PAINLEVÉ'S PROBLEM AND THE SEMIADDITIVITY OF ANALYTIC CAPACITY

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Abstract. Let $\gamma(E)$ be the analytic capacity of a compact set E and let $\gamma_+(E)$ be the capacity of E originated by Cauchy transforms of positive measures. In this paper we prove that $\gamma(E) \approx \gamma_+(E)$ with estimates independent of E. As a corollary, we characterize removable singularities for bounded analytic functions in terms of curvature of measures, and we deduce that γ is semiadditive.

1. Introduction

The *analytic capacity* of a compact set $E \subset \mathbb{C}$ is defined as

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$. For a general set $F \subset \mathbb{C}$, we set $\gamma(F) = \sup\{\gamma(E) : E \subset F,\ \text{E compact}\}$.

The notion of analytic capacity was first introduced by Ahlfors [Ah] in the 1940's in order to study the removability of singularities of bounded analytic functions. A compact set $E \subset \mathbb{C}$ is said to be removable (for bounded analytic functions) if for any open set Ω containing E, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to Ω. In [Ah] Ahlfors showed that E is removable if and only if $\gamma(E) = 0$. However, this result doesn't characterize removable singularities for bounded analytic functions in a geometric way, since the definition of γ is purely analytic.

Analytic capacity was rediscovered by Vitushkin in the 1950's in connection with problems of uniform approximation of analytic functions by rational functions (see [Vi], for example). He showed that analytic capacity plays a central role in this type of problems. This fact motivated a renewed interest in analytic capacity. The main drawback of Vitushkin's techniques arises from the fact that there is not a complete description of analytic capacity in metric or geometric terms.

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On the other hand, the *analytic capacity* γ_+ (or *capacity* γ_+) of a compact set E is

$$\gamma_+(E) = \sup_\mu \mu(E),$$

where the supremum is taken over all positive Radon measures μ supported on E such that the Cauchy transform $f = \frac{1}{z} * \mu$ is an $L^\infty(\mathbb{C})$ function with $\|f\|_\infty \leq 1$. Since $\left(\frac{1}{z} * \mu\right)'(\infty) = \mu(E)$, we have

$$\gamma_+(E) \leq \gamma(E). \tag{1.1}$$

To the best of our knowledge, the capacity γ_+ was introduced by Murai [Mu, pp. 71-72]. He showed that some estimates on γ_+ are related to the L^2 boundedness of the Cauchy transform.

In this paper we prove the converse of inequality (1.1) (modulo a multiplicative constant):

Theorem 1.1. *There exists an absolute constant A such that*

$$\gamma(E) \leq A\gamma_+(E)$$

for any compact set E.

Therefore, we deduce $\gamma(E) \approx \gamma_+(E)$ (where $a \approx b$ means that there exists an absolute positive constant C such that $C^{-1}b \leq a \leq Cb$), which was a quite old question concerning analytic capacity (see for example [DØ] or [Ve1, p.435]).

To describe the consequences of Theorem 1.1 for Painlevé's problem (that is, the problem of characterizing removable singularities for bounded analytic functions in a geometric way) and for the semiadditivity of analytic capacity, we need to introduce some additional notation and terminology.

Given a complex Radon measure ν on $\mathbb{C}$, the *Cauchy transform* of ν is

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi - z}\, d\nu(\xi).$$

This definition does not make sense, in general, for $z \in \operatorname{supp}(\nu)$, although one can easily see that the integral above is convergent at a.e. $z \in \mathbb{C}$ (with respect to Lebesgue measure). This is the reason why one considers the *truncated Cauchy transform* of ν, which is defined as

$$\mathcal{C}_\varepsilon\nu(z) = \int_{|\xi - z| > \varepsilon} \frac{1}{\xi - z}\, d\nu(\xi),$$

for any $\varepsilon > 0$ and $z \in \mathbb{C}$. Given a μ-measurable function f on $\mathbb{C}$ (where μ is some fixed positive Radon measure on $\mathbb{C}$), we write $\mathcal{C}f \equiv \mathcal{C}(f\, d\mu)$ and $\mathcal{C}_\varepsilon f \equiv \mathcal{C}_\varepsilon(f\, d\mu)$ for any $\varepsilon > 0$. It is said that the Cauchy transform is bounded on $L^2(\mu)$ if the operators $\mathcal{C}_\varepsilon$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon > 0$.

A positive Radon measure μ is said to have linear growth if there exists some constant C such that $\mu(B(x,r)) \leq Cr$ for all $x \in \mathbb{C}$, $r > 0$.

Given three pairwise different points $x, y, z \in \mathbb{C}$, their *Menger curvature* is

$$c(x, y, z) = \frac{1}{R(x, y, z)},$$

where $R(x, y, z)$ is the radius of the circumference passing through x, y, z (with $R(x, y, z) = \infty$, $c(x, y, z) = 0$ if x, y, z lie on a same line). If two among these points coincide, we let $c(x, y, z) = 0$. For a positive Radon measure μ, we set

$$c_\mu^2(x) = \iint c(x, y, z)^2 \, d\mu(y) d\mu(z),$$

and we define the *curvature of* μ as

$$c^2(\mu) = \int c_\mu^2(x) \, d\mu(x) = \iiint c(x, y, z)^2 \, d\mu(x) d\mu(y) d\mu(z). \tag{1.2}$$

The notion of curvature of measures was introduced by Melnikov [Me2] when he was studying a discrete version of analytic capacity, and it is one of the ideas which is responsible for the big recent advances in connection with analytic capacity. On the one hand, the notion of curvature is connected to the Cauchy transform. This relationship comes from the following remarkable identity found by Melnikov and Verdera [MV] (assuming that μ has linear growth):

$$\|\mathcal{C}_\varepsilon \mu\|_{L^2(\mu)}^2 = \frac{1}{6} c_\varepsilon^2(\mu) + O(\mu(\mathbb{C})), \tag{1.3}$$

where $c_\varepsilon^2(\mu)$ is an ε-truncated version of $c^2(\mu)$ (defined as in the right hand side of (1.2), but with the integrals over $\{x, y, z \in \mathbb{C} : |x-y|, |y-z|, |x-z| > \varepsilon\}$). On the other hand, the curvature of a measure encodes metric and geometric information from the support of the measure and is related to rectifiability (see [Lé]). In fact, there is a close relationship between $c^2(\mu)$ and the coefficients β which appear in Jones' traveling salesman result [Jo].

Using the identity (1.3), it has been shown in [To2] that the capacity γ_+ has a rather precise description in terms of curvature of measures (see (2.2) and (2.4)). As a consequence, from Theorem 1.1 we get a characterization of analytic capacity with a definite metric-geometric flavour. In particular, in connection with Painlevé's problem we obtain the following result, previously conjectured by Melnikov (see [Da3] or [Ma3]).

Theorem 1.2. *A compact set $E \subset \mathbb{C}$ is non removable for bounded analytic functions if and only if it supports a positive Radon measure with linear growth and finite curvature.*

It is easy to check that this result follows from the comparability between γ and γ_+. In fact, it can be considered as a qualitative version of Theorem 1.1.

From Theorem 1.1 and [To4, Corollary 4] we also deduce the following result, which in a sense can be considered as the dual of Theorem 1.2.

Theorem 1.3. *A compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if and only if there exists a finite positive Radon measure μ on $\mathbb{C}$ such that for all $x \in E$ either $\Theta^*_\mu(x) = \infty$ or $c^2_\mu(x) = \infty$.*

In this theorem, $\Theta^*_\mu(x)$ stands for the upper linear density of μ at x, i.e. $\Theta^*_\mu(x) = \limsup_{r\to 0} \mu(B(x,r))\, r^{-1}$.

Theorem 1.1 has another important corollary. Up to now, it was not known if analytic capacity is semiadditive, that is, if there exists an absolute constant C such that

$$\gamma(E \cup F) \leq C(\gamma(E) + \gamma(F)). \tag{1.4}$$

This question was raised by Vitushkin in the early 1960's (see [Vi] and [VM]) and was known to be true only in some particular cases (see [Me1] and [Su] for example, and [Dve] and [DØ] for some related results). On the other hand, a positive answer to the semiadditivity problem would have interesting applications to rational approximation (see [Ve1] and [Vi]). Theorem 1.1 implies that, indeed, analytic capacity is semiadditive because γ_+ is semiadditive [To2]. In fact, the following stronger result holds.

Theorem 1.4. *Let $E \subset \mathbb{C}$ be compact. Let E_i, $i \geq 1$, be Borel sets such that $E = \bigcup_{i=1}^\infty E_i$. Then,*

$$\gamma(E) \leq C \sum_{i=1}^\infty \gamma(E_i),$$

where C is an absolute constant.

Several results dealing with analytic capacity have been obtained recently. Curvature of measures plays an essential role in most of them. G. David proved in [Da1] (using [DM] and [Lé]) that a compact set E with finite length, i.e. with $\mathcal{H}^1(E) < \infty$ (where $\mathcal{H}^s$ stands for the s-dimensional Hausdorff measure), has vanishing analytic capacity if and only if it is purely unrectifiable, that is, if $\mathcal{H}^1(E \cap \Gamma) = 0$ for all rectifiable curves Γ. This result had been known as Vitushkin's conjecture for a long time. Let us also mention that in [MMV] the same result had been proved previously under an additional regularity assumption on the set E.

David's theorem is a very remarkable result. However, it only applies to sets with finite length. Indeed, Mattila [Ma1] showed that the natural generalization of Vitushkin's conjecture to sets with non σ-finite length does not hold (see also [JM]).

After David's solution of Vitushkin's conjecture, Nazarov, Treil and Volberg [NTV1] proved a $T(b)$ theorem useful for dealing with analytic capacity. Their theorem also solves (the last step of) Vitushkin's conjecture. Moreover, they obtained some quantitative results which imply the following estimate:

$$\gamma_+(E) \geq C^{-1}\gamma(E)\left(1 + \left(\frac{\operatorname{diam}(E)}{\gamma(E)}\right)^2 \left(\frac{\mathcal{H}^1(E)}{\gamma(E)}\right)^{38}\right)^{-1/2}. \tag{1.5}$$

Notice that if $\mathcal{H}^1(E) = \infty$, then the right hand side equals 0, and so this inequality is not useful in this case.

For a compact connected set E, P. Jones proved around 1999 that $\gamma(E) \approx \gamma_+(E)$. This proof can be found in [Pa]. The arguments used by P. Jones (of geometric type) are very different from the ones in the present paper.

Other problems related to the capacity γ_+ have been studied recently. Some density estimates for γ_+ (among other results) have been obtained in [MP2], while in [To4] it has been shown that γ_+ verifies some properties which usually hold for other capacities generated by positive potentials and energies, such as Riesz capacities. Now all these results apply automatically to analytic capacity, by Theorem 1.1. See also [MP1] and [VMP] for other questions related to γ_+.

Let us mention some additional consequences of Theorem 1.1. Up to now it was not even known if the class of sets with vanishing analytic capacity was invariant under affine maps such as $x + iy \mapsto x + i2y$, $x, y \in \mathbb{R}$ (this question was raised by O'Farrell, as far as we know). However, this is true for γ_+ (and so for γ), because its characterization in terms of curvature of measures. Indeed, it is quite easy to check that the class of sets with vanishing capacity γ_+ is invariant under $\mathcal{C}^{1+\varepsilon}$ diffeomorphisms (see [To1], for example). The analogous fact for $\mathcal{C}^1$ or bilipschitz diffeomorphisms is an open problem.

Also, our results imply that David's theorem can be extended to sets with σ-finite length. That is, if E has σ-finite length, then $\gamma(E) = 0$ if and only if E is purely unrectifiable. This fact, which also remained unsolved, follows directly either from Theorem 1.1 or 1.4.

The proof of Theorem 1.1 in this paper is inspired by the recent arguments of [MTV], where it is shown that γ is comparable to γ_+ for a big class of Cantor type sets. One essential new idea from [MTV] is the "induction on scales" technique, which can be also adapted to general sets, as we shall see. Let us also remark that another important ingredient of the proof of Theorem 1.1 is the $T(b)$ theorem of [NTV1].

Theorems 1.2 and 1.3 follow easily from Theorem 1.1 and known results about γ_+. Also, to prove Theorem 1.4, one only has to use Theorem 1.1 and the fact that γ_+ is countably semiadditive. This has been shown in [To2] under the additional assumption that the sets E_i in Theorem 1.4 are compact. With some minor modifications, the proof in [To2] is also valid if the sets E_i are Borel. For the sake of completeness, the detailed arguments are shown in Remark 2.1.

The plan of the paper is the following. In Section 2 we introduce some notation and recall some preliminary results. In Section 3, for the reader's convenience, we sketch the ideas involved in the proof of Theorem 1.1. In Section 4 we prove a preliminary lemma which will be necessary for Theorem 1.1. The rest of the paper is devoted to the proof of this theorem, which we have split into three parts. The first one corresponds to the First Main Lemma 5.1, which is stated in Section 6 and proved in Sections 7–8. The second one is the Second Main Lemma 9.1, stated in Section 9 and proved in Sections 10–11. The last part of the proof of Theorem 1.1 is in Section 12 and consists of the induction argument on scales.

2. Notation and background

We denote by $\Sigma(E)$ the set of all positive Radon measures μ supported on $E \subset \mathbb{C}$ such that $\mu(B(x,r)) \leq r$ for all $x \in E$, $r > 0$.

As mentioned in the Introduction, curvature of measures was introduced by Melnikov in [Me2]. In this paper he proved the following inequality:

$$\gamma(E) \geq C \sup_{\mu \in \Sigma(E)} \frac{\mu(E)^2}{\mu(E) + c^2(\mu)}, \tag{2.1}$$

where $C > 0$ is some absolute constant. In [To2] it was shown that inequality (2.1) also holds if one replaces $\gamma(E)$ by $\gamma_+(E)$ on the left hand side, and then one obtains

$$\gamma_+(E) \approx \sup_{\mu \in \Sigma(E)} \frac{\mu(E)^2}{\mu(E) + c^2(\mu)}. \tag{2.2}$$

Let M be the maximal radial Hardy-Littlewood operator:

$$M\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r}$$

[if μ were a complex measure, we would replace $\mu(B(x,r))$ by $|\mu|(B(x,r))$], and let $c_\mu(x) = \left(c_\mu^2(x)\right)^{1/2}$. The following potential was introduced by Verdera in [Ve2]:

$$U_\mu(x) := M\mu(x) + c_\mu(x), \tag{2.3}$$

It turns out that γ_+ can also be characterized in terms of this potential (see [To4], and also [Ve2] for a related result):

$$\gamma_+(E) \approx \sup\{\mu(E) : \operatorname{supp}(\mu) \subset E,\, U_\mu(x) \leq 1 \,\forall x \in E\}. \tag{2.4}$$

Let us also mention that the potential U_μ will be very important for the proof of Theorem 1.1.

Remark 2.1. Let us see that Theorem 1.4 follows easily from Theorem 1.1 and the characterization (2.4) of γ_+. Indeed, if $E \subset \mathbb{C}$ is compact and $E = \bigcup_{i=1}^\infty E_i$, where E_i, $i \geq 1$, are Borel sets, then we take a Radon measure μ such that $\gamma_+(E) \approx \mu(E)$ and $U_\mu(x) \leq 1$ for all $x \in E$. For each $i \geq 1$, let $F_i \subset E_i$ be compact and such that $\mu(F_i) \geq \mu(E_i)/2$. Since $U_{\mu|F_i}(x) \leq 1$ for all $x \in F_i$, we deduce $\gamma_+(F_i) \geq C^{-1}\mu(F_i)$. Then, from Theorem 1.1 we get

$$\begin{aligned} \gamma(E) &\approx \gamma_+(E) \approx \mu(E) \leq C \sum_i \mu(F_i) \\ &\leq C \sum_i \gamma_+(F_i) \approx C \sum_i \gamma(F_i) \leq C \sum_i \gamma(E_i). \end{aligned}$$

Let us recall the definition of the maximal Cauchy transform of a complex measure ν:

$$\mathcal{C}_*\nu(x) = \sup_{\varepsilon>0} |\mathcal{C}_\varepsilon\nu(x)|.$$

Let ψ be a $\mathcal{C}^\infty$ radial function supported on $B(0,1)$, with $0 \leq \psi \leq 2$, $\|\nabla\psi\|_\infty \leq 100$, and $\int \psi d\mathcal{L}^2 = 1$ (where $\mathcal{L}^2$ stands for the Lebesgue measure). We denote $\psi_\varepsilon(x) = \varepsilon^{-2}\psi(x/\varepsilon)$. The regularized operators $\widetilde{\mathcal{C}}_\varepsilon$ are defined as

$$\widetilde{\mathcal{C}}_\varepsilon\nu := \psi_\varepsilon * \mathcal{C}\nu = \psi_\varepsilon * \frac{1}{z} * \nu.$$

Let $r_\varepsilon = \psi_\varepsilon * \dfrac{1}{z}$. It is easily seen that $r_\varepsilon(z) = 1/z$ if $|z| > \varepsilon$, $\|r_\varepsilon\|_\infty \leq C/\varepsilon$, and $|\nabla r_\varepsilon(z)| \leq C|z|^{-2}$. Further, since r_ε is a uniformly continuous kernel, $\widetilde{\mathcal{C}}_\varepsilon\nu$ is a continuous function on $\mathbb{C}$. Notice also that if $|\mathcal{C}\nu| \leq B$ a. e. with respect to Lebesgue measure, then $|\widetilde{\mathcal{C}}_\varepsilon(\nu)(z)| \leq B$ for all $z \in \mathbb{C}$.

Moreover, we have

$$|\widetilde{\mathcal{C}}_\varepsilon\nu(x) - \mathcal{C}_\varepsilon\nu(x)| = \left|\int_{|y-x|\leq\varepsilon} r_\varepsilon(y-x)d\nu(y)\right| \leq C\, M\nu(x). \tag{2.5}$$

By a square Q we mean a closed square with sides parallel to the axes.

The constant A in Theorem 1.1 will be fixed at the end of the proof. Throughout all the paper, the letter C will stand for an absolute constant that may change at different occurrences. Constants with subscripts, such as C_1, will retain its value, in general. On the other hand, the constants $C, C_1, \ldots$ do **not** depend on A.

3. Outline of the arguments for the proof of Theorem 1.1

In this section we will sketch the arguments involved in the proof of Theorem 1.1.

In the rest of the paper, unless stated otherwise, **we will assume that** E **is a finite union of compact disjoint segments.** We will prove Theorem 1.1 for this type of sets. The general case follows from this particular instance by a discretization argument, such as in [Me2, Lemma 1]. Moreover, we will assume that the segments make an angle of $\pi/4$, say, with the x axis. In this way, the intersection of E with any parallel line to one the coordinate axes will always have zero length. This fact will avoid some technical problems.

To prove Theorem 1.1 we want to apply some kind of $T(b)$ theorem, as David in [Da1] for the proof of Vitushkin's conjecture. Because of the definition of analytic capacity, there exists a complex Radon measure ν_0 supported on E such that

$$\|\mathcal{C}\nu_0\|_\infty \leq 1, \tag{3.1}$$

$$|\nu_0(E)| = \gamma(E), \tag{3.2}$$

$$d\nu_0 = b_0\, d\mathcal{H}^1|E, \quad \text{with } \|b_0\|_\infty \leq 1. \tag{3.3}$$

We would like to show that there exists some Radon measure μ supported on E with $\mu \in \Sigma(E)$, $\mu(E) \approx \gamma(E)$, and such that the Cauchy transform is bounded on $L^2(\mu)$ with absolute constants. Then, using (2.2) for example, we would get

$$\gamma_+(E) \geq C^{-1}\mu(E) \geq C^{-1}\gamma(E),$$

and we would be done.

However, by a more or less direct application of a $T(b)$ theorem we cannot expect to prove that the Cauchy transform is bounded with respect to such a measure μ with absolute constants. Let us explain the reasons in some detail. Suppose for example that there exists some function b such that $d\nu_0 = b\,d\mu$ and we use the information about ν_0 given by (3.1), (3.2) and (3.3). From (3.1) and (3.2) we derive

$$\|\mathcal{C}(b\,d\mu)\|_\infty \leq 1 \tag{3.4}$$

and

$$\left|\int b\,d\mu\right| \approx \mu(E). \tag{3.5}$$

The estimate (3.4) is very good for our purposes. In fact, most classical $T(b)$-type theorems require only the $BMO(\mu)$ norm of b to be bounded, which is a weaker assumption. On the other hand, (3.5) is a global paraaccretivity condition, and with some technical difficulties (which may involve some kind of stopping time argument, like in [Ch], [Da1] or [NTV1]), one can hope to be able to prove that the local paraaccretivity condition

$$\left|\int_Q b\,d\mu\right| \approx \mu(Q\cap E)$$

holds for many squares Q.

Our problems arise from (3.3). Notice that (3.3) implies that $|\nu_0|(E) \leq \mathcal{H}^1(E)$, where $|\nu_0|$ stands for the variation of ν_0. This is a very bad estimate since we don't have any control on $\mathcal{H}^1(E)$ (we only know $\mathcal{H}^1(E) < \infty$ because our assumptions on E). However, as far as we know, all $T(b)$-type theorems require the estimate

$$|\nu_0|(E) \leq C\mu(E) \tag{3.6}$$

(and often stronger assumptions involving the L^∞ norm of b). So by a direct application of a $T(b)$-type theorem we will obtain bad results when $\gamma(E) \ll \mathcal{H}^1(E)$, and at most we will get estimates which involve the ratio $\mathcal{H}^1(E)/\gamma(E)$, such as (1.5).

To prove Theorem 1.1, we need to work with a measure "better" than ν_0, which we call ν. This new measure will be a suitable modification of ν_0 with the required estimate for its variation. To construct ν, we operate as in [MTV]. We consider a set F containing E made up of a finite disjoint union of squares: $F = \bigcup_{i\in I} Q_i$. One should think that the squares Q_i approximate E at some "intermediate scale". For example, in the case of the usual 1/4 planar Cantor set of generation n studied in [MTV], the squares Q_i are the squares of generation $n/2$. For each square Q_i, we take a complex measure ν_i supported on Q_i such that $\nu_i(Q_i) = \nu_0(Q_i)$ and $|\nu_i|(Q_i) = |\nu_i(Q_i)|$ (that is, ν_i will be a constant multiple of a positive measure). Then we set $\nu = \sum_i \nu_i$. So, if the squares Q_i are big enough, the variation $|\nu|$ will be sufficiently small. On the other hand, the squares Q_i cannot be too big, because we will need

$$\gamma_+(F) \leq C\gamma_+(E). \tag{3.7}$$

In this way, we will have constructed a complex measure ν supported on F satisfying

$$|\nu|(F) \approx |\nu(F)| = \gamma(E). \tag{3.8}$$

Taking a suitable measure μ such that $\mathrm{supp}(\mu) \supset \mathrm{supp}(\nu)$ and $\mu(F) \approx \gamma(E)$, we will be ready for the application of a $T(b)$ theorem. Indeed, notice that (3.8) implies that ν satisfies a global paraaccretivity condition and that also the variation $|\nu|$ is controlled. On the other hand, if we have been careful enough, we will have also some useful estimates on $|\mathcal{C}\nu|$, since ν is an approximation of ν_0 (in fact, when defining ν in the paragraph above, the measures ν_i have to be constructed in a smoother way). Using the $T(b)$ theorem of [NTV1], we will deduce

$$\gamma_+(F) \geq C^{-1}\mu(E),$$

and so, $\gamma_+(E) \geq C^{-1}\gamma(E)$, by (3.7), and we will be done.

Several difficulties arise in the implementation of the arguments above. In order to obtain the right estimates on the measures ν and μ we will need to assume that $\gamma(E\cap Q_i) \approx \gamma_+(E\cap Q_i)$ for each square Q_i. For this reason, we will use an induction argument involving the sizes of the squares Q_i, as in [MTV]. Further, the choice of the right squares Q_i which approximate E at an intermediate scale is more complicated than in [MTV]. A careful examination of the arguments in [MTV] shows the following. Let σ be an extremal measure for the right hand side of (2.2), and so for γ_+ in a sense (now E is some precise planar Cantor set). It is not difficult to check that $U_\sigma(x) \approx 1$ for all $x \in E$ (remember (2.3)). Moreover, one can also check that the corresponding squares Q_i satisfy

$$U_{\sigma|2Q_i}(x) \approx U_{\sigma|\mathbb{C}\setminus 2Q_i}(x) \approx 1 \quad \text{for all } x \in Q_i. \tag{3.9}$$

In the general situation of E given a finite union of disjoint compact segments, the choice of the squares Q_i will be also determined by the potential U_σ, where now σ is corresponding maximal measure for the right hand side of (2.2). We will not ask the squares Q_i to satisfy (3.9). Instead we will use a variant of this idea.

Let us mention that the First Main Lemma 5.1 below deals with the construction of the measures ν and μ and the estimates involved in this construction. The Second Main Lemma 9.1 is devoted to the application of a suitable $T(b)$ theorem.

4. A preliminary lemma

In next lemma we show a property of the capacity γ_+ and its associated potential which will play an important role in the choice of the squares Q_i mentioned in the preceding section.

Lemma 4.1. *There exists a measure $\sigma \in \Sigma(E)$ such that $\sigma(E) \approx \gamma_+(E)$ and $U_\sigma(x) \geq \alpha$ for all $x \in E$, where $\alpha > 0$ is an absolute constant.*

Let us remark that a similar result has been proved in [To4, Theorem 3.3], but without the assumption $\sigma \in \Sigma(E)$.

Proof. We will see first that there exists a Radon measure $\sigma \in \Sigma(E)$ such that the supremum on the right hand side of (2.2) is attained by σ. That is,

$$g(E) := \sup_{\mu \in \Sigma(E)} \frac{\mu(E)^2}{\mu(E) + c^2(\mu)} = \frac{\sigma(E)^2}{\sigma(E) + c^2(\sigma)}.$$

This measure will fulfill the required properties.

It is easily seen that any measure $\mu \in \Sigma(E)$ can be written as $d\mu = f\, d\mathcal{H}^1|E$, with $\|f\|_{L^\infty(\mathcal{H}^1|E)} \leq 1$, by the Radon-Nikodym theorem. Take a sequence of functions $\{f_n\}_n$, with $\|f_n\|_{L^\infty(\mathcal{H}^1|E)} \leq 1$, converging weakly in $L^\infty(\mu)$ to some function $f \in L^\infty(\mu)$ and such that

$$\lim_{n\to\infty} \frac{\mu_n(E)^2}{\mu_n(E) + c^2(\mu_n)} = g(E),$$

with $d\mu_n = f_n\, d\mathcal{H}^1|E$, $\mu_n \in \Sigma(E)$. Consider the measure $d\sigma = f\, d\mathcal{H}^1|E$. Because of the weak convergence, $\mu_n(E) \to \sigma(E)$ as $n \to \infty$, and moreover $\sigma \in \Sigma(E)$. On the other hand, it is an easy exercise to check that $c^2(\sigma) \leq \liminf_{n\to\infty} c^2(\mu_n)$. So we get

$$g(E) = \frac{\sigma(E)^2}{\sigma(E) + c^2(\sigma)}.$$

Let us see that $\sigma(E) \approx \gamma_+(E)$. Since σ is maximal and $\sigma/2$ is also in $\Sigma(E)$, we have

$$\frac{\sigma(E)^2}{\sigma(E) + c^2(\sigma)} \geq \frac{\sigma(E)^2/4}{\sigma(E)/2 + c^2(\sigma)/8}.$$

Therefore,

$$\frac{\sigma(E)}{2} + \frac{c^2(\sigma)}{8} \geq \frac{\sigma(E)}{4} + \frac{c^2(\sigma)}{4}.$$

That is, $c^2(\sigma) \leq 2\sigma(E)$. Thus,

$$\gamma_+(E) \approx g(E) \approx \sigma(E).$$

It remains to show that there exists some $\alpha > 0$ such that $U_\sigma(x) \geq \alpha$ for all $x \in E$. Suppose that $M\sigma(x) \leq 1/1000$ for some $x \in E$, and let $B := B(x, R)$ be some fixed ball. We will prove the following:

Claim. *If $R>0$ is small enough, then there exists some set $A\subset B(x,R)\cap E$, with $\mathcal{H}^1(A)>0$, such that the measure $\sigma_\lambda := \sigma+\lambda\mathcal{H}^1|A$ belongs to $\Sigma(E)$ for $0\leq\lambda\leq 1/100$.*

Proof of the claim. Since E is made up of a finite number of disjoint compact segments, we may assume that $R>0$ is so small that $\mathcal{H}^1(B(y,r)\cap E)\leq 2r$ for all $y\in B$, $0<r\leq 4R$, and also that $\mathcal{H}^1(B(x,R)\cap E)\geq R$. These assumptions imply that for any subset $A\subset B$ we have

$$\mathcal{H}^1(A\cap B(y,r))\leq \mathcal{H}^1(E\cap B(y,r)\cap B)\leq 2r \qquad \text{for all } y\in B,\ r>0.$$

Thus, $\mathcal{H}^1(A\cap B(y,r))\leq 4r$ for all $y\in\mathbb{C}$ and so

$$M(\mathcal{H}^1|A)(y)\leq 4 \qquad \text{for all } y\in\mathbb{C}. \tag{4.1}$$

We define A as

$$A=\{y\in B: M\sigma(y)\leq 1/4\}.$$

Let us check that $\mathcal{H}^1(A)>0$. Notice that

$$\sigma(2B)\leq 2R\,M\sigma(x)\leq \frac{2R}{1000}\leq\frac{1}{500}\mathcal{H}^1(B\cap E). \tag{4.2}$$

Let $D=B\setminus A$. If $y\in D$, then $M\sigma(y)>1/4$. If $r>0$ is such that $\sigma(B(y,r))/r>1/4$, then $r\leq R/10$. Otherwise, $B(y,r)\subset B(x,12r)$ and so

$$\frac{\sigma(B(y,r))}{r}\leq\frac{\sigma(B(x,12r))}{r}\leq 12M\sigma(x)\leq\frac{12}{1000}.$$

Therefore,

$$D\subset\{y\in B: M(\sigma|2B)(y)>1/4\}.$$

For each $y\in D$, take r_y with $0<r_y\leq R/10$ such that

$$\frac{\sigma(B(y,r_y))}{r_y}>\frac{1}{4}.$$

By Vitali's $5r$-covering Theorem there are some disjoint balls $B(y_i,r_{y_i})$ such that $D\subset\bigcup_i B(y_i,5r_{y_i})$. Since we must have $r_{y_i}\leq R/10$, we get $\mathcal{H}^1(B(y_i,5r_{y_i})\cap E)\leq 15r_{y_i}$. Then, by (4.2) we deduce

$$\begin{aligned}\mathcal{H}^1(D\cap E) &\leq \sum_i \mathcal{H}^1(B(y_i,5r_{y_i})\cap E)\\ &\leq \sum_i 15r_{y_i}\leq 60\sum_i\sigma(B(y_i,r_{y_i}))\\ &\leq 60\sigma(2B)\leq\frac{60}{500}\mathcal{H}^1(B\cap E).\end{aligned}$$

Thus, $\mathcal{H}^1(A)>0$.

Now we have to show that $M\sigma_\lambda(y)\leq 1$ for all $y\in E$. If $y\in A$, then $M\sigma(y)\leq 1/4$, and then by (4.1) we have

$$M\sigma_\lambda(y)\leq\frac{1}{4}+\lambda M(\mathcal{H}^1|A)(y)\leq\frac{1}{4}+\frac{4}{100}<1.$$

If $y\notin A$ and $B(y,r)\cap A=\varnothing$, then we obviously have

$$\frac{\sigma_\lambda(B(y,r))}{r}=\frac{\sigma(B(y,r))}{r}\leq 1.$$

Suppose $y\notin A$ and $B(y,r)\cap A\neq\varnothing$. Let $z\in B(y,r)\cap A$. Then,

$$\frac{\sigma(B(y,r))}{r}\leq\frac{\sigma(B(z,2r))}{r}\leq 2M\sigma(z)\leq\frac{1}{2}.$$

Thus,

$$\frac{\sigma_\lambda(B(y,r))}{r}\leq\frac{1}{2}+\lambda M(\mathcal{H}^1|A)(y)\leq\frac{1}{2}+\frac{4}{100}<1.$$

So we always have $M\sigma_\lambda(y)\leq 1$. $\square$

Let us continue the proof of Lemma 4.1 and let us see that $U_\sigma(x)\geq\alpha$. Consider the function

$$\varphi(\lambda)=\frac{\sigma_\lambda(E)^2}{\sigma_\lambda(E)+c^2(\sigma_\lambda)}.$$

Since σ is a maximal measure for $g(E)$ and $\sigma_\lambda\in\Sigma(E)$ for some $\lambda>0$, we must have $\varphi'(0)\leq 0$. Observe that

$$\begin{aligned}\varphi(\lambda)\;&=\;\big[\sigma(E)+\lambda\mathcal{H}^1(A)\big]^2\\&\quad\times\Big[\sigma(E)+\lambda\mathcal{H}^1(A)+c^2(\sigma)+3\lambda\,c^2(\mathcal{H}^1|A,\sigma,\sigma)\\&\quad+3\lambda^2c^2(\sigma,\mathcal{H}^1|A,\mathcal{H}^1|A)+\lambda^3c^2(\mathcal{H}^1|A)\Big]^{-1}.\end{aligned}$$

So,

$$\varphi'(0)=\frac{2\sigma(E)\mathcal{H}^1(A)(\sigma(E)+c^2(\sigma))-\sigma(E)^2\,(\mathcal{H}^1(A)+3\,c^2(\mathcal{H}^1|A,\sigma,\sigma))}{(\sigma(E)+c^2(\sigma))^2}.$$

Therefore, $\varphi'(0)\leq 0$ if and only if

$$2\mathcal{H}^1(A)(\sigma(E)+c^2(\sigma))\leq\sigma(E)\,(\mathcal{H}^1(A)+3\,c^2(\mathcal{H}^1|A,\sigma,\sigma)).$$

That is,

$$\frac{\sigma(E)+2c^2(\sigma)}{\sigma(E)}\leq\frac{3\,c^2(\mathcal{H}^1|A,\sigma,\sigma)}{\mathcal{H}^1(A)}.$$

Therefore, $c^2(\mathcal{H}^1|A,\sigma,\sigma)/\mathcal{H}^1(A)\geq\frac{1}{3}$. So there exists some $x_0\in A$ such that

$$c^2(x_0,\sigma,\sigma)\geq\frac{1}{3}. \tag{4.3}$$

We write

$$\begin{aligned}c^2(x_0,\sigma,\sigma)\;&=\;c^2(x_0,\sigma|2B,\sigma|2B)+c^2(x_0,\sigma|2B,\sigma|\mathbb{C}\setminus 2B)\\&\quad+c^2(x_0,\sigma|\mathbb{C}\setminus 2B,\sigma|\mathbb{C}\setminus 2B).\end{aligned} \tag{4.4}$$

If R is chosen small enough, then $B \cap E$ coincides with a segment and so $c^2(x_0, \sigma|2B, \sigma|2B) = 0$. On the other hand,

$$\begin{aligned} c^2(x_0, \sigma|2B, \sigma|\mathbb{C} \setminus 2B) &\leq C \int_{y\in 2B} \int_{z \in \mathbb{C}\setminus 2B} \frac{1}{|x-z|^2}\, d\sigma(y)\, d\sigma(z) \\ &\leq C_2 M\sigma(x)^2. \end{aligned}$$

Thus, if $M\sigma(x)^2 \leq 1/(6C_2)$, then by (4.3) and (4.4) we obtain

$$c^2_{\sigma|\mathbb{C}\setminus 2B}(x_0) = c^2(x_0, \sigma|\mathbb{C} \setminus 2B, \sigma|\mathbb{C} \setminus 2B) \geq \frac{1}{3} - \frac{1}{6} = \frac{1}{6}.$$

Also, it is easily checked that

$$|c_{\sigma|\mathbb{C}\setminus 2B}(x) - c_{\sigma|\mathbb{C}\setminus 2B}(x_0)| \leq C_3 M\sigma(x). \tag{4.5}$$

This follows easily from the inequality

$$|c(x,y,z) - c(x_0,y,z)| \leq C \frac{R}{|x-y|\,|x-z|},$$

for x, x_0, y, z such that $|x - x_0| \leq R$ and $|x-y|$, $|x-z| \geq 2R$ (see Lemma 2.4 of [To2], for example) and some standard estimates. Therefore, if we suppose $M\sigma(x) \leq 1/(100C_3)$, then we obtain

$$c_{\sigma|\mathbb{C}\setminus 2B}(x) \geq c_{\sigma|\mathbb{C}\setminus 2B}(x_0) - \frac{1}{100} \geq \frac{1}{10}.$$

So we have proved that if $M\sigma(x) \leq \min(1/1000,\, 1/(6C_2)^{1/2},\, 1/(100C_3))$, then $c_\sigma(x) > 1/10$. This implies that in any case we have $U_\sigma(x) \geq \alpha$, for some $\alpha > 0$. $\square$

5. The First Main lemma

The proof of Theorem 1.1 uses an induction argument on scales, analogous to the one in [MTV]. Indeed, if Q is a sufficiently small square, then $E \cap Q$ either coincides with a segment or it is void, and so

$$\gamma_+(E \cap Q) \approx \gamma(E \cap Q). \tag{5.1}$$

Roughly speaking, the idea consists of proving (5.1) for squares[1] Q of any size, by induction. To prove that (5.1) holds for some fixed square Q_0, we will take into account that (5.1) holds for squares with side length $\leq \ell(Q_0)/5$.

Our next objective consists of proving the following result.

Lemma 5.1 (First Main Lemma). *Suppose that $\gamma_+(E) \leq C_4 \operatorname{diam}(E)$, with $C_4 > 0$ small enough. Then there exists a compact set $F = \bigcup_{i\in I} Q_i$, with $\sum_{i\in I} \chi_{10Q_i} \leq C$, such that*

(a) *$E \subset F$ and $\gamma_+(F) \leq C\gamma_+(E)$,*
(b) *$\sum_{i\in I} \gamma_+(E \cap 2Q_i) \leq C\gamma_+(E)$,*
(c) *$\operatorname{diam}(Q_i) \leq \frac{1}{10}\operatorname{diam}(E)$ for every $i \in I$.*

[1] Actually, in the induction argument we will use rectangles instead of squares.

Let $A \geq 1$ be some fixed constant and $\mathcal{D}$ any fixed dyadic lattice. Suppose that $\gamma(E \cap 2Q_i) \leq A\gamma_+(E \cap 2Q_i)$ for all $i \in I$. If $\gamma(E) \geq A\gamma_+(E)$, then there exist a positive Radon measure μ and a complex Radon measure ν, both supported on F, and a subset $H_{\mathcal{D}} \subset F$, such that

(d) $C_a^{-1}\gamma(E) \leq \mu(F) \leq C_a\,\gamma(E)$,
(e) $d\nu = b\,d\mu$, *with* $\|b\|_{L^\infty(\mu)} \leq C_b$,
(f) $|\nu(F)| = \gamma(E)$,
(g) $\int_{F\setminus H_{\mathcal{D}}} \mathcal{C}_*\nu\,d\mu \leq C_c\mu(F)$,
(h) *If $\mu(B(x,r)) > C_0 r$ (for some big constant C_0), then $B(x,r) \subset H_{\mathcal{D}}$. In particular, $\mu(B(x,r)) \leq C_0 r$ for all $x \in F \setminus H_{\mathcal{D}}$, $r > 0$.*
(i) *$H_{\mathcal{D}} = \bigcup_{k\in I_H} R_k$, where R_k, $k \in I_H$, are disjoint dyadic squares from the lattice $\mathcal{D}$, with $\sum_{k\in I_H} \ell(R_k) \leq \varepsilon\mu(F)$, for $0 < \varepsilon < 1/10$ arbitrarily small (choosing C_0 big enough).*
(j) $|\nu(H_{\mathcal{D}})| \leq \varepsilon|\nu(F)|$.
(k) $\mu(H_{\mathcal{D}}) \leq \delta\mu(F)$, *with* $\delta = \delta(\varepsilon) < 1$.

The constants C_4, C, C_a, C_b, C_c, C_0, ε, δ do not depend on A. They are absolute constants.

Let us remark that the construction of the set $H_{\mathcal{D}}$ depends on the chosen dyadic lattice $\mathcal{D}$. On the other hand, the construction of F, μ, ν and b is independent of $\mathcal{D}$.

We also insist on the fact that all the constants different from A which appear in the lemma do not depend on A. This fact will be essential for the proof of Theorem 1.1 in Section 12. We have preferred to use the notation C_a, C_b, C_c instead of C_5, C_6, C_7, say, because these constants will play an important role in the proof of Theorem 1.1. Of course, the constant C_b does not depend on b (it is an absolute constant).

Remember that we said that we assumed the squares to be closed. This is not the case for the squares of the dyadic squares of the lattice $\mathcal{D}$. We suppose that these squares are half open - half closed (i.e. of the type $(a,b] \times (c,d]$).

For the reader's convenience, before going on we will make some comments on the lemma. As we said in Section 3, the set F has to be understood as an approximation of E at an intermediate scale. The first part of the lemma, which deals with the construction of F and the properties (a)–(c), is proved in Section 6. The choice of the squares Q_i which satisfy (a) and (b) is one of the keys of the proof Theorem 1.1. Notice that (a) implies that the squares Q_i are not too big and (b) that they are not too small. That is, they belong to some intermediate scale. The property (b) will be essential for the proof of (d). On the other hand, the assertion (c) will only be used in the induction argument, in Section 12.

The properties (d), (e), (f) and (g) are proved in Section 7. These are the basic properties which must satisfy μ and ν in order to apply a $T(b)$ theorem with absolute constants, as explained in Section 3. To prove (d) we will need the assumptions in the paragraph after (c) in the lemma. In (g) notice that instead of the $L^\infty(\mu)$ or $BMO(\mu)$ norm of $\mathcal{C}\nu$, we estimate the $L^1(\mu)$ norm of $\mathcal{C}_*\nu$ out of the set $H_{\mathcal{D}}$.

Roughly speaking, the *exceptional set* $H_{\mathcal{D}}$ contains the part of μ without linear growth. The properties (h), (i), (j) and (k) describe $H_{\mathcal{D}}$ and are proved in Section 8. Observe that (i), (j) and (k) mean that $H_{\mathcal{D}}$ is a rather small set.

6. Proof of (a)–(c) in First Main Lemma

6.1. **The construction of F and the proof of (a).** Let $\sigma \in \Sigma(E)$ be a measure satisfying $\sigma(E) \approx \gamma_+(E)$ and $U_\sigma(x) \geq \alpha > 0$ for all $x \in E$ (recall Lemma 4.1). Let λ be some constant with $0 < \lambda \leq 10^{-8}\alpha$ which will be fixed below. Let $\Omega \subset \mathbb{C}$ be the **open** set

$$\Omega := \{x \in \mathbb{C} : U_\sigma(x) > \lambda\}.$$

Notice that $E \subset \Omega$, and by [To4, Theorem 3.1] we have

$$\gamma_+(\Omega) \leq C\lambda^{-1}\sigma(E) \leq C\lambda^{-1}\gamma_+(E). \tag{6.1}$$

Let $\Omega = \bigcup_{i\in J} Q_i$ be a Whitney decomposition of Ω, where $\{Q_i\}_{i\in J}$ is the usual family of Whitney squares with disjoint interiors satisfying $20Q_i \subset \Omega$, $RQ_i \cap (\mathbb{C} \setminus \Omega) \neq \varnothing$ (where R is some fixed absolute constant), and $\sum_{i\in J} \chi_{10Q_i} \leq C$.

Let $\{Q_i\}_{i\in I}$, $I \subset J$, be the subfamily of squares such that $2Q_i \cap E \neq \varnothing$. We set

$$F := \bigcup_{i\in I} Q_i.$$

Observe that the property (a) of First Main Lemma is a consequence of (6.1) and the geometry of the Whitney decomposition.

To see that F is compact it suffices to check that the family $\{Q_i\}_{i\in I}$ is finite. Notice that $E \subset \bigcup_{i\in J}(1.1\mathring{Q}_i)$. Since E is compact, there exists a finite covering

$$E \subset \bigcup_{1\leq k\leq n} (1.1\mathring{Q}_{i_k}).$$

Each square $2Q_i$, $i \in I$, intersects some square $1.1Q_{i_k}$, $k = 1, \ldots, n$. Because of the geometric properties of the Whitney decomposition, the number of squares $2Q_i$ which intersect some fixed square $1.1Q_{i_k}$ is bounded above by some constant C_5. Thus, the family $\{Q_i\}_{i\in I}$ has at most $C_5 n$ squares.

6.2. **Proof of (b).** Let us see now that (b) holds if λ has been chosen small enough. We will show below that if $x \in E \cap 2Q_i$ for some $i \in I$, then

$$U_{\sigma|4Q_i}(x) > \alpha/4, \tag{6.2}$$

assuming that λ is small enough. This implies $E \cap 2Q_i \subset \{U_{\sigma|4Q_i} > \alpha/4\}$ and then, by [To4, Theorem 3.1], we have

$$\gamma_+(E \cap 2Q_i) \leq C\alpha^{-1}\sigma(4Q_i).$$

Using the finite overlap of the squares $4Q_i$, we deduce

$$\sum_{i\in I}\gamma_+(E\cap 2Q_i)\leq C\alpha^{-1}\sum_{i\in I}\sigma(4Q_i)\leq C\alpha^{-1}\sigma(E)\leq C\gamma_+(E).$$

Notice that in the last inequality, the constant α^{-1} has been absorbed by the constant C.

Now we have to show that (6.2) holds for $x\in E\cap 2Q_i$. Let $z\in RQ_i\setminus\Omega$, so that $\operatorname{dist}(z,Q_i)\approx\operatorname{dist}(\partial\Omega,Q_i)\approx\ell(Q_i)$ (where $\ell(Q_i)$ stands for the side length of Q_i). Since $M\sigma(z)\leq U_\sigma(z)\leq\lambda$, we deduce that for any square P with $\ell(P)\geq\ell(Q_i)/4$ and $P\cap 2Q_i\neq\varnothing$, we have

$$\frac{\sigma(P)}{\ell(P)}\leq C_6\lambda\leq 10^{-6}\alpha, \tag{6.3}$$

where the constant C_6 depends on the Whitney decomposition (in particular, on the constant R), and we assume that λ has been chosen so small that the last inequality holds.

Remember that $U_\sigma(x)>\alpha$. If $M\sigma(x)>\alpha/2$, then

$$\frac{\sigma(Q)}{\ell(Q)}>\alpha/2$$

for some "small" square Q contained in $4Q_i$, because the "big" squares P satisfy (6.3). So, $U_{\sigma|4Q_i}(x)>\alpha/2$.

Assume now that $M\sigma(x)\leq\alpha/2$. In this case, $c_\sigma(x)>\alpha/2$. We decompose $c_\sigma^2(x)=:c^2(x,\sigma,\sigma)$ as follows:

$$\begin{aligned}c^2(x,\sigma,\sigma) &= c^2(x,\sigma|4Q_i,\sigma|4Q_i)+2c^2(x,\sigma|4Q_i,\sigma|\mathbb{C}\setminus 4Q_i)\\ &\quad + c^2(x,\sigma|\mathbb{C}\setminus 4Q_i,\sigma|\mathbb{C}\setminus 4Q_i).\end{aligned}$$

We want to see that

$$c_{\sigma|4Q_i}(x)>\alpha/4. \tag{6.4}$$

So it is enough to show that the last two terms in the equation above are sufficiently small. First we deal with $c^2(x,\sigma|4Q_i,\sigma|\mathbb{C}\setminus 4Q_i)$:

$$\begin{aligned}c^2(x,\sigma|4Q_i,\sigma|\mathbb{C}\setminus 4Q_i) &\leq C\int_{y\in 4Q_i}\int_{t\in\mathbb{C}\setminus 4Q_i}\frac{1}{|t-x|^2}\,d\sigma(y)d\sigma(t)\\ &= C\sigma(4Q_i)\int_{t\in\mathbb{C}\setminus 4Q_i}\frac{1}{|t-x|^2}\,d\sigma(t)\\ &\leq C\sigma(4Q_i)\frac{M\sigma(z)}{\ell(4Q_i)}\leq C\,M\sigma(z)^2\leq C\lambda^2. \qquad (6.5)\end{aligned}$$

For the term $c^2(x,\sigma|\mathbb{C}\setminus 4Q_i,\sigma|\mathbb{C}\setminus 4Q_i)$ we write

$$\begin{aligned}c^2(x,\sigma|\mathbb{C}\setminus 4Q_i,\sigma|\mathbb{C}\setminus 4Q_i) &= c^2(x,\sigma|\mathbb{C}\setminus 2RQ_i,\sigma|\mathbb{C}\setminus 2RQ_i)\\ &\quad + 2c^2(x,\sigma|\mathbb{C}\setminus 2RQ_i,\sigma|2RQ_i\setminus 4Q_i)\\ &\quad + c^2(x,\sigma|2RQ_i\setminus 4Q_i,\sigma|2RQ_i\setminus 4Q_i).\end{aligned}$$

Arguing as in (6.5), it easily follows that the last two terms are bounded above by $CM\sigma(z)^2 \leq C\lambda^2$ again. So we get,

$$c^2_{\sigma|4Q_i}(x) \geq c^2_\sigma(x) - c^2_{\sigma|\mathbb{C}\setminus 2RQ_i}(x) - C\lambda^2. \tag{6.6}$$

We are left with the term $c^2_{\sigma|\mathbb{C}\setminus 2RQ_i}(x)$. Since $x, z \in RQ_i$, it is not difficult to check that

$$|c_{\sigma|\mathbb{C}\setminus 2RQ_i}(x) - c_{\sigma|\mathbb{C}\setminus 2RQ_i}(z)| \leq CM\sigma(z) \leq C_6\lambda$$

[this is proved like (4.5)]. Taking into account that $c_\sigma(z) \leq \lambda$, we get

$$c_{\sigma|\mathbb{C}\setminus 2RQ_i}(x) \leq (1 + C_6)\lambda.$$

Thus, by (6.6), we obtain

$$c^2_{\sigma|4Q_i}(x) \geq \frac{\alpha^2}{4} - C\lambda^2 \geq \frac{\alpha^2}{16},$$

if λ is small enough. That is, we have proved (6.4), and so in this case (6.2) holds too.

6.3. **Proof of (c).** Now we have to show that

$$\operatorname{diam}(Q_i) \leq \frac{1}{10}\operatorname{diam}(E). \tag{6.7}$$

This will allow the application of our induction argument.

It is immediate to check that

$$U_\sigma(x) \leq \frac{100\sigma(E)}{\operatorname{dist}(x, E)}$$

for all $x \notin E$ (of course, 100 is not the best constant here). Thus, for $x \in \Omega \setminus E$ we have

$$\lambda < U_\sigma(x) \leq \frac{100\sigma(E)}{\operatorname{dist}(x, E)}.$$

Therefore,

$$\operatorname{dist}(x, E) \leq 100\lambda^{-1}\sigma(E) \leq C\lambda^{-1}\gamma_+(E) \leq \frac{1}{100}\operatorname{diam}(E),$$

taking the constant C_4 in First Main Lemma small enough. As a consequence, $\operatorname{diam}(\Omega) \leq \frac{11}{10}\operatorname{diam}(E)$. Since $20Q_i \subset \Omega$ for each $i \in I$, we have

$$20\operatorname{diam}(Q_i) \leq \operatorname{diam}(\Omega) \leq \frac{11}{10}\operatorname{diam}(E),$$

which implies (6.7).

7. Proof of (d)–(g) in First Main Lemma

7.1. **The construction of μ and ν and the proof of (d)–(f).** It is easily seen that there exists a family of $\mathcal{C}^\infty$ functions $\{g_i\}_{i\in J}$ such that, for each $i \in J$, $\operatorname{supp}(g_i) \subset 2Q_i$, $0 \leq g_i \leq 1$, and $\|\nabla g_i\|_\infty \leq C/\ell(Q_i)$, so that $\sum_{i\in J} g_i = 1$ on Ω. Notice that by the definition of I in Subsection 6.1, we also have $\sum_{i\in I} g_i = 1$ on E.

Let $f(z)$ be the Ahlfors function of E, and consider the complex measure ν_0 such that $f(z) = \mathcal{C}\nu_0(z)$ for $z \notin E$, with $|\nu_0(B(z,r))| \leq r$ for all $z \in \mathbb{C}$, $r > 0$ (see [Ma2, Theorem 19.9], for example). So we have

$$|\mathcal{C}\nu_0(z)| \leq 1 \quad \text{ for all } z \notin E,$$

and

$$\nu_0(E) = \gamma(E).$$

The measure ν will be a suitable modification of ν_0. As we explained in Section 3, the main drawback of ν_0 is that the only information that we have about its variation $|\nu_0|$ is that $|\nu_0| = b_0\, d\mathcal{H}^1_E$, with $\|b_0\|_\infty \leq 1$. This is a very bad estimate if we try to apply some kind of $T(b)$ theorem in order to show that the Cauchy transform is bounded (with absolute constants). The main advantage of ν over ν_0 is that we will have a much better estimate for the variation $|\nu|$.

First we define the measure μ. For each $i \in I$, let Γ_i be a circumference concentric with Q_i and radius $\gamma(E \cap 2Q_i)/10$. Observe that $\Gamma_i \subset \frac{1}{2}Q_i$ for each i. We set

$$\mu = \sum_{i\in I} \mathcal{H}^1 | \Gamma_i.$$

Let us define ν now:

$$\nu = \sum_{i\in I} \frac{1}{\mathcal{H}^1(\Gamma_i)} \int g_i\, d\nu_0 \cdot \mathcal{H}^1|\Gamma_i.$$

Notice that $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu) \subset F$. Moreover, we have $\nu(Q_i) = \int g_i\, d\nu_0$, and since $\sum_{i\in I} g_i = 1$ on E, we also have $\nu(F) = \sum_{i\in I} \nu(Q_i) = \nu_0(E) = \gamma(E)$ (which yields (f)).

We have $d\nu = b\, d\mu$, with $b = \frac{\int g_i\, d\nu_0}{\mathcal{H}^1(\Gamma_i)}$ on Γ_i. To estimate $\|b\|_{L^\infty(\mu)}$, notice that

$$|\mathcal{C}(g_i\nu_0)(z)| \leq C \quad \text{ for all } z \notin E \cap 2Q_i. \tag{7.1}$$

This follows easily from the formula

$$\mathcal{C}(g_i\nu_0)(\xi) = \mathcal{C}\nu_0(\xi)\, g_i(\xi) + \frac{1}{\pi}\int \frac{\mathcal{C}\nu_0(z)}{z-\xi}\, \bar\partial g_i(z)\, d\mathcal{L}^2(z), \tag{7.2}$$

where $\mathcal{L}^2$ stands for the planar Lebesgue measure on $\mathbb{C}$. Let us remark that this identity is used often to split singularities in Vitushkin's way. Inequality

(7.1) implies that

$$\left|\int g_i\,d\nu_0\right| = \left|(\mathcal{C}(g_i\,\nu_0))'\,(\infty)\right| \leq C\gamma(E\cap 2Q_i) = C\mathcal{H}^1(\Gamma_i). \tag{7.3}$$

As a consequence, $\|b\|_{L^\infty(\mu)} \leq C$, and (e) is proved.

It remains to check that (d) also holds. Using (7.3), the assumption $\gamma(E\cap 2Q_i) \leq A\gamma_+(E\cap 2Q_i)$, (b), and the hypothesis $A\gamma_+(E) \leq \gamma(E)$, we obtain the following inequalities:

$$\begin{aligned}
\gamma(E) = |\nu_0(E)| = \left|\sum_{i\in I}\int g_i\,d\nu_0\right| \;&\leq\; \sum_{i\in I}\left|\int g_i\,d\nu_0\right| \\
&\leq\; C\sum_{i\in I}\gamma(E\cap 2Q_i) = C\mu(F) \\
&\leq\; C\,A\sum_{i\in I}\gamma_+(E\cap 2Q_i) \\
&\leq\; C\,A\gamma_+(E) \leq C\gamma(E),
\end{aligned}$$

which gives (d) (with constants independent of A).

Notice, by the way, that the preceding inequalities show that $\gamma(E) \leq CA\gamma_+(E)$. This is not very useful for us, because if we try to apply induction, at each step of the induction the constant A will be multiplied by the constant C.

On the other hand, since for each square Q_i we have $\mu(F\cap Q_i) \leq CA\gamma_+(E\cap 2Q_i) \leq CA\sigma(2Q_i)$, with $\sigma\in\Sigma(E)$, it follows easily that

$$\mu(B(x,r)) \leq C\,A\,r \quad \text{for all } x\in F,\ r>0. \tag{7.4}$$

Unfortunately, for our purposes this is not enough. We would like to obtain the same estimate without the constant A on the right hand side, but we will not be able. Instead, we will get it for all $x\in F$ out of a rather small exceptional set H.

7.2. **The exceptional set H.** Before constructing the dyadic exceptional set $H_{\mathcal{D}}$, we will consider a non dyadic version, which we will denote by H.

Let $C_0 \geq 100C_a$ be some fixed constant. Following [NTV1], given $x\in F$, $r>0$, we say that $B(x,r)$ is a **non Ahlfors disk** if $\mu(B(x,r)) > C_0 r$. For a fixed $x\in F$, if there exists some $r>0$ such that $B(x,r)$ is a non Ahlfors disk, then we say that x is a **non Ahlfors point**. For any $x\in F$, we denote

$$\mathcal{R}(x) = \sup\{r>0:\ B(x,r) \text{ is a non Ahlfors disk}\}.$$

If $x\in F$ is an Ahlfors point, we set $\mathcal{R}(x)=0$. We say that $\mathcal{R}(x)$ is the **Ahlfors radius** of x.

Observe (d) implies that $\mu(F) \leq C_a\gamma(E) \leq C_a\gamma(F) \leq C_a\mathrm{diam}(F)$. Therefore,

$$\mu(B(x,r)) \leq \mu(F) \leq C_a\mathrm{diam}(F) \leq 100C_a r$$

for $r \geq \mathrm{diam}(F)/100$. Thus $\mathcal{R}(x) \leq \mathrm{diam}(F)/100$ for all $x\in F$.

We denote

$$H_0 = \bigcup_{x\in F,\,\mathcal{R}(x)>0} B(x,\mathcal{R}(x)).$$

By Vitali's $5r$-Covering Theorem there is a disjoint family $\{B(x_h,\mathcal{R}(x_h))\}_h$ such that $H_0 \subset \bigcup_h B(x_h, 5\mathcal{R}(x_h))$. We denote

$$H = \bigcup_h B(x_h, 5\mathcal{R}(x_h)). \tag{7.5}$$

Since $H_0 \subset H$, all non Ahlfors disks are contained in H and then,

$$\operatorname{dist}(x, F\setminus H) \geq \mathcal{R}(x) \tag{7.6}$$

for all $x \in F$.

Since $\mu(B(x_h,\mathcal{R}(x_h))) \geq C_0\mathcal{R}(x_h)$ for every h, we get

$$\sum_h \mathcal{R}(x_h) \leq \frac{1}{C_0}\sum_h \mu(B(x_h,\mathcal{R}(x_h))) \leq \frac{1}{C_0}\mu(F), \tag{7.7}$$

with $\frac{1}{C_0}$ arbitrarily small (choosing C_0 big enough).

7.3. **Proof of (g).** The dyadic exceptional set $H_\mathcal{D}$ will be constructed in Section 8. We will have $H_\mathcal{D} \supset H$ for any choice of $\mathcal{D}$. In this subsection we will show that

$$\int_{F\setminus H} \mathcal{C}_*\nu \, d\mu \leq C_c\mu(F), \tag{7.8}$$

which implies (g), provided $H_\mathcal{D} \supset H$.

We will work with the regularized operators $\widetilde{\mathcal{C}}_\varepsilon$ introduced at the end of Section 2. Remember that $|\mathcal{C}\nu_0(z)| \leq 1$ for all $z \notin E$. Since $\mathcal{L}^2(E) = 0$, the same inequality holds $\mathcal{L}^2$-a.e. $z \in \mathbb{C}$. Thus, $|\widetilde{\mathcal{C}}_\varepsilon\nu_0(z)| \leq 1$ and $\widetilde{\mathcal{C}}_*\nu_0(z) \leq 1$ for all $z\in\mathbb{C}$, $\varepsilon > 0$.

To estimate $\widetilde{\mathcal{C}}_*\nu$, we will deal with the term $\widetilde{\mathcal{C}}_*(\nu-\nu_0)$. This will be the main point for the proof of (7.8).

We denote $\nu_i := \nu|Q_i$.

Lemma 7.1. *For every $z \in \mathbb{C}\setminus 4Q_i$, we have*

$$\widetilde{\mathcal{C}}_*(\nu_i - g_i\nu_0)(z) \leq \frac{C\ell(Q_i)\,\mu(Q_i)}{\operatorname{dist}(z,2Q_i)^2}. \tag{7.9}$$

Notice that $\int(d\nu_i - g_i\,d\nu_0) = 0$. Then, using the smoothness of the kernels of the operators $\widetilde{\mathcal{C}}_\varepsilon$, $\varepsilon > 0$, by standard estimates it easily follows

$$\widetilde{\mathcal{C}}_*(\nu_i - g_i\nu_0)(z) \leq \frac{C\ell(Q_i)\,(|\nu|(Q_i) + |\nu_0|(2Q_i))}{\operatorname{dist}(z,2Q_i)^2}.$$

This inequality is not useful for our purposes because to estimate $|\nu_0|(2Q_i)$ we only can use $|\nu_0|(2Q_i) \leq \mathcal{H}^1(E\cap 2Q_i)$. However, we don't have any control over $\mathcal{H}^1(E\cap 2Q_i)$ (we only know that it is finite, by our assumptions on E). The estimate (7.9) is much sharper.

Proof of the lemma. We set $\alpha_i = \nu_i - g_i\nu_0$. To prove the lemma, we have to show that

$$|\widetilde{\mathcal{C}}_\varepsilon \alpha_i(z)| \leq \frac{C\ell(Q_i)\,\mu(Q_i)}{\operatorname{dist}(z,2Q_i)^2} \tag{7.10}$$

for all $\varepsilon > 0$.

Assume first $\varepsilon \leq \operatorname{dist}(z,2Q_i)/2$. Since $|\mathcal{C}\alpha_i(w)| \leq C$ for all $w \notin \operatorname{supp}(\alpha_i)$ and $\alpha_i(\mathbb{C}) = 0$, we have

$$|\mathcal{C}\alpha_i(w)| \leq \frac{C\operatorname{diam}(\operatorname{supp}(\alpha_i))\,\gamma(\operatorname{supp}(\alpha_i))}{\operatorname{dist}(w,\operatorname{supp}(\alpha_i))^2}$$

(see [Gar, p.12-13]). Remember that

$$\operatorname{supp}(\alpha_i) \subset \Gamma_i \cup (E \cap 2Q_i) \subset 2Q_i.$$

Then we get

$$|\mathcal{C}\alpha_i(w)| \leq \frac{C\,\ell(Q_i)\gamma(\Gamma_i \cup (E \cap 2Q_i))}{\operatorname{dist}(w,2Q_i)^2}. \tag{7.11}$$

Moreover, we have

$$\gamma(\Gamma_i \cup (E \cap 2Q_i)) \leq C(\gamma(\Gamma_i) + \gamma(E \cap 2Q_i)),$$

because semiadditivity holds for the special case $\Gamma_i \cup (E \cap 2Q_i)$. This fact follows easily from Melnikov's result about semiadditivity of analytic capacity for two compacts which are separated by a circumference [Me1]. Therefore, by the definition of Γ_i, we get

$$\gamma(\Gamma_i \cup (E \cap 2Q_i)) \leq C\gamma(E \cap 2Q_i) = C\mu(Q_i). \tag{7.12}$$

If $w \in B(z,\varepsilon)$, then $\operatorname{dist}(w,2Q_i) \approx \operatorname{dist}(z,2Q_i)$. By (7.11) and (7.12) we obtain

$$|\mathcal{C}\alpha_i(w)| \leq \frac{C\,\ell(Q_i)\mu(Q_i)}{\operatorname{dist}(z,2Q_i)^2}.$$

Making the convolution with ψ_ε, (7.10) follows for $\varepsilon \leq \operatorname{dist}(z,2Q_i)/2$.

Suppose now that $\varepsilon > \operatorname{dist}(z,2Q_i)/2$. We denote $h = \psi_\varepsilon * \alpha_i$. Then we have

$$\widetilde{\mathcal{C}}_\varepsilon \alpha_i = \psi_\varepsilon * \frac{1}{z} * \alpha_i = \mathcal{C}(h\,d\mathcal{L}^2).$$

Therefore,

$$|\widetilde{\mathcal{C}}_\varepsilon \alpha_i(z)| \leq \int \frac{|h(\xi)|}{|\xi - z|}\,d\mathcal{L}^2(\xi) \leq \|h\|_\infty\,[\mathcal{L}^2(\operatorname{supp}(h))]^{1/2}. \tag{7.13}$$

We have to estimate $\|h\|_\infty$ and $\mathcal{L}^2(\operatorname{supp}(h))$. Observe that, if we write $\ell_i = \ell(Q_i)$ and we denote the center of Q_i by z_i, we have

$$\operatorname{supp}(h) \subset \operatorname{supp}(\psi_\varepsilon) + \operatorname{supp}(\alpha_i) \subset B(0,\varepsilon) + B(z_i,2\ell_i) = B(z_i,\varepsilon + 2\ell_i).$$

Thus, $\mathcal{L}^2(\operatorname{supp}(h)) \leq C\varepsilon^2$, since $\ell_i \leq \varepsilon$.

Let us deal with $\|h\|_\infty$ now. Let η_i be a $\mathcal{C}^\infty$ function supported on $3Q_i$ which is identically 1 on $2Q_i$ and such that $\|\nabla\eta_i\|_\infty \leq C/\ell_i$. Taking into account that $\alpha_i(2Q_i) = 0$, we have

$$h(w) = \int \psi_\varepsilon(\xi - w)\, d\alpha_i(\xi) = \int \big(\psi_\varepsilon(\xi - w) - \psi_\varepsilon(z_i - w)\big)\, d\alpha_i(\xi)$$
$$= \frac{\ell_i}{\varepsilon^3}\int \frac{\varepsilon^3}{\ell_i}\big(\psi_\varepsilon(\xi-w)-\psi_\varepsilon(z_i-w)\big)\,\eta_i(\xi)\, d\alpha_i(\xi) =: \frac{\ell_i}{\varepsilon^3}\int \varphi_w(\xi)\eta_i(\xi)\, d\alpha_i(\xi).$$

We will show below that

$$\|\mathcal{C}(\varphi_w\eta_i\, d\alpha_i)\|_{L^\infty(\mathbb{C})} \leq C. \tag{7.14}$$

Let us assume this estimate for the moment. Since $\mathcal{C}(\varphi_w\eta_i d\alpha_i)$ is analytic in $\mathbb{C}\setminus \mathrm{supp}(\alpha_i)$, using (7.12) we deduce

$$\left|\frac{\ell_i}{\varepsilon^3}\int \varphi_w(\xi)\eta_i(\xi)\, d\alpha_i(\xi)\right| \leq \frac{C\ell_i}{\varepsilon^3}\gamma(\Gamma_i \cup (E\cap 2Q_i)) \leq \frac{C\ell_i\mu(Q_i)}{\varepsilon^3}.$$

Therefore,

$$\|h\|_\infty \leq \frac{C\ell_i\mu(Q_i)}{\varepsilon^3}.$$

By (7.13) and the estimates on $\|h\|_\infty$ and $\mathcal{L}^2(\mathrm{supp}(h))$, we obtain

$$|\widetilde{\mathcal{C}}_\varepsilon\alpha_i(z)| \leq \frac{C\ell(Q_i)\mu(Q_i)}{\varepsilon^2} \leq \frac{C\ell(Q_i)\mu(Q_i)}{\mathrm{dist}(z,2Q_i)^2}.$$

It remains to prove (7.14). Remember that $\mathcal{C}\alpha_i$ is a bounded function. By the identity (7.2), since $\mathrm{supp}(\varphi_w\eta_i) \subset 3Q_i$, it is enough to show that

$$\|\varphi_w\eta_i\|_\infty \leq C \tag{7.15}$$

and

$$\|\nabla(\varphi_w\eta_i)\|_\infty \leq \frac{C}{\ell_i}, \tag{7.16}$$

For $\xi \in 3Q_i$, we have

$$|\varphi_w(\xi)| = \frac{\varepsilon^3}{\ell_i}\,\big|\psi_\varepsilon(\xi - w) - \psi_\varepsilon(z_i - w)\big| \leq \varepsilon^3\,\|\nabla\psi_\varepsilon\|_\infty \leq C,$$

which yields (7.15). Finally, (7.16) follows easily too:

$$\|\nabla(\varphi_w\eta_i)\|_\infty \leq \|\nabla\varphi_w\|_\infty + \|\varphi_w\|_\infty\|\eta_i\|_\infty \leq \frac{C}{\ell_i}.$$

We are done. □

Now we are ready to prove (7.8). We write

$$\begin{aligned}\int_{F\setminus H}\widetilde{\mathcal{C}}_*\nu\, d\mu &\leq \int_{F\setminus H}\widetilde{\mathcal{C}}_*\nu_0\, d\mu + \int_{F\setminus H}\widetilde{\mathcal{C}}_*(\nu - \nu_0)\, d\mu \\ &\leq C\mu(F\setminus H) + \sum_{i\in I}\int_{F\setminus H}\widetilde{\mathcal{C}}_*(\nu_i - g_i\nu_0)\, d\mu. \end{aligned}\tag{7.17}$$

To estimate the last integral we use Lemma 7.1 and recall that $\|\widetilde{\mathcal{C}}_*(\nu_i - g_i\nu_0)\|_{L^\infty(\mu)} \leq C$:

$$\int_{F\setminus H} \widetilde{\mathcal{C}}_*(\nu_i - g_i\nu_0)\, d\mu \leq C\mu(4Q_i) + \int_{F\setminus(4Q_i\cup H)} \frac{C\ell(Q_i)\,\mu(Q_i)}{\operatorname{dist}(z,2Q_i)^2}\, d\mu(z). \tag{7.18}$$

Let $N \geq 1$ be the least integer such that $(4^{N+1}Q_i \setminus 4^N Q_i) \setminus H \neq \varnothing$, and take some fixed $z_0 \in (4^{N+1}Q_i \setminus 4^N Q_i) \setminus H$. We have

$$\begin{aligned}
\int_{F\setminus(4Q_i\cup H)} \frac{1}{\operatorname{dist}(z,2Q_i)^2}\, d\mu(z) &= \sum_{k=N}^{\infty} \int_{(4^{k+1}Q_i\setminus 4^k Q_i)\setminus H} \\
&\leq C \sum_{k=N}^{\infty} \frac{\mu(4^{k+1}Q_i)}{\ell(4^{k+1}Q_i)^2} \\
&\leq C \sum_{k=N}^{\infty} \frac{\mu(B(z_0, 2\ell(4^{k+1}Q_i)))}{\ell(4^{k+1}Q_i)^2} \\
&\leq C \sum_{k=N}^{\infty} \frac{C_0\ell(4^{k+1}Q_i)}{\ell(4^{k+1}Q_i)^2} \\
&\leq C\,C_0 \frac{1}{\ell(4^N Q_i)} \leq C\,C_0 \frac{1}{\ell(Q_i)}.
\end{aligned}$$

Notice that in the second inequality we have used that $z_0 \in F \setminus H$, and so $\mu(B(z_0,r)) \leq C_0 r$ for all r. By (7.18), we obtain

$$\int_{F\setminus H} \widetilde{\mathcal{C}}_*(\nu_i - g_i\nu_0)\, d\mu \leq C\mu(4Q_i).$$

Thus, by the finite overlap of the squares $4Q_i$, $i \in I$, and (7.17), we get

$$\int_{F\setminus H} \widetilde{\mathcal{C}}_*\nu\, d\mu \leq C\mu(F\setminus H) + C\sum_{i\in I} \mu(4Q_i) \leq C\mu(F). \tag{7.19}$$

Now, (2.5) relates $\mathcal{C}_*\nu$ with $\widetilde{\mathcal{C}}_*\nu$:

$$|\widetilde{\mathcal{C}}_*\nu(z) - \mathcal{C}_*\nu(z)| \leq C\, M\nu(z). \tag{7.20}$$

By (e), if $z \in F \setminus H$, we have $M\nu(z) \leq CM\mu(z) \leq C$. Thus (7.19) and (7.20) imply

$$\int_{F\setminus H} \mathcal{C}_*\nu(z)\, d\mu(z) \leq C\mu(F).$$

8. The exceptional set $H_{\mathcal{D}}$

8.1. **The construction of $H_{\mathcal{D}}$ and the proof of (h)–(i).** Remember that in (7.5) we defined $H = \bigcup_h B(x_h, 5\mathcal{R}(x_h))$, where $\{B(x_h, \mathcal{R}(x_h))\}_h$ is some precise family of non Ahlfors disks. Consider the family of dyadic squares $\mathcal{D}_H \subset \mathcal{D}$ such that $R \in \mathcal{D}_H$ if there exists some ball $B(x_h, 5\mathcal{R}(x_h))$ satisfying

$$B(x_h, 5\mathcal{R}(x_h)) \cap R \neq \varnothing \tag{8.1}$$

and

$$10\mathcal{R}(x_h) < \ell(R) \le 20\mathcal{R}(x_h). \tag{8.2}$$

Notice that

$$\bigcup_h B(x_h, 5\mathcal{R}(x_h)) \subset \bigcup_{R\in\mathcal{D}_H} R. \tag{8.3}$$

We take a subfamily of disjoint maximal squares $\{R_k\}_{k\in I_H}$ from $\mathcal{D}_H$ such that

$$\bigcup_{R\in\mathcal{D}_H} R = \bigcup_{k\in I_H} R_k,$$

and we define the **dyadic exceptional set** $H_{\mathcal{D}}$ as

$$H_{\mathcal{D}} = \bigcup_{k\in I_H} R_k.$$

Observe that (8.3) implies $H \subset H_{\mathcal{D}}$ and, since for each ball $B(x_h, 5\mathcal{R}(x_h))$ there are at most four squares $R \in \mathcal{D}_H$ satisfying (8.1) and (8.2), by (7.7), we obtain

$$\sum_{k\in I_H} \ell(R_k) \le 80 \sum_h \mathcal{R}(x_h) \le \frac{80}{C_0}\, \mu(F) \le \varepsilon\mu(F),$$

assuming $C_0 \ge 80\varepsilon^{-1}$.

8.2. **Proof of (j).** Remember that the squares from the lattice $\mathcal{D}$ are half open - half closed. The other squares, such as the squares $\{Q_i\}_{i\in I}$ which form F, are supposed to be closed. From the point of view of the measures μ and ν, there is no difference between both choices because $\mu(\partial Q) = |\nu|(\partial Q) = 0$ for any square Q (remember that μ is supported on a finite union of circumferences).

We have

$$|\nu(H_{\mathcal{D}})| \le \sum_{k\in I_H} |\nu(R_k)|,$$

because the squares R_k, $k \in I_H$, are pairwise disjoint. On the other hand, from (i), we deduce

$$\sum_{k\in I_H} \ell(R_k) \le \varepsilon\mu(F) \le C_a\varepsilon|\nu(F))|,$$

with $\varepsilon \to 0$ as $C_0 \to \infty$. So (j) follows from next lemma.

Lemma 8.1. *For all squares $R \subset \mathbb{C}$, we have*

$$|\nu(R)| \le C\ell(R),$$

where C is some absolute constant.

To prove this result we will need a couple of technical lemmas.

Lemma 8.2. *Suppose that $\bar{C}_0$ is some big enough constant. Let $R \subset \mathbb{C}$ be a square such that $\mu(R) > \bar{C}_0 \ell(R)$. If Q_i is a Whitney square such that $2Q_i \cap R \neq \varnothing$, then $\ell(Q_i) \leq \ell(R)/4$.*

Proof. Let us see that if $\ell(Q_i) > \ell(R)/4$, then $\mu(R) \leq \bar{C}_0 \ell(R)$. We may assume $\mu(R) \geq 100\ell(R)$. Notice that $R \subset 9Q_i$ and, by Whitney's construction, we have $\#\{j : Q_j \cap 9Q_i \neq \varnothing\} \leq C$. Further, $\ell(Q_j) \approx \ell(Q_i)$ for this type of squares. Recall also that the measure μ on each Whitney square Q_j coincides with $\mathcal{H}^1|\Gamma_j$, where Γ_j is a circumference contained in $\frac{1}{2}Q_j$, and so $\mu(Q_j) \leq C\ell(Q_j)$ for each j. Therefore,

$$\mu(R) \leq \sum_{j:Q_j \cap 9Q_i \neq \varnothing} \mu(Q_j) \leq C \sum_{j:Q_j \cap 9Q_i \neq \varnothing} \ell(Q_j) \leq C\ell(Q_i).$$

So we only have to show that $\ell(Q_i) \leq C\ell(R)$.

Since $2Q_i \cap R \neq \varnothing$, there exists some Whitney square Q_j such that $Q_j \cap R \neq \varnothing$ and $Q_j \cap 2Q_i \neq \varnothing$. Since we are assuming $\mu(R) \geq 100\ell(R)$, we have $\ell(R) \geq \varepsilon_0 \ell(Q_j)$, where $\varepsilon_0 > 0$ is some absolute constant (for instance, $\varepsilon_0 = 1/100$ would possibly work). Thus, $\ell(Q_i) \approx \ell(Q_j) \leq C\ell(R)$. □

Lemma 8.3. *Let $R \subset \mathbb{C}$ be a square such that $\ell(Q_i) \leq \ell(R)/4$ for each Whitney square Q_i with $2Q_i \cap R \neq \varnothing$. Let $L_R = \{h \in I : 2Q_h \cap \partial R \neq \varnothing\}$. Then,*

$$\sum_{h \in L_R} \ell(Q_h) \leq C\ell(R).$$

Proof. Let L be one of the sides of R. Let $\{Q_h\}_{h \in I_L}$ be the subfamily of Whitney squares such that $2Q_h \cap L \neq \varnothing$. Since $\ell(Q_h) \leq \ell(R)/4$, we have $\mathcal{H}^1(4Q_h \cap L) \geq C^{-1}\ell(Q_h)$. Then, by the bounded overlap of the squares $4Q_h$, we obtain

$$\sum_{h \in I_L} \ell(Q_h) \leq C \sum_{h \in I_L} \mathcal{H}^1(4Q_h \cap L) \leq C\ell(R). \tag{8.4}$$

□

Proof of Lemma 8.1. By Lemma 8.2, we may assume $\ell(Q_i) \leq \ell(R)/4$ if $2Q_i \cap R \neq \varnothing$. Otherwise, $|\nu(R)| \leq C_b \mu(R) \leq C_b \bar{C}_0 \ell(R)$.

From the fact that $\|\mathcal{C}\nu_0\|_{L^\infty(\mathbb{C})} \leq 1$, we deduce $|\nu_0(R)| \leq C\ell(R)$. So we only have to estimate the difference $|\nu(R) - \nu_0(R)|$.

Let $\{Q_i\}_{i \in I_R}$, $I_R \subset I$, be the subfamily of Whitney squares such that $Q_i \cap R \neq \varnothing$, and let $\{Q_i\}_{i \in J_R}$, $J_R \subset I$, be the Whitney squares such that

$Q_i \subset R$. We write

$$\begin{aligned}
|\nu(R)-\nu_0(R)| &= \Big|\nu\Big(\bigcup_{i\in I_R}(Q_i\cap R)\Big)-\nu_0\Big(\bigcup_{i\in I_R}(Q_i\cap R)\Big)\Big| \\
&\leq \Big|\nu\Big(\bigcup_{i\in I_R\setminus J_R}(Q_i\cap R)\Big)-\nu_0\Big(\bigcup_{i\in I_R\setminus J_R}(Q_i\cap R)\Big)\Big| \\
&\quad +\Big|\nu\Big(\bigcup_{i\in J_R}Q_i\Big)-\nu_0\Big(\bigcup_{i\in J_R}Q_i\Big)\Big| \\
&= A+B.
\end{aligned}$$

First we deal with the term A. We have

$$\begin{aligned}
A &= \Big|\sum_{i\in I_R\setminus J_R}\frac{\mathcal{H}^1(\Gamma_i\cap R)}{\mathcal{H}^1(\Gamma_i)}\int g_i\,d\nu_0-\sum_{i\in I_R\setminus J_R}\nu_0(Q_i\cap R)\Big| \\
&\leq \sum_{i\in I_R\setminus J_R}\Big|\int g_i\,d\nu_0\Big|+\sum_{i\in I_R\setminus J_R}|\nu_0(Q_i\cap R)|.
\end{aligned}$$

Since $|\mathcal{C}(g_i\nu_0)|\leq C$ and $|\mathcal{C}\nu_0|\leq C$, we have

$$\Big|\int g_i\,d\nu_0\Big|+|\nu_0(Q_i\cap R)|\leq C\ell(Q_i)+C\mathcal{H}^1(\partial(Q_i\cap R))\leq C\ell(Q_i).$$

Thus, $A\leq C\sum_{i\in I_R\setminus J_R}\ell(Q_i)$. Notice now that if $i\in I_R\setminus J_R$, then $Q_i\cap R\neq\varnothing$ and $Q_i\not\subset R$. Therefore, $Q_i\cap\partial R\neq\varnothing$. From Lemma 8.3 we deduce $A\leq C\ell(R)$.

Let us turn our attention to B:

$$\begin{aligned}
B &= \Big|\sum_{i\in J_R}\int g_i\,d\nu_0-\int_{\bigcup_{i\in J_R}Q_i}d\nu_0\Big| \\
&= \Big|\int\Big(\sum_{i\in J_R}g_i-\chi_{\bigcup_{i\in J_R}Q_i}\Big)\,d\nu_0\Big| \\
&\leq \sum_{j\in J_R}\Big|\int_{Q_j}\Big(\sum_{i\in J_R}g_i-1\Big)\,d\nu_0\Big|+\Big|\int_{\mathbb{C}\setminus\bigcup_{j\in J_R}Q_j}\sum_{i\in J_R}g_i\,d\nu_0\Big| \\
&= B_1+B_2.
\end{aligned}$$

We consider first B_1. If $\sum_{i\in J_R}g_i\not\equiv 1$ on Q_j, we write $j\in M_R$. In this case there exists some $h\in I\setminus J_R$ such that $g_h\not\equiv 0$ on Q_j. So $2Q_h\cap Q_j\neq\varnothing$, with $Q_h\not\subset R$. Thus, $2Q_h\cap\partial R\neq\varnothing$. That is, $h\in L_R$.

For each $h\in L_R$ there are at most C_8 squares Q_j such that $2Q_h\cap Q_j\neq\varnothing$. Moreover, for these squares Q_j we have $\ell(Q_j)\leq C\ell(Q_h)$. Then, by Lemma 8.3, we get

$$\sum_{j\in M_R}\ell(Q_j)\leq C\,C_8\sum_{h\in L_R}\ell(Q_h)\leq C\ell(R). \tag{8.5}$$

Now we set

$$\begin{aligned} B_1 &= \sum_{j\in M_R} \left| \int_{Q_j} \Bigl(\sum_{i\in J_R} g_i - 1 \Bigr)\, d\nu_0 \right| \\ &\leq \sum_{j\in M_R} \left(\left| \int_{Q_j} \sum_{i\in J_R} g_i \, d\nu_0 \right| + |\nu_0(Q_j)| \right). \end{aligned}$$

We have $|\nu_0(Q_j)| \leq C\ell(Q_j)$ and also

$$\left| \int_{Q_j} \sum_{i\in J_R} g_i \, d\nu_0 \right| \leq \sum_{i\in J_R} \left| \int_{Q_j} g_i \, d\nu_0 \right| \leq C\ell(Q_j),$$

because $\#\{i \in J_R : \operatorname{supp}(g_i) \cap Q_j \neq \varnothing\} \leq C$ and $|\mathcal{C}(g_i\nu_0)| \leq C$ for each i. Thus, by (8.5), we deduce

$$B_1 \leq C\ell(R).$$

Finally we have to estimate B_2. We have

$$B_2 \leq \sum_{i\in J_R} \left| \int_{\mathbb{C}\setminus \bigcup_{j\in J_R} Q_j} g_i \, d\nu_0 \right| = \sum_{i\in J_R} B_{2,i}.$$

Observe that if $B_{2,i} \neq 0$, then $\operatorname{supp}(g_i) \cap \operatorname{supp}(\nu_0) \cap \mathbb{C} \setminus \bigcup_{j\in J_R} Q_j \neq \varnothing$. As a consequence, $2Q_i \cap Q_h \neq \varnothing$ for some $h \in I \setminus J_R$. Since $Q_i \subset R$ and $Q_h \not\subset R$, we deduce that either $2Q_i \cap \partial R \neq \varnothing$ or $Q_h \cap \partial R \neq \varnothing$. So either $i \in L_R$ or $h \in L_R$. Taking into account that $\ell(Q_i) \approx \ell(Q_h)$, arguing as above we get

$$\begin{aligned} B_2 &\leq C \sum_{i\in L_R} \ell(Q_i) + C \sum_{i\in J_R} \sum_{h\in L_R : Q_h\cap 2Q_i \neq \varnothing} \ell(Q_i) \\ &\leq C\ell(R) + C \sum_{h\in L_R} \sum_{i\in I : Q_h \cap 2Q_i \neq\varnothing} \ell(Q_h) \leq C\ell(R). \end{aligned}$$

□

8.3. **Proof of (k).** Let us see that (k) is a direct consequence of (j). We have

$$|\nu(F \setminus H_{\mathcal{D}}) \geq |\nu(F)| - |\nu(H_{\mathcal{D}})| \geq (1-\varepsilon)|\nu(F)|.$$

By (d) and (f), we get

$$\mu(F) \leq C|\nu(F)| \leq \frac{C}{1-\varepsilon}\, |\nu(F \setminus H_{\mathcal{D}})|.$$

Since $\|b\|_{L^\infty(\mu)} \leq C$, we have $|\nu(F \setminus H_{\mathcal{D}})| \leq C\mu(F \setminus H_{\mathcal{D}})$. Thus, $\mu(F) \leq \frac{C_9}{1-\varepsilon}\, \mu(F \setminus H_{\mathcal{D}})$. That is, $\mu(H_{\mathcal{D}}) \leq \delta\mu(F)$, with $\delta = 1 - \frac{1-\varepsilon}{C_9}$.

9. The Second Main Lemma

The second part of the proof of Theorem 1.1 is based on the $T(b)$ theorem of Nazarov, Treil and Volberg in [NTV1]. The precise result that we will prove is the following. We use the same notation of First Main Lemma 5.1.

Lemma 9.1 (**Second Main Lemma).** *Assume that $\gamma_+(E) \leq C_4 \mathrm{diam}(E)$, $\gamma(E) \geq A\gamma_+(E)$, and $\gamma(E\cap 2Q_i) \leq A\gamma_+(E\cap 2Q_i)$ for all $i \in I$. Then there exists some subset $G \subset F$, with $\mu(F) \leq C_{10}\mu(G)$, such that $\mu(G\cap B(x,r)) \leq C_0 r$ for all $x \in G$, $r > 0$, and the Cauchy transform is bounded on $L^2(\mu|G)$ with $\|\mathcal{C}\|_{L^2(\mu|G),L^2(\mu|G)} \leq C_{11}$, where C_{11} is some absolute constant. The constants C_0, C_4, C_{10}, C_{11} are absolute constants, and do not depend on A.*

We will prove this lemma in the next two sections. First, in Section 10 we will introduce two exceptional sets S and $T_{\mathcal{D}}$ such that $\mathcal{C}_*\nu$ will be uniformly bounded on $F \setminus S$ and b will behave as a paraacretive function out of $T_{\mathcal{D}}$. In the same section we will introduce the "suppressed" operators of Nazarov, Treil and Volberg. In Section 11 we will describe which modifications are required in the $T(b)$ theorem of [NTV1] to prove Second Main Lemma.

10. The exceptional sets S and $T_{\mathcal{D}}$ and the suppressed operators $\mathcal{C}_\Theta$

10.1. **The exceptional set S.** The arguments in this subsection will be similar to the ones in [To3].

We set

$$S_0 = \{x \in F : \mathcal{C}_*\nu > \alpha\},$$

where α is some big constant which will be chosen below. For the moment, let us say that $\alpha \gg C_0C_b, C_c$. For $x \in S_0$, let

$$\varepsilon(x) = \sup\{\varepsilon : \varepsilon > 0,\, |\mathcal{C}_\varepsilon\nu(x)| > \alpha\}.$$

Otherwise, we set $\varepsilon(x) = 0$. We define the **exceptional set** S as

$$S = \bigcup_{x\in S_0} B(x, \varepsilon(x)).$$

To show that $\mu(S \setminus H_{\mathcal{D}})$ is small we will use the following result.

Lemma 10.1. *If $y \in S \setminus H_{\mathcal{D}}$, then $\mathcal{C}_*\nu(y) > \alpha - 8C_0C_b$.*

Proof. Observe that if $y \in S\setminus H_{\mathcal{D}}$, then $y \in B(x, \varepsilon(x))$ for some $x \in S_0$. Let $\varepsilon_0(x)$ be such that $|\mathcal{C}_{\varepsilon_0(x)}\nu(x)| > \alpha$ and $y \in B(x, \varepsilon_0(x))$. We will show that

$$|\mathcal{C}_{\varepsilon_0(x)}\nu(x) - \mathcal{C}_{\varepsilon_0(x)}\nu(y)| \leq 8C_0C_b, \tag{10.1}$$

and we will be done. We have

$$\begin{aligned}
&|\mathcal{C}_{\varepsilon_0(x)}\nu(x) - \mathcal{C}_{\varepsilon_0(x)}\nu(y)| \leq \\
&\qquad |\mathcal{C}_{\varepsilon_0(x)}(\nu|B(y,2\varepsilon_0(x)))(x)| + |\mathcal{C}_{\varepsilon_0(x)}(\nu|B(y,2\varepsilon_0(x)))(y)| \\
&\quad + |\mathcal{C}_{\varepsilon_0(x)}(\nu|\mathbb{C}\setminus B(y,2\varepsilon_0(x)))(x) - \mathcal{C}_{\varepsilon_0(x)}(\nu|\mathbb{C}\setminus B(y,2\varepsilon_0(x)))(y)|. \qquad (10.2)
\end{aligned}$$

Notice that the first two terms on the right hand side are bounded above by

$$\frac{|\nu|(B(y,2\varepsilon_0(x)))}{\varepsilon_0(x)} \leq \frac{C_b\mu(B(y,2\varepsilon_0(x)))}{\varepsilon_0(x)} \leq 2C_0C_b,$$

since $y \notin H_{\mathcal{D}}$. The last term on the right hand side of (10.2) is bounded above by

$$\begin{aligned}\int_{\mathbb{C}\setminus B(y,2\varepsilon_0(x))} \left|\frac{1}{z-x} - \frac{1}{z-y}\right| d|\nu|(z) &= \int_{\mathbb{C}\setminus B(y,2\varepsilon_0(x))} \frac{|x-y|}{|z-x||z-y|}\, d|\nu|(z)\\ &\leq 2C_b\varepsilon_0(x)\int_{\mathbb{C}\setminus B(y,2\varepsilon_0(x))} \frac{1}{|z-y|^2}\, d\mu(z),\end{aligned}$$

where we have applied that $|x-y| \leq \varepsilon_0(x)$ and $|z-x| \geq |z-y|/2$ in the last inequality. As $y \notin H_{\mathcal{D}}$, we have the following standard estimate:

$$\begin{aligned}&2C_b\varepsilon_0(x)\int_{\mathbb{C}\setminus B(y,2\varepsilon_0(x))} \frac{1}{|z-y|^2}\, d\mu(z)\\ &= 2C_b\varepsilon_0(x)\sum_{k=1}^{\infty}\int_{2^k\varepsilon_0(x)\leq|z-y|<2^{k+1}\varepsilon_0(x)} \frac{1}{|z-y|^2}\, d\mu(z)\\ &\leq 2C_b\varepsilon_0(x)\sum_{k=1}^{\infty}\frac{\mu(B(y,2^{k+1}\varepsilon_0(x)))}{2^{2k}\varepsilon_0(x)^2}\\ &\leq 4C_0C_b.\end{aligned}$$

So we get

$$|\mathcal{C}_{\varepsilon_0(x)}(\nu|\mathbb{C}\setminus B(y,2\varepsilon_0(x)))(x) - \mathcal{C}_{\varepsilon_0(x)}(\nu|\mathbb{C}\setminus B(y,2\varepsilon_0(x)))(y)| \leq 4C_0C_b,$$

and (10.1) holds. □

Choosing α big enough, we will have $\alpha/2 \geq 8C_0C_b$. Then, from the preceding lemma, we deduce

$$\mu(S\setminus H_{\mathcal{D}}) \leq \frac{2}{\alpha}\int_{F\setminus H_{\mathcal{D}}} \mathcal{C}_*\nu\, d\mu \leq \frac{2C_c}{\alpha}\,\mu(F), \tag{10.3}$$

which tends to 0 as $\alpha \to \infty$.

10.2. **The suppressed operators $\mathcal{C}_\Theta$.** Let $\Theta : \mathbb{C}\longrightarrow[0,\infty)$ be a Lipschitz function with Lipschitz constant ≤ 1. We denote

$$K_\Theta(x,y) = \frac{\overline{x-y}}{|x-y|^2 + \Theta(x)\Theta(y)}.$$

It is not difficult to check that K_Θ is Calderón-Zygmund kernel [NTV1]. Indeed, we have

$$|K_\Theta(x,y)| \leq \frac{1}{|x-y|},$$

and

$$|\nabla_x K_\Theta(x,y)| + |\nabla_y K_\Theta(x,y)| \leq \frac{8}{|x-y|^2}.$$

The following estimate also holds:

$$|K_\Theta(x,y)| \leq \frac{1}{\max\{\Theta(x),\Theta(y)\}}. \tag{10.4}$$

We set

$$\mathcal{C}_{\Theta,\varepsilon}\nu(x) = \int_{\mathbb{C}\setminus B(x,\varepsilon)} K_\Theta(x,y)\,d\nu(y).$$

The operator $\mathcal{C}_{\Theta,\varepsilon}$ is the (ε-truncated) **Θ-suppressed Cauchy transform**. We also denote

$$\mathcal{C}_{\Theta,*}\nu(x) = \sup_{\varepsilon>0} \mathcal{C}_{\Theta,\varepsilon}\nu(x).$$

The following lemma is a variant of some estimates which appear in [NTV1]. It is also very similar to [To3, Lemma 2.3].

Lemma 10.2. *Let $x \in \mathbb{C}$ and $r_0 \geq 0$ be such that $\mu(B(x,r)) \leq C_0\, r$ for $r \geq r_0$ and $|\mathcal{C}_\varepsilon\nu(x)| \leq \alpha$ for $\varepsilon \geq r_0$. If $\Theta(x) \geq \eta r_0$ for some $\eta > 0$, then*

$$|\mathcal{C}_{\Theta,\varepsilon}\nu(x)| \leq C_\eta \tag{10.5}$$

for all $\varepsilon > 0$, with C_η depending only on C_0, C_b, α and η.

Proof. If $\varepsilon \geq \eta^{-1}\Theta(x)$, then

$$\begin{aligned}
|\mathcal{C}_{\Theta,\varepsilon}\nu(x) - \mathcal{C}_\varepsilon\nu(x)| &\leq \int_{|x-y|>\varepsilon} \left| \frac{\overline{x-y}}{|x-y|^2 + \Theta(x)\Theta(y)} - \frac{\overline{x-y}}{|x-y|^2} \right| d|\nu|(y) \\
&\leq C_b \int_{|x-y|>\varepsilon} \left| \frac{(\overline{x-y})\,\Theta(x)\,\Theta(y)}{|x-y|^2\,(|x-y|^2 + \Theta(x)\Theta(y))} \right| d\mu(y) \\
&\leq C_b \int_{|x-y|>\varepsilon} \frac{\Theta(x)\Theta(y)}{|x-y|^3} d\mu(y) \\
&\leq C_b \int_{|x-y|>\varepsilon} \frac{\Theta(x)(\Theta(x) + |x-y|)}{|x-y|^3} d\mu(y) \\
&= C_b\Theta(x)^2 \int_{|x-y|>\varepsilon} \frac{1}{|x-y|^3} d\mu(y) \\
&\qquad + C_b\Theta(x) \int_{|x-y|>\varepsilon} \frac{1}{|x-y|^2} d\mu(y).
\end{aligned}$$

Since $\mu(B(x,r)) \leq C_0\, r$ for $r \geq \varepsilon$, it is easily checked that

$$\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^3} d\mu(y) \leq \frac{C}{\varepsilon^2}$$

and

$$\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^2} d\mu(y) \leq \frac{C}{\varepsilon},$$

where C depends only on C_0. Therefore

$$|\mathcal{C}_{\Theta,\varepsilon}\nu(x) - \mathcal{C}_\varepsilon\nu(x)| \leq \frac{C\Theta(x)^2}{\varepsilon^2} + \frac{C\Theta(x)}{\varepsilon} \leq 2C,$$

and so (10.5) holds for $\varepsilon \geq \eta^{-1}\Theta(x)$.

If $\varepsilon < \eta^{-1}\Theta(x)$, then

$$\begin{aligned}|\mathcal{C}_{\Theta,\varepsilon}\nu(x)| \;&\leq\; C_b \int_{B(x,\eta^{-1}\Theta(x))} |K_\Theta(x,y)|d\mu(y) \\ &\quad + \left|\int_{\mathbb{C}\setminus B(x,\eta^{-1}\Theta(x))} K_\Theta(x,y)d\nu(y)\right|. \end{aligned} \tag{10.6}$$

To estimate the first integral on the right hand side we use the inequality (10.4) and the fact that

$$\mu(B(x,\eta^{-1}\Theta(x))) \leq C_0\eta^{-1}\Theta(x),$$

because $\eta^{-1}\Theta(x) \geq r_0$. The second integral on the right hand side of (10.6) equals $\mathcal{C}_{\Theta,\eta^{-1}\Theta(x)}\nu(x)$. This term is bounded by some constant, as shown in the preceding case. □

We denote $\Phi_{0,\mathcal{D}}(x) = \operatorname{dist}(x, \mathbb{C} \setminus (H_\mathcal{D} \cup S))$. Obviously, $\Phi_{0,\mathcal{D}}(x) = 0$ if $x \notin H_\mathcal{D} \cup S$. Moreover, $\Phi_{0,\mathcal{D}}$ is a Lipschitz function with Lipschitz constant 1. On the other hand, $H_\mathcal{D} \cup S$ contains all non Ahlfors disks and all the balls $B(x, \varepsilon(x))$, $x \in F$, and so

$$\Phi_{0,\mathcal{D}}(x) \geq \max(\mathcal{R}(x), \varepsilon(x)).$$

From the construction of S and the preceding lemma we deduce:

Lemma 10.3. *Let $\Theta : \mathbb{C} \longrightarrow [0, +\infty)$ be a Lipschitz function with Lipschitz constant 1 such that $\Theta(x) \geq \eta\, \Phi_{0,\mathcal{D}}(x)$ for all $x \in \mathbb{C}$ (where $\eta > 0$ is some fixed constant). Then, $\mathcal{C}_{\Theta,*}\nu(x) \leq C_\eta$ for all $x \in F$.*

10.3. **The exceptional set $T_\mathcal{D}$.** Looking at conditions (d), (e) and (f) of First Main Lemma 5.1 one can guess that the function b will behave as a paraacretive function on many squares from the dyadic lattice $\mathcal{D}$. We deal with this question in this subsection.

Let us define the exceptional set $T_\mathcal{D}$. If a dyadic square $R \in \mathcal{D}$ satisfies

$$\mu(R) \geq C_d|\nu(R)|, \tag{10.7}$$

where C_d is some big constant which will be chosen below, we write $R \in \mathcal{D}_T$. Let $\{R_k\}_{k\in I_T} \subset \mathcal{D}_T$ be the subfamily of disjoint maximal dyadic squares from $\mathcal{D}_T$. The **exceptional set** $T_\mathcal{D}$ is

$$T_\mathcal{D} = \bigcup_{k\in I_T} R_k.$$

We are going to show that $\mu(F \setminus (H_\mathcal{D} \cup S \cup T_\mathcal{D}))$ is big. That is, that it is comparable to $\mu(F)$. We need to deal with the sets $H_\mathcal{D}$ and $T_\mathcal{D}$ simultaneously. Both $H_\mathcal{D}$ and $T_\mathcal{D}$ have been defined as a union of dyadic squares satisfying some precise conditions (remember the property (i) for the dyadic squares R_k, $k \in I_H$).

Let $\{R_k\}_{k\in I_{HT}}$ be the subfamily of different maximal (and thus *disjoint*) squares from

$$\{R_k\}_{k\in I_H} \cup \{R_k\}_{k\in I_T},$$

so that

$$H_{\mathcal{D}} \cup T_{\mathcal{D}} = \bigcup_{k \in I_{HT}} R_k.$$

From Lemma 8.1, (10.7) and the property (i) in First Main Lemma 5.1, we get

$$\begin{aligned}
|\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})| &\leq \sum_{k \in I_{HT}} |\nu(R_k)| \\
&\leq \sum_{k \in I_H} |\nu(R_k)| + \sum_{k \in I_T} |\nu(R_k)| \\
&\leq C \sum_{k \in I_H} \ell(R_k) + C_d^{-1} \sum_{k \in I_T} \mu(R_k) \\
&\leq C_{12}\varepsilon\mu(F) + C_d^{-1}\mu(F) \\
&\leq C_a(C_{12}\varepsilon + C_d^{-1})|\nu(F)|.
\end{aligned}$$

So if we choose ε small enough and C_d big enough, we obtain

$$|\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})| \leq \frac{1}{2}\,|\nu(F)|.$$

Now we argue as in Subsection 8.3 for proving (k). We have

$$|\nu(F \setminus (H_{\mathcal{D}} \cup T_{\mathcal{D}}))| \geq |\nu(F)| - |\nu(H_{\mathcal{D}} \cup T_{\mathcal{D}})| \geq \frac{1}{2}\,|\nu(F)|.$$

Therefore,

$$\mu(F) \leq C_a|\nu(F)| \leq 2C_a|\nu(F \setminus (H_{\mathcal{D}} \cup T_{\mathcal{D}}))| \leq 2C_aC_b\mu(F \setminus (H_{\mathcal{D}} \cup T_{\mathcal{D}})).$$

Thus, $\mu(H_{\mathcal{D}} \cup T_{\mathcal{D}}) \leq \delta_1\mu(F)$, with $\delta_1 = 1 - \frac{1}{2C_aC_b} < 1$.

Let us remark that the estimates above are not valid if we argue with the non dyadic exceptional set H. We would have troubles for estimating $\nu(H \cup T_{\mathcal{D}})$, because H and $T_{\mathcal{D}}$ are not disjoint in general. This is the main reason for considering the dyadic version $H_{\mathcal{D}}$ of the exceptional set H in First Main Lemma.

Now we turn our attention to the set S. In (10.3) we obtained an estimate for $\mu(S \setminus H_{\mathcal{D}})$ in terms of the constant α. We set $\delta_2 = (\delta_1 + 1)/2$. Then we choose α such that

$$\mu(H_{\mathcal{D}} \cup T_{\mathcal{D}}) + \mu(S \setminus H_{\mathcal{D}}) \leq \delta_2\mu(F).$$

10.4. **Summary.** In next lemma we summarize what we have shown in this section.

Lemma 10.4. *Assume that* $\gamma_+(E) \leq C_4\mathrm{diam}(E)$, $\gamma(E) \geq A\gamma_+(E)$, *and* $\gamma(E \cap Q) \leq A\gamma_+(E \cap Q)$ *for all squares* Q *with* $\mathrm{diam}(Q) \leq \mathrm{diam}(E)/5$. *Let* $\mathcal{D}$ *be any fixed dyadic lattice. There are subsets* $H_{\mathcal{D}}$, S, $T_{\mathcal{D}} \subset F$ *(with* $H_{\mathcal{D}}$ *and* $T_{\mathcal{D}}$ *depending on* $\mathcal{D}$*) such that*

(a) $\mu(H_{\mathcal{D}} \cup S \cup T_{\mathcal{D}}) \leq \delta_2\mu(F)$ *for some absolute constant* $\delta_2 < 1$.

(b) *All non Ahlfors disks (with respect to some constant C_0 big enough) are contained in $H_{\mathcal{D}}$.*
(c) *If $\Theta:\mathbb{C}\longrightarrow[0,+\infty)$ is any Lipschitz function with Lipschitz constant 1 such that $\Theta(x)\geq\eta\,\mathrm{dist}(x,\mathbb{C}\setminus H_{\mathcal{D}}\cup S)$, for all $x\in\mathbb{C}$ (where $\eta>0$ is some fixed constant), then $\mathcal{C}_{\Theta,*}\nu(x)\leq C_\eta$ for all $x\in F$.*
(d) *All dyadic squares $R\in\mathcal{D}$ such that $R\not\subset T_{\mathcal{D}}$ satisfy $\mu(R)<C_d|\nu(R)|$.*

11. The proof of Second Main Lemma

Throughout all this section we will assume that all the hypotheses in Second Main Lemma 9.1 hold.

11.1. **Random dyadic lattices.**

We are going to introduce random dyadic lattices. We follow the construction of [NTV1].

Suppose that $F\subset B(0,2^{N-3})$, where N is a big enough integer. Consider the random square $Q^0(w)=w+\left[-2^N,\,2^N\right)^2$, with $w\in\left[-2^{N-1},\,2^{N-1}\right)^2=:\Omega$. We take $Q^0(w)$ as the starting square of the dyadic lattice $\mathcal{D}(w)$. Observe that $F\subset Q^0(w)$ for all $w\in\Omega$. Only the dyadic squares which are contained in $Q^0(w)$ will play some role in the arguments below. For the moment, we don't worry about the other squares.

We take a uniform probability on Ω. So we let the probability measure P be the normalized Lebesgue measure on the square Ω.

A square $Q\in\mathcal{D}\equiv\mathcal{D}(w)$ contained in Q^0 is called **terminal** if $Q\subset H_{\mathcal{D}}\cup T_{\mathcal{D}}$. Otherwise, it is called **transit**. The set of terminal squares is denoted by $\mathcal{D}^{term}$, and the set of transit squares by $\mathcal{D}^{tr}$. It is easy to check that Q^0 is always transit.

11.2. **The dyadic martingale decomposition.**

For $f\in L^1_{loc}(\mu)$ (we assume always f real, for simplicity) and any square Q with $\mu(Q)\neq 0$, we set

$$\langle f\rangle_Q=\frac{1}{\mu(Q)}\int_Q f\,d\mu.$$

We define the operator Ξ as

$$\Xi f=\frac{\langle f\rangle_{Q^0}}{\langle b\rangle_{Q^0}}\,b,$$

where b is the complex function that we have constructed in Main Lemma 5.1. It follows easily that $\Xi f\in L^2(\mu)$ if $f\in L^2(\mu)$, and $\Xi^2=\Xi$. Moreover, the definition of Ξ does not depend on the choice of the lattice $\mathcal{D}$. The adjoint of Ξ is

$$\Xi^* f=\frac{\langle fb\rangle_{Q^0}}{\langle b\rangle_{Q^0}}.$$

Let $Q\in\mathcal{D}$ be some fixed dyadic square. The set of the four children of Q is denoted as $\mathcal{C}h(Q)$. In this subsection we will also write $\mathcal{C}h(Q)=\{Q_j : j=1,2,3,4\}$.

For any square $Q \in \mathcal{D}^{tr}$ and any $f \in L^1_{loc}(\mu)$, we define the function $\Delta_Q f$ as follows:

$$\Delta_Q f = \begin{cases} 0 & \text{in } \mathbb{C} \setminus Q, \\ \left(\dfrac{\langle f \rangle_{Q_j}}{\langle b \rangle_{Q_j}} - \dfrac{\langle f \rangle_Q}{\langle b \rangle_Q} \right) b & \text{in } Q_j \text{ if } Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{tr}, \\ f - \dfrac{\langle f \rangle_Q}{\langle b \rangle_Q}\, b & \text{in } Q_j \text{ if } Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{term}. \end{cases}$$

The operators Δ_Q satisfy the following properties.

Lemma 11.1. *For all $f \in L^2(\mu)$ and all $Q \in \mathcal{D}^{tr}$,*

(a) $\Delta_Q f \in L^2(\mu)$,
(b) $\int \Delta_Q f \, d\mu = 0$,
(c) *Δ_Q is a projection, i.e.* $\Delta_Q^2 = \Delta_Q$,
(d) $\Delta_Q \, \Xi = \Xi \, \Delta_Q = 0$,
(e) *If $R \in \mathcal{D}^{tr}$ and $R \neq Q$, then $\Delta_Q \Delta_R = 0$.*
(f) *The adjoint of Δ_Q is*

$$\Delta_Q^* f = \begin{cases} 0 & \textit{in } \mathbb{C} \setminus Q, \\ \dfrac{\langle f b \rangle_{Q_j}}{\langle b \rangle_{Q_j}} - \dfrac{\langle f b \rangle_Q}{\langle b \rangle_Q} & \textit{in } Q_j \textit{ if } Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{tr}, \\ f - \dfrac{\langle f b \rangle_Q}{\langle b \rangle_Q} & \textit{in } Q_j \textit{ if } Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{term}. \end{cases}$$

The properties (a)–(e) are stated in [NTV1, Section XII] and are easily checked. The property (f) is also immediate (although it does not appear in [NTV1]).

Now we have:

Lemma 11.2. *For any $f \in L^2(\mu)$, we have the decomposition*

$$f = \Xi f + \sum_{Q \in \mathcal{D}^{tr}} \Delta_Q f, \tag{11.1}$$

with the sum convergent in $L^2(\mu)$. Moreover, there exists some absolute constant C_{13} such that

$$C_{13}^{-1} \|f\|^2_{L^2(\mu)} \leq \|\Xi f\|^2_{L^2(\mu)} + \sum_{Q \in \mathcal{D}^{tr}} \|\Delta_Q f\|^2_{L^2(\mu)} \leq C_{13} \|f\|^2_{L^2(\mu)}. \tag{11.2}$$

This lemma has been proved in [NTV1, Section XII] under the assumption that the paraacretivity constant C_d (see (10.7)) is sufficiently close to $\|b\|^{-1}_{L^\infty(\mu)}$. The arguments in [NTV1] are still valid in our case for the $L^2(\mu)$ decomposition of f in (11.1) and for the second inequality in (11.2). However, they don't work for the first inequality in (11.2). We will show below that this estimate follows from the second inequality by duality. The arguments are of the same type as the ones in [Da2] and [NTV2] (see also [NTV3]). However, some additional work is necessary due to the presence of terminal squares and because we cannot assume b^{-1} to be a bounded function in our case, since b since may vanish in sets of positive measure.

We will need the Dyadic Carleson Imbedding Theorem:

Theorem 11.3. *Let $\mathcal{D}$ be some dyadic lattice and let $\{a_Q\}_{Q\in\mathcal{D}}$ be a family of non negative numbers. Suppose that for every square $R \in \mathcal{D}$ we have*

$$\sum_{Q\in\mathcal{D}:Q\subset R} a_Q \leq C_{14}\mu(R). \tag{11.3}$$

Then, for all $f \in L^2(\mu)$, we have

$$\sum_{Q\in\mathcal{D}:\mu(Q)\neq 0} a_Q \, |\langle f\rangle_Q|^2 \leq 4C_{14}\|f\|^2_{L^2(\mu)}.$$

See [NTV1, Section XII], for example, for the proof.

Proof of the first inequality in (11.2). By (11.1) and the fact that Ξ and Δ_Q are projections, we have

$$f = \Xi f + \sum_{Q\in\mathcal{D}^{tr}} \Delta_Q f = \Xi^2 f + \sum_{Q\in\mathcal{D}^{tr}} \Delta_Q^2 f.$$

Then we deduce

$$\begin{aligned}
\int f^2\,d\mu &= \int \Big(\Xi^2 f + \sum_{Q\in\mathcal{D}^{tr}} \Delta_Q^2 f\Big) f\,d\mu \\
&= \int (\Xi f)(\Xi^* f)\,d\mu + \sum_{Q\in\mathcal{D}^{tr}} \int (\Delta_Q f)(\Delta_Q^* f)\,d\mu \\
&\leq \Big(\|\Xi f\|^2_{L^2(\mu)} + \sum_{Q\in\mathcal{D}^{tr}} \|\Delta_Q f\|^2_{L^2(\mu)}\Big)^{1/2} \\
&\qquad \times \Big(\|\Xi^* f\|^2_{L^2(\mu)} + \sum_{Q\in\mathcal{D}^{tr}} \|\Delta_Q^* f\|^2_{L^2(\mu)}\Big)^{1/2}.
\end{aligned} \tag{11.4}$$

So if we show that

$$\|\Xi^* f\|^2_{L^2(\mu)} + \sum_{Q\in\mathcal{D}^{tr}} \|\Delta_Q^* f\|^2_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)}, \tag{11.5}$$

we will be done. Notice, by the way, that the second inequality in (11.2) and (11.4) imply

$$\|f\|^2_{L^2(\mu)} \leq C\|\Xi^* f\|^2_{L^2(\mu)} + C\sum_{Q\in\mathcal{D}^{tr}} \|\Delta_Q^* f\|^2_{L^2(\mu)}.$$

Let us see that (11.5) holds. It is straightforward to check that

$$\|\Xi^* f\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}.$$

So we only have to estimate $\sum_{Q\in\mathcal{D}^{tr}} \|\Delta_Q^* f\|^2_{L^2(\mu)}$. To this end we need to introduce the operators D_Q. They are defined as follows:

$$D_Q f = \begin{cases} 0 & \text{in } \mathbb{C}\setminus Q, \\ \langle f\rangle_{Q_j} - \langle f\rangle_Q & \text{in } Q_j. \end{cases}$$

We also define $Ef = \langle f\rangle_{Q^0}$. Then it is well known that

$$\|Ef\|^2_{L^2(\mu)} + \sum_{Q\in\mathcal{D}} \|D_Q f\|^2_{L^2(\mu)} = \|f\|_{L^2(\mu)}. \tag{11.6}$$

If $Q_j \in \mathcal{C}h(Q)$ is a transit square, then we have (using (f) from Lemma 11.1)

$$\begin{aligned} \Delta_Q^* f|_{Q_j} &= \frac{\langle fb\rangle_{Q_j} - \langle fb\rangle_Q}{\langle b\rangle_Q} + \langle fb\rangle_{Q_j}\left(\frac{1}{\langle b\rangle_{Q_j}} - \frac{1}{\langle b\rangle_Q}\right) \\ &= \frac{1}{\langle b\rangle_Q} D_Q(fb)|_{Q_j} - \frac{\langle fb\rangle_{Q_j}}{\langle b\rangle_{Q_j}\langle b\rangle_Q} D_Q b|_{Q_j}. \end{aligned}$$

Since $|\langle b\rangle_Q|$, $|\langle b\rangle_{Q_j}| \geq C_d^{-1}$, we obtain

$$\sum_{Q\in\mathcal{D}^{tr}} \sum_{Q_j\in\mathcal{C}h(Q)\cap\mathcal{D}^{tr}} \|\Delta_Q^* f\|^2_{L^2(\mu|Q_j)} \leq$$

$$C\sum_{Q\in\mathcal{D}} \|D_Q(fb)\|^2_{L^2(\mu)} + C\sum_{Q\in\mathcal{D}} \sum_{Q_j\in\mathcal{C}h(Q)} \|\langle fb\rangle_{Q_j} D_Q b|_{Q_j}\|^2_{L^2(\mu|Q_j)}. \tag{11.7}$$

From (11.6) we deduce

$$\sum_{Q\in\mathcal{D}} \|D_Q(fb)\|^2_{L^2(\mu)} \leq \|fb\|^2_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)}.$$

Now observe that the last term in (11.7) can be rewritten as

$$\sum_{Q\in\mathcal{D}} |\langle fb\rangle_Q|^2 \|\chi_Q D_{\widehat{Q}} b\|^2_{L^2(\mu)} =: B,$$

where $\widehat{Q}$ stands for the father of Q. To estimate this term we will apply the Dyadic Carleson Imbedding Theorem. Let us check the numbers $a_Q := \|\chi_Q D_{\widehat{Q}} b\|^2_{L^2(\mu)}$ satisfy the packing condition (11.3). Taking into account that b is bounded and (11.6), for each square $R\in\mathcal{D}$ we have

$$\begin{aligned} \sum_{Q\subset R} \|\chi_Q D_{\widehat{Q}} b\|^2_{L^2(\mu)} &= \|D_{\widehat{R}} b\|^2_{L^2(\mu|R)} + \sum_{Q\subset R,\, Q\neq R} \|D_{\widehat{Q}} b\|^2_{L^2(\mu)} \\ &\leq C\mu(R) + \sum_{Q\subset R} \|D_Q(b\chi_R)\|^2_{L^2(\mu)} \leq C\mu(R). \end{aligned}$$

So (11.3) holds and then

$$B \leq C\|fb\|^2_{L^2(\mu)} \leq C\|f\|^2_{L^2(\mu)}.$$

Now we have to deal with the terminal squares. If $Q \in \mathcal{D}^{tr}$ and $Q_j \in \mathcal{D}^{term}$, then we have

$$\Delta_Q^* f|_{Q_j} = \left(f - \frac{\langle fb \rangle_{Q_j}}{\langle b \rangle_Q} \right) + \left(\frac{\langle fb \rangle_{Q_j}}{\langle b \rangle_Q} - \frac{\langle fb \rangle_Q}{\langle b \rangle_Q} \right).$$

Since b is bounded and $|\langle b \rangle_Q| \geq C_d^{-1}$, we get

$$\begin{aligned} |\Delta_Q^* f|_{Q_j}| &\leq C\big(|f| + \langle |f| \rangle_{Q_j}\big) + C\big|\langle fb \rangle_{Q_j} - \langle fb \rangle_Q\big| \\ &= C\big(|f| + \langle |f| \rangle_{Q_j}\big) + C\big|D_Q(fb)|_{Q_j}\big|. \end{aligned}$$

Therefore,

$$\sum_{Q \in \mathcal{D}^{tr}} \sum_{Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{term}} \|\Delta_Q^* f\|_{L^2(\mu|Q_j)}^2 \leq$$

$$C \sum_{Q \in \mathcal{D}^{tr}} \sum_{Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{term}} \int_{Q_j} |f|^2 \, d\mu + C \sum_{Q \in \mathcal{D}} \|D_Q(fb)\|_{L^2(\mu)}^2. \tag{11.8}$$

For the first sum on the right hand side above, notice that the squares $Q_j \in \mathcal{D}^{term}$ whose father is a transit square are pairwise disjoint. For the last sum, we only have to use (11.6). Then we obtain

$$\sum_{Q \in \mathcal{D}^{tr}} \sum_{Q_j \in \mathcal{C}h(Q) \cap \mathcal{D}^{term}} \|\Delta_Q^* f\|_{L^2(\mu|Q_j)}^2 \leq C\|f\|_{L^2(\mu)}^2.$$

Since the left hand side of (11.7) is also bounded above by $C\|f\|_{L^2(\mu)}^2$, (11.5) follows. □

11.3. **Good and bad squares.** Following [NTV1], we say that a square Q has M**-negligible boundary** if

$$\mu\{x \in \mathbb{C} : \operatorname{dist}(x, \partial Q) \leq r\} \leq Mr$$

for all $r \geq 0$.

We now define bad squares as in [NTV1] too. Let $\mathcal{D}_1 = \mathcal{D}(w_1)$ and $\mathcal{D}_2 = \mathcal{D}(w_2)$, with $w_1, w_2 \in \Omega$, be two dyadic lattices. We say that a transit square $Q \in \mathcal{D}_1^{tr}$ is bad (with respect to $\mathcal{D}_2$) if either

(a) there exists a square $R \in \mathcal{D}_2$ such that $\operatorname{dist}(Q, \partial R) \leq 16\, \ell(Q)^{1/4} \ell(R)^{3/4}$ and $\ell(R) \geq 2^m \ell(Q)$ (where m is some fixed positive integer), or
(b) there exists a square $R \in \mathcal{D}_2$ such that $R \subset (2^{m+2}+1)Q$, $\ell(R) \geq 2^{-m+1}\ell(Q)$, and ∂R is not M-negligible.

Of course, if Q is not bad, then we say that it is good.

Let us remark that in the definition above we consider all the squares of $\mathcal{D}_2$, not only the squares contained in $Q^0(w_2) \in \mathcal{D}_2$, which was the case up to now. On the other hand, observe that the definition depends on the constants m and M. So strictly speaking, bad squares should be called (m, M)-bad squares.

Bad squares don't appear very often in dyadic lattices. To be precise, we have the following result.

Lemma 11.4 ([NTV1]). *Let $\varepsilon_b > 0$ be any fixed (small) number. Suppose that the constants m and M are big enough (depending only on ε_b). Let $\mathcal{D}_1 = \mathcal{D}(w_1)$ be any fixed dyadic lattice. For each fixed $Q \in \mathcal{D}_1$, the probability that it is bad with respect to a dyadic lattice $\mathcal{D}_2 = \mathcal{D}(w_2)$, $w_2 \in \Omega$, is $\leq \varepsilon_b$. That is,*

$$P\{w_2 : Q \in \mathcal{D}_1 \text{ is bad with respect to } \mathcal{D}(w_2)\} \leq \varepsilon_b.$$

The notion of good and bad squares allows now to introduce the definition of good functions. Remember that given any fixed dyadic lattice $\mathcal{D}_1 = \mathcal{D}(w_1)$, every function $\varphi \in L^2(\mu)$ can be written as

$$\varphi = \Xi\varphi + \sum_{Q \in \mathcal{D}_1^{tr}} \Delta_Q \varphi.$$

We say that φ is $\mathcal{D}_1$**-good** with respect to $\mathcal{D}_2$ (or simply, good) if $\Delta_Q \varphi = 0$ for all bad squares $Q \in \mathcal{D}_1^{tr}$ (with respect to $\mathcal{D}_2$).

11.4. **Estimates on good functions.** We define the function $\Phi_{\mathcal{D}}$ as

$$\Phi_{\mathcal{D}}(x) = \operatorname{dist}(x, \mathbb{C} \setminus (H_{\mathcal{D}} \cup S \cup T_{\mathcal{D}})).$$

Notice that $\Phi_{\mathcal{D}}$ is a Lipschitz function with Lipschitz constant 1 which equals zero in $\mathbb{C} \setminus (H_{\mathcal{D}} \cup S \cup T_{\mathcal{D}})$. Observe also that $\Phi_{\mathcal{D}} \geq \Phi_{0,\mathcal{D}}$ (this function was introduced at the end of Subsection 10.2).

Now we have the following result.

Lemma 11.5. *Let $\mathcal{D}_1 = \mathcal{D}(w_1)$ and $\mathcal{D}_2 = \mathcal{D}(w_2)$, with $w_1, w_2 \in \Omega$, be two dyadic lattices. Let $\Theta : \mathbb{C} \longrightarrow [0, +\infty)$ be a Lipschitz function with Lipschitz constant 1 such that $\inf_{x \in \mathbb{C}} \Theta(x) > 0$ and $\Theta(x) \geq \eta \max(\Phi_{\mathcal{D}_1}(x), \Phi_{\mathcal{D}_2}(x))$ for all $x \in \mathbb{C}$ (where $\eta > 0$ is some fixed constant). If φ is $\mathcal{D}_1$-good with respect to $\mathcal{D}_2$, and ψ is $\mathcal{D}_2$-good with respect to $\mathcal{D}_1$, then*

$$|\langle \mathcal{C}_\Theta \varphi, \psi \rangle| \leq C_{15} \|\varphi\|_{L^2(\mu)} \|\psi\|_{L^2(\mu)},$$

where C_{15} is some constant depending on η.

This lemma follows by the same estimates and arguments of the corresponding result in [NTV1].

11.5. **The probabilistic argument.** Following some ideas from [NTV1], we are going to show that the estimates for good functions from Lemma 11.5 imply that there exists a set $G \subset \mathbb{C} \setminus H$ with $\mu(G) \geq C^{-1}\mu(F)$ such that the Cauchy transform is bounded on $L^2(\mu|G)$. The probabilistic arguments of [NTV1, Section V] don't work in our case because we would need $\mu(H_{\mathcal{D}} \cup S \cup T_{\mathcal{D}})$ to be very small (choosing some adequate parameters), but we only have been able to show that $\mu(H_{\mathcal{D}} \cup S \cup T_{\mathcal{D}}) \leq \delta_2 \mu(F)$, for some fixed $\delta_2 < 1$. Nevertheless, the approach of [NTV1, Section XXIII] doesn't need the preceding assumption and is well suited for our situation.

Let us describe briefly the ideas from [NTV1, Section XXIII] that we need. We denote $W_{\mathcal{D}} = H_{\mathcal{D}} \cup S \cup T_{\mathcal{D}}$, and we call it the **total exceptional set**.

Let $W_{\mathcal{D}_1}$, $W_{\mathcal{D}_2}$ be the total exceptional sets corresponding to two independent dyadic lattices $\mathcal{D}_1 = \mathcal{D}(w_1)$, $\mathcal{D}_2 = \mathcal{D}(w_2)$. We have shown that

$$\mu(F \setminus W_{\mathcal{D}(w)}) \geq (1 - \delta_2)\mu(F),$$

with $0 < \delta_2 < 1$ for all $w \in \Omega$. For each $x \in F$ we consider the probabilities

$$p_1(x) = P\{w \in \Omega : x \in F \setminus W_{\mathcal{D}(w)}\},$$

and

$$p(x) = P\{(w_1, w_2) \in \Omega \times \Omega : x \in F \setminus (W_{\mathcal{D}(w_1)} \cup W_{\mathcal{D}(w_2)})\}.$$

Since the sets $F \setminus W_{\mathcal{D}(w_1)}$ and $F \setminus W_{\mathcal{D}(w_2)}$ are independent, we deduce $p(x) = p_1(x)^2$. Now we have

$$\int_F p_1(x)\, d\mu(x) = \mathcal{E} \int \chi_{F \setminus W_{\mathcal{D}(w)}}(x)\, d\mu(x) = \mathcal{E}\mu(F \setminus W_{\mathcal{D}(w)}) \geq (1 - \delta_2)\mu(F),$$

where $\mathcal{E}$ denotes the mathematical expectation. Let $G = \{x \in F : p_1(x) > (1 - \delta_2)/2\}$, and $B = F \setminus G$. We have

$$\begin{aligned} \mu(B) &\leq \frac{2}{1 + \delta_2} \int_F (1 - p_1(x))\, d\mu(x) \\ &= \frac{2}{1 + \delta_2} \left(\mu(F) - \int_F p_1(x)\, d\mu(x) \right) \leq \frac{2\delta_2}{1 + \delta_2}\, \mu(F). \end{aligned}$$

Thus,

$$\mu(G) \geq \frac{1 - \delta_2}{1 + \delta_2}\, \mu(F).$$

Observe that for every $x \in G$ we have $p(x) = p_1(x)^2 > (1 - \delta_2)^2/4 =: \beta$. Now we define $\Phi_{(w_1,w_2)}(x) = \operatorname{dist}\big(x, F \setminus (W_{\mathcal{D}(w_1)} \cup W_{\mathcal{D}(w_2)})\big)$. From the preceding calculations, we deduce

$$\mu\{x \in F : p(x) > \beta\} \geq \mu(G) \geq \frac{1 - \delta_2}{1 + \delta_2}\, \mu(F).$$

That is,

$$\mu\big\{x \in F : P\{(w_1, w_2) : \Phi_{(w_1,w_2)}(x) = 0\} > \beta\big\} \geq \frac{1 - \delta_2}{1 + \delta_2}\, \mu(F).$$

Let us define

$$\Phi(x) = \inf_{B \subset \Omega \times \Omega, P(B) = \beta}\ \sup_{(w_1,w_2) \in B} \Phi_{(w_1,w_2)}(x).$$

Notice that Φ is a 1-Lipschitz function such that $\Phi(x) = 0$ for all $x \in G$. Moreover, $\Phi(x) \geq \mathcal{R}(x), \varepsilon(x)$ for all $x \in F$, because $\Phi_{(w_1,w_2)}(x) \geq \mathcal{R}(x), \varepsilon(x)$ for all $x \in F$, $(w_1, w_2) \in \Omega \times \Omega$, since all non Ahlfors disks are contained in $\mathcal{H}_D$ for any choice of the lattice $\mathcal{D}$, and S does not depend on $\mathcal{D}$.

Finally, from Lemmas 11.4 and 11.5, and [NTV1, Main Lemma (Section XXIII)], we deduce that $\mathcal{C}_\Phi$ is bounded on $L^2(\mu)$, and all the constants involved are absolute constants. Since $\Phi(x) = 0$ on G, **the Cauchy transform is bounded on** $L^2(\mu|G)$. On the other hand, the fact that $\Phi(x) = 0$

on G also implies that $\mathcal{R}(x) = 0$ on G, which is equivalent to say that $\mu(B(x,r)) \leq C_0 r$ for all $r > 0$ if $x \in G$.

Now Second Main Lemma is proved.

12. The proof of Theorem 1.1

From First Main Lemma and Second Main Lemma we get:

Lemma 12.1. *There exists some absolute constant B such that if $A \geq 1$ is any fixed constant and*

(a) $\gamma_+(E) \leq C_4 \mathrm{diam}(E)$,
(b) $\gamma(E \cap Q) \leq A\gamma_+(E \cap Q)$ *for all squares Q with* $\mathrm{diam}(Q) \leq \mathrm{diam}(E)/5$,
(c) $\gamma(E) \geq A\gamma_+(E)$,

then $\gamma(E) \leq B\gamma_+(E)$.

Proof. By First Main Lemma 5.1 and Second Main Lemma 9.1, there are sets F, G and a measure μ supported on F such that

(1) $E \subset F$ and $\gamma_+(E) \approx \gamma_+(F)$,
(2) $\mu(F) \approx \gamma(E)$,
(3) $G \subset F$ and $\mu(G) \geq C_{10}^{-1}\mu(F)$,
(4) $\mu(G \cap B(x,r)) \leq C_0 r$ for all $x \in G$, $r > 0$, and $\|\mathcal{C}\|_{L^2(\mu|G),L^2(\mu|G)} \leq C_{11}$.

From (4) and (3), we get

$$\gamma_+(F) \geq C^{-1}\mu(G) \geq C^{-1}\mu(F).$$

Then, by (2), the preceding inequality, and (1),

$$\gamma(E) \leq C\mu(F) \leq C\gamma_+(F) \leq B\gamma_+(E).$$

□

As a corollary we deduce:

Lemma 12.2. *There exists some absolute constant A_0 such that if $\gamma(E \cap Q) \leq A_0\gamma_+(E \cap Q)$ for all squares Q with* $\mathrm{diam}(Q) \leq \mathrm{diam}(E)/5$, *then $\gamma(E) \leq A_0\gamma_+(E)$.*

Proof. We take $A_0 = \max(1, C_4^{-1}, B)$. If $\gamma_+(E) > C_4\mathrm{diam}(E)$, then we get $\gamma_+(E) > C_4\gamma(E)$ and we are done. If $\gamma_+(E) \leq C_4\mathrm{diam}(E)$, then we also have $\gamma(E) \leq A_0\gamma_+(E)$. Otherwise, we apply Lemma 12.1 and we deduce $\gamma(E) \leq B\gamma_+(E) \leq A_0\gamma_+(E)$, which is a contradiction. □

Notice, by the way, that any constant $A_0 \geq \max(1, C_4^{-1}, B)$ works in the argument above. So Lemma 12.2 holds for any constant A_0 sufficiently big.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Remember that we are assuming that E is a finite union of disjoint compact segments L_j. We set

$$d := \frac{1}{10} \min_{j \neq k} \mathrm{dist}(L_j, L_k).$$

We will prove by induction on n that if R is a closed rectangle with sides parallel to the axes and diameter $\leq 4^n d$, $n \geq 0$, then

$$\gamma(R \cap E) \leq A_0 \gamma_+(R \cap E). \tag{12.1}$$

Notice that if $\operatorname{diam}(R) \leq d$, then R can intersect at most one segment L_j. So either $R \cap E = \varnothing$ or $R \cap E$ coincides with a segment, and in any case, (12.1) follows (assuming A_0 sufficiently big).

Let us see now that if (12.1) holds for all rectangles R with diameter $\leq 4^n d$, then it also holds for a rectangle R_0 with diameter $\leq 4^{n+1} d$. We only have to apply Lemma 12.2 to the set $R_0 \cap E$, which is itself a finite union of disjoint compact segments. Indeed, take a square Q with diameter $\leq \operatorname{diam}(R_0 \cap E)/5$. By the induction hypothesis we have

$$\gamma(Q \cap R_0 \cap E) \leq A_0 \gamma_+(Q \cap R_0 \cap E),$$

because $Q \cap R_0$ is a rectangle with diameter $\leq 4^n d$. Therefore,

$$\gamma(R_0 \cap E) \leq A_0 \gamma_+(R_0 \cap E)$$

by Lemma 12.2. □

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