1/2$ small enough, which will be fixed below, consider a function $\varphi \in \mathcal{C}^2(0, \infty)$ satisfying:

(i) $\varphi(r) = r^{(s+1)/2}$ if $0 \leq r \leq 1$,
(ii) $\varphi(r) = -\frac{r}{\rho} + 1 + \rho + \frac{1}{\rho}$ if $1 + \rho^2 \leq r \leq 1 + \rho^2 + \rho$,
(iii) $\mathrm{supp}(\varphi) \subset [0, 1 + \rho + 2\rho^2]$, $|\varphi(r)| \leq C$, $|\varphi'(r)| \leq 1/\rho$ and $|\varphi''(r)| \leq C_\rho$ for all $r > 0$, where C_ρ depends on ρ.

See fig. 1.

Given $\varepsilon > 0$, consider the operator:

$$R^s_{\varphi,\varepsilon}\mu(x) = \int \varphi\left(\frac{|x-y|^2}{\varepsilon^2}\right)\frac{x-y}{|x-y|^{s+1}}d\mu(y) = \int k_{\varphi,\varepsilon}(x-y)d\mu(y).$$

Notice that $k_{\varphi,\varepsilon}$ is a kernel supported on $B(0, 3\varepsilon)$ satisfying $\|k_{\varphi,\varepsilon}\|_\infty \leq C/\varepsilon^s$ and

$$\|\nabla k_{\varphi,\varepsilon}\|_\infty \leq C(\rho)/\varepsilon^{s+1}. \tag{3}$$

Also observe that

$$R^s_{\varphi,\varepsilon}\mu(x) = \int \int_{0<t<\frac{|x-y|^2}{\varepsilon^2}} \varphi'(t)dt\frac{x-y}{|x-y|^{s+1}}d\mu(y) = \int \varphi'(t)R^s_{\varepsilon\sqrt{t}}\mu(x)dt.$$

Using the fact that $\int \varphi'(t)dt < \infty$ and $\sup_{\varepsilon>0} |R^s_\varepsilon\mu(x)| < \infty$ for μ-almost all $x \in \mathbb{R}^m$, we conclude that if $\lim_{\varepsilon\to 0} R^s_\varepsilon\mu(x)$ exists, then $\lim_{\varepsilon\to 0} R^s_{\varphi,\varepsilon}\mu(x)$ also exists.

Given $C_0, r_0, \varepsilon_0 > 0$ and $0 < \delta < 1$, set

$$F_\delta := \big\{x \in \mathbb{R}^m : \mu(B(x,r)) \le 2\theta_\mu^{s,*}(x) r^s \text{ for all } r \le r_0, \\ |R^s_{\varphi,\varepsilon}\mu(x) - R^s_{\varphi,\varepsilon'}\mu(x)| \le \delta \text{ for all } \varepsilon, \varepsilon' \le \varepsilon_0, \text{ and } \theta_\mu^{s,*}(x) \le C_0\big\}. \tag{4}$$

If r_0 and ε_0 are small enough and C_0 is big enough, we have $\mu(F_\delta) > 0$. Also observe that if $x \in F_\delta$, for all $r > 0$,

$$\mu(B(x,r)) \le M r^s, \tag{5}$$

where $M = \max\{2C_0, \mu(\mathbb{R}^m)/r_0^s\}$.

Lemma 1. *Suppose that $0 \in F_\delta$ and let $x \in B(0, \varepsilon/4)$, then:*

$$R^s_{\varphi,\varepsilon}\mu(x) - R^s_{\varphi,\varepsilon}\mu(0) = T^\varepsilon(x) + E(x), \tag{6}$$

where

$$T^\varepsilon(x) = \int \frac{1}{|y|^{s+1}} \left[\varphi\left(\frac{|y|^2}{\varepsilon^2}\right)\left(x - \frac{(s+1)(x\cdot y)y}{|y|^2}\right) + \varphi'\left(\frac{|y|^2}{\varepsilon^2}\right)\frac{2(x\cdot y)y}{\varepsilon^2}\right] d\mu(y), \tag{7}$$

and

$$|E(x)| \le C_1 \theta^s(3\varepsilon)\frac{|x|^2}{\varepsilon^2}.$$

The constant C_1 only depends on ρ (and also C_ρ).

Proof. We will prove equality (6) as in [To3]. Applying Taylor's formula to the function $g(t) = \varphi(t)/t^{(s+1)/2}$ at a point $t_0 > 0$ we have

$$\frac{\varphi(t)}{t^{(s+1)/2}} = \frac{\varphi(t_0)}{t_0^{(s+1)/2}} + \frac{t_0\varphi'(t_0) - (s+1)\varphi(t_0)/2}{t_0^{(s+3)/2}}(t - t_0) + g''(\xi)\frac{(t-t_0)^2}{2},$$

for some $\xi \in [t, t_0]$. Notice that if $0 < t \le 1$, then $\varphi(t)/t^{(s+1)/2} = 1$. Setting $t = |x-y|^2/\varepsilon^2$ and $t_0 = |y|^2/2$, and multiplying by the vector $(x-y)$ we get

$$\begin{aligned}\varphi\left(\frac{|x-y|^2}{\varepsilon^2}\right)\frac{x-y}{|x-y|^{s+1}} =& \varphi\left(\frac{|y|^2}{\varepsilon^2}\right)\frac{-y}{|y|^{s+1}} + \varphi\left(\frac{|y|^2}{\varepsilon^2}\right)\frac{x}{|y|^{s+1}} \\ &+ \frac{|y|^2\varphi'\left(\frac{|y|^2}{\varepsilon^2}\right)/\varepsilon^2 - (s+1)\varphi\left(\frac{|y|^2}{\varepsilon^2}\right)/2}{|y|^{s+3}}(|x|^2 - 2x\cdot y)(x-y) \\ &+ g''(\xi_{x,y})\frac{(|x|^2 - 2x\cdot y)^2}{2\varepsilon^{s+5}}(x-y),\end{aligned}$$

where $\xi_{x,y} \in [|x-y|^2/\varepsilon^2, |y|^2/\varepsilon^2]$. Integrating with respect to y we obtain (6) with

$$E(x) = \int E(x,y)\,d\mu(y),$$

where

$$\begin{aligned}E(x,y) =& \frac{1}{|y|^{s+3}}\left[\frac{|y|^2}{\varepsilon^2}\varphi'\left(\frac{|y|^2}{\varepsilon^2}\right) - \frac{s+1}{2}\varphi\left(\frac{|y|^2}{\varepsilon^2}\right)\right][|x|^2 x - |x|^2 y - 2(x\cdot y)x] \\ &+ g''(\xi_{x,y})\frac{(|x|^2 - 2x\cdot y)^2}{2\varepsilon^{s+5}}(x-y) = E_1(x,y) + E_2(x,y).\end{aligned}$$

For $i = 1, 2$ consider the decomposition

$$\int E_i(x,y)d\mu(y) = \left(\int_{|y|<\varepsilon/2} + \int_{\varepsilon/2\le|y|\le 3\varepsilon}\right) E_i(x,y)d\mu(y) = A_i + B_i.$$

Let us estimate E_1 first.

(a) If $|y| \le \varepsilon/2$, using that $\varphi(r) = r^{(s+1)/2}$ for $0 < r \le 1$,

$$\frac{|y|^2}{\varepsilon^2}\varphi'\left(\frac{|y|^2}{\varepsilon^2}\right) - \frac{s+1}{2}\varphi\left(\frac{|y|^2}{\varepsilon^2}\right) = 0,$$

thus $A_1 = 0$.

(b) If $|y| > \varepsilon/2$, we have

$$\left|\frac{|y|^2}{\varepsilon^2}\varphi'\left(\frac{|y|^2}{\varepsilon^2}\right) - \frac{s+1}{2}\varphi\left(\frac{|y|^2}{\varepsilon^2}\right)\right| \le C = C(\rho).$$

Since $|x| < \varepsilon/4$, then $||x|^2x - |x|^2y - 2(x\cdot y)x| \le C|y||x|^2$. Moreover, recall that $\mathrm{supp}(\varphi) \subset [0,3]$. As a consequence,

$$|B_1| \le C|x|^2 \int_{\varepsilon/2\le|y|\le 3\varepsilon} \frac{1}{|y|^{s+2}}d\mu(y) \le C\theta^s(3\varepsilon)\frac{|x|^2}{\varepsilon^2}.$$

We now estimate E_2. Recall that

$$E_2(x,y) = g''(\xi_{x,y})\frac{(|x|^2 - 2x\cdot y)^2}{2\varepsilon^4}(x-y),$$

with $\xi_{x,y} \in [|y|^2/\varepsilon^2, |x-y|^2/\varepsilon^2]$ and

$$g''(r) = \frac{r^2\varphi''(r) - (s+1)r\varphi'(r) + \frac{(s+1)(s+3)}{4}\varphi(r)}{r^{(s+5)/2}}.$$

Denote $t = \max\{|y|, |x-y|\}$.

(a) If $|y| \le \varepsilon/2$, we have $|\xi_{x,y}| < 1$ and thus $|g''(\xi_{x,y})| = 0$. So $A_2 = 0$.

(b) If $|y| > \varepsilon/2$, we have $\xi_{x,y} \approx \frac{|y|^2}{\varepsilon^2}$, and so $|g''(\xi_{x,y})| \le C(\rho)(\varepsilon/|y|)^{s+5}$. Moreover, if $|y| > 3\varepsilon$, then $|x-y| > 2\varepsilon$ and so $\xi_{x,y} > 4$, which implies that $g''(\xi_{x,y}) = 0$. On the other hand,

$$|(|x|^2 - 2x\cdot y)^2(x-y)| \le C|(|x|^2 + |x||y|)^2(|x|+|y|)| \le C|x|^2|y|^3.$$

Therefore,

$$\begin{aligned}|B_2| &\le C\frac{|x|^2}{\varepsilon^{s+5}}\int_{\varepsilon/2\le|y|\le 3\varepsilon}\left(\frac{\varepsilon}{|y|}\right)^{s+5}|y|^3d\mu(y)\\ &\le C|x|^2\int_{\varepsilon/2\le|y|\le 3\varepsilon}\frac{1}{|y|^{s+2}}d\mu(y) \le C\theta^s(3\varepsilon)\frac{|x|^2}{\varepsilon^2}.\end{aligned}$$

□

To prove theorem 1, we will find a ball with high average density and an n-dimensional hyperplane L such that all the points in the ball are close to L. Estimating densities from above and below, we will get a contradiction. We need the following auxiliary result.

Lemma 2. *Suppose that $\mu(B(x_0,r)\cap F_\delta)\geq C_2 r^s$ and $n<s\leq n+1\leq m$. Then there exist a constant $C_3>0$ depending on n,s,C_2 and M (from the equation (5)), and $n+2$ points $y_0,\dots,y_{n+1}\in B(x_0,r)\cap F_\delta$ such that for $j=1,\dots,n+1$*

$$d(y_j,L_{j-1})\geq C_3 r, \tag{8}$$

where L_j stands for the j-dimensional hyperplane that contains $y_0,\dots,y_j$

Proof. The proof of this lemma can be found in [DS] (chapter 5, p. 28). For completeness we recall the arguments. We will use induction. Take $1\leq j\leq n$ and suppose that there exist $y_0,\dots,y_j\in B(x_0,r)\cap F_\delta$ satisfying (8) and such that for all $y\in B(x_0,r)\cap F_\delta$, denoting $L_j=\langle y_0,\dots,y_j\rangle$,

$$d(y,L_j)<\nu r$$

with $\nu>0$ to be chosen below. Then $B(x_0,r)\cap F_\delta$ can be covered by C/ν^j balls with radius νr, so using the polynomial growth of degree s of the measure,

$$C_2 r^s\leq \mu(B(x_0,r)\cap F_\delta)\leq \frac{CM}{\nu^j}(\nu r)^s.$$

Taking $\nu<C(C_2/M)^{1/(s-j)}$ we get a contradiction. □

Below we will use the following notation. Given points $y_0,\dots,y_k$, the k-dimensional hyperplane which contains these points is $\langle y_0,\dots,y_k\rangle$. On the other hand, given vectors $u_1,\dots,u_k$, the subspace spanned by $u_1,\dots,u_k$ is denoted by $[u_1,\dots,u_k]$. So we have $\langle y_0,\dots,y_k\rangle=y_0+[y_1-y_0,\dots,y_k-y_0]$.

Lemma 3. *Suppose that $\mu(B(x_0,r)\cap F_\delta)\geq C_2 r^s$ and $r\leq \varepsilon/20$. Consider points $y_0,\dots,y_{n+1}\in B(x_0,r)\cap F_\delta$ and hyperplanes $L_0,\dots,L_{n+1}$ satisfying (8), like in lemma 2 (in particular $L_n=\langle y_0,\dots,y_n\rangle$ and $L_{n+1}=\langle y_0,\dots,y_{n+1}\rangle$). Then we have*

$$d(y_{n+1},L_n)|U^\varepsilon(y_0)|\leq C_4\left(\sum_{j=1}^{n+1}|R^s_{\varphi,\varepsilon}(y_j)-R^s_{\varphi,\varepsilon}(y_0)|+\theta^s(y_0,3\varepsilon)\frac{r^2}{\varepsilon^2}\right), \tag{9}$$

where C_4 depends on C_2 and M, and denoting by $\Pi_{L_{n+1}}(z)$ the orthogonal projection of z onto L_{n+1},

$$\begin{aligned}U^\varepsilon(y_0)=\int\frac{1}{|z-y_0|^{s+1}}\Bigg[&\varphi\left(\frac{|z-y_0|^2}{\varepsilon^2}\right)\left((n+1)-(s+1)\frac{|\Pi_{L_{n+1}}(z-y_0)|^2}{|z-y_0|^2}\right)\\&+2\varphi'\left(\frac{|z-y_0|^2}{\varepsilon^2}\right)\frac{|\Pi_{L_{n+1}}(z-y_0)|^2}{\varepsilon^2}\Bigg]d\mu(z).\end{aligned}$$

Proof. Suppose without loss of generality that $y_0=0$. Consider orthonormal vectors $e_1,\dots,e_{n+1}$ such that $L_k=[e_1,\dots,e_k]$ for $k=1,\dots,n+1$. Moreover, take $e_{n+1}=(y_{n+1}-u)/|y_{n+1}-u|$, where u denotes the orthogonal projection of y_{n+1} onto L_n.

Observe that, denoting $z_{(i)}=z\cdot e_i$ for $i=1,\dots,n+1$,

$$\begin{aligned}U^\varepsilon(0)=\int\frac{1}{|z|^{s+1}}\Bigg[&\varphi\left(\frac{|z|^2}{\varepsilon^2}\right)\left((n+1)-(s+1)\frac{\sum_{k=1}^{n+1}z_{(k)}^2}{|z|^2}\right)\\&+2\varphi'\left(\frac{|z|^2}{\varepsilon^2}\right)\frac{\sum_{k=1}^{n+1}z_{(k)}^2}{\varepsilon^2}\Bigg]d\mu(z)=\sum_{k=1}^{n+1}T^\varepsilon(e_k)\cdot e_k.\end{aligned}$$

To show (9), we will estimate $U^\varepsilon(y_0)$ from above using lemma 1. Let us prove it by induction on k ($k \leq n$):

$$|T^\varepsilon(e_k)\cdot e_k| \leq |T^\varepsilon(e_k)| \lesssim \frac{1}{r}\sum_{j=1}^{k}|T^\varepsilon(y_j)|. \tag{10}$$

For $k=1$ we write $e_1 = y_1/|y_1|$. Since $|y_1| = \operatorname{dist}(y_1, 0) \geq Cr$,

$$|T^\varepsilon(e_1)| \lesssim \frac{1}{r}|T^\varepsilon(y_1)|.$$

Now suppose that equation (10) holds for $k-1$. There exist $\lambda_j, \tilde{\lambda}_j \in \mathbb{R}$, with $\lambda_k \neq 0$, such that

$$e_k = \lambda_k y_k + \sum_{j=1}^{k-1}\lambda_j y_j = \lambda_k y_k + \sum_{j=1}^{k-1}\tilde{\lambda}_j e_j,$$

and so

$$y_k = \frac{1}{\lambda_k}e_k - \sum_{j=1}^{k-1}\frac{\tilde{\lambda}_j}{\lambda_k}e_j.$$

Then,

$$\frac{1}{|\lambda_k|} = |y_k\cdot e_k| = \operatorname{dist}(y_k, L_{k-1}) \geq Cr,$$

so

$$|\lambda_k| \lesssim 1/r.$$

On the other hand, for $j = 1, \ldots, k-1$,

$$0 = e_k\cdot e_j = \lambda_k y_k\cdot e_j + \tilde{\lambda}_j,$$

and so

$$|\tilde{\lambda}_j| = |\lambda_k y_k\cdot e_j| \leq C.$$

Finally,

$$|T^\varepsilon(e_k)| \lesssim \frac{1}{r}|T^\varepsilon(y_k)| + \Big|\sum_{j=1}^{k-1}T^\varepsilon(e_j)\Big| \lesssim \frac{1}{r}\sum_j |T^\varepsilon(y_j)|.$$

Now, since $u \in L_n = [e_1, \ldots, e_n]$ there exist $\overline{\lambda}_1, \ldots, \overline{\lambda}_n$ with $|\overline{\lambda}_i| \leq Cr$ for $i = 1, \ldots, n$ such that $u = \sum_{i=1}^n \overline{\lambda}_i e_i$. Therefore,

$$\begin{aligned}|T^\varepsilon(e_{n+1})| &= \frac{1}{\operatorname{dist}(y_{n+1}, L_n)}|T(y_{n+1}) - T(u)| \\ &\lesssim \frac{1}{\operatorname{dist}(y_{n+1}, L_n)}\left(\sum_{j=1}^{n}|T^\varepsilon(y_j)| + |T^\varepsilon(y_{n+1})|\right).\end{aligned}$$

Applying lemma 1, since $|y_j| \leq r$ for $i = 1, \ldots, n+1$, we finally have

$$|U^\varepsilon(0)| = |\sum_{k=1}^{n+1}T^\varepsilon(e_k)e_k| \lesssim \frac{1}{\operatorname{dist}(y_{n+1}, L_n)}\left(\sum_{j=1}^{n+1}|R^s_{\varphi,\varepsilon}(y_j) - R^s_{\varphi,\varepsilon}(0)| + \theta^s(3\varepsilon)\frac{r^2}{\varepsilon^2}\right).$$

□

The following key lemma gives us a estimate from below of the term $|U^\varepsilon(y_0)|$.

Lemma 4. *Suppose that $\mu(B(x_0,r)) \geq C_2 r^s$ and consider points $y_0,\dots,y_{n+1} \in B(x_0,r)\cap F_\delta$ as in Lemma 3, and let $\varepsilon_1 = r/\tau$ with $\tau < 1/4$. If $\rho > 0$ is a constant small enough (depending only on s), then there exists an $\omega_0 = \omega_0(\tau,s,\rho,M,C_2) \geq 1$ such that we can find an $\varepsilon > 0$ satisfying $\varepsilon_1 \leq \varepsilon \leq \omega_0\varepsilon_1$, $\theta^s(y_0,4\varepsilon) \leq C\theta^s(y_0,\varepsilon)$, $\theta^s(y_0,\varepsilon) \geq C_2\tau^s/2$, and*

$$|U^\varepsilon(y_0)| \geq \frac{7}{10}\frac{\theta^s(y_0,\varepsilon)(n+1-s)}{\varepsilon}.$$

Remark 1. *Notice that this lemma is useful only when s is non integer, that is when $n < s < n+1$. This is one of the key steps of the proof of theorem 1, where there are differences between the integer and the non integer case.*

Proof of lemma 4. Clearly we may assume $s \neq n+1$. Also, we suppose that $y_0 = 0$. For $k \geq 0$, let us denote

$$\delta_k = \sup_{\varepsilon_1 \leq t \leq 4^k\varepsilon_1} \frac{\mu(B(0,t))}{t^s}.$$

Suppose that for all $k \geq 0$ we have $\delta_k \leq \delta_{k+1}/(1+\rho^2/4)$. Then, since $\delta_0 \geq C_2\tau^s$,

$$C_2\tau^s(1+\rho^2/4)^k \leq \delta_k \leq M,$$

which leads to contradiction for k big enough. Thus, there exists $\omega_0 = \omega_0(\tau,s,\rho,M,C_2) > 0$ and there exists $1 \leq k \leq \log_4 \omega_0$ such that $\delta_k \geq \delta_{k+1}/(1+\rho^2/4)$. Take $\varepsilon \in [\varepsilon_1, 4^k\varepsilon_1]$ such that $\delta_k \leq \theta^s(\varepsilon)(1+\rho^2/4)$. Then, $\theta^s(\varepsilon) \geq \tau^s\mu(B(x_0,r))/(2r^s) \geq C_2\tau^s/2$, and also for all t such that $\varepsilon \leq t \leq 4\varepsilon$ we have

$$\theta^s(t) = \frac{\mu(B(0,t))}{t^s} \leq \delta_{k+1} \leq \delta_k(1+\rho^2/4) \leq \theta^s(\varepsilon)(1+\rho^2). \tag{11}$$

Given orthonormal vectors $\{e_i\}_{i=1}^{n+1}$ such that $[e_1,\dots,e_{n+1}] = [y_1-y_0,\dots,y_{n+1}-y_0]$, we denote

$$g_\varepsilon(z) = \frac{1}{|z|^{s+1}}\left[\varphi\left(\frac{|z|^2}{\varepsilon^2}\right)\left((n+1)-(s+1)\frac{\sum_{i=1}^{n+1} z_{(i)}^2}{|z|^2}\right) + 2\varphi'\left(\frac{|z|^2}{\varepsilon^2}\right)\frac{\sum_{i=1}^{n+1} z_{(i)}^2}{\varepsilon^2}\right],$$

where $z_{(i)} = z\cdot e_i$. Consider the following domains:

- $A_1 := \{z \in \mathbb{R}^m : |z| < \varepsilon\}$,
- $A_2 := \{z \in \mathbb{R}^m : \varepsilon \leq |z| \leq \varepsilon\sqrt{1+\rho^2}\}$,
- $A_3 := \{z \in \mathbb{R}^m : \varepsilon\sqrt{1+\rho^2} < |z| < \varepsilon\sqrt{1+\rho+\rho^2}\}$,
- $A_4 := \{z \in \mathbb{R}^m : \varepsilon\sqrt{1+\rho+\rho^2} < |z| < \varepsilon\sqrt{1+\rho+2\rho^2}\}$.

Then,

$$U^\varepsilon(0) = \sum_{i=1}^4 \int_{A_i} g_\varepsilon(z)d\mu(z) =: I_1+I_2+I_3+I_4 \geq I_1 - |I_2| - |I_3| - |I_4|.$$

First we consider I_1:

$$I_1 = \frac{1}{\varepsilon^{s+1}} \int_{|z|<\varepsilon} \frac{1}{|z|^{s+1}} \left[|z|^{s+1} \left((n+1) - (s+1) \frac{\sum_{j=1}^{n+1} z_{(j)}^2}{|z|^2} \right) + (s+1)|z|^{s-1} \sum_{j=1}^{n+1} z_{(j)}^2 \right] d\mu(z)$$
$$= \frac{n+1}{\varepsilon^{s+1}} \int_{|z|<\varepsilon} d\mu(z) = \frac{(n+1)\theta^s(\varepsilon)}{\varepsilon}.$$

Now we estimate I_2 using the fact that for all $r > 0$, $|\varphi(r)| \leq C$ and $|\varphi'(r)| \leq 1/\rho$:

$$|I_2| \leq C \frac{1}{\rho \varepsilon^{s+1}} \mu(B(0, \varepsilon(1+2\rho^2)) \backslash B(0,\varepsilon)) \leq C \frac{\theta^s(\varepsilon)}{\varepsilon} \left(\frac{(1+\rho^2)(1+2\rho^2)^s}{\rho} - \frac{1}{\rho} \right).$$

So, if ρ is small enough,

$$|I_2| \leq \frac{(n+1-s)\theta^s(\varepsilon)}{10\varepsilon}.$$

Let us deal with I_3. Recall that in A_3, $|\varphi(\frac{|z|^2}{\varepsilon^2})| = |-\frac{|z|^2}{\varepsilon^2 \rho} + 1 + \rho + \frac{1}{\rho}| \leq 1$ and $|\varphi'(\frac{|z|^2}{\varepsilon^2})| = \frac{1}{\rho}$. Using that for $z \in A_3$, $\sum_{i=1}^{n+1} z_{(i)}^2 \leq \varepsilon^2(1+\rho+\rho^2)$ and $|z|^2 \geq \varepsilon^2(1+\rho^2)$, we obtain

$$|I_3| \leq \int_{A_3} \frac{1}{|z|^{s+1}} \left| (n+1) - (s+1) \frac{\sum_{i=1}^{n+1} z_{(i)}^2}{|z|^2} \right| + \frac{2}{\rho |z|^{s+1}} \left| \frac{\sum_{i=1}^{n+1} z_{(i)}^2}{\varepsilon^2} \right| d\mu(z)$$
$$\leq \frac{(n+s+2)\mu(A_3)}{\varepsilon^{s+1}(1+\rho^2)^{(s+1)/2}} + \frac{1+\rho+\rho^2}{\varepsilon^{s+1}(1+\rho^2)^{(s+1)/2}} \frac{2\mu(A_3)}{\rho} := I_3^1 + I_3^2.$$

Observe that, by (11), we have

$$\frac{\mu(A_3)}{\varepsilon^{s+1}} \leq \frac{1}{\varepsilon^{s+1}} \left(\mu(B(0, \varepsilon\sqrt{1+\rho+\rho^2}) \backslash B(0,\varepsilon)) \right)$$
$$\leq \frac{\theta^s(\varepsilon)}{\varepsilon} \left((1+\rho^2)(1+\rho+\rho^2)^{s/2} - 1 \right).$$

So

$$|I_3^1| \leq \frac{(n+1-s)\theta^s(\varepsilon)}{10\varepsilon},$$

provided by ρ is small enough. On the other hand, by (11) again,

$$\frac{2\mu(A_3)}{\varepsilon^s \rho} \leq \frac{2}{\varepsilon^s \rho} \left(\mu(B(0, \varepsilon\sqrt{1+\rho+\rho^2}) \backslash B(0,\varepsilon)) \right)$$
$$\leq \frac{2\theta^s(\varepsilon)}{\rho} ((1+\rho)(\sqrt{1+\rho+\rho^2})^s - 1)$$

Since $\lim_{\rho \to 0} \frac{2(1+\rho+\rho^2)^{s/2}-1}{\rho} = s$, we deduce

$$|I_3^2| \leq \frac{s\theta^s(\varepsilon)}{\varepsilon} + \frac{(n+1-s)\theta^s(\varepsilon)}{10\varepsilon},$$

for ρ small enough.

Using similar arguments to the ones used to estimate $|I_2|$ and $|I_3|$ we deduce that

$$|I_4| \leq \frac{(n+1-s)\theta^s(\varepsilon)}{10\varepsilon},$$

for ρ small enough.

We conclude that

$$U^\varepsilon(0) \geq \frac{\theta^s(\varepsilon)}{\varepsilon}\left(n+1-s-\frac{3}{10}(n+1-s)\right) = \frac{7}{10}\frac{(n+1-s)\theta^s(\varepsilon)}{\varepsilon},$$

so taking ρ small enough we are done. □

Remark 2. *In the proof of the preceding lemma the special form of the function φ plays an important role. The choice of this function is one of the key points in our arguments.*

In the following lemma we are strongly using the hypothesis that $\lim_{\varepsilon\to 0} R^s_{\varphi,\varepsilon}\mu(x)$ exists μ-a.e.

Lemma 5. *Given $0<\delta<1/4$, $x_0\in\mathbb{R}^m$ and $r>0$. If $\varepsilon, r/\delta<\varepsilon_0$, then for all $x,z\in B(x_0,r)\cap F_\delta$ we have*

$$|R^s_{\varphi,\varepsilon}\mu(x) - R^s_{\varphi,\varepsilon}\mu(z)| \leq C_6\delta,$$

with C_6 depending on ρ, s and M.

Proof. Take $x,z\in B(x_0,r)\cap F_\delta$ and denote $\eta = r/\delta$. By (3) and using that $B(z,3\eta), B(x,3\eta)\subset B(x_0,4\eta)$,

$$\begin{aligned}|R^s_{\varphi,\eta}\mu(x) - R^s_{\varphi,\eta}\mu(z)| &\leq \int |k_{\varphi,\eta}(x-y)-k_{\varphi,\eta}(z-y)|d\mu(y) \leq |z-x|\|\nabla k_{\varphi,\eta}\|_\infty \mu(B(x_0,4\eta))\\ &\leq \frac{C(\rho)M|z-x|}{\eta} \leq C(\rho)M\delta.\end{aligned}$$

Now, since $x,z\in F_\delta$,

$$\begin{aligned}|R^s_{\varphi,\varepsilon}\mu(x) - R^s_{\varphi,\varepsilon}\mu(z)| &\leq |R^s_{\varphi,\varepsilon}\mu(x) - R^s_{\varphi,\eta}\mu(x)| + |R^s_{\varphi,\eta}\mu(x) - R^s_{\varphi,\eta}\mu(z)|\\ &\quad + |R^s_{\varphi,\eta}\mu(z) - R^s_{\varphi,\varepsilon}\mu(z)| \leq C\delta.\end{aligned}$$

□

3. Proof of theorem 1

Let $0<\delta,\tau<1/4$ to be chosen below and ρ and $\omega_0=\omega_0(\tau,\rho)$ as in lemma 4. Consider the modified Riesz transform $R^s_{\varphi,\varepsilon}$ depending on ρ. Suppose that s is non integer, and so $n<s<n+1\leq m$. Let $x_0\in F_\delta$ be a density point of F_δ with respect to μ. Replacing μ by $\mu/\theta^{s,*}_\mu(x_0)$ if necessary, we may assume that $\theta^{s,*}_\mu(x_0)=1$. Take

$$r<\varepsilon_0\tau^{s+2}/\omega_0 \text{ such that } \mu(B(x_0,r)\cap F_\delta)\geq r^s/2. \tag{12}$$

Applying lemma 2 we can find $n+2$ points $y_0,\dots,y_{n+1}\in B(x_0,r)\cap F_\delta$ such that

$$\operatorname{dist}(y_k, L_{k-1}) \geq Cr \text{ for } k=1,\dots,n+1, \tag{13}$$

where L_k stands for the k-dimensional hyperplane that contains $y_0,\dots,y_k$, and C depends on $s,m,\theta^{s,*}_\mu(x_0)$ and the constant M in (5) (which, in its turn, depends on the constants r_0 and C_0 in the definition of F_δ in (4), but not on ε_0!). Without loss of generality we suppose that $y_0=0$. Taking

$$\varepsilon_1 = r/\tau,$$

by lemma 4, we can find $\varepsilon > 0$ such that

$$\varepsilon_1 \leq \varepsilon \leq \omega_0 \varepsilon_1, \tag{14}$$

$$|U^\varepsilon(0)| \geq C\theta^s(\varepsilon)/\varepsilon, \tag{15}$$

and

$$\theta^s(\varepsilon) \geq C\tau^s \text{ and } \theta^s(4\varepsilon) \leq C\theta^s(\varepsilon). \tag{16}$$

If we take

$$\delta = \tau^{s+2}/\omega_0,$$

(notice that ω_0 is a large number, and so if τ is small enough, $\delta < 1/4$), then we have

$$\frac{r}{\delta} < \varepsilon_0 \text{ by (12)},$$

and

$$\varepsilon \leq \omega_0 \varepsilon_1 = \frac{\omega_0 r}{\tau} < \frac{\varepsilon_0 \tau^{s+2}}{\tau} < \varepsilon_0.$$

By (13) and (15), and lemmas 3 and 5, we obtain

$$\theta^s(\varepsilon) r \lesssim \varepsilon |U^\varepsilon(0)| \mathrm{dist}(y_{n+1}, L_n) \lesssim \varepsilon\delta + \theta^s(3\varepsilon)\frac{r^2}{\varepsilon}. \tag{17}$$

By the definition of δ and ε_1, and by (14) and (16), we get

$$\varepsilon\delta < \omega_0 \varepsilon_1 \frac{\tau^{s+2}}{\omega_0} = \varepsilon_1 \tau^{s+2} < r\theta^s(\varepsilon)\tau,$$

and by (14) and (16), and the definition of ε_1

$$\theta^s(3\varepsilon)\frac{r^2}{\varepsilon} \lesssim \theta^s(\varepsilon)\frac{r^2}{\varepsilon_1} = \theta^s(\varepsilon)\tau r.$$

Thus, by (17),

$$\theta^s(\varepsilon) r \lesssim \tau\theta^s(\varepsilon) r. \tag{18}$$

Finally, taking τ small enough we get a contradiction.

References

[DS] G. David, S. Semmes, *Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs* , Astérisque No. 193 (1991), 152 pp.

[Mar] J.M. Marstrand, *The (φ, s) regular subsets of n-space,* Trans. Amer. Math. Soc. 113 1964 369–392.

[Mat] P. Mattila, *Cauchy singular integrals and rectifiability in measures of the plane*, Adv. Math. 115 (1995), no. 1, 1–34.

[MM] P. Mattila, M.Melnikov, *Existence and weak-type inequalities for Cauchy integrals of general measures on rectifiable curves and sets*, Proc. Amer. Math. Soc. 120(1994), no 1, 143-149.

[MP] P. Mattila, D. Preiss, *Rectifiable measures in R^n and existence of principal values for singular integrals*, J. London Math. Soc. (2) 52 (1995), no. 3, 482–496.

[Pr1] L. Prat, *Potential theory of signed Riesz kernels: capacity and Hausdorff measure,* Int. Math. Res. Not. 2004, no. 19, 937–981.

[Pr2] L. Prat, *Principal values for the signed Riesz kernels of non-integer dimension*, Preprint (2006). To appear in Rocky Mountain J. Math.

[Pre] D. Preiss, *Geometry of measures in R^n: distribution, rectifiability, and densities*, Ann. of Math. (2) 125 (1987), no. 3, 537–643.

[To1] X. Tolsa, *Principal values for the Cauchy integral and rectifiability,* Proc. Amer. Math. Soc. 128 (2000) 2111-2119.

[To2] X. Tolsa, *Principal values for Riesz transforms and rectifiability*, J. Funct. Anal. 254 (2008), no. 7, 1811–1863.

[To3] X. Tolsa, *Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality*, Preprint (2006). To appear in Proc. London Math. Soc.

[Vi] M. Vihtilä, *The boundedness of Riesz s-transforms of measures in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. 124 (1996), no. 12, 3797–3804.

Aleix Ruiz de Villa, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Catalunya

E-mail address: aleixrv@mat.uab.cat

Xavier Tolsa, Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Catalunya

E-mail address: xtolsa@mat.uab.cat