THE SEMIADDITIVITY OF CONTINUOUS ANALYTIC CAPACITY AND THE INNER BOUNDARY CONJECTURE

XAVIER TOLSA

ABSTRACT. Let $\alpha(E)$ be the continuous analytic capacity of a compact set $E \subset \mathbb{C}$. In this paper we obtain a characterization of α in terms of curvature of measures with zero linear density, and we deduce that α is countably semiadditive. This result has important consequences for the theory of uniform rational approximation on compact sets. In particular, it implies the so called inner boundary conjecture.

1. INTRODUCTION

The continuous analytic capacity α was introduced by Erokhin and Vitushkin (see [Vi]) in the 1950's in order to study problems of uniform rational approximation in compact subsets of the complex plane. Up to now, the geometric properties of the capacity α have not been well understood. It has been an open question if α is semiadditive as a set function, that is to say, if

$$\alpha(E \cup F) \leq C(\alpha(E) + \alpha(F)),$$

for arbitrary compact sets $E, F \subset \mathbb{C}$, where C is an absolute constant. It was known that an affirmative answer to this question would imply important results on uniform rational approximation, such as the so called “inner boundary conjecture” (see [VM, Conjecture 2]). In this paper we will show that α is indeed semiadditive. As a consequence, the inner boundary conjecture is true. Our proof of the semiadditivity of α does not follow from the recently proved semiadditivity of analytic capacity γ in [To5]. However, some of the ideas and techniques of [To5] are essential for the results of this paper.

To state these results in more detail, we need to introduce some notation and terminology. The continuous analytic capacity of a compact set $E \subset \mathbb{C}$ is defined as

$$\alpha(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all complex functions which are continuous in $\mathbb{C}$, analytic on $\mathbb{C} \setminus E$, and satisfy $|f(z)| \leq 1$ for all $z \in \mathbb{C}$; and $f'(\infty) = \lim_{z \rightarrow \infty} z(f(z) - f(\infty))$. For a general set F , we set $\alpha(F) = \sup\{\alpha(E) : E \subset F, E \text{ compact}\}$.

Date: May, 2002.

Supported by the program Ramón y Cajal (Spain). Also partially supported by grants DGICYT BFM2000-0361 (Spain) and 2001/SGR/00431 (Generalitat de Catalunya).

If in the supremum above we don't ask the continuity on $\mathbb{C}$ for f (we only ask f to be analytic on $\mathbb{C} \setminus E$ and $|f(z)| \leq 1$ for all $z \in \mathbb{C} \setminus E$), we obtain the analytic capacity γ of the compact set E . For a general set F , we set $\gamma(F) = \sup\{\gamma(E) : E \subset F, E \text{ compact}\}$.

Our main result is the following.

Theorem 1.1. *Let E_i , $i \geq 1$, be Borel sets in $\mathbb{C}$. Then,*

$$\alpha\left(\bigcup_{i=1}^{\infty} E_i\right) \leq C \sum_{i=1}^{\infty} \alpha(E_i),$$

where C is an absolute constant.

The semiadditivity of continuous analytic capacity was already known in some special cases. Melnikov [Me1] proved it for two compact sets E, F separated by an analytic curve. Vitushkin [Vi] extended the result to the case in which the curve which separates E and F is piecewise Lyapunov. Davie [Dve] showed that it is enough to assume that the curve is hypo-Lyapunov. On the other hand, Davie also proved in [Dve] that the semiadditivity of γ for arbitrary compact sets (recently proved in [To5]) implies the semiadditivity of α for *disjoint* compact sets. However, as far as we know, the semiadditivity of α for arbitrary compact sets (which is needed for the proof of the inner boundary conjecture) cannot be derived directly from the semiadditivity of γ . Melnikov, Paramonov and Verdera [VMP] also proved the semiadditivity of α_+ (which is a variant of α originated by Cauchy transforms of positive measures) in several cases (in particular, for bounded Borel sets E and F which are relatively open in the topology of $E \cup F$).

We need now to introduce some additional terminology in connection with uniform rational approximation. Given $E \subset \mathbb{C}$ compact, we denote by $R(E)$ the algebra of complex functions on E which are uniform limits on E of functions analytic in a neighborhood of E (i.e. each function is analytic in a neighborhood of E). $A(E)$ is the algebra of those complex functions on E which are continuous on E and analytic on $\overset{\circ}{E}$. Vitushkin proved that the following result is a consequence of the semiadditivity of α (see [Vi, p.187]).

Theorem 1.2. *Suppose that for all $z \in \partial E$, with the exception of a set with zero continuous analytic capacity, we have*

$$\limsup_{r \rightarrow 0} \frac{\alpha(\bar{B}(z, r) \setminus \overset{\circ}{E})}{\alpha(\bar{B}(z, r) \setminus E)} < \infty.$$

Then, $R(E) = A(E)$.

The inner boundary of E , denoted by $\partial_i E$, is the set of boundary points which do not belong to the boundary of any connected component of $\mathbb{C} \setminus E$. The inner boundary conjecture is a corollary of the preceding theorem:

Theorem 1.3 (Inner boundary conjecture). *If $\alpha(\partial_i E) = 0$, then $R(E) = A(E)$.*

In the special case where $\dim(\partial_i E) < 1$ (where $\dim$ is the Hausdorff dimension), Davie and Øksendal [DØ] had already proved that $R(E) = A(E)$. Another partial result was obtained in [MTV2] as a straightforward consequence of the semiadditivity of γ . Namely, it was shown that $R(E) = A(E)$ also holds if $\gamma(\partial_i E) = 0$.

The semiadditivity of α follows from a more precise result which asserts that α and α_+ are comparable. Further, α_+ (and also α , because of its comparability with α) can be characterized in terms of the so called curvature of measures, or equivalently in terms of L^2 estimates for the Cauchy transform. We now proceed to define the main notions involved in these results.

Given a complex Radon measure ν on $\mathbb{C}$, the *Cauchy transform* of ν is

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi - z} d\nu(\xi), \quad z \notin \text{supp}(\nu).$$

This definition does not make sense, in general, for $z \in \text{supp}(\nu)$, although one can easily see that the integral above is convergent at a.e. $z \in \mathbb{C}$ (with respect to the Lebesgue measure $\mathcal{L}^2$). This is the reason why one considers the *truncated Cauchy transform* of ν , which is defined as

$$\mathcal{C}_\varepsilon \nu(z) = \int_{|\xi - z| > \varepsilon} \frac{1}{\xi - z} d\nu(\xi),$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$. Given a μ -measurable function f on $\mathbb{C}$ (where μ is some fixed positive Radon measure on $\mathbb{C}$), we write $\mathcal{C}f \equiv \mathcal{C}(f d\mu)$ and $\mathcal{C}_\varepsilon f \equiv \mathcal{C}_\varepsilon(f d\mu)$ for any $\varepsilon > 0$. It is said that the Cauchy transform is bounded on $L^2(\mu)$ if the operators $\mathcal{C}_\varepsilon$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

The capacity α_+ of a bounded set $E \subset \mathbb{C}$ is defined as

$$\alpha_+(E) = \sup \mu(E),$$

where the supremum is taken over all positive Radon measures μ supported on E such that $\mathcal{C}\mu$ is a continuous function on $\mathbb{C}$ (i.e. it coincides $\mathcal{L}^2$ -a.e. with a continuous function on $\mathbb{C}$), with $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$. Notice that we clearly have $\alpha_+(E) \leq \alpha(E)$, because $(\mathcal{C}\mu)'(\infty) = \mu(E)$.

If in the definition of $\alpha_+(E)$ we don't ask $\mathcal{C}\mu$ to be continuous on $\mathbb{C}$, we obtain $\gamma_+(E)$. That is, $\gamma_+(E) = \sup \mu(E)$, with the supremum taken over all positive Radon measures μ supported on E such that $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$.

A positive Radon measure μ is said to have linear growth if there exists some constant C such that $\mu(B(x, r)) \leq Cr$ for all $x \in \mathbb{C}$, $r > 0$. The linear density of μ at $x \in \mathbb{C}$ is (if it exists)

$$\Theta_\mu(x) = \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{r}.$$

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x, y, z) = \frac{1}{R(x, y, z)},$$

where $R(x, y, z)$ is the radius of the circumference passing through x, y, z (with $R(x, y, z) = \infty$, $c(x, y, z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we let $c(x, y, z) = 0$. For a positive Radon measure μ , we set

$$c_\mu^2(x) = \iint c(x, y, z)^2 d\mu(y) d\mu(z),$$

and we define the *curvature of μ* as

$$(1.1) \quad c^2(\mu) = \int c_\mu^2(x) d\mu(x) = \iiint c(x, y, z)^2 d\mu(x) d\mu(y) d\mu(z).$$

The notion of curvature of measures was introduced by Melnikov [Me2] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible of the big recent advances in connection with analytic capacity. On the one hand, the notion of curvature is connected to the Cauchy transform. This relationship comes from the following identity found by Melnikov and Verdera [MV] (assuming that μ has linear growth):

$$(1.2) \quad \|\mathcal{C}_\varepsilon \mu\|_{L^2(\mu)}^2 = \frac{1}{6} c_\varepsilon^2(\mu) + O(\mu(\mathbb{C})),$$

where $c_\varepsilon^2(\mu)$ is an ε -truncated version of $c^2(\mu)$ (defined as in the right hand side of (1.1), but with the triple integral over $\{x, y, z \in \mathbb{C} : |x - y|, |y - z|, |x - z| > \varepsilon\}$).

Next theorem contains the aforementioned characterization of α and α_+ .

Theorem 1.4. *For all compact sets $E \subset \mathbb{C}$, we have*

$$(1.3) \quad \alpha(E) \approx \alpha_+(E) \\ \approx \sup\{\mu(E) : \text{supp}(\mu) \subset E, \Theta_\mu(x) = 0 \ \forall x \in E, c^2(\mu) \leq \mu(E)\},$$

with absolute constants.

The notation $A \approx B$ means that A is comparable to B , that is to say, that there exists a positive absolute constant C such that $C^{-1}A \leq B \leq CA$.

In (2.3) in next section, the reader will find another characterization of α and α_+ which involves the L^2 norm of the Cauchy transform.

Let us remark that, by the results of [To4, Section 5], it follows that the supremum in (1.3) is comparable to

$$\sup\{\mu(E) : \text{supp}(\mu) \subset E, \Theta_\mu(x) = 0 \ \forall x \in E, \\ \mu(B(x, r)) \leq r \ \forall x \in E, r > 0, c^2(\mu) \leq \mu(E)\}.$$

See Remark 2.2 below for more details. Notice also that the latter supremum without the condition “ $\Theta_\mu(x) = 0 \ \forall x \in E$ ” is comparable to $\gamma_+(E)$ and $\gamma(E)$ (see [To4] and [To5]).

The first step for proving Theorem 1.4 consists of showing that α_+ is comparable to the supremum in (1.3). This is done in Section 3. The difficult inequality is $\alpha_+(E) \geq C^{-1} \sup \dots$. We will derive it from the fact that if $\Theta_\mu(x) = 0$ μ -a.e. in $\mathbb{C}$, then the $L^2(\mu)$ norm of the Cauchy transform

restricted to a ball $B(x, r)$ tends to 0 as $r \rightarrow 0$ (assuming some uniformity conditions concerning the linear density and curvature of μ). Let us remark that the same idea was used in [To1] to prove that the $L^2(\mu)$ boundedness of the Cauchy transform implies the existence of principal values μ -a.e. when the linear density of μ is zero.

The second step for Theorem 1.4 is the proof of the comparability between α and α_+ . A possible approach consists of proving that there exists a measure μ supported on E with zero linear density, such that $\mu(E) \approx \alpha(E)$, with the Cauchy transform bounded in $L^2(\mu)$ with absolute constants. If we try to use arguments similar to the ones of [To5] to construct μ , several difficulties arise. For example, as far as we know, α_+ does not admit a dual characterization such as the one of γ_+ in [To4, Theorem 3.3]. This characterization plays a key role in the proof of $\gamma \approx \gamma_+$. To overcome these difficulties we have introduced in Section 4 some auxiliary capacities γ^h and γ_+^h associated to a gauge function h that measures how fast $\mu(B(x, r))/r$ tends to 0. The comparability of α and α_+ is then a consequence of the comparability between γ^h and γ_+^h for each h . Some properties of γ^h and γ_+^h are shown in Sections 5 and 6. They will be used in Section 7 to show that $\gamma^h \approx \gamma_+^h$, by arguments analogous to the ones of the proof of $\gamma \approx \gamma_+$ in [To5].

2. PRELIMINARIES

Let M be the maximal radial Hardy-Littlewood operator:

$$M\mu(x) = \sup_{r>0} \frac{\mu(B(x, r))}{r}$$

[if μ were a complex measure, we would replace $\mu(B(x, r))$ by $|\mu|(B(x, r))$], and let $c_\mu(x) = (c_\mu^2(x))^{1/2}$. The following potential was introduced by Verdera in [Ve2]:

$$(2.1) \quad U_\mu(x) := M\mu(x) + c_\mu(x),$$

We recall that γ_+ and γ can be characterized in terms of this potential:

$$\gamma(E) \approx \gamma_+(E) \approx \sup\{\mu(E) : \text{supp}(\mu) \subset E, U_\mu(x) \leq 1 \forall x \in \mathbb{C}\}.$$

From Theorem 1.4 (see also Remark 2.2 below) we infer that α and α_+ also can be described using the potential U_μ :

$$(2.2) \quad \alpha(E) \approx \alpha_+(E) \approx \sup\{\mu(E) : \text{supp}(\mu) \subset E, \Theta_\mu(x) = 0, U_\mu(x) \leq 1 \forall x \in \mathbb{C}\}.$$

Moreover, using the identity (1.2), with arguments analogous to the ones for γ_+ in [To2] and [To4], it can be shown that (1.3) and (2.2) are equivalent to

$$(2.3) \quad \alpha(E) \approx \alpha_+(E) \approx \sup\{\mu(E) : \text{supp}(\mu) \subset E, \Theta_\mu(x) = 0, \|\mathcal{C}\|_{L^2(\mu), L^2(\mu)} \leq 1\}.$$

Given two positive Radon measure ν and μ , we set

$$c^2(x, \nu, \mu) = \iint c(x, y, z)^2 d\nu(y) d\mu(z).$$

Remember the following maximum principle for curvature of measures (see [To2] and [To4]).

Lemma 2.1. *Let μ and ν be positive Radon measures such that*

$$c^2(x, \nu, \mu) \leq \beta \quad \text{for all } x \in \text{supp}(\nu).$$

Then,

$$c^2(x, \nu, \mu) \leq 2\beta + C_1 M\nu(x) \cdot \sup_{z \in \text{supp}(\nu)} M\mu(z) \quad \text{for all } x \in \mathbb{C}.$$

If $c_\mu^2(x) \leq \beta$ for all $x \in \text{supp}(\mu)$, then

$$c_\mu^2(x) \leq 2\beta + C_1 M\mu(x)^2 \quad \text{for all } x \in \mathbb{C}.$$

Remark 2.2. If μ is such that $\Theta_\mu^*(x) := \limsup_{r \rightarrow 0} \mu(B(x, r))/r \leq 1$ and $c_\mu(x) \leq 1$ for all $x \in \text{supp}(\mu)$, then there exists some absolute constant C such that $U_\mu(x) \leq C$ for all $x \in \mathbb{C}$. This follows from Lemma 5.2 in [To4], which asserts that if ν is a positive Radon measure supported on $B(x_0, R)$ with $\Theta_\nu^*(x) \leq 1$ ν -a.e. and $c^2(\nu) \leq \nu(B(x_0, R))$, then there is some absolute constant M_0 such that $\nu(B(x_0, R)) \leq M_0 R$. Indeed, from the condition

$$c_\mu(x) \leq 1 \quad \text{for all } x \in \text{supp}(\mu)$$

we deduce

$$c^2(\mu|B(x, r)) \leq \mu(B(x, r)) \quad \text{for all } x \in \text{supp}(\mu), r > 0,$$

and so $M\mu(x) \leq M_0$ for all $x \in \text{supp}(\mu)$. Thus $U_\mu(x) \leq M_0 + 1$ for all $x \in \text{supp}(\mu)$, and then, by the maximum principle above, $U_\mu(x) \leq C$ for all $x \in \mathbb{C}$.

This is the reason why the statement “ $\mu(B(x, r)) \leq r$ ” can be avoided in (1.3).

Let us recall the definition of the maximal Cauchy transform of a complex measure ν :

$$\mathcal{C}_*\nu(x) = \sup_{\varepsilon > 0} |\mathcal{C}_\varepsilon\nu(x)|.$$

If f is a μ -measurable function (for a fixed measure μ), we set $\mathcal{C}_*f \equiv \mathcal{C}_*(f d\mu)$.

Let ψ be a $\mathcal{C}^\infty$ radial function supported on $B(0, 1)$, with $0 \leq \psi \leq 2$, $\|\nabla\psi\|_\infty \leq 10$, and $\int \psi d\mathcal{L}^2 = 1$ (where $\mathcal{L}^2$ stands for the Lebesgue measure). We denote $\psi_\varepsilon(x) = \varepsilon^{-2}\psi(x/\varepsilon)$. The regularized operators K_ε are defined as

$$K_\varepsilon\nu := \psi_\varepsilon * \mathcal{C}\nu = \psi_\varepsilon * \frac{1}{z} * \nu.$$

Let $k_\varepsilon = \psi_\varepsilon * \frac{1}{z}$. It is easily seen that $k_\varepsilon(z) = 1/z$ if $|z| > \varepsilon$, $\|k_\varepsilon\|_\infty \leq C/\varepsilon$, and $|\nabla k_\varepsilon(z)| \leq C|z|^{-2}$. Further, since k_ε is a uniformly continuous kernel, $K_\varepsilon \nu$ is a continuous function on $\mathbb{C}$.

Notice that if $|\mathcal{C}\nu| \leq B$ a. e. with respect to Lebesgue measure, then $|K_\varepsilon \nu(z)| \leq B$ for all $z \in \mathbb{C}$. Moreover, we have

$$(2.4) \quad |K_\varepsilon \nu(x) - \mathcal{C}_\varepsilon \nu(x)| = \left| \int_{|y-x| \leq \varepsilon} k_\varepsilon(y-x) d\nu(y) \right| \leq C M\nu(x).$$

We will also use the notation $k_\varepsilon(x, y) := k_\varepsilon(y-x)$.

The corresponding maximal operator associated to K_ε is

$$K_* \nu(x) = \sup_{\varepsilon > 0} |K_\varepsilon \nu(x)|.$$

Given a μ -measurable function f on $\mathbb{C}$, we write $K_\varepsilon f \equiv K_\varepsilon(f d\mu)$ and $K_* f \equiv K_*(f d\mu)$ for any $\varepsilon > 0$.

Observe that the definitions of $K_\varepsilon \nu$ and $K_* \nu$ also make sense if ν is a distribution with compact support instead of a measure.

By a square (rectangle) Q we mean a closed square (rectangle) with sides parallel to the axes, unless otherwise stated.

Throughout all the paper, the letter C will stand for an absolute constant that may change at different occurrences. Constants with subscripts, such as C_1 , will retain its value, in general.

3. THE CAPACITY α_+

In this section we will prove:

Theorem 3.1. *For all compact sets $E \subset \mathbb{C}$, we have*

$$\alpha_+(E) \approx \sup\{\mu(E) : \text{supp}(\mu) \subset E, \Theta_\mu(x) = 0 \forall x \in E, U_\mu(x) \leq 1 \forall x \in \mathbb{C}\},$$

with absolute constants.

We denote

$$\alpha_0(E) := \sup\{\mu(E) : \text{supp}(\mu) \subset E, \Theta_\mu(x) = 0 \forall x \in E, U_\mu(x) \leq 1 \forall x \in \mathbb{C}\}.$$

Next lemma deals with the easy inequality of Theorem 3.1.

Lemma 3.2. *There exists an absolute constant C such that*

$$\alpha_+(E) \leq C\alpha_0(E)$$

for any compact set $E \subset \mathbb{C}$.

Proof. Let μ be a Radon measure supported on E such that $\mathcal{C}\mu$ is a continuous function on $\mathbb{C}$, with $\|\mathcal{C}\mu\|_{L^\infty(\mathbb{C})} \leq 1$, and $\mu(E) \geq \alpha_+(E)/2$. Then, for all $x \in \mathbb{C}$ and $r > 0$ we have

$$(3.1) \quad \mu(B(x, r)) = \frac{-1}{2\pi i} \int_{|z-x|=r} \mathcal{C}\mu(z) dz \leq r$$

(see [Gar, p.40], for example). Also,

$$c^2(\mu) \leq C_2 \mu(E),$$

by (1.2). By Chebishev, there exists some compact set $F \subset E$ such that $c_\mu^2(x) \leq 2C_2$ and $\mu(F) \geq 4\mu(E)$. We set $\nu = \mu|_F$. Since $c_\nu^2(x) \leq 2C_2$ for all $x \in F$, by the maximum principle for curvature of measures (see Lemma 2.1), we get

$$c_\nu^2(x) \leq C_3 \quad \text{for all } x \in \mathbb{C}.$$

Thus,

$$U_\nu(x) \leq 1 + C_3^{1/2} =: C_4 \quad \text{for all } x \in \mathbb{C}.$$

To see that $\Theta_\nu(x) = 0$ for all $x \in E$ it suffices to show that the same condition holds for μ . Given $\varepsilon > 0$, let $\delta > 0$ be such that $|\mathcal{C}\mu(z) - \mathcal{C}\mu(x)| \leq \varepsilon$ for $x, z \in \mathbb{C}$ with $|z - x| \leq \delta$. For $r \leq \delta$ we deduce

$$\begin{aligned} \mu(B(x, r)) &= \frac{-1}{2\pi i} \int_{|z-x|=r} \mathcal{C}\mu(z) dz \\ &= \frac{-1}{2\pi i} \int_{|z-x|=r} (\mathcal{C}\mu(z) - \mathcal{C}\mu(x)) dz \leq \varepsilon r. \end{aligned}$$

That is, $\mu(B(x, r))/r \leq \varepsilon$ if $r \leq \delta$, and so $\Theta_\mu(x) = 0$.

Therefore, we have

$$\alpha_+(E) \leq C\nu(E) \leq C\alpha_0(E).$$

□

The proof of the inequality $\alpha_0(E) \leq C\alpha_+(E)$ involves the regularized operators K_ε introduced in Section 2. The following lemma concentrates the main technical difficulties of the proof.

Lemma 3.3. *Suppose that $\lim_{r \rightarrow 0} \mu(B(x, r))/r = 0$ and $\lim_{r \rightarrow 0} c_{\mu|B(x, r)}^2(x) = 0$ both uniformly on E . Let f be a positive bounded function supported on E , such that $\|K_*f\|_{L^\infty(\mathbb{C})} < \infty$. Given any $\tau > 0$, there exists some $\delta > 0$ and some function g supported on E such that $\int g d\mu = \int f d\mu$, $0 \leq g \leq \|f\|_{L^\infty(\mu)} + \tau$, $\|K_*g\|_{L^\infty(\mathbb{C})} \leq \|K_*f\|_{L^\infty(\mathbb{C})} + \tau$, and moreover*

$$(3.2) \quad |K_\varepsilon g(x) - K_\varepsilon g(y)| \leq \tau \quad \text{for } |x - y| \leq \delta \text{ and all } \varepsilon > 0,$$

and

$$(3.3) \quad |K_\varepsilon g(x) - K_\varepsilon g(y)| \leq \sup_{\varepsilon' > 0} |K_{\varepsilon'} f(x) - K_{\varepsilon'} f(y)| + \tau \quad \text{for all } x, y \in \mathbb{C}, \varepsilon > 0.$$

In a sense, this lemma asserts that under the appropriate assumptions, given a positive bounded function f such that K_*f is bounded, one can construct another positive bounded function g with $\int g d\mu = \int f d\mu$, with the L^∞ norms of g and K_*g very close to the corresponding L^∞ norms of f and K_*f , such that the term $|K_\varepsilon g(x) - K_\varepsilon g(y)|$ is very small when x and y are close enough (by (3.2)), and such that the same term is also under control if we don't assume x and y to be close (by (3.3)).

Before proving Lemma 3.3, we will show how one can deduce the inequality $\alpha_0(E) \leq C\alpha_+(E)$ from it.

Lemma 3.4. *There exists an absolute constant C such that*

$$\alpha_0(E) \leq C\alpha_+(E)$$

for any compact set $E \subset \mathbb{C}$.

Proof. Let μ be a Radon measure supported on E such that $\mu(E) \geq \alpha_0(E)/2$, with $\Theta_\mu(x) = 0$ for all $x \in E$, and $U_\mu(x) \leq 1$ for all $x \in \mathbb{C}$. Let $F \subset E$ a compact set such that $\lim_{r \rightarrow 0} \mu(B(x, r))/r = 0$ and $\lim_{r \rightarrow 0} c_{\mu|B(x, r)}^2(x) = 0$ both uniformly on F , with $\mu(F) \geq \mu(E)/2$. Since the Cauchy transform is bounded on $L^2(\mu)$, there exists a function f_1 supported on F , with $0 \leq f_1 \leq 1$, $\|K_* f_1\|_{L^\infty(\mathbb{C})} \leq 1$, and $\int f_1 d\mu \geq C^{-1}\mu(F)$. We set $\delta_1 = 1$.

For $n \geq 1$, we set $\tau_n = 2^{-n}$, and given a positive bounded function f_n supported on F and $\delta_n > 0$, by means of Lemma 3.3 we construct a function f_{n+1} also supported on F , so that $\int f_{n+1} d\mu = \int f_n d\mu$, $0 \leq f_{n+1} \leq \|f_n\|_{L^\infty(\mu)} + \tau_n$, $\|K_* f_{n+1}\|_{L^\infty(\mathbb{C})} \leq \|K_* f_n\|_{L^\infty(\mathbb{C})} + \tau_n$, and moreover

$$(3.4) \quad |K_\varepsilon f_{n+1}(x) - K_\varepsilon f_{n+1}(y)| \leq \tau_n \quad \text{for } |x - y| \leq \delta_{n+1} \text{ and all } \varepsilon > 0$$

(where $\delta_{n+1} \leq \delta_n$ is some constant small enough), and

$$(3.5) \quad |K_\varepsilon f_{n+1}(x) - K_\varepsilon f_{n+1}(y)| \leq \sup_{\varepsilon' > 0} |K_{\varepsilon'} f_n(x) - K_{\varepsilon'} f_n(y)| + \tau_n$$

for all $x, y \in \mathbb{C}$, $\varepsilon > 0$.

Let f be a weak $*$ limit in $L^\infty(\mu)$ of a subsequence $\{f_{n_k}\}_k$. Clearly, f is a positive bounded function such that

$$\int f d\mu = \int f_1 d\mu \geq C^{-1}\alpha_0(E).$$

Also, for each $\varepsilon > 0$ and $x \in \mathbb{C}$,

$$K_\varepsilon f_{n_k}(x) \rightarrow K_\varepsilon f(x) \quad \text{as } k \rightarrow \infty.$$

Since

$$\|K_\varepsilon f_n\|_{L^\infty(\mathbb{C})} \leq \|K_* f_1\|_{L^\infty(\mathbb{C})} + \sum_{j=1}^{n-1} 2^{-j} \leq 2$$

for all n , we deduce $\|K_\varepsilon f\|_{L^\infty(\mathbb{C})} \leq 2$.

On the other hand, by (3.4), if $|x - y| \leq \delta_n$, then

$$|K_\varepsilon f_n(x) - K_\varepsilon f_n(y)| \leq 2^{-n}$$

for all $\varepsilon > 0$. From (3.5), for $k \geq n$ we get

$$|K_\varepsilon f_k(x) - K_\varepsilon f_k(y)| \leq \sup_{\varepsilon' > 0} |K_{\varepsilon'} f_n(x) - K_{\varepsilon'} f_n(y)| + \sum_{j=n}^{k-1} 2^{-j} \leq 2^{-n+2},$$

assuming $|x - y| \leq \delta_n$. Thus,

$$(3.6) \quad |K_\varepsilon f(x) - K_\varepsilon f(y)| \leq 2^{-n+2} \quad \text{if } |x - y| \leq \delta_n.$$

Consider now the family of functions $\{K_\varepsilon f\}_{\varepsilon>0}$ on $\bar{B}(0, R)$, where R is big enough so that $E \subset B(0, R-1)$. This is a family of functions which is uniformly bounded and equicontinuous on $\bar{B}(0, R)$, by (3.6). By the Ascoli-Arzelà theorem, there exists a sequence $\{\varepsilon_n\}_n$, with $\varepsilon_n \rightarrow 0$, such that $K_{\varepsilon_n} f$ converges uniformly on $\bar{B}(0, R)$ to some continuous function g . It is easily seen that g coincides with $\mathcal{C}f$ $\mathcal{L}^2$ -a.e. in $\bar{B}(0, R)$ and $\|g\|_{L^\infty(\bar{B}(0, R))} \leq 2$. Since $\mathcal{C}f$ is also continuous in $\mathbb{C} \setminus \bar{B}(0, R)$, we deduce that $\mathcal{C}f$ can be extended continuously to the whole complex plane. Therefore,

$$\alpha_+(E) \geq C^{-1} \int f d\mu \geq C^{-1} \alpha_0(E).$$

□

To prove Lemma 3.3 we will need some preliminary results.

Lemma 3.5. *Suppose that $\mu(B(x, r)) \leq C_0 r$ for all $r > r_0$. Then,*

$$\int_{|x-y| \geq r_0} \frac{1}{|x-y|^2} d\mu(y) \leq C_5 \frac{C_0}{r_0},$$

where C_5 is an absolute constant.

The proof of this standard estimate follows easily if we split the complex plane in rings centered at x , for example.

Next lemma deals with a now quite well known dualization of the usual weak $(1, 1)$ inequality for the Cauchy transform, which is a consequence of its $L^2(\mu)$ boundedness (see either [To2], [NTV1] or [To3], for example).

Lemma 3.6. *Suppose that $\mu(B(x, r)) \leq \varepsilon_0 r$ for all $x \in \mathbb{C}$, $r > 0$, and $\|\mathcal{C}\|_{L^2(\mu), L^2(\mu)} \leq \varepsilon_0$. Given $\tau > 0$, for all $F \subset \mathbb{C}$ there exists some function φ supported on F , such that $0 \leq \varphi \leq 1$, $\int \varphi d\mu \geq (1 + \tau)^{-1} \mu(F)$, and $\|K_\varepsilon \varphi\|_{L^\infty(\mu)} \leq C(\tau, \varepsilon_0)$ for all $\varepsilon > 0$. Moreover, for any fixed $\tau > 0$, $C(\tau, \varepsilon_0) \rightarrow 0$ as $\varepsilon_0 \rightarrow 0$.*

For the proof, see [Ch, p.107-108]. Regarding the fact that φ can be chosen independently of ε , an argument similar to the one in [To2, Lemma 4.4], for example, works.

Proof of Lemma 3.3. Consider the grid of all dyadic squares $\{Q_i\}_{i \in I}$ with side length ℓ . Notice that since $\Theta_\mu(x) = 0$ for all x , we have $\mu(\partial Q_i) = 0$ for all i . Let N be some big integer and $\varepsilon_1 \ll \tau / \|f\|_{L^\infty(\mu)}$ some small constant both to be chosen later. We will take ℓ small enough so that

$$(3.7) \quad \frac{\mu(B(x, r))}{r} \leq \varepsilon_1 \quad \text{for all } x \in E, 0 < r \leq N\ell.$$

We also assume that ℓ has been chosen so small that

$$(3.8) \quad c_{\mu|_{Q_i}}(x) \leq \varepsilon_1 \quad \text{for all } x \in E \cap Q_i.$$

Notice that (3.7) and (3.8) imply that

$$\|\mathcal{C}\|_{L^2(\mu|_{Q_i}), L^2(\mu|_{Q_i})} \leq \varepsilon_2,$$

with $\varepsilon_2 \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$ (using Theorem 1.1 of [To2], for example). This will be the main ingredient of the proof.

Now we are going to start the construction of g . Let Q_i be some fixed square of the grid such that $\mu(Q_i) \neq 0$. Since the Cauchy transform is bounded on $L^2(\mu|_{Q_i})$ with a very small norm (assuming ε_2 very small), by Lemma 3.6 there exists some function φ_i supported on Q_i (not depending on ε), with $0 \leq \varphi_i \leq 1$, such that

$$(3.9) \quad \int \varphi_i d\mu \geq (1 + \tau)^{-1} \mu(Q_i)$$

and

$$\|K_\varepsilon \varphi_i\|_{L^\infty(\mathbb{C})} \leq \varepsilon_3,$$

with $\varepsilon_3 \rightarrow 0$ as $\|\mathcal{C}\|_{L^2(\mu|_{Q_i})} \rightarrow 0$. We set

$$g_i = \frac{\int_{Q_i} f d\mu}{\int \varphi_i d\mu} \varphi_i.$$

Then, by (3.9), we have

$$(3.10) \quad \|K_\varepsilon g_i\|_{L^\infty(\mathbb{C})} \leq \frac{\int_{Q_i} f d\mu}{\int \varphi_i d\mu} \varepsilon_3 \leq \varepsilon_3 (1 + \tau) \|f\|_{L^\infty(\mu)} \leq 2\varepsilon_3 \|f\|_{L^\infty(\mu)}$$

(assuming $\tau \leq 1$). We denote $g = \sum_{i \in I} g_i$. By construction $g \geq 0$, $\int g d\mu = \int f d\mu$ and, by (3.9),

$$\|g\|_{L^\infty(\mu)} = \sup_{i \in I} \|g_i\|_{L^\infty(\mu)} \leq (1 + \tau) \|f\|_{L^\infty(\mu)}.$$

Proof of (3.2). We set $\delta = \ell/2$. Let Q_i be some fixed square of the grid. Notice that if $x \in Q_i$ and $|x - y| \leq \delta$, then $y \in 2Q_i$. We set $g_a = g\chi_{3Q_i}$ and $g_b = g\chi_{\mathbb{C} \setminus 3Q_i}$.

First we deal with g_a . We have

$$(3.11) \quad \begin{aligned} |K_\varepsilon g_a(x) - K_\varepsilon g_a(y)| &\leq |K_\varepsilon g_a(x)| + |K_\varepsilon g_a(y)| \\ &\leq \sum_{k: Q_k \subset 3Q_i} (|K_\varepsilon g_k(x)| + |K_\varepsilon g_k(y)|) \leq 36\varepsilon_3 \|f\|_{L^\infty(\mu)}. \end{aligned}$$

Now we consider the term originated by g_b . We have

$$\begin{aligned} |K_\varepsilon g_b(x) - K_\varepsilon g_b(y)| &\leq \|g\|_{L^\infty(\mu)} \int_{\mathbb{C} \setminus 3Q_i} |k_\varepsilon(x, z) - k_\varepsilon(y, z)| d\mu(z) \\ &\leq C\ell \|f\|_{L^\infty(\mu)} \int_{\mathbb{C} \setminus 3Q_i} \frac{1}{|x - z|^2} d\mu(z). \end{aligned}$$

To estimate the last integral we use that the ratio $\mu(B(x, r))/r$ is very small if $r \leq N\ell$. Indeed, by Lemma 3.5 and (3.7) we get

$$\begin{aligned} \int_{\mathbb{C} \setminus 3Q_i} \frac{1}{|x - z|^2} d\mu(z) &\leq \left(\int_{\ell \leq |x - z| \leq N\ell} + \int_{|x - z| > N\ell} \right) \frac{1}{|x - z|^2} d\mu(z) \\ &\leq C_5 \left(\frac{\varepsilon_1}{\ell} + \frac{1}{N\ell} \right). \end{aligned}$$

Therefore, if ε_1 is small enough and N big enough, then

$$|K_\varepsilon g_b(x) - K_\varepsilon g_b(y)| \leq \frac{1}{2} \tau.$$

Thus, from the preceding estimate and (3.11), we deduce

$$|K_\varepsilon g(x) - K_\varepsilon g(y)| \leq \tau,$$

assuming that $\varepsilon_3 \ll \tau/\|f\|_{L^\infty(\mu)}$.

Proof of (3.3). Suppose now that $x \in Q_i$ and $y \in Q_j$. We write

$$\begin{aligned} |K_\varepsilon g(x) - K_\varepsilon g(y)| &\leq |K_\varepsilon(g\chi_{\mathbb{C} \setminus 3Q_i})(x) - K_\varepsilon(g\chi_{\mathbb{C} \setminus 3Q_j})(y)| \\ (3.12) \quad &\quad + |K_\varepsilon(g\chi_{3Q_i})(x)| + |K_\varepsilon(g\chi_{3Q_j})(y)|. \end{aligned}$$

By (3.10), each one of the last two terms is bounded above by $18\varepsilon_3\|f\|_{L^\infty(\mu)} \leq \tau/10$. So it only remains to estimate the first term on the right hand side.

We set

$$\begin{aligned} (3.13) \quad &|K_\varepsilon(g\chi_{\mathbb{C} \setminus 3Q_i})(x) - K_\varepsilon(g\chi_{\mathbb{C} \setminus 3Q_j})(y)| \leq |K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_i})(x) - K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_j})(y)| \\ &\quad + |K_\varepsilon(g\chi_{\mathbb{C} \setminus 3Q_i})(x) - K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_i})(x)| \\ &\quad + |K_\varepsilon(g\chi_{\mathbb{C} \setminus 3Q_j})(y) - K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_j})(y)| \\ &= I + II + III. \end{aligned}$$

Let us estimate I . We set $\varepsilon' = \max(\varepsilon, \ell)$. First we are going to show that $K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_i})(x)$ is very close to $K_{\varepsilon'}f(x)$. We have

$$\begin{aligned} (3.14) \quad &|K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_i})(x) - K_{\varepsilon'}f(x)| \\ &\leq \|f\|_{L^\infty(\mu)} \int |k_{\varepsilon'}(x, z) - k_\varepsilon(x, z)\chi_{\mathbb{C} \setminus 3Q_i}(z)| d\mu(z). \end{aligned}$$

If $\varepsilon \geq \ell$, then $\varepsilon' = \varepsilon$, and so the integral above equals

$$\int_{3Q_i} |k_\varepsilon(x, z)| d\mu(z) \leq \frac{\mu(3Q_i)}{\ell} \leq C\varepsilon_1.$$

If $\varepsilon < \ell$, then $k_{\varepsilon'}(x, z) = k_\varepsilon(x, z)$ for $z \notin 3Q_i$, and so the last integral in (3.14) now equals

$$\int_{3Q_i} |k_{\varepsilon'}(x, z)| d\mu(z) \leq C \frac{\mu(3Q_i)}{\ell} \leq C\varepsilon_1.$$

Thus in any case we have

$$|K_\varepsilon(f\chi_{\mathbb{C} \setminus 3Q_i})(x) - K_{\varepsilon'}f(x)| \leq C\varepsilon_1\|f\|_{L^\infty(\mu)}.$$

The same estimate holds interchanging Q_i by Q_j and x by y . Therefore,

$$I \leq \sup_{\varepsilon' > 0} |K_{\varepsilon'} f(x) - K_{\varepsilon'} f(y)| + \tau/4,$$

since $\varepsilon_1 \ll \tau/\|f\|_{L^\infty(\mu)}$.

Let us turn our attention to II . We have

$$\begin{aligned} II &= \left| \int_{\mathbb{C} \setminus 3Q_i} k_\varepsilon(x, z) [g(z) - f(z)] d\mu(z) \right| \\ &\leq \sum_{k=1}^{\infty} \left| \int_{3^{k+1} \setminus 3^k Q_i} k_\varepsilon(x, z) [g(z) - f(z)] d\mu(z) \right| =: \sum_{k=1}^{\infty} A_k. \end{aligned}$$

We denote by z_h the center of each square Q_h . Then we get

$$\begin{aligned} A_k &\leq \sum_{h \in I: Q_h \subset 3^{k+1} Q_i \setminus 3^k Q_i} \left| \int_{Q_h} k_\varepsilon(x, z) [g(z) - f(z)] d\mu(z) \right| \\ &\leq \sum_{h \in I: Q_h \subset 3^{k+1} Q_i \setminus 3^k Q_i} \int_{Q_h} |k_\varepsilon(x, z) - k(x, z_h)| |g(z) - f(z)| d\mu(z) \\ &\leq C \sum_{h \in I: Q_h \subset 3^{k+1} Q_i \setminus 3^k Q_i} \frac{\ell}{(3^k \ell)^2} \mu(Q_h) \|f\|_{L^\infty(\mu)} \\ &\leq C \frac{\mu(3^{k+1} Q_i)}{3^k \ell (3^{k+1} Q_i)} \|f\|_{L^\infty(\mu)}. \end{aligned}$$

Therefore, assuming that we take $N := 3^M$, where M is some big integer, we get

$$\begin{aligned} II &\leq C \left(\sum_{k=1}^M 3^{-k} \varepsilon_1 + C \sum_{k=M+1}^{\infty} 3^{-k} \right) \|f\|_{L^\infty(\mu)} \\ &\leq C \left(\varepsilon_1 + \frac{1}{N} \right) \|f\|_{L^\infty(\mu)}. \end{aligned}$$

Thus, $II \leq \tau/4$ if ε_1 and N have been chosen appropriately.

The term III is estimated in the same way as II , and so we also have $III \leq \tau/4$. Now, gathering the estimates corresponding to I, II, III and to the last two terms in (3.12), we obtain (3.3).

Proof of $\|K_* g\|_{L^\infty(\mathbb{C})} \leq \|K_* f\|_{L^\infty(\mathbb{C})} + \tau$. For $x \in Q_i$, we have

$$\begin{aligned} |K_\varepsilon g(x)| &\leq |K_\varepsilon(g \chi_{3Q_i})(x)| + |K_\varepsilon(g \chi_{\mathbb{C} \setminus 3Q_i})(x) - K_\varepsilon(f \chi_{\mathbb{C} \setminus 3Q_i})(x)| \\ &\quad + K_*(f \chi_{\mathbb{C} \setminus 3Q_i})(x). \end{aligned}$$

By (3.10), the first term is bounded above by $18\varepsilon_3 \|f\|_{L^\infty(\mu)} \leq \tau/10$. The second one is estimated like the term II in (3.13). $\square$

From Theorem 3.1 and the characterization of γ in terms of curvature of measures, we deduce the following result.

Corollary 3.7. *Let $E \subset \mathbb{C}$ be compact. Suppose that $\gamma(F) = 0$ for any compact set $F \subset E$ with $\mathcal{H}^1(F) < \infty$. Then, $\alpha(E) \approx \gamma(E)$.*

Proof. Let μ be supported on E such that $\mu(E) \approx \gamma(E)$ and $U_\mu(x) \leq 1$ for all $x \in \mathbb{C}$. Let

$$E_n = \{x \in E : \Theta_\mu^*(x) > 1/n\}.$$

It is easily seen that $\mathcal{H}^1(E_n) < \infty$. Since $\gamma(E_n) = 0$, we have $\mu(E_n) = 0$. Thus, $\Theta(x) = 0$ for μ -a.e $x \in E$. By Theorem 3.1, this implies that $\alpha_+(E) \geq C^{-1}\mu(E) \approx \gamma(E)$, and so $\alpha(E) \approx \gamma(E)$. $\square$

4. THE CAPACITIES γ^h AND γ_+^h

Let us define the capacity γ^h of a compact set $E \subset \mathbb{C}$. We consider a continuous function $h : (0, +\infty) \rightarrow (0, +\infty)$ such that $h(r)/r$ is non decreasing in r ,

$$(4.1) \quad h(r) \leq r \quad \text{and} \quad h(2r) \leq 4h(r) \quad \text{for all } r > 0,$$

and moreover

$$(4.2) \quad \lim_{r \rightarrow 0+} \frac{h(r)}{r} = 0.$$

We set

$$\gamma^h(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all functions $f \in L^\infty(\mathbb{C})$ which are analytic in $\mathbb{C} \setminus E$, with $f(\infty) = 0$, $\|f\|_{L^\infty(\mathbb{C})} \leq 1$, such that

$$\left| \int f \bar{\partial} \varphi d\mathcal{L}^2 \right| \leq h(r)r \|\nabla \varphi\|_\infty$$

for any real function $\varphi \in \mathcal{C}_c^\infty$ supported on some ball of radius r . If f satisfies all these properties we say that f is admissible for γ^h and E , and we write $f \in A^h(E)$.

The capacity $\gamma_+^h(E)$ is defined in an analogous way, but we ask an additional condition on the functions in the supremum above. Namely, f must be the Cauchy transform of some positive Radon measure supported on E . We say that f is admissible for γ_+^h and E , and we write $f \in A_+^h(E)$.

Let us remark that the doubling property $h(2r) \leq 4h(r)$, for $r > 0$, implies that

$$(4.3) \quad h(\lambda t) \leq 4\lambda^2 h(t) \quad \text{for all } \lambda > 1 \text{ and } t > 0.$$

Lemma 4.1. *There exists an absolute constant C such that for any compact set E ,*

$$\alpha(E) \leq C \sup_h \gamma^h(E),$$

with the supremum over all continuous functions $h : (0, +\infty) \rightarrow (0, +\infty)$ satisfying (4.1) and (4.2), with $h(r)/r$ non decreasing.

Proof. Let f be a continuous function on $\mathbb{C}$, analytic in $\mathbb{C} \setminus E$, such that $\|f\|_{L^\infty(\mathbb{C})} \leq 1$, and $|f'(\infty)| \geq \alpha(E)/2$. Let $w_f(r)$ be the modulus of continuity of f , that is

$$w_f(r) := \sup_{x,y:|x-y|\leq r} |f(x) - f(y)|.$$

Notice that $|w_f(r)| \leq 2$ for all $r > 0$, w_f is continuous, non decreasing, and by the triangle inequality $w_f(2r) \leq 2w_f(r)$. We set $h(r) = rw_f(r)/2$, and so h fulfills the properties (4.1) and (4.2). Then, if φ is a real $\mathcal{C}^\infty$ function supported on $B(x_0, r)$, we have

$$\begin{aligned} \left| \int f \bar{\partial} \varphi d\mathcal{L}^2 \right| &= \left| \int (f(x) - f(x_0)) \bar{\partial} \varphi(x) d\mathcal{L}^2(x) \right| \\ &\leq \pi r^2 w_f(r) \|\nabla \varphi\|_\infty = 2\pi h(r) r \|\nabla \varphi\|_\infty. \end{aligned}$$

Thus $(2\pi)^{-1}f \in A^h(E)$, and so $\alpha(E) \leq C\gamma^h(E)$. $\square$

To deal with the capacity γ_+^h it is useful to introduce the potential U_μ^h of a measure μ . We consider the maximal operator M^h :

$$M^h \mu(x) = \sup_{r>0} \frac{\mu(B(x, r))}{h(r)}.$$

Then we set

$$U_\mu^h(x) = M^h \mu(x) + c_\mu(x).$$

Notice that, since $M^h \mu(x) \geq M\mu(x)$, we have $U_\mu^h(x) \geq U_\mu(x)$.

Lemma 4.2. *For any compact set E , we have*

$$(4.4) \quad \gamma_+^h(E) \approx \sup\{\mu(E) : \text{supp}(\mu) \subset E, U_\mu^h(x) \leq 1 \ \forall x \in \mathbb{C}\},$$

with absolute constants (independent of h). Also,

$$(4.5) \quad \alpha_+(E) \approx \sup_h \gamma_+^h(E),$$

with the supremum over all continuous functions $h : (0, +\infty) \rightarrow (0, +\infty)$ satisfying (4.1) and (4.2), with $h(r)/r$ non decreasing.

Proof. First we will show that

$$\gamma_+^h(E) \geq C^{-1} \sup\{\mu(E) : \text{supp}(\mu) \subset E, U_\mu^h(x) \leq 1 \ \forall x \in \mathbb{C}\}.$$

Take μ supported on E such that $U_\mu^h(x) \leq 1$ for all $x \in \mathbb{C}$. Since the Cauchy transform is bounded on $L^2(\mu)$ (with absolute constants), there exists some function f , with $0 \leq f \leq 1$, $\int f d\mu \geq \mu(E)/2$, such that $\|\mathcal{C}(f d\mu)\|_{L^\infty(\mathbb{C})} \leq C_6$. Then, $C^{-1}\mathcal{C}(f d\mu) \in A_+^h(E)$ for some constant C , since for any real function $\varphi \in \mathcal{C}^\infty$ supported on $B(x_0, r)$ we have

$$\left| \int \mathcal{C}(f d\mu) \bar{\partial} \varphi d\mathcal{L}^2 \right| = \left| \pi \int f \varphi d\mu \right| \leq \pi \mu(B(x_0, r)) \|\varphi\|_\infty \leq Ch(r)r \|\nabla \varphi\|_\infty.$$

To see the opposite inequality in (4.4), take μ supported on E such that $\mu(E) \geq \gamma_+^h(E)/2$, with $\mathcal{C}\mu \in A_+^h(E)$. Let φ be a $\mathcal{C}^\infty$ function supported on $B(x_0, 2r)$, with $\chi_{B(x_0, r)} \leq \varphi \leq 1$, and $\|\nabla\varphi\|_\infty \leq Cr^{-1}$. Then,

$$\mu(B(x_0, r)) \leq \int \varphi d\mu = \left| \pi^{-1} \int \mathcal{C}\mu \bar{\partial}\varphi d\mathcal{L}^2 \right| \leq \pi^{-1} h(2r) 2r \|\nabla\varphi\|_\infty \leq C_7 h(r).$$

Thus, $M^h\mu(x_0) \leq C_7$. Also, from (1.2) we deduce $c^2(\mu) \leq C_8\mu(E)$. Then, by Chebishev, there exists a compact set $F \subset E$ such that $\mu(F) \geq \mu(E)/4$ and $c_{\mu|F}^2(x) \leq c_\mu^2(x) \leq 2C_8$ for all $x \in F$. By the maximum principle (2.1), we get

$$c_{\mu|F}^2(x) \leq C_9 \quad \text{for all } x \in \mathbb{C}.$$

Thus, $U_{\mu|F}^h(x) \leq C_7 + C_9^{1/2}$ for all $x \in \mathbb{C}$. So some appropriate multiple of $\mu|_F$ fulfills the properties in the supremum on the right hand side of (4.4). This completes the proof of (4.4).

Let us turn our attention to (4.5) now. By Theorem 3.1 and (4.4) it is clear that

$$\alpha_+(E) \geq C^{-1} \sup_h \gamma_+^h(E).$$

The opposite inequality follows like the proof of Lemma 4.1. We have to change α by α_+ , γ^h by γ_+^h , f by $\mathcal{C}\mu$ (where μ is some Radon measure supported on E), $|f'(\infty)|$ by $\mu(E)$, etc. $\square$

We will show in Section 7 that $\gamma^h \approx \gamma_+^h$ with constants independent of h . Then, from the Lemmas 4.1 and 4.2 we will get

$$\alpha(E) \leq C \sup_h \gamma^h(E) \approx \sup_h \gamma_+^h(E) \approx \alpha_+(E).$$

5. SOME PROPERTIES OF γ^h

Lemma 5.1. (a) *For any compact set $E \subset \mathbb{C}$ we have*

$$\gamma^h(E) \leq C_{10} h(\text{diam}(E)).$$

(b) *For any closed disk D_r of radius r we have*

$$\gamma^h(D_r) \approx \gamma_+^h(D_r) \approx h(r).$$

(c) *If R is a rectangle with side lengths L and ℓ , with $L \geq \ell$, then*

$$\gamma^h(R) \approx \gamma_+^h(R) \approx L \frac{h(\ell)}{\ell}.$$

Proof. (a) Suppose that $E \subset B(x_0, r)$. Take a function $\varphi \in \mathcal{C}^\infty$ supported on $B(x_0, 2r)$, with $\chi_{B(x_0, r)} \leq \varphi \leq \chi_{B(x_0, 2r)}$, and $\|\nabla\varphi\|_\infty \leq Cr^{-1}$. Then, if $f \in A^h(E)$,

$$|f'(\infty)| = \left| C \int f \bar{\partial}\varphi d\mathcal{L}^2 \right| \leq Ch(2r)r \|\nabla\varphi\|_\infty \leq Ch(r).$$

(b) We only have to show that $\gamma_+^h(D_r) \geq C^{-1}h(r)$. Consider the measure

$$d\mu = \frac{h(r)}{r^2} d\mathcal{L}_{|D_r}^2.$$

Then, for all $x \in \mathbb{C}$ and $0 \leq s \leq r$,

$$\mu(B(x, s)) \leq \frac{s^2}{r^2} h(r) \leq Ch(s).$$

The last inequality follows from (4.3). In the case $s > r$, we have

$$\mu(B(x, s)) \leq \mu(D_r) = \pi h(r) \leq \pi h(s).$$

It is also straightforward to check that $\mathcal{C}\mu$ is a bounded function (with absolute constants). So there exists some constant C such that $C^{-1}\mathcal{C}\mu \in A_+^h(D_r)$, and then $\gamma_+^h(D_r) \geq C^{-1}\mu(D_r) = C^{-1}h(r)$.

(c) This statement will be proved after Lemma 5.2 below. $\square$

Notice that, unlike for the usual analytic capacity γ , in general, for $\lambda > 0$,

$$\gamma^h(\lambda E) \neq \lambda \gamma^h(E).$$

In next lemma we show that Vitushkin's localization operator V_φ , defined by

$$V_\varphi f := \frac{1}{\pi z} * (\varphi \bar{\partial} f) = \varphi f - \frac{1}{\pi z} * (f \bar{\partial} \varphi),$$

where φ is a compactly supported $\mathcal{C}^\infty$ function, behaves well with respect to γ^h .

Lemma 5.2. *Let $E \subset \mathbb{C}$ be compact and $f \in A^h(E)$. Let φ be a $\mathcal{C}^\infty$ function supported on $\bar{B}(x_0, r)$, such that $\|\varphi\|_\infty \leq C_{11}$ and $\|\nabla \varphi\|_\infty \leq C_{11}r^{-1}$. Then there exists some constant C depending on C_{11} such that $C^{-1}V_\varphi f \in A^h(E \cap \bar{B}(x_0, r))$.*

Proof. It is well known that if $f \in A^h(E)$, then $V_\varphi f$ is a bounded function analytic in $\mathbb{C} \setminus (E \cap \bar{B}(x_0, r))$. So we only have to check that

$$(5.1) \quad \left| \int V_\varphi f \bar{\partial} \psi d\mathcal{L}^2 \right| \leq Ch(s)s \|\nabla \psi\|_\infty,$$

for any $\mathcal{C}^\infty$ function ψ supported on a ball $B(y_0, s)$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between the space of distributions $\mathcal{D}'$ and $\mathcal{D} := \mathcal{C}_c^\infty(\mathbb{R}^n)$. We have

$$\int V_\varphi f \bar{\partial} \psi d\mathcal{L}^2 = -\langle \bar{\partial}(V_\varphi f), \psi \rangle = -\langle \bar{\partial} f, \varphi \psi \rangle = \int f \bar{\partial}(\varphi \psi) d\mathcal{L}^2.$$

Thus, if we denote $d = \text{diam}(\text{supp}(\varphi \psi))$, we get

$$\left| \int V_\varphi f \bar{\partial} \psi d\mathcal{L}^2 \right| \leq Ch(d)d \|\nabla(\varphi \psi)\|_\infty.$$

We have

$$\begin{aligned} \|\nabla(\varphi\psi)\|_\infty &\leq \|\varphi\|_\infty \|\nabla\psi\|_\infty + \|\psi\|_\infty \|\nabla\varphi\|_\infty \\ &\leq C\|\nabla\psi\|_\infty + C(\|\nabla\psi\|_\infty s) \frac{1}{r} \leq C\left(1 + \frac{s}{r}\right) \|\nabla\psi\|_\infty. \end{aligned}$$

Notice that $d \leq \min(2s, 2r)$, and since h is non decreasing, we get $h(d) \leq h(2s) \leq 4h(s)$, and then

$$\left| \int V_\varphi f \bar{\partial}\psi d\mathcal{L}^2 \right| \leq C\left(h(d)d + h(d) \frac{ds}{r}\right) \|\nabla\psi\|_\infty \leq Ch(s)s.$$

□

Proof of Lemma 5.1 (c). Given a rectangle R with side lengths L and ℓ , with $L \geq \ell$, we have to show that

$$\gamma^h(R) \approx \gamma_+^h(R) \approx L \frac{h(\ell)}{\ell}.$$

If R is a square, then from (b) in Lemma 5.1, it easily follows that $\gamma^h(R) \approx h(\ell)$, and we are done.

Suppose now that $L = N\ell$, where N is some positive integer. So $R = \bigcup_{i=1}^N Q_i$, where each Q_i is a square of side length ℓ , and different squares Q_i have disjoint interiors. Let φ_i , $1 \leq i \leq N$, be non negative $\mathcal{C}^\infty$ functions such that, for each i , $\text{supp}(\varphi_i) \subset 2Q_i$, $\|\nabla\varphi_i\|_\infty \leq C\ell^{-1}$, and $\sum_{i=1}^N \varphi_i = 1$ in a neighborhood of R . If $f \in A^h(R)$, then $C^{-1}V_{\varphi_i}f \in A^h(2Q_i)$ and so

$$|(V_{\varphi_i}f)'(\infty)| \leq C\gamma^h(2Q_i) \approx h(\ell).$$

Thus,

$$|f'(\infty)| \leq \sum_{i=1}^N |(V_{\varphi_i}f)'(\infty)| \leq CNh(\ell),$$

which yields $\gamma^h(R) \leq CNh(\ell)$.

For a general rectangle R , let N be the least integer such that $N \geq L/\ell$. Let R_0 be a rectangle containing R with side lengths $N\ell$ and ℓ . Then,

$$\gamma^h(R) \leq \gamma^h(R_0) \leq CNh(\ell) \leq CL \frac{h(\ell)}{\ell}.$$

Let us see now that $\gamma_+^h(R) \geq C^{-1}Lh(\ell)/\ell$. We will use (4.4). Consider the measure

$$\mu = \frac{h(\ell)}{\ell^2} \mathcal{L}_{|R}^2.$$

Then $\mu(R) = Lh(\ell)/\ell$, and also it is easy to check that $c_\mu(x) \leq C$ for all $x \in \mathbb{C}$. Further, for any ball $B(x, r)$ we have

$$\mu(B(x, r)) = \frac{h(\ell)}{\ell^2} \mathcal{L}^2(B(x, r) \cap R) \leq \frac{Ch(\ell)}{\ell^2} \min(r^2, r\ell).$$

Thus, if $r \geq \ell$,

$$\mu(B(x, r)) \leq \frac{Ch(\ell)r}{\ell} \leq \frac{Ch(r)r}{r} = Ch(r),$$

since $h(t)/t$ is a non decreasing function of t . If $r < \ell$, we get

$$\mu(B(x, r)) \leq \frac{Ch(\ell)r^2}{\ell^2} \leq Ch(r),$$

by (4.3). So $M^h\mu(x) \leq C$ for all $x \in \mathbb{C}$. Therefore,

$$\gamma_+^h(R) \geq C^{-1}\mu(R) = C^{-1}Lh(\ell)/\ell.$$

□

Using Lemma 5.2, we can prove the semiadditivity of γ^h in some special cases:

Lemma 5.3. (a) *For a compact set $E \subset \mathbb{C}$ and a closed disk $D \subset \mathbb{C}$, we have*

$$\gamma^h(E \cup D) \leq C(\gamma^h(E) + \gamma^h(D)).$$

(b) *Given two rectangles $R, T \subset \mathbb{C}$ (we do not assume their sides to be parallel to the axes),*

$$\gamma^h(R \cup T) \leq C(\gamma^h(R) + \gamma^h(T)).$$

Proof. Take $f \in A^h(E \cup D)$ such that $\gamma^h(E \cup D) \leq 2|f'(\infty)|$. Let $\varphi \in \mathcal{C}^\infty$ be supported on $2D$ such that $\varphi = 1$ in a neighborhood of D , with $\|\nabla\varphi\|_\infty \leq Cr^{-1}$, where r is the radius of D . Now set $f_1 = V_\varphi f$ and $f_2 = f - f_1$. By Lemma 5.2, $C^{-1}f_1 \in A^h(2D)$ for some constant C . So we deduce that $C^{-1}f_2 \in A^h(E)$ for another constant C . Therefore,

$$\gamma^h(E \cup D) \leq 2(|f_1'(\infty)| + |f_2'(\infty)|) \leq C(\gamma^h(2D) + \gamma^h(E)).$$

Then (a) follows from the fact that $\gamma^h(2D) \approx h(2r) \approx h(r) \approx \gamma^h(D)$.

Consider now the rectangles R and T . Let L_R, ℓ_R be the side lengths of R , and L_T, ℓ_T the side lengths of T . Suppose $\ell_R \leq L_R$ and $\ell_T \leq L_T$. Assume first that $L_R = M\ell_R$ and $L_T = N\ell_T$ with $M, N \in \mathbb{Z}$. So we can write $R = \bigcup_{i=1}^M P_i$ and $T = \bigcup_{j=1}^N Q_j$, where P_i and Q_j are squares of side lengths ℓ_R and ℓ_T respectively. Take positive functions $\varphi_i \in \mathcal{C}^\infty$, $1 \leq i \leq M$, with $\text{supp}(\varphi_i) \subset 2P_i$ for each i , satisfying $\sum_{i=1}^M \varphi_i \leq 1$ on $\mathbb{C}$, $\sum_{i=1}^M \varphi_i = 1$ on R , and $\|\nabla\varphi_i\|_\infty \leq C\ell_R^{-1}$. Let ψ_j , $1 \leq j \leq N$ be the analogous functions for T . Suppose that $\ell_R \geq \ell_T$. Set $\tilde{\psi}_j = \psi_j(1 - \sum_{i=1}^M \varphi_i)$. Then,

$$\sum_{i=1}^M \varphi_i + \sum_{j=1}^N \tilde{\psi}_j = 1$$

on $R \cup T$ and so, given $f \in A^h(R \cup T)$, we have

$$f = \sum_{i=1}^M V_{\varphi_i} f + \sum_{j=1}^N V_{\tilde{\psi}_j} f.$$

Moreover, for some $C > 0$, $CV_{\varphi_i}f \in A^h(2P_i)$, and also $CV_{\tilde{\psi}_j}f \in A^h(2Q_j)$ (because $\|\tilde{\psi}_j\|_\infty \leq 1$ and $\|\nabla\tilde{\psi}_j\|_\infty \leq C\ell_T^{-1}$). Thus,

$$\begin{aligned} |f'(\infty)| &\leq \sum_{i=1}^M \gamma^h(2P_i) + \sum_{j=1}^N \gamma^h(2Q_j) \\ &\leq C(Mh(\ell_R) + Nh(\ell_T)) \leq C(\gamma^h(R) + \gamma^h(T)). \end{aligned}$$

Suppose now that R and T are arbitrary rectangles. Let M be the least integer such that $M \geq L_R/\ell_R$, and N the least integer such that $N \geq L_T/\ell_T$. Let R_0 be a rectangle containing R with side lengths ℓ_R and $M\ell_R$, and T_0 another rectangle containing T with side lengths ℓ_T and $N\ell_T$. Then,

$$\gamma^h(R \cup T) \leq \gamma^h(R_0 \cup T_0) \leq C(\gamma^h(R_0) + \gamma^h(T_0)) \leq C(\gamma^h(R) + \gamma^h(T)).$$

□

Another property of γ^h is the following.

Lemma 5.4. *Suppose that $E \subset B(x_0, \delta)$. Let $f \in A^h(E)$ and consider the Laurent expansion of f centered at x_0 :*

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{(z - x_0)^n}, \quad |z - x_0| > \delta.$$

We have $|a_n| \leq Cn(\frac{3}{2}\delta)^{n-1}\gamma^h(E)$.

Proof. We argue as in [Ve1, p.444]. Suppose for simplicity that $x_0 = 0$. The function

$$g(z) = z^{n-1} \left(f(z) - \sum_{j=1}^{n-1} \frac{a_j}{z^j} \right)$$

is analytic outside E and satisfies $g(\infty) = 0$. We are going to show that $g/(Cn(\frac{3}{2}\delta)^{n-1}) \in A^h(E)$ for some constant C . Observe that this fact implies that

$$|a_n| = |g'(\infty)| \leq Cn\left(\frac{3}{2}\delta\right)^{n-1}\gamma^h(E).$$

To show that $g/(Cn\frac{3}{2}\delta^{n-1}) \in A^h(E)$, we intend to apply Lemma 5.2. Notice that

$$(5.2) \quad \bar{\partial}g = z^{n-1}\bar{\partial}f = z^{n-1}\rho\bar{\partial}f,$$

where ρ is a C^∞ function supported on $B(0, \frac{3}{2}\delta)$, identically 1 on $B(0, \delta)$, with $\|\rho\|_\infty \leq 1$ and $\|\nabla\rho\|_\infty \leq C/\delta$. We denote $\varphi(z) := \frac{1}{n(\frac{3}{2}\delta)^{n-1}}z^{n-1}\rho(z)$.

From (5.2), it follows that

$$\frac{1}{n(\frac{3}{2}\delta)^{n-1}}g = V_\varphi f.$$

We have $\text{supp}(\varphi) \subset B(0, \frac{3}{2}\delta)$, and also

$$\|\varphi\|_\infty = \frac{1}{n \left(\frac{3}{2}\delta\right)^{n-1}} \|z^{n-1}\rho\|_\infty \leq \frac{1}{n} \leq 1.$$

Now it only remains to estimate $\|\nabla\varphi\|_\infty$:

$$\begin{aligned} \|\nabla\varphi\|_\infty &\leq \frac{1}{n \left(\frac{3}{2}\delta\right)^{n-1}} (\|\nabla(z^{n-1})\|_{\infty, B(0, \frac{3}{2}\delta)} \|\rho\|_\infty + \|z^{n-1}\|_{\infty, B(0, \frac{3}{2}\delta)} \|\nabla\rho\|_\infty) \\ &\leq \frac{1}{n \left(\frac{3}{2}\delta\right)^{n-1}} Cn \left(\frac{3}{2}\delta\right)^{n-2} = \frac{C}{\delta}. \end{aligned}$$

Hence by Lemma 5.2 we are done. $\square$

From the preceding lemma we deduce the following property of γ^h , which will be very useful in Section 7.

Lemma 5.5. *Suppose that $f \in A^h(E)$ satisfies $f'(\infty) = 0$. Then, for all $z \in \mathbb{C}$ such that $\text{dist}(z, E) \geq 2\text{diam}(E)$, we have*

$$|f(z)| \leq C \frac{\text{diam}(E) \gamma^h(E)}{\text{dist}(z, E)^2}.$$

Proof. Set $\delta = \text{diam}(E)$. Take $x_0 \in E$. Then, $E \subset B(x_0, \delta)$. We consider the Laurent expansion of f centered at x_0 :

$$f(z) = \sum_{n=2}^{\infty} \frac{a_n}{(z - x_0)^n}, \quad |z - x_0| > \delta.$$

Notice that if $\text{dist}(z, E) \geq 2\text{diam}(E)$, then $|z - x_0| \geq 2\delta$ and, from the preceding lemma, we deduce

$$|f(z)| \leq \sum_{n=2}^{\infty} \frac{Cn \left(\frac{3}{2}\delta\right)^{n-1} \gamma^h(E)}{|z - x_0|^n} \leq \frac{C\delta \gamma^h(E)}{|z - x_0|^2}.$$

$\square$

6. SOME PROPERTIES OF γ_+^h

We denote

$$\Sigma^h(E) = \{\mu \in M_+(\mathbb{C}) : \text{supp}(\mu) \subset E, M^h\mu(x) \leq 1 \forall x \in \mathbb{C}\}.$$

Lemma 6.1. *The following properties of γ_+^h hold:*

(a) *For any compact set $E \subset \mathbb{C}$,*

$$\gamma_+^h(E) \approx \sup_{\mu \in \Sigma^h(E)} \frac{\mu(E)^2}{\mu(E) + c^2(\mu)}.$$

(b) *For any compact set $E \subset \mathbb{C}$,*

$$\gamma_+^h(E) \leq C\mathcal{H}_\infty^h(E).$$

(c) If $E, F \subset \mathbb{C}$ are compact, then

$$\gamma_+^h(E \cup F) \leq C(\gamma_+^h(E) + \gamma_+^h(F)).$$

(d) For any finite Radon measure μ on $\mathbb{C}$ and any $\lambda > 0$,

$$\gamma_+^h\{x \in \mathbb{C} : U_\mu^h(x) > \lambda\} \leq \frac{C\mu(\mathbb{C})}{\lambda}.$$

(e) Let $\{E_n\}_{n \geq 1}$ be a sequence of compact sets in $\mathbb{C}$ such that $E_{n+1} \subset E_n$ for all n . Set $E = \bigcap_n E_n$. Then, $\gamma^h(E) = \lim_{n \rightarrow \infty} \gamma^h(E_n)$.

In (b), $\mathcal{H}_\infty^h$ stands for the h -Hausdorff content. We recall that $\mathcal{H}_\infty^h$ fulfills the following weak $(1, 1)$ inequality:

$$(6.1) \quad \mathcal{H}_\infty^h\{x \in \mathbb{C} : M^h\mu(x) > \lambda\} \leq \frac{C\mu(\mathbb{C})}{\lambda},$$

for any Radon measure μ and all $\lambda > 0$. This can be proved by means of the $5r$ covering theorem of Vitali (using the doubling property of h). Notice also that the capacitary version (for γ_+^h) of (6.1) is (d) in the lemma above.

Proof of Lemma 6.1. (a) follows from the characterization (4.4) of γ_+^h . The arguments are analogous to the ones for γ_+ in [To2, Theorem 5.1].

The proof of (b) is also derived easily from (4.4), and is left for the reader again.

Also, (c) is another straightforward consequence of (4.4).

The property (d) is proved as the analogous inequality for γ_+ in [To4, Theorem 3.1]. One has to interchange γ_+ by γ_+^h , the maximal operator M by M^h , etc. Notice by the way that the maximum principle (2.1) (which is needed in the proof of [To4, Theorem 3.1]) also holds with M^h instead of M , since $M\mu \leq M^h\mu$ for any measure μ .

The proof of (e) follows by standard arguments and is left for the reader. $\square$

The capacities γ_+^h also satisfy the following property.

Lemma 6.2. *For $E \subset \mathbb{C}$ compact, we have*

$$(6.2) \quad \gamma_+^h(E) \approx \inf\{\mu(\mathbb{C}) : \mu \in M_+(\mathbb{C}), U_\mu^h(x) \geq 1 \forall x \in E\}.$$

The constants involved in (6.2) do not depend on h . The lemma is proved by arguments analogous to the ones of [To4, Theorem 3.3]. It follows from (c) in Lemma 6.1 and next lemma applied to a suitable discrete version $\tilde{E}$ of E .

Lemma 6.3. *Consider a grid of squares of side length $\delta > 0$ in $\mathbb{C}$ with sides parallel to the axes. Take a finite collection of closed squares $\{Q_i\}_{i \in I}$ of the grid. For each $i \in I$, let P_i be the closed square of side length $\delta/4$ concentric with Q_i (and sides parallel to the axes). Set $\tilde{E} = \bigcup_{i \in I} Q_i$. Let $\Sigma_0^h(\tilde{E})$ be the*

subset of $\Sigma_h(\tilde{E})$ of measures μ of the form $\mu = \sum_{i \in I} a_i \mathcal{L}_{P_i}^2$, $a_i \geq 0$. Then there exists a measure $\sigma \in \Sigma_0^h(\tilde{E})$ such that

$$\frac{\|\sigma\|^2}{\|\sigma\| + c^2(\sigma)} = \sup_{\mu \in \Sigma_0^h(E)} \frac{\|\mu\|^2}{\|\mu\| + c^2(\mu)}.$$

The maximal measure σ satisfies $c^2(\sigma) \leq 2\|\sigma\|$ and $U_\sigma^h(x) \geq C_{12}$ for all $x \in \tilde{E}$, where C_{12} is some absolute constant.

This lemma follows from a variational argument analogous to the one of [To4, Lemma 3.6]. The details of the proofs of this lemma and the preceding one are left for the reader again.

7. COMPARABILITY BETWEEN γ^h AND γ_+^h

In this section we will prove the following result. This will be the last step of the proof of Theorem 1.4.

Theorem 7.1. *There exists an absolute constant C (not depending on h) such that*

$$\gamma^h(E) \leq C\gamma_+^h(E)$$

for any compact set $E \subset \mathbb{C}$.

The proof of this theorem is quite similar to the proof of the comparability between γ and γ_+ in [To5, Theorem 1.1]. We must show that there exists some measure μ supported on E , with $\mu(E) \approx \gamma^h(E)$, such that $\mu(B(x, r)) \leq h(r)$, and with the Cauchy transform is bounded on $L^2(\mu)$ with absolute constants. In order to construct μ , we will need to apply an induction argument: We will prove that $\gamma^h(E \cap R) \approx \gamma_+^h(E \cap R)$ for all the rectangles R , by induction on the size of R .

7.1. The First Main lemma. Next lemma, which is one of the main steps for the proof of Theorem 7.1, deals with the construction of an intermediate set F and some measures μ (non negative) and ν (complex) suitable for the application of a Tb theorem. Unlike in [To5, Lemma 5.1], we do not assume E to be made up a finite union of segments, or squares, or rectangles, etc. Now E is supposed to be an arbitrary compact set.

Lemma 7.2 (First Main Lemma). *Suppose that $\gamma_+^h(E) \leq C_{13}h(\text{diam}(E))$, with $C_{13} > 0$ small enough. Then there exists a compact set $F = \bigcup_{i \in I} Q_i$, with $\sum_{i \in I} \chi_{10Q_i} \leq C$, such that*

- (a) $E \subset F$ and $\gamma_+^h(F) \leq C\gamma_+^h(E)$,
- (b) $\sum_{i \in I} \gamma_+^h(E \cap 2Q_i) \leq C\gamma_+^h(E)$,
- (c) $\text{diam}(Q_i) \leq \frac{1}{10}\text{diam}(E)$ for every $i \in I$.

Let $A \geq 1$ be some fixed constant. Suppose that $\gamma^h(E \cap 2Q_i) \leq A\gamma_+^h(E \cap 2Q_i)$ for all $i \in I$. If $\gamma^h(E) \geq A\gamma_+^h(E)$, then there exist a positive Radon measure μ and a complex Radon measure ν , both supported on F , and a subset $H \subset F$, such that

- (d) $C_a^{-1}\gamma^h(E) \leq \mu(F) \leq C_a\gamma^h(E)$,
- (e) $d\nu = b\,d\mu$, with $\|b\|_{L^\infty(\mu)} \leq C_b$,
- (f) $|\nu(F)| \geq \gamma^h(E)/2$,
- (g) $\int_{F \setminus H} C_*\nu\,d\mu \leq C_c\mu(F)$,
- (h) $|\int \varphi\,d\nu| \leq Ch(r)r\|\nabla\varphi\|_\infty$ for any function $\varphi \in C^\infty$ supported on a ball of radius r .
- (i) If $\mu(B(x, r)) > C_h h(r)$ (for some big constant C_h), then $B(x, r) \cap F \subset H$. In particular, $\mu(B(x, r)) \leq C_h h(r)$ for all $x \in F \setminus H$, $r > 0$.
- (j) H is of the form $H = \bigcup_{k \in I_H} B(x_k, r_k) \cap F$, with $\sum_{k \in I_H} h(r_k) \leq \varepsilon\mu(F)$, for $0 < \varepsilon < 1/10$ arbitrarily small (choosing C_h big enough).

The constants C_{13} , C , C_a , C_b , C_c , C_h , ε , δ do not depend on A . They are absolute constants.

Let us insist on the fact that all the constants different from A which appear in the lemma do not depend on A . This will be essential for the proof of Theorem 7.1. We have preferred to use the notation C_a, C_b, C_c, C_h instead of $C_{14}, \dots, C_{17}$, say, because these constants will play an important role in the proof of Theorem 7.1. Of course, the constants C_b and C_h do not depend either on b or h (they are absolute constants).

The set F has to be understood as an approximation of E at an intermediate scale. The first part of the lemma, which deals with the construction of F and the properties (a)–(c), is proved in Subsections 7.2–7.4. The choice of the squares Q_i which satisfy (a) and (b) is one of the key steps of the proof Theorem 7.1. Notice that (a) implies that the squares Q_i are not too big and (b) that they are not too small. That is, they belong to some intermediate scale. The property (b) will be essential for the proof of (d). On the other hand, the assertion (c) will only be used in the final induction argument, in Subsection 7.14.

The properties (d), (e), (f) and (g) are proved in Subsections 7.5 and 7.7. These are the basic properties which must satisfy μ and ν in order to apply a $T(b)$ theorem with absolute constants. To prove (d) we will need the assumptions in the paragraph after (c) in the lemma. In (g) notice that instead of the $L^\infty(\mu)$ or $BMO(\mu)$ norm of $C\nu$, we estimate the $L^1(\mu)$ norm of $C_*\nu$ outside H . The statement in (h) prevents ν from being too much concentrated, in a sense. It is a smooth version of the inequality $|\nu(B(x, r))| \leq Ch(r)$, which we don't know if it holds.

Roughly speaking, the *exceptional set* H contains the part of μ where the measure μ is too much concentrated. The properties (i) and (j) describe H and are proved in Subsection 7.6. Observe that (j) means that H is a rather small set, in a sense.

7.2. The construction of F and the proof of (a) in First Main Lemma. Let $\sigma \in M_+(\mathbb{C})$ be a measure satisfying $\sigma(E) \approx \gamma_+^h(E)$ and $U_\sigma^h(x) \geq 1$ for all $x \in E$ (recall Lemma 6.2). Let λ be some constant

with $0 < \lambda \leq 10^{-8}$ which will be fixed below. Let $\Omega_\lambda \subset \mathbb{C}$ be the **open** set

$$\Omega_\lambda := \{x \in \mathbb{C} : U_\sigma^h(x) > \lambda\}.$$

Notice that $E \subset \Omega_\lambda$, and by (d) in Lemma 6.1 we have

$$(7.1) \quad \gamma_+^h(\Omega_\lambda) \leq C\lambda^{-1}\sigma(E) \leq C\lambda^{-1}\gamma_+^h(E).$$

Let $\Omega_\lambda = \bigcup_{i \in J} Q_i$ be a Whitney decomposition of Ω_λ , where $\{Q_i\}_{i \in J}$ is the usual family of Whitney squares with disjoint interiors, satisfying $20Q_i \subset \Omega_\lambda$, $RQ_i \cap (\mathbb{C} \setminus \Omega_\lambda) \neq \emptyset$ (where $R > 20$ is some fixed absolute constant), and $\sum_{i \in J} \chi_{10Q_i} \leq C$.

Let $\{Q_i\}_{i \in I}$, $I \subset J$, be the subfamily of squares such that $2Q_i \cap E \neq \emptyset$. We set

$$F := \bigcup_{i \in I} Q_i.$$

Observe that the property (a) of First Main Lemma is a consequence of (7.1).

On the other hand, since E is compact, it is easy to check that the collection of squares Q_i , $i \in I$, is finite, and as a consequence, F is also compact.

7.3. Proof of (b) in First Main Lemma. Let us see now that (b) holds if λ has been chosen small enough. Arguing as in [To5, Subsection 5.2], one can check that if $x \in E \cap 2Q_i$ for some $i \in I$, then

$$(7.2) \quad U_{\sigma|4Q_i}^h(x) > 1/4,$$

assuming that λ is small enough. This implies $E \cap 2Q_i \subset \{U_{\sigma|4Q_i}^h > 1/4\}$ and then, by (d) in Lemma 6.1, we have

$$\gamma_+^h(E \cap 2Q_i) \leq C\sigma(4Q_i).$$

Using the finite overlap of the squares $4Q_i$, we deduce

$$\sum_{i \in I} \gamma_+^h(E \cap 2Q_i) \leq C \sum_{i \in I} \sigma(4Q_i) \leq C\sigma(E) \leq C\gamma_+^h(E).$$

7.4. Proof of (c) in First Main Lemma. Now we have to show that

$$(7.3) \quad \text{diam}(Q_i) \leq \frac{1}{10} \text{diam}(E).$$

It is immediate to check that

$$U_\sigma^h(x) \leq \frac{10\sigma(E)}{h(\text{dist}(x, E))}$$

for all $x \notin E$. Thus, for $x \in \Omega_\lambda \setminus E$ we have

$$\lambda < U_\sigma^h(x) \leq \frac{10\sigma(E)}{h(\text{dist}(x, E))}.$$

Therefore,

$$\begin{aligned} h(\text{dist}(x, E)) &\leq 10\lambda^{-1}\sigma(E) \leq C\lambda^{-1}\gamma_+^h(E) \\ &\leq \frac{1}{401} h(\text{diam}(E)) < h\left(\frac{\text{diam}(E)}{10}\right), \end{aligned}$$

assuming C_{13} in First Main Lemma small enough for the third inequality, and using (4.3) in the last one. Since h is non decreasing, we deduce $\text{dist}(x, E) < \text{diam}(E)/10$.

As a consequence, $\text{diam}(\Omega_\lambda) \leq \frac{6}{5}\text{diam}(E)$. Since $20Q_i \subset \Omega_\lambda$ for each $i \in I$, we have

$$20 \text{diam}(Q_i) \leq \text{diam}(\Omega_\lambda) \leq \frac{6}{5}\text{diam}(E),$$

which implies (7.3).

7.5. The construction of μ and ν and the proof of (d)–(f) in First Main Lemma. It is easily seen that there exists a family of C^∞ functions $\{g_i\}_{i \in J}$ such that, for each $i \in J$, $\text{supp}(g_i) \subset 2Q_i$, $0 \leq g_i \leq 1$, and $\|\nabla g_i\|_\infty \leq C/\ell(Q_i)$, so that $\sum_{i \in J} g_i = 1$ on Ω_λ . Notice that by the definition of I in Subsection 7.2, we also have $\sum_{i \in I} g_i = 1$ on E .

Let $f \in A^h(E)$ be such that $f'(\infty) \geq \gamma^h(E)/2$. In a sense, the measure ν will be a suitable approximation of the distribution $\pi^{-1}\bar{\partial}f$ [remember that $f = \mathcal{C}(\pi^{-1}\bar{\partial}f)$].

First we introduce the measure μ . For each $i \in I$, let Δ_i be a disk concentric with Q_i , of radius r_i such that $h(r_i) = \gamma^h(E \cap 2Q_i)/(10^3 C_{10})$ (this radius exists by continuity). Using Lemma 5.1 (a), it is easy to check that $\Delta_i \subset \frac{1}{2}Q_i$ for each i . We set

$$\mu = \sum_{i \in I} \frac{h(r_i)}{\mathcal{L}^2(\Delta_i)} \mathcal{L}^2|_{\Delta_i}.$$

Observe that

$$\gamma^h(\Delta_i) \approx h(r_i) = \frac{1}{10^3 C_{10}} \gamma^h(E \cap 2Q_i) = \mu(\Delta_i).$$

Let us define ν now:

$$\nu = \sum_{i \in I} \frac{1}{\mathcal{L}^2(\Delta_i)} \langle \pi^{-1}\bar{\partial}f, g_i \rangle \cdot \mathcal{L}^2|_{\Delta_i} = \sum_{i \in I} \frac{-1}{\pi \mathcal{L}^2(\Delta_i)} \int f \bar{\partial}g_i d\mathcal{L}^2 \cdot \mathcal{L}^2|_{\Delta_i}.$$

Notice that $\text{supp}(\nu) \subset \text{supp}(\mu) \subset F$. Moreover, we have $\nu(Q_i) = \langle \pi^{-1}\bar{\partial}f, g_i \rangle$, and since $\sum_{i \in I} g_i = 1$ on E , we also have $\nu(F) = \sum_{i \in I} \nu(Q_i) = f'(\infty) \geq \gamma^h(E)/2$ (which yields (f)).

We have $d\nu = b d\mu$, with $b = \langle \pi^{-1}\bar{\partial}f, g_i \rangle$ on Δ_i . To estimate $\|b\|_{L^\infty(\mu)}$, notice that $\|V_{g_i}f\|_\infty \leq C$, and so

$$(7.4) \quad |\langle \pi^{-1}\bar{\partial}f, g_i \rangle| = |(V_{g_i}f)'(\infty)| \leq C\gamma^h(E \cap 2Q_i) = Ch(r_i) = C\mu(\Delta_i).$$

As a consequence, $\|b\|_{L^\infty(\mu)} \leq C$, and (e) is proved.

It remains to check that (d) also holds. Using (7.4), the assumption $\gamma^h(E \cap 2Q_i) \leq A\gamma_+^h(E \cap 2Q_i)$, (b), and the hypothesis $A\gamma_+^h(E) \leq \gamma^h(E)$, we obtain the following inequalities:

$$\begin{aligned} \gamma^h(E) \leq 2|f'(\infty)| &= \left| \sum_{i \in I} \langle \pi^{-1} \bar{\partial} f, g_i \rangle \right| \leq \sum_{i \in I} |\langle \pi^{-1} \bar{\partial} f, g_i \rangle| \\ &\leq C \sum_{i \in I} \gamma^h(E \cap 2Q_i) = C\mu(F) \\ &\leq C A \sum_{i \in I} \gamma_+^h(E \cap 2Q_i) \leq C A \gamma_+^h(E) \leq C \gamma^h(E), \end{aligned}$$

which gives (d) (with constants independent of A).

7.6. Construction of the exceptional set H and proof of (i)–(j). Let C_h be some big constant to be fixed below. Following [NTV2], given $x \in F$, $r > 0$, we say that $B(x, r)$ is a **non Ahlfors disk** if $\mu(B(x, r)) > C_h h(r)$. For a fixed $x \in F$, if there exists some $r > 0$ such that $B(x, r)$ is a non Ahlfors disk, then we say that x is a **non Ahlfors point**. For any $x \in F$, we denote

$$\mathcal{R}(x) = \sup\{r > 0 : B(x, r) \text{ is a non Ahlfors disk}\}.$$

If $x \in F$ is an Ahlfors point, we set $\mathcal{R}(x) = 0$. We say that $\mathcal{R}(x)$ is the **Ahlfors radius** of x .

Observe (d) implies that $\mu(F) \leq C_a \gamma^h(E) \leq C_a \gamma^h(F) \leq C_{14} h(\text{diam}(F))$. Therefore,

$$\mu(B(x, r)) \leq \mu(F) \leq C_{14} h(\text{diam}(F)) \leq C_{15} h(r)$$

for $r \geq \text{diam}(F)/100$. Thus $\mathcal{R}(x) \leq \text{diam}(F)/100$ for all $x \in F$, if we choose $C_h \geq C_{15}$.

We denote

$$H_0 = \bigcup_{x \in F, \mathcal{R}(x) > 0} B(x, \mathcal{R}(x)).$$

By Vitali's $5r$ -Covering Theorem there is a disjoint subfamily $\{B(x_k, \mathcal{R}(x_k))\}_k$ such that $H_0 \subset \bigcup_k B(x_k, 5\mathcal{R}(x_k))$. We denote

$$(7.5) \quad H = \bigcup_k B(x_k, 5\mathcal{R}(x_k)).$$

Since $H_0 \subset H$, all non Ahlfors disks are contained in H and then,

$$(7.6) \quad \text{dist}(x, F \setminus H) \geq \mathcal{R}(x)$$

for all $x \in F$.

Since $\mu(B(x_k, \mathcal{R}(x_k))) \geq C_h h(\mathcal{R}(x_k))$ for every k , we get

$$(7.7) \quad \sum_k h(5\mathcal{R}(x_k)) \leq C \sum_k h(\mathcal{R}(x_k)) \leq \frac{C}{C_h} \sum_k \mu(B(x_k, \mathcal{R}(x_k))) \leq \frac{C}{C_h} \mu(F),$$

with $\frac{C}{C_h}$ arbitrarily small (choosing C_h big enough).

7.7. Proof of (g) in First Main Lemma. In this subsection we will show that

$$(7.8) \quad \int_{F \setminus H} \mathcal{C}_* \nu \, d\mu \leq C_c \mu(F).$$

We will work with the regularized operators K_ε introduced in Section 2. Remember that $|\mathcal{C}(\pi^{-1}\bar{\partial}f)(z)| \leq 1$ $\mathcal{L}^2$ -a.e. $z \in \mathbb{C}$. Thus,

$$\|K_\varepsilon(\pi^{-1}\bar{\partial}f)\|_{L^\infty} = \|\psi_\varepsilon * \mathcal{C}(\pi^{-1}\bar{\partial}f)\|_{L^\infty} \leq 1$$

and so $K_*(\pi^{-1}\bar{\partial}f)(z) \leq 1$ for all $z \in \mathbb{C}$, $\varepsilon > 0$.

To estimate $K_*\nu$, we will deal with the term $K_*(\nu - \pi^{-1}\bar{\partial}f)$. This will be the main point for the proof of (7.8).

We denote $\nu_i := \nu|_{Q_i}$.

Lemma 7.3. *For every $z \in \mathbb{C} \setminus 10Q_i$, we have*

$$(7.9) \quad K_*(\nu_i - \pi^{-1}g_i\bar{\partial}f)(z) \leq \frac{C\ell(Q_i)\mu(Q_i)}{\text{dist}(z, 2Q_i)^2}.$$

Proof. We set $\alpha_i = \nu_i - \pi^{-1}g_i\bar{\partial}f$. Notice that α_i is a distribution supported on $(E \cap 2Q_i) \cup \Delta_i$. We have to show that

$$(7.10) \quad |K_\varepsilon\alpha_i(z)| \leq \frac{C\ell(Q_i)\mu(Q_i)}{\text{dist}(z, 2Q_i)^2}$$

for all $\varepsilon > 0$.

Assume first $\varepsilon \leq \text{dist}(z, 2Q_i)/2$. Since $|\mathcal{C}\alpha_i(w)| \leq C$ for all $w \notin \text{supp}(\alpha_i)$ and $\langle \alpha_i, 1 \rangle = 0$, we have

$$(7.11) \quad |\mathcal{C}\alpha_i(w)| \leq \frac{C \text{diam}(\text{supp}(\alpha_i)) \gamma^h(\text{supp}(\alpha_i))}{\text{dist}(w, \text{supp}(\alpha_i))^2},$$

by Lemma 5.5. On the other hand, by Lemma 5.3, we have

$$\gamma^h(\text{supp}(\alpha_i)) \leq \gamma^h((E \cap 2Q_i) \cup \Delta_i) \leq C(\gamma^h(\Delta_i) + \gamma^h(E \cap 2Q_i)).$$

Therefore, by the definition of Δ_i , we get

$$(7.12) \quad \gamma^h(\Delta_i \cup (E \cap 2Q_i)) \leq C\gamma^h(E \cap 2Q_i) = C\mu(Q_i).$$

If $w \in B(z, \varepsilon)$, then $\text{dist}(w, 2Q_i) \approx \text{dist}(z, 2Q_i)$. By (7.11) and (7.12) we obtain

$$|\mathcal{C}\alpha_i(w)| \leq \frac{C\ell(Q_i)\mu(Q_i)}{\text{dist}(z, 2Q_i)^2}.$$

Making the convolution with ψ_ε , (7.10) follows for $\varepsilon \leq \text{dist}(z, 2Q_i)/2$.

Suppose now that $\varepsilon > \text{dist}(z, 2Q_i)/2$. We denote $h_i = \psi_\varepsilon * \alpha_i$. Then we have

$$K_\varepsilon\alpha_i = \psi_\varepsilon * \frac{1}{z} * \alpha_i = \mathcal{C}(h_i d\mathcal{L}^2).$$

Therefore,

$$(7.13) \quad |K_\varepsilon\alpha_i(z)| \leq \int \frac{|h_i(\xi)|}{|\xi - z|} d\mathcal{L}^2(\xi) \leq C\|h_i\|_\infty [\mathcal{L}^2(\text{supp}(h_i))]^{1/2}.$$

We have to estimate $\|h_i\|_\infty$ and $\mathcal{L}^2(\text{supp}(h_i))$. Observe that, if we write $\ell_i = \ell(Q_i)$ and we denote the center of Q_i by z_i , we get

$$\text{supp}(h_i) \subset \text{supp}(\psi_\varepsilon) + \text{supp}(\alpha_i) \subset B(0, \varepsilon) + B(z_i, 2\ell_i) = B(z_i, \varepsilon + 2\ell_i).$$

Thus, $\mathcal{L}^2(\text{supp}(h_i)) \leq C\varepsilon^2$, since $\ell_i \leq \varepsilon$ (recall $z \notin 10Q_i$, $\varepsilon > \text{dist}(z, 2Q_i)/2$).

Let us deal with $\|h_i\|_\infty$ now. Let η_i be a $\mathcal{C}^\infty$ function supported on $3Q_i$ which is identically 1 on $2Q_i$ and such that $\|\nabla\eta_i\|_\infty \leq C/\ell_i$. Since $\langle \alpha_i, 1 \rangle = 0$, we have

$$\begin{aligned} h_i(w) &= (\psi_\varepsilon * \alpha_i)(w) = \langle \alpha_i, \psi_\varepsilon(\cdot - w) \rangle = \langle \alpha_i, \psi_\varepsilon(\cdot - w) - \psi_\varepsilon(z_i - w) \rangle \\ &= \frac{\ell_i}{\varepsilon^3} \left\langle \alpha_i, \frac{\varepsilon^3}{\ell_i} (\psi_\varepsilon(\cdot - w) - \psi_\varepsilon(z_i - w)) \eta_i \right\rangle =: \frac{\ell_i}{\varepsilon^3} \langle \alpha_i, \varphi_w \eta_i \rangle. \end{aligned}$$

We will show below that $C^{-1}\mathcal{C}(\varphi_w \eta_i \alpha_i) \in A^h(\text{supp}(\alpha_i))$ for some constant C . Assuming this fact for the moment, by (7.12) we deduce

$$\left| \frac{\ell_i}{\varepsilon^3} \langle \alpha_i, \varphi_w \eta_i \rangle \right| = \left| \frac{\ell_i}{\varepsilon^3} \mathcal{C}(\varphi_w \eta_i \alpha_i)'(\infty) \right| \leq \frac{C\ell_i}{\varepsilon^3} \gamma^h(\Delta_i \cup (E \cap 2Q_i)) \leq \frac{C\ell_i \mu(Q_i)}{\varepsilon^3}.$$

Therefore,

$$\|h_i\|_\infty \leq \frac{C\ell_i \mu(Q_i)}{\varepsilon^3}.$$

By (7.13) and the estimates on $\|h_i\|_\infty$ and $\mathcal{L}^2(\text{supp}(h_i))$, we obtain

$$|K_\varepsilon \alpha_i(z)| \leq \frac{C\ell(Q_i)\mu(Q_i)}{\varepsilon^2} \leq \frac{C\ell(Q_i)\mu(Q_i)}{\text{dist}(z, 2Q_i)^2}.$$

It remains to prove that $C^{-1}\mathcal{C}(\varphi_w \eta_i \alpha_i) \in A^h(\text{supp}(\alpha_i))$. Observe that $\mathcal{C}(\varphi_w \eta_i \alpha_i) = V_{(\varphi_w \eta_i)}(\mathcal{C}\alpha_i)$ and remember that $\mathcal{C}\alpha_i$ is a bounded function. By Lemma 5.2, since $\text{supp}(\varphi_w \eta_i) \subset 3Q_i$, it is enough to show that

$$(7.14) \quad \|\varphi_w \eta_i\|_\infty \leq C$$

and

$$(7.15) \quad \|\nabla(\varphi_w \eta_i)\|_\infty \leq \frac{C}{\ell_i},$$

For $\xi \in 3Q_i$, we have

$$|\varphi_w(\xi)| = \frac{\varepsilon^3}{\ell_i} |\psi_\varepsilon(\xi - w) - \psi_\varepsilon(z_i - w)| \leq \varepsilon^3 \|\nabla\psi_\varepsilon\|_\infty \leq C,$$

which yields (7.14). Finally, (7.15) follows easily too:

$$\|\nabla(\varphi_w \eta_i)\|_\infty \leq \|\nabla\varphi_w\|_\infty + \|\varphi_w\|_{\infty, 3Q_i} \|\nabla\eta_i\|_\infty \leq \frac{C}{\ell_i}.$$

We are done. □

Now we are ready to prove (7.8). We write

$$\begin{aligned}
 \int_{F \setminus H} K_* \nu \, d\mu &\leq \int_{F \setminus H} K_*(\pi^{-1} \bar{\partial} f) \, d\mu + \int_{F \setminus H} K_*(\nu - \pi^{-1} \bar{\partial} f) \, d\mu \\
 (7.16) \qquad &\leq C\mu(F \setminus H) + \sum_{i \in I} \int_{F \setminus H} K_*(\nu_i - \pi^{-1} g_i \bar{\partial} f) \, d\mu.
 \end{aligned}$$

To estimate the last sum of integrals we argue as in [To5]. Using Lemma 7.3 and recalling that $\|K_*(\nu_i - \pi^{-1} g_i \bar{\partial} f)\|_{L^\infty(\mu)} \leq C$, we get

$$\int_{F \setminus H} K_*(\nu_i - \pi^{-1} g_i \bar{\partial} f) \, d\mu \leq C\mu(4Q_i)$$

(see the last part of Subsection 7.3 in [To5] for the details). Thus, by the finite overlap of the squares $4Q_i$, $i \in I$, and (7.16), we get

$$(7.17) \qquad \int_{F \setminus H} K_* \nu \, d\mu \leq C\mu(F \setminus H) + C \sum_{i \in I} \mu(4Q_i) \leq C\mu(F).$$

We also have

$$(7.18) \qquad |K_* \nu(z) - C_* \nu(z)| \leq C M \nu(z).$$

By (e), if $z \in F \setminus H$, we have $M \nu(z) \leq C M \mu(z) \leq C M^h \mu(z) \leq C$. Thus (7.17) and (7.18) imply

$$\int_{F \setminus H} C_* \nu(z) \, d\mu(z) \leq C\mu(F).$$

7.8. Proof of (h) in First Main Lemma. For a given function $\varphi \in C^\infty$ supported on some ball B_r of radius r , we have to show that

$$(7.19) \qquad \left| \int \varphi \, d\nu \right| \leq C h(r) r \|\nabla \varphi\|.$$

First, assume that there exists some square Q_{i_0} , $i_0 \in I$, with side length $\ell(Q_{i_0}) \geq r$ such that $Q_{i_0} \cap B_r \neq \emptyset$. Then we have $B_r \subset 5Q_{i_0}$. Thus,

$$(7.20) \qquad \#\{i : Q_i \cap \text{supp}(\varphi) \neq \emptyset\} \leq C.$$

Remember that for each square Q_i , $\mu(Q_i) = C\gamma^h(E \cap 2Q_i) \leq Ch(\ell(Q_i))$. Then, since $\nu = b \, d\mu$, with b bounded, we get

$$\left| \int_{Q_i} \varphi \, d\nu \right| \leq C\mu(Q_i \cap B_r) \|\varphi\|_\infty \leq C\mu(Q_i \cap B_r) r \|\nabla \varphi\|_\infty.$$

Remember that $\mu|_{Q_i} = h(r_i) \mathcal{L}^2|_{\Delta_i/\mathcal{L}^2(\Delta_i)}$, where Δ_i is a closed disk of radius r_i . Using $\mu(Q_i \cap B_r) \leq C \min(r^2, r_i^2) h(r_i)/r_i^2$, it follows easily that

$$\left| \int_{Q_i} \varphi \, d\nu \right| \leq C h(r) r \|\nabla \varphi\|_\infty.$$

From (7.20), we deduce that (7.19) holds in this case.

Suppose now that all the squares Q_i , $i \in I$, that intersect B_r satisfy $\ell(Q_i) < r$. We will prove below that for any collection of points $\{z_i\}_{i \in I}$ with $z_i \in Q_i$,

$$(7.21) \quad \left| \sum_{i \in I} \varphi(z_i) \nu(Q_i) + \frac{1}{\pi} \int f \bar{\partial} \varphi d\mathcal{L}^2 \right| \leq Ch(r)r \|\nabla \varphi\|_\infty.$$

Notice that this estimate implies that

$$\left| \sum_{i \in I} \varphi(z_i) \nu(Q_i) \right| \leq Ch(r)r \|\nabla \varphi\|_\infty$$

because $f \in A^h(E)$. Since $\nu|_{Q_i}$ is a constant multiple of a positive measure, by continuity there exists some point $z_i \in Q_i$ such that $\varphi(z_i) \nu(Q_i) = \int_{Q_i} \varphi d\nu$. From this fact and the preceding inequality, (7.19) follows.

To prove (7.21), we set

$$(7.22) \quad \left| \sum_{i \in I} \varphi(z_i) \nu(Q_i) + \frac{1}{\pi} \int f \bar{\partial} \varphi d\mathcal{L}^2 \right| \leq \sum_{i \in I} \left| \varphi(z_i) \nu(Q_i) + \frac{1}{\pi} \int V_{g_i} f \bar{\partial} \varphi d\mathcal{L}^2 \right|.$$

For each i we have

$$(7.23) \quad \begin{aligned} \left| \varphi(z_i) \nu(Q_i) + \int V_{g_i} f \bar{\partial} \varphi d\mathcal{L}^2 \right| &= \frac{1}{\pi} \left| -\varphi(z_i) \int f \bar{\partial} g_i d\mathcal{L}^2 + \int f \bar{\partial}(\varphi g_i) d\mathcal{L}^2 \right| \\ &= \frac{1}{\pi} \left| \int f \bar{\partial}([\varphi - \varphi(z_i)]g_i) d\mathcal{L}^2 \right|. \end{aligned}$$

We denote $\psi_i := [\varphi - \varphi(z_i)]g_i$ and $\ell_i = \ell(Q_i)$. Because $f \in A^h(E)$, the left side of (7.23) is bounded above by

$$\begin{aligned} Ch(\ell_i)\ell_i \|\nabla \psi_i\|_\infty &\leq Ch(\ell_i)\ell_i \left(\|\nabla g_i\|_\infty \|\varphi - \varphi(z_i)\|_{\infty, 2Q_i} + \|g_i\|_\infty \|\nabla \varphi\|_\infty \right) \\ &\leq Ch(\ell_i)\ell_i \|\nabla \varphi\|_\infty. \end{aligned}$$

Taking into account that $h(t)/t$ is non decreasing in t we obtain

$$\begin{aligned} \left| \sum_{i \in I} \varphi(z_i) \nu(Q_i) + \frac{1}{\pi} \int f \bar{\partial} \varphi d\mathcal{L}^2 \right| &\leq C \sum_{i: Q_i \cap B_r \neq \emptyset} h(\ell_i)\ell_i \|\nabla \varphi\|_\infty \\ &\leq C \sum_{i: Q_i \cap B_r \neq \emptyset} \frac{h(r)}{r} \ell_i^2 \|\nabla \varphi\|_\infty \\ &\leq C \frac{h(r)}{r} \mathcal{L}^2(B_{3r}) \|\nabla \varphi\|_\infty \\ &= Ch(r)r \|\nabla \varphi\|_\infty, \end{aligned}$$

and we are done.

7.9. The dyadic exceptional set $H_{\mathcal{D}}$. Remember that in (7.5) we defined $H = \bigcup_k B(x_k, 5\mathcal{R}(x_k))$, where $\{B(x_k, \mathcal{R}(x_k))\}_k$ is some precise family of non Ahlfors disks. For technical reasons, it is convenient to introduce a dyadic version of H made up of dyadic squares. Let $\mathcal{D}$ be a fixed dyadic lattice and consider the family $\mathcal{D}_H \subset \mathcal{D}$ of dyadic squares such that $R \in \mathcal{D}_H$ if there exists some ball $B(x_k, 5\mathcal{R}(x_k))$ satisfying

$$(7.24) \quad B(x_k, 5\mathcal{R}(x_k)) \cap R \neq \emptyset$$

and

$$(7.25) \quad 10\mathcal{R}(x_k) < \ell(R) \leq 20\mathcal{R}(x_k).$$

Notice that

$$(7.26) \quad \bigcup_k B(x_k, 5\mathcal{R}(x_k)) \subset \bigcup_{R \in \mathcal{D}_H} R.$$

We take a subfamily of disjoint maximal squares $\{R_j\}_{j \in I_H}$ from $\mathcal{D}_H$ such that

$$\bigcup_{R \in \mathcal{D}_H} R = \bigcup_{j \in I_H} R_j,$$

and we define the **dyadic exceptional set $H_{\mathcal{D}}$** as

$$H_{\mathcal{D}} = \bigcup_{j \in I_H} R_k.$$

Observe that (7.26) implies $H \subset H_{\mathcal{D}}$ and, since for each ball $B(x_k, 5\mathcal{R}(x_k))$ there are at most four squares $R \in \mathcal{D}_H$ satisfying (7.24) and (7.25), by (7.7), we obtain

$$(7.27) \quad \sum_{j \in I_H} h(\ell(R_j)) \leq 4 \sum_k h(20\mathcal{R}(x_k)) \leq C_{15} \sum_k h(\mathcal{R}(x_k)) \leq \frac{C_{15}}{C_h} \mu(F) \leq \varepsilon \mu(F),$$

assuming $C_h \geq C_{15}\varepsilon^{-1}$.

7.10. The accretivity condition and the exceptional set $T_{\mathcal{D}}$. We say that a square $R \subset \mathbb{C}$ is accretive if

$$(7.28) \quad \mu(R) \leq C_d |\nu(R)|,$$

where C_d is some big constant to be chosen later. We denote by $\mathcal{D}_T$ the collection of **non** accretive squares from $\mathcal{D}$. Let $\{R_k\}_{k \in I_T} \subset \mathcal{D}_T$ be the subfamily of disjoint maximal dyadic squares from $\mathcal{D}_T$. The **exceptional set $T_{\mathcal{D}}$** is

$$T_{\mathcal{D}} = \bigcup_{k \in I_T} R_k.$$

From the properties (d) and (f), one can easily infer that $|\nu(T_{\mathcal{D}})| \leq \frac{1}{2} |\nu(F)|$ if the constant C_d in (7.28) is chosen big enough. However, for technical reasons (as in [To5]), we need to estimate the size of both $H_{\mathcal{D}}$ and $T_{\mathcal{D}}$ simultaneously. Roughly speaking, $H_{\mathcal{D}}$ is a small set because the measure ν

is “not too concentrated”, that is, because of (h) in First Main Lemma. If we knew that

$$(7.29) \quad |\nu(Q)| \leq Ch(\ell(Q)) \quad \text{for every square } Q \subset \mathbb{C},$$

then we would try to argue as in Section 8 of [To5] in order to show that $|\nu(H_{\mathcal{D}})|$ and $\mu(H_{\mathcal{D}})$ are small. However, instead of (7.29), we only know (h), which is a weaker condition. In Section 7.12, instead of estimating $|\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})|$ and $\mu(H_{\mathcal{D}} \cup T_{\mathcal{D}})$ for a particular dyadic lattice $\mathcal{D}$, we will prove that the average values of $|\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})|$ and $\mu(H_{\mathcal{D}} \cup T_{\mathcal{D}})$ are small when we take different dyadic lattices $\mathcal{D}$.

7.11. Random dyadic lattices. We are going to introduce random dyadic lattices. We follow the construction of [NTV2].

Suppose that $F \subset B(0, 2^{N-3})$, where N is a big enough integer. Consider the random square $Q^0(w) = w + [-2^N, 2^N]^2$, with $w \in [-2^{N-1}, 2^{N-1}]^2 =: \Omega$. We take $Q^0(w)$ as the starting square of the dyadic lattice $\mathcal{D}(w)$. Observe that $F \subset Q^0(w)$ for all $w \in \Omega$. Only the dyadic squares which are contained in $Q^0(w)$ will play some role in the arguments below. For the moment, we don't worry about the other squares.

We take a uniform probability on Ω . So we let the probability measure P be the normalized Lebesgue measure on the square Ω .

A square $Q \in \mathcal{D} \equiv \mathcal{D}(w)$ contained in $Q^0(w)$ is called **terminal** if $Q \subset H_{\mathcal{D}} \cup T_{\mathcal{D}}$. Otherwise, it is called **transit**. The set of terminal squares is denoted by $\mathcal{D}^{term}$, and the set of transit squares by $\mathcal{D}^{tr}$. It is easy to check that Q^0 is always transit.

7.12. The size of $H_{\mathcal{D}} \cup T_{\mathcal{D}}$. The sets $H_{\mathcal{D}}$ and $T_{\mathcal{D}}$ are made up of dyadic squares. For each dyadic lattice $\mathcal{D}(w)$, $w \in \Omega$, we obtain the corresponding sets $H_{\mathcal{D}}(w)$ and $T_{\mathcal{D}}(w)$. Notice that for all $w \in \Omega$, we have $H \subset H_{\mathcal{D}}(w)$.

Lemma 7.4. *If C_h and C_d have been chosen big enough, then*

$$(7.30) \quad \int_{w \in \Omega} |\nu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))| dP(w) \leq \frac{1}{2} |\nu(F)|,$$

and

$$(7.31) \quad \int_{w \in \Omega} \mu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)) dP(w) \leq \delta_0 \mu(F),$$

where $\delta_0 < 1$ is an absolute constant.

Proof. First we will prove (7.30). Let $\{R_k(w)\}_{k \in I_{HT}(w)}$ be the subfamily of different maximal (and thus *disjoint*) squares from

$$\{R_k(w)\}_{k \in I_H(w)} \cup \{R_k(w)\}_{k \in I_T(w)},$$

so that

$$H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w) = \bigcup_{k \in I_{HT}(w)} R_k(w).$$

For each $w \in \Omega$, we have

$$\begin{aligned}
 |\nu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))| &\leq \sum_{k \in I_{HT}(w)} |\nu(R_k(w))| \\
 (7.32) \qquad \qquad \qquad &\leq \sum_{k \in I_H(w)} |\nu(R_k(w))| + \sum_{k \in I_T(w)} |\nu(R_k(w))|.
 \end{aligned}$$

The last sum above is easy to estimate using (7.28):

$$\begin{aligned}
 \sum_{k \in I_T(w)} |\nu(R_k(w))| &\leq C_d^{-1} \sum_{k \in I_T(w)} \mu(R_k(w)) \\
 (7.33) \qquad \qquad \qquad &\leq C_d^{-1} \mu(F) \leq 2C_a C_d^{-1} |\nu(F)| \leq \frac{1}{4} |\nu(F)|,
 \end{aligned}$$

for C_d big enough.

To estimate $|\nu(H_{\mathcal{D}}(w))|$, we consider for each square $R_k(w)$, $k \in I_H(w)$, a $\mathcal{C}^\infty$ function φ_k^w supported on $(1 + \eta)R_k(w)$ (with $0 < \eta \leq 1/10$ to be chosen later), such that $0 \leq \varphi_k^w \leq 1$, φ_k^w identically 1 on $R_k(w)$, satisfying $\|\nabla \varphi_k^w\|_\infty \leq C\eta^{-1}\ell(R_k(w))^{-1}$. Using the fact that $d\nu = b d\mu$, with b bounded, and (h), we get

$$\begin{aligned}
 |\nu(R_k(w))| &\leq \left| \int \varphi_k^w d\nu \right| + C\mu((1 + \eta)R_k(w) \setminus R_k(w)) \\
 &\leq \frac{C}{\eta} h(\ell(R_k(w))) + C\mu((1 + \eta)R_k(w) \setminus R_k(w)).
 \end{aligned}$$

Therefore, by (7.27) we obtain

$$\sum_{k \in I_H(w)} |\nu(R_k(w))| \leq \frac{C}{C_h \eta} \mu(F) + C \sum_{k \in I_H(w)} \mu((1 + \eta)R_k(w) \setminus R_k(w)).$$

We will show in Lemma 7.5 below that

$$\int_{w \in \Omega} \sum_{k \in I_H(w)} \mu((1 + \eta)R_k(w) \setminus R_k(w)) \leq C_{16} \eta \mu(F),$$

where C_{16} is an absolute constant. As a consequence,

$$\int_{w \in \Omega} \sum_{k \in I_H(w)} |\nu(R_k(w))| dP(w) \leq C_{17} \left(\frac{1}{\eta C_h} + \eta \right) |\nu(F)|.$$

Therefore, if $\eta \leq 1/8C_{17}$ and C_h is such that $C_{17}/(\eta C_h) \leq 1/8$, then

$$\int_{w \in \Omega} \sum_{k \in I_H(w)} |\nu(R_k(w))| dP(w) \leq |\nu(F)|/4.$$

Inequality (7.30) follows from this estimated and (7.33).

Let us see now how (7.31) is derived from (7.30). We have

$$\begin{aligned} \int_{w \in \Omega} |\nu(F \setminus (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)))| dP(w) \\ \geq \int_{w \in \Omega} (|\nu(F)| - |\nu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))|) dP(w) \geq \frac{1}{2} |\nu(F)|. \end{aligned}$$

Thus,

$$\begin{aligned} \mu(F) \leq C\gamma^h(E) \leq C|\nu(F)| \leq C \int_{w \in \Omega} |\nu(F \setminus (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)))| dP(w) \\ \leq C_{18} \int_{w \in \Omega} \mu(F \setminus (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))) dP(w). \end{aligned}$$

Therefore,

$$\int_{w \in \Omega} \mu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)) dP(w) \leq \left(1 - \frac{1}{C_{18}}\right) \mu(F) =: \delta_0 \mu(F).$$

□

It remains to prove the following result.

Lemma 7.5. *Let η be some constant with $0 < \eta \leq 1/10$. Then,*

$$(7.34) \quad \int_{w \in \Omega} \sum_{k \in I_H(w)} \mu((1 + \eta)R_k(w) \setminus R_k(w)) \leq C_{16}\eta\mu(F).$$

Proof. Let $B(x_0, r_0)$ be one of the balls $B(x_j, r_j)$ that form the exceptional set H , as explained in (j) from First Main Lemma. Denote by $R_1(w), \dots, R_4(w)$ the squares from $\mathcal{D}(w)$ such that $B(x_0, r_0) \subset \bigcup_{k=1}^4 R_k(w)$, with $2r_0 < \ell(R_k(w)) \leq 4r_0$ (remember (7.24) and (7.25)). Notice that $\bigcup_{k=1}^4 (1 + \eta)R_k(w) \subset B(x_0, 10r_0)$, and since $B(x_0, \lambda r_0)$ is an Ahlfors disk for $\lambda > 1$, unlike $B(x_0, r_0/5)$, we have

$$(7.35) \quad \mu(B(x_0, \lambda r_0)) \leq 100\lambda^2 h(r_0/5) \leq 100\lambda^2 \mu(B(x_0, r_0/5)).$$

We will show that

$$(7.36) \quad \int_{w \in \Omega} \sum_{k=1}^4 \mu((1 + \eta)R_k(w) \setminus R_k(w)) dP(w) \leq C\eta \mu(B(x_0, r_0/5)).$$

The lemma follows from this estimate and the fact that the balls $B(x_j, r_j/5)$ are disjoint.

Let us prove (7.36) now. Let S_0 be a square centered at x_0 of side length 2^m , with $m \in \mathbb{Z}$ such that

$$B(x_0, 10r_0) \subset S_0 \subset B(x_0, 30r_0).$$

Let $L_{V,t}$ be the vertical line $\{(x, y) \in \mathbb{C} : x = t\}$. Let $(a, b) \in \mathbb{C}$ be the lower left corner of S_0 . By Fubini, it easily follows that for all ε with $0 < \varepsilon \leq 2^m$,

$$\frac{1}{2^m} \int_{t \in (a, a+2^m]} \mu(U_\varepsilon(L_{V,t}) \cap S_0) dt \leq \frac{C\varepsilon}{2^m} \mu(2S_0),$$

where $U_\varepsilon(L_{V,t})$ stands for the ε -neighborhood of $L_{V,t}$. The analogous estimate holds for horizontal lines $L_{H,t}$. Setting $\varepsilon = \eta\ell(R_k(w))$, which depends neither on k nor on w , we obtain

$$\begin{aligned} & \int_{w \in \Omega} \sum_{k=1}^4 \mu((1+\eta)R_k(w) \setminus R_k(w)) dP(w) \\ & \leq \frac{C}{2^m} \int_{t \in (a, a+2^m]} \mu(U_\varepsilon(L_{V,t}) \cap S_0) dt + \frac{C}{2^m} \int_{t \in (b, b+2^m]} \mu(U_\varepsilon(L_{H,t}) \cap S_0) dt \\ & \leq \frac{C\eta\ell(R_k(w))}{2^m} \mu(2S_0) \leq C\eta\mu(B(x_0, r_0/5)), \end{aligned}$$

by (7.35). $\square$

7.13. The Second Main Lemma and the Tb Theorem of Nazarov, Treil and Volberg. The next step consists of proving the following result. We use the same notation of First Main Lemma.

Lemma 7.6 (Second Main Lemma). *Assume $\gamma_+^h(E) \leq C_{13}h(\text{diam}(E))$, $\gamma^h(E) \geq A\gamma_+^h(E)$, and $\gamma^h(E \cap 2Q_i) \leq A\gamma_+^h(E \cap 2Q_i)$ for all $i \in I$. Then there exists some subset $G \subset F$, with $\mu(F) \leq C_{19}\mu(G)$, such that $\mu(G \cap B(x, r)) \leq C_h h(r)$ for all $x \in G$, $r > 0$, and the Cauchy transform is bounded on $L^2(\mu|_G)$ with $\|\mathcal{C}\|_{L^2(\mu|_G), L^2(\mu|_G)} \leq C_{20}$. The constants C_h , C_{13} , C_{19} , C_{20} are absolute constants, and do not depend on A .*

To prove this lemma we only have to apply a suitable version of a Tb type theorem of Nazarov, Treil and Volberg [NTV2] to the measure μ and the function b constructed in First Main Lemma. The precise version of the Tb theorem of Nazarov, Treil and Volberg that we need uses the random dyadic lattices $\mathcal{D}(w)$, $w \in \Omega$, introduced in Section 7.11 and reads as follows.

Theorem 7.7 ([NTV2]). *Let μ be a measure supported on $F \subset \mathbb{C}$. Suppose that there exist a complex measure ν and, for each $w \in \Omega$, two exceptional sets $H_{\mathcal{D}}(w)$ and $T_{\mathcal{D}}(w)$ made up of dyadic squares from $\mathcal{D}(w)$ such that*

- (a) *Every ball B_r of radius r such that $\mu(B_r) > C_h r$ is contained in $H_{\mathcal{D}}(w)$, for all $w \in \Omega$.*
- (b) *$d\nu = b d\mu$, with $\|b\|_{L^\infty(\mu)} \leq C$.*
- (c) *$\int_{\mathbb{C} \setminus H_{\mathcal{D}}(w)} \mathcal{C}_* \nu d\mu \leq C \mu(F)$, for all $w \in \Omega$.*
- (d) *If $Q \in \mathcal{D}(w)$ is such that $Q \not\subset T_{\mathcal{D}}(w)$, then $\mu(Q) \leq C_d |\nu(Q)|$ (i.e. Q is an accretive square).*
- (e) *$\int_{w \in \Omega} \mu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)) dP(w) \leq \delta_0 \mu(F)$, for some $\delta_0 < 1$.*

Then there exists a subset $G \subset F \setminus \bigcap_{w \in \Omega} (H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w))$ such that the Cauchy transform is bounded on $L^2(\mu|_G)$, with the L^2 norm bounded above by some constant depending on the constants above.

This theorem is not stated in [NTV2]. However, it can be deduced from the arguments in [NTV2]. Theorem 7.7, changing the statement (e) by

- (e') $\mu(H_{\mathcal{D}}(w) \cup T_{\mathcal{D}}(w)) \leq \delta_0 \mu(F)$, for all $w \in \Omega$ and some $\delta_0 < 1$,

was used in [To5]. In this article, the reader will find the proof of the necessary modifications in order to derive Theorem 7.7 (changing (e) by (e')) from the results of [NTV2].

The only change needed in the arguments of [To5] in order to prove Theorem 7.7 with the statement (e) instead of (e') is a minor modification in the probabilistic argument explained in [To5, Section 11.5] (which, in its turn, describes the arguments of [NTV2, Section XXIII]). The reader can easily verify that the estimates of [To5, Section 11.5] (or [NTV2, Section XXIII]) hold if one assumes (e) instead of (e'). We will not go into more details.

Proof of Second Main Lemma. This is a direct consequence of First Main Lemma and Theorem 7.7.

Notice that the set G obtained in Theorem 7.7 fulfills

$$G \subset \mathbb{C} \setminus \bigcap_{w \in \Omega} H_D \subset \mathbb{C} \setminus H,$$

and so $\mu(B(x, r)) \leq C_h h(r)$ for all $x \in G$. $\square$

7.14. The induction argument for the proof of Theorem 7.1. The arguments in this subsection are quite similar to the ones of [To5, Section 12]. The reader can check that the main differences with respect to [To5, Section 12] appear after Lemmas 7.8 and 7.9, and are due to the fact that now we are assuming that E is an arbitrary compact set, and not a finite union of segments, say, as in [To5].

Lemma 7.8. *There exists some absolute constant B such that if $A \geq 1$ is any fixed constant and*

- (a) $\gamma_+^h(E) \leq C_{13} h(\text{diam}(E))$,
- (b) $\gamma^h(E \cap Q) \leq A \gamma_+^h(E \cap Q)$ for all squares Q with $\text{diam}(Q) \leq \text{diam}(E)/5$,
- (c) $\gamma^h(E) \geq A \gamma_+^h(E)$,

then $\gamma^h(E) \leq B \gamma_+^h(E)$.

Proof. By First Main Lemma 7.2 and Second Main Lemma 7.6, there are sets F , G and a measure μ supported on F such that

- (i) $E \subset F$ and $\gamma_+^h(E) \approx \gamma_+^h(F)$,
- (ii) $\mu(F) \approx \gamma^h(E)$,
- (iii) $G \subset F$ and $\mu(G) \geq C_{19}^{-1} \mu(F)$,
- (iv) $\mu(G \cap B(x, r)) \leq C_h h(r)$ for all $x \in G$, $r > 0$, and $\|\mathcal{C}\|_{L^2(\mu|_G), L^2(\mu|_G)} \leq C_{20}$.

From (iv) and (iii), we get

$$\gamma_+^h(F) \geq C^{-1} \mu(G) \geq C^{-1} \mu(F).$$

Then, by (ii), the preceding inequality, and (i),

$$\gamma^h(E) \leq C \mu(F) \leq C \gamma_+^h(F) \leq B \gamma_+^h(E).$$

$\square$

As a corollary we deduce:

Lemma 7.9. *There exists some absolute constant A_0 such that if $\gamma^h(E \cap Q) \leq A_0 \gamma_+^h(E \cap Q)$ for all squares Q with $\text{diam}(Q) \leq \text{diam}(E)/5$, then $\gamma^h(E) \leq A_0 \gamma_+^h(E)$.*

Proof. If $\gamma_+^h(E) > C_{13}h(\text{diam}(E))$, then we get $\gamma_+^h(E) > C_{21}\gamma^h(E)$, and we are done provided that $A_0 \geq C_{21}^{-1}$. We set $A_0 = \max(1, C_{21}^{-1}, B)$. If $\gamma_+^h(E) \leq C_{13}h(\text{diam}(E))$, then we also have $\gamma^h(E) \leq A_0 \gamma_+^h(E)$. Otherwise, we apply Lemma 7.8 and we deduce $\gamma^h(E) \leq B\gamma_+^h(E) \leq A_0 \gamma_+^h(E)$, which is a contradiction. $\square$

Notice that any constant $A_0 \geq \max(1, C_{21}^{-1}, B)$ works in the argument above. So Lemma 7.9 holds for any constant A_0 sufficiently big.

Proof of Theorem 7.1. By the approximation property (e) of Lemma 6.1, if $d > 0$ is small enough, then $\gamma_+^h(\overline{U_d(E)}) \leq \gamma_+^h(E) + \varepsilon$, where $\varepsilon > 0$ is some arbitrarily small constant. Consider the compact set E_0 made up of all the closed dyadic squares (from the usual standard dyadic lattice, say) of side length $d/2$ that intersect E . We will show below that $\gamma^h(E_0) \leq C\gamma_+^h(E_0)$. From this inequality, since $E \subset E_0 \subset \overline{U_d(E)}$, we get

$$\gamma^h(E) \leq \gamma^h(E_0) \leq C\gamma_+^h(E_0) \leq C(\gamma_+^h(E) + \varepsilon),$$

and then we are done.

To prove that $\gamma^h(E_0) \leq C\gamma_+^h(E_0)$, we are going to show by induction on n that if R is a closed rectangle with sides parallel to the axes and diameter $\leq 4^{n-1}d$, $n \geq 0$, then

$$(7.37) \quad \gamma^h(R \cap E_0) \leq A_0 \gamma_+^h(R \cap E_0).$$

Notice that if $\text{diam}(R) \leq d/4$, then R can intersect at most four of the dyadic squares of side length d that form E_0 . In any case it is easy to check that $R \cap E_0$ is either a rectangle or a polygon which can be decomposed as the union of two closed rectangles R_1, R_2 (with $R_1 \cap R_2 \neq \emptyset$). In the first case, (7.37) follows from Lemma 5.1 (assuming A_0 sufficiently big); and in the second one, by Lemmas 5.3 and 5.1, we have

$$\gamma^h(R_1 \cup R_2) \leq C(\gamma^h(R_1) + \gamma^h(R_2)) \leq C(\gamma_+^h(R_1) + \gamma_+^h(R_2)) \leq 2C\gamma_+^h(R_1 \cup R_2).$$

Let us see now that if (7.37) holds for all rectangles R with diameter $\leq 4^n d$, then it also holds for a rectangle $\tilde{R}$ with diameter $\leq 4^{n+1}d$. We only have to apply Lemma 7.9 to the set $\tilde{R} \cap E_0$. Indeed, take a square Q with diameter $\leq \text{diam}(\tilde{R} \cap E_0)/5$. By the induction hypothesis we have

$$\gamma^h(Q \cap \tilde{R} \cap E_0) \leq A_0 \gamma_+^h(Q \cap \tilde{R} \cap E_0),$$

because $Q \cap \tilde{R}$ is a rectangle with diameter $\leq 4^n d$. Therefore,

$$\gamma^h(\tilde{R} \cap E_0) \leq A_0 \gamma_+^h(\tilde{R} \cap E_0)$$

by Lemma 7.9. $\square$

REFERENCES

- [Ch] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy Integral*. Colloq. Math. 60/61(2) (1990), 601-628.
- [Da] G. David. *Analytic capacity, Calderón-Zygmund operators, and rectifiability*. Publ. Mat. 43 (1999), 3-25.
- [Dve] A.M. Davie. *Analytic capacity and approximation problems*. Trans. Amer. Math. Soc. 171 (1972), 409-444.
- [DØ] A.M. Davie and B. Øksendal. *Analytic capacity and differentiability properties of finely harmonic functions*. Acta Math. 149 (1982), 127-152.
- [Gar] J. Garnett. *Analytic capacity and measure*. Lecture Notes in Math. 297, Springer-Verlag, 1972.
- [MTV1] J. Mateu, X. Tolsa and J. Verdera. *The planar Cantor sets of zero analytic capacity and the local $T(b)$ -Theorem*. Preprint (2001). J. Amer. Math. Soc. 16 (2003), 19-28.
- [MTV2] J. Mateu, X. Tolsa and J. Verdera. *On the semiadditivity of analytic capacity and planar cantor sets*. Preprint (2002). To appear in Contemporary Math.
- [Ma] P. Mattila. *Geometry of sets and measures in Euclidean spaces*. Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.
- [Me1] M.S. Melnikov. *Estimate of the Cauchy integral over an analytic curve*. (Russian) Mat. Sb. 71(113) (1966), 503-514. Amer. Math. Soc. Translation 80(2) (1969), 243-256.
- [Me2] M.S. Melnikov. *Analytic capacity: discrete approach and curvature of a measure*. Sbornik: Mathematics 186(6) (1995), 827-846.
- [MV] M.S. Melnikov and J. Verdera. *A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs*. Internat. Math. Res. Notices (1995), 325-331.
- [NTV1] F. Nazarov, S. Treil and A. Volberg. *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators in nonhomogeneous spaces*. Int. Math. Res. Not. **9** (1998), 463-487.
- [NTV2] F. Nazarov, S. Treil and A. Volberg. *How to prove Vitushkin's conjecture by pulling ourselves up by the hair*. Preprint (2000).
- [To1] X. Tolsa. *Cotlar's inequality and existence of principal values for the Cauchy integral without the doubling condition*. J. Reine Angew Math. 502 (1998), 199-235.
- [To2] X. Tolsa. *L^2 -boundedness of the Cauchy integral operator for continuous measures*. Duke Math. J. 98(2) (1999), 269-304.
- [To3] X. Tolsa. *A proof of the weak $(1,1)$ inequality for singular integrals with non doubling measures based on a Calderón-Zygmund decomposition*. Publ. Mat. 45:1 (2001), 163-174.
- [To4] X. Tolsa. *On the analytic capacity γ_+* . Indiana Univ. Math. J. 51(2) (2002), 317-344.
- [To5] X. Tolsa. *Painlevé's problem and the semiadditivity of analytic capacity*. Preprint (2001). To appear in Acta Math.
- [Ve1] J. Verdera. *Removability, capacity and approximation*. In "Complex Potential Theory", (Montreal, PQ, 1993), NATO Adv. Sci. Int. Ser. C Math. Phys. Sci. 439, Kluwer Academic Publ., Dordrecht, 1994, pp. 419-473.
- [Ve2] J. Verdera. *On the $T(1)$ -theorem for the Cauchy integral*. Ark. Mat. 38 (2000), 183-199.
- [VMP] J. Verdera, M.S. Melnikov, and P.V. Paramonov, *C^1 -approximation and extension of subharmonic functions*, Sbornik: Mathematics 192:4, 515-535.

- [Vi] A. G. Vitushkin, *The analytic capacity of sets in problems of approximation theory*. Uspekhi Mat. Nauk. 22(6) (1967), 141-199 (Russian); in Russian Math. Surveys 22 (1967), 139-200.
- [VM] A. G. Vitushkin and M. S. Melnikov. *Analytic capacity and rational approximation*, Linear and complex analysis, Problem book, Lecture Notes in Math. 1403, Springer-Verlag, Berlin, 1984.

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193
BELLATERRA (BARCELONA), SPAIN

E-mail address: `xtolsa@mat.uab.es`