PERTURBATIONS OF L-FUNCTIONS WITH OR WITHOUT NON-TRIVIAL ZEROS OFF THE CRITICAL LINE

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Abstract. There exist small perturbations of L-functions, satisfying the appropriate functional equation, for which the analogue of the Riemann hypothesis fails radically. Moreover, this phenomenon is generic. However, there also exist small perturbations, for which the analogue of the Riemann hypothesis holds.

1. Introduction

In 2003, Lev D. Pustyl'nikov [18] showed that the Riemann zeta-function $\zeta(s)$ can be approximated by a function $\zeta_\epsilon(s)$ which fails to satisfy the Riemann hypothesis. That is, $\zeta_\epsilon(s)$ has non-trivial zeros off the critical axis. Moreover, the function $\zeta_\epsilon(s)$ shares the following important properties of the Riemann zeta-function. 1) $\zeta_\epsilon(s)$ is a meromorphic function with a unique pole at $s = 1$ and assumes real values for real values of s. 2) The real zeros of $\zeta_\epsilon(s)$ are the negative even integers. 3) The function $\zeta_\epsilon(s)$ satisfies the functional equation of the Riemann zeta-function. In 2004, Paul M. Gauthier and Eduardo S. Zeron [11] considerably improved the result of Pustyl'nikov and in 2007, Markus Niess [17] showed that there exist such functions $\zeta_\epsilon(s)$ which also have interesting universality properties.

L-functions are generalizations of the Riemann zeta-function and have led to the Langlands programme and to fundamental open questions such as the Birch and Swinnerton-Dyer conjecture. In the present paper, we establish results for L-functions analogous to those established in [11] for the Riemann zeta function. Namely, we approximate L-functions by functions for which the grand Riemann hypothesis fails. In addition, we also establish two results for L-functions which were not yet known for the Riemann zeta function. First of all, we show that the preceding result on approximation of L-functions by functions which fail to satisfy the Riemann Hypothesis is generic. For the Riemann zeta-function, this has been shown simultaneously and independently by Niess [17]. Secondly, despite this genericity of functions which fail to satisfy the Riemann Hypothesis, we also establish the possibility of approximating by functions for which the grand Riemann hypothesis *does* hold.

Many L-functions $f(s)$ appearing naturally in number theory satisfy a functional equation

$$\Lambda_Q(f, s) := Q(s)f(s) = \Lambda_Q(f, 1 - s),$$

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where $Q = Q_f$ is a meromorphic function. Examples of such functions are the L-functions associated to some Dirichlet Characters, Hecke characters, modular forms and more general automorphic forms. In this case, the function $Q(s)$ is always of the form $K^s \prod_{j=1}^{n} \Gamma(\lambda_j s + \mu_j)$, where n is a natural number, K and the λ_j's are positive real numbers and the μ_j's are complex numbers with non-negative real part.

The Grand Riemann Hypothesis for this type of L-funtions asserts that all zeros of $\Lambda_Q(f,s)$ lie on the critical axis $\Re s = 1/2$. We shall show that the conclusion of the Riemann Hypothesis fails for most functions satisfying such a functional equation, and, in particular, for small perturbations of such L-functions.

We shall say that a function Λ is symmetric with respect to the point $s = 1/2$, if $\Lambda(1-s) = \Lambda(s)$, for all $s \in \mathbb{C}$.

If $Q(s)$ is a meromorphic function on $\mathbb{C}$, not identically zero, we denote by M_Q the family of meromorphic functions f on $\mathbb{C}$, not identically zero and satisfying the functional equation

$$\Lambda_Q(f,s) := Q(s)f(s) = \Lambda_Q(f,1-s), \tag{1}$$

and we denote by H_Q the subfamily of entire functions in M_Q. We shall say that a function $f \in M_Q$ satisfies the Riemann hypothesis, if all zeros of $\Lambda_Q(f,s)$ lie on the critical axis $\Re s = 1/2$.

We endow M_Q with the usual topology of uniform convergence on compacta. Thus, a sequence f_n of functions in M_Q converges to a function f in M_Q, if for each compact $K \subset \mathbb{C}$ and each $\epsilon > 0$, there is an n_0 such that, for $n > n_0$, the functions f and f_n have the same poles with same principal parts on K and $|f(z) - f_n(z)| < \epsilon$, for all $z \in K$. Denote by RM_Q and RH_Q the functions f in M_Q and H_Q respectively, for which all zeros of $\Lambda_Q(f,s)$ lie on the critical axis $\Re s = 1/2$. Thus, RM_Q and RH_Q are respectively the meromorphic and holomorphic functions satisfying the functional equation associated to Q and for which the associated Riemann hypothesis holds.

Let us say that a function Λ defined for $s \in \mathbb{C}$ is symmetric with respect to the real axis, if $\overline{\Lambda(\overline{s})} = \Lambda(s)$, for all $s \in \mathbb{C}$. In the previous papers on this topic ([18],[11] and [17]), the function approximating the Riemann zeta-function also shared with the zeta-function the property of symmetry with respect to the real axis. In the present paper, if the L-function under consideration has this symmetry, then the approximating functions can also be chosen to have this same symmetry. We have not included this in the statements of our results in order to treat more L-functions - not just those having this additional symmetry. In particular, Theorem 1 and Theorem 3, which for the Riemann zeta-function are new, are also valid with this "real" symmetry.

2. Approximation by functions failing to satisfy the analogue of the grand Riemann hypothesis

If $f \in M_Q$ and $\nu \not\equiv 0$ is meromorphic on $\mathbb{C}$ and symmetric with respect to the point $1/2$, then the product νf is again in M_Q. It follows that, if M_Q is not empty, then there exists a function in the class RM_Q. That is, there exists a function in M_Q which satisfies the Grand Riemann Hypothesis. On the other hand the following result shows that, generically, the Grand Riemann Hypothesis fails.

Theorem 1. *In the spaces M_Q and H_Q of meromorphic and holomorphic functions satisfying the functional equation (1), the classes $M_Q \setminus RM_Q$ and $H_Q \setminus RH_Q$, for which the Riemann Hypothesis fails are open and dense.*

The following version of the Walsh Lemma on simultaneous approximation and interpolation is due to Frank Deutsch [4].

Lemma 1. *Let E be a dense subspace of a normed linear space F. Then, for each $f \in F$, for each $\epsilon > 0$, and for each finite choice of continuous linear functionals $L_1, \cdots, L_m$ on F, there is an $e \in E$ such that $\|e - f\| < \epsilon$ and*

$$L_j(e) = L_j(f), \; j = 1, \cdots, m.$$

Proof of Theorem 1. Let $\{f_n\}$ be a sequence of functions in RM_Q and suppose $f_n \to f$, where $f \in M_Q$. Suppose $f \notin RM_Q$. Then, $\Lambda_Q(f, \cdot)$ has a zero at a point s_0 not on the critical axis. Let d be less than the distance of s_0 from the critical axis and also less than the distance from s_0 to the nearest pole of f. Then, by Hurwitz's theorem, there is an n_0 such that for all $n > n_0$, the function $\Lambda_Q(f_n, \cdot)$ has a zero in the disc $|s - s_0| < d$. Thus, $f_n \notin RM_Q$, which is a contradiction. Thus, RM_Q is closed. The proof that RH_Q is closed is similar.

To show that RM_Q is nowhere dense, let $f \in M_Q$, and consider a basic neighborhood of f:

$$N(f, K, \epsilon) = \{g \in M_Q : \max_{s \in K} |f(s) - g(s)| < \epsilon\},$$

where K be a compact subset of $\mathbb{C}$ and ϵ a positive number. We may assume that K is a closed disc centered at $s = 1/2$. And that f has no poles on the boundary of K. Choose a point a such that the set $A = \{a, 1 - a, \overline{a}, 1 - \overline{a}\}$ is disjoint from K, the critical axis, the real axis and the zeros and poles of $\Lambda_Q(f, \cdot)$. Let B be a finite set in the interior K^0 of K which is symmetric with respect to the point $1/2$ and the real axis and includes all of the poles of f on K.

Fix $\epsilon_0 > 0$ and let m be the maximum order of the poles of f at the points of B. By Lemma 1, given $\epsilon_1 > 0$, there is a polynomial p, such that $|p - 1| < \epsilon_1$ on K, for each $b \in B$, $p(b) = 1, p^{(j)}(b) = 0, j = 1, \cdots, m - 1$, and $p(a) = 0$, for each point $a \in A$. We may choose ϵ_1 so small that, for the polynomial

$$\nu(s) = p(s)\overline{p(\overline{s})}p(1 - s)\overline{p(1 - \overline{s})},$$

$|\nu - 1| < \epsilon_0$ on K. The polynomial ν is clearly symmetric with respect to the point $1/2$ and the real axis, and assumes the value 0 at the points of A. Moreover, we claim that ν continues to assume the value 1 with multiplicity at least m at each point of B. Since each of the four factors of ν has this property, it is sufficient to verify that, if each of two polynomials, say F and G, assumes the value 1 at a point b with multiplicity at least m, then the same is true of the product FG. For each $k = 1, \cdots, m - 1$, the k-th derivative of FG evaluated at b is a finite sum, each of whose terms is a product, one of whose factors is a derivative of either F or G of order between 1 and $m - 1$. But these derivatives all vanish at the point b. Hence, FG assumes the value 1 at the point b with multiplicity at least m. Thus, the polynomial ν verifies the claim.

Set $M = \max |f|$ on ∂K. Now, choose $\epsilon_0 = \epsilon/M$ and set $g = \nu f$. Then, g satisfies the functional equation and $\Lambda_Q(g, \cdot)$ has zeros at the points of A. On ∂K we have the estimate $|f - g| < \epsilon$. Since, $1 - \nu$ has a zero of order at least m at each pole of

f in K^o, it follows that $f - g = (1-\nu)f$ is holomorphic in K^o. By the maximum principle, the estimate $|f - g| < \epsilon$ holds on all of K.

Thus, g is in the neighborhood $N(f, K, \epsilon)$ of f and not in RM_Q. Since f was arbitrary in M_Q, the complement of RM_Q in M_Q is open. Thus the closed set RM_Q is nowhere dense in M_Q.

If f were initially chosen in H_Q, then the corresponding g would be in RH_Q and so the same proof shows that also that the closed set RH_Q is nowhere dense in H_Q. □

In the proof of Theorem 1, we invoked a Walsh type lemma in order to simultaneously approximate on a compact set and interpolate at finitely many points. In the sequel, we shall need to approximate on a (possibly unbounded) closed set and simultaneously interpolate on a (possibly infinite) discrete subset thereof.

For a closed subset $X \subset \mathbb{C}$, we denote by $A(X)$ the space of functions continuous on X and holomorphic on X^0. Every function $f : X \to \mathbb{C}$, which can be uniformly approximated on X by entire functions, is necessarily in the class $A(X)$. A closed set X is said to be a set of *uniform* approximation if, for every $f \in A(X)$ and every positive *constant* ϵ, there is an entire function g such that $|f(z) - g(z)| < \epsilon$, for all $z \in X$. A theorem of Norair U. Arakelian (see [7]) asserts that a closed subset $E \subset \mathbb{C}$ is a set of uniform approximation if and only if $\overline{\mathbb{C}} \setminus E$ is connected and locally connected, where $\overline{\mathbb{C}}$ denotes the closed complex plane $\mathbb{C} \cup \{\infty\}$.

A closed set X is said to be a set of *tangential* approximation if, for every $f \in A(X)$ and every positive continuous *function* ϵ on X, there is an entire function g such that $|f(z) - g(z)| < \epsilon(z)$, for all $z \in X$. By the Tietze extension theorem for closed sets [5], it makes no difference whether we take the function ϵ to be defined on X or on $\mathbb{C}$. Of course, a set of tangential approximation is necessarily a set of uniform approximation. Let us say that a family $\{E_j\}$ of subsets of $\mathbb{C}$ has *no long islands*, if, for each $r > 0$, there is a (larger) r' such that no E_j meets both circles $|z| = r$ and $|z| = r'$. This condition was introduced by the first author [8], who showed that, for a closed set E of uniform approximation to be a set of tangential approximation, a necessary condition is that the family of components of the interior E^0 have no long islands. Ashot H. Nersessian [16] showed that this condition is also sufficient. For an overview of uniform and tangential approximation, see the book of Dieter Gaier [7].

An example of a set of uniform approximation, would be the set E_0 consisting of the union of the critical strip $0 \le \Re s \le 1$ and the real axis $\Im s = 0$. However, this is not a set of tangential approximation, since the interior has an unbounded component. Let $\{a_j\}$ and $\{b_j\}$ be sequences of positive numbers strictly increasing to infinity, with $a_j < b_j < a_{j+1}$, for each j. For each j, let A_j be the closed *double rectangle* $A_j = \{z : a_j \le |\Im z| \le b_j, |\Re z| \le j\}$. Let E_1 be the union of the real axis, the critical axis and all of the double rectangles:

$$E_1 = \{z : \Im z = 0\} \cup \{z : \Re z = 1/2\} \cup \bigcup_j A_j.$$

The set E_1 is an example of a set of tangential approximation.

The following lemma was given in [11], where it was claimed that the proof followed from another paper. For completeness, we provide the proof.

Lemma 2. *Let X be a closed set of tangential approximation in $\mathbb{C}$. Let B be a finite subset of X and for each $b \in B$, let m_b be a natural number, with the* *restriction that $m_b = 1$ if $b \in \partial X$. Then, for every $f \in A(X)$ and for every positive continuous function $\epsilon : X \to \mathbb{R}$, there exists an entire function F, such that the following simultaneous approximation and interpolation holds: $|f(z) - F(z)| < \epsilon(z)$ for every $z \in X$, and $f - F$ has a zero of multiplicity (at least) m_b, for every $b \in B$.*

Proof. For $g : X \to \mathbb{C}$, set

$$\|g\|_\epsilon = \sup_{z \in X} \frac{|g(z)|}{\epsilon(z)}$$

and

$$A_\epsilon(X) = \{g \in A(X) : \|g\|_\epsilon < \infty\}.$$

Then, $A_\epsilon(X)$ is a normed linear space. Moreover, for each $z \in X$, the mapping $g \longmapsto g^{(j)}(z)$ is a continuous linear functional on $A_\epsilon(X)$, for all $j = 0, 1, 2, \cdots$, if $z \in X^o$, and for $j = 0$, if $z \in \partial X$.

Now, let $f \in A(X)$. Since X is a set of tangential approximation, there exists an entire function h such that $|h - f| < \epsilon$ on X. Thus, $g = f - h$ is in $A_\epsilon(X)$. Since, X is a set of tangential approximation, the entire functions are dense in $A_\epsilon(X)$. By Lemma 1 there is an entire function G such that $|G - g| < \epsilon$ on X and $G^{(j)}(b) = g^{(j)}(b)$, for $b \in B$ and $j = 0, \cdots, m_b - 1$. Set $F = G + h$. □

We shall make use of the above lemma and an induction process to approximate while simultaneously interpolating on an *infinite* set B. First, we introduce a type of set on which this is possible.

Let us say that a closed set $X \subset \mathbb{C}$ is a *chaplet* if there is a strictly increasing sequence $\{r_n\}$ of positive numbers tending to ∞, such that the exhaustion of $\mathbb{C}$ by the discs $K_n = \{z : |z| \leq r_n\}$ has the following compatibility conditions with the set X.

Condition (i). Setting $K_0 = \emptyset$, each set $Y_n = X \cup K_{n-1}$ is a set of tangential approximation.

Condition (ii). $X \cap \partial K_n$ is finite, for each n.

Lemma 3. *Let X be a chaplet in $\mathbb{C}$. Let B be a discrete subset of X and for each $b \in B$, let m_b be a natural number, with the restriction that $m_b = 1$ if $b \in \partial X$.*

Then, for every $f \in A(X)$ and for every positive continuous function $\epsilon : X \to \mathbb{R}$, there exists an entire function h, such that the following simultaneous approximation and interpolation holds: $|f(z) - h(z)| < \epsilon(z)$ for every $z \in X$, and $f - h$ has a zero of multiplicity (at least) m_b, for every $b \in B$.

Proof. A similar result on simultaneous approximation and interpolation was obtained in [9], but for uniform approximation. Here, we need the stronger tangential approximation and so we shall imitate the proof in [9], but with appropriate modifications to obtain tangential approximation. We also adapt a technique from [12]

We may assume that ϵ is defined on all of $\mathbb{C}$ and, in fact, is a strictly decreasing function of $|s|$, for $s \in \mathbb{C}$. Let X, B, f and m_b be as in the hypotheses of the lemma and $\{K_n\}$ be an exhaustion of $\mathbb{C}$ compatible with the chaplet X. We may assume that B includes the finite sets $X \cap \partial K_n$, for each n.

Since, X is a set of tangential approximation, the entire functions are dense in $A_\epsilon(X)$. Set $F_0 = f$. It follows from Lemma 2 that there is an entire function F_1 such that

$$|F_1(z) - f(z)| < \epsilon(z) \cdot 2^{-1}, \quad z \in X,$$

$$|F_1(z) - F_0(z)| < \epsilon(z) \cdot 2^0, \quad z \in K_0,$$

$$F_1^{(j)}(b) = f^{(j)}(b), \quad b \in B \cap K_1, \ j = 0, \cdots, m_b - 1,$$

Note that, the second equation is vacuous. We proceed by induction. Given an entire function F_k such that

$$|F_k(z) - f(z)| < \epsilon(z) \cdot \sum_{j=1}^{k} 2^{-j}, \quad z \in X, \tag{2}$$

$$|F_k(z) - F_{k-1}(z)| < \epsilon(z) \cdot 2^{-k}, \quad z \in K_{k-1}, \tag{3}$$

$$F_k^{(j)}(b) = f^{(j)}(b), \quad b \in B \cap K_k, \ j = 0, \cdots, m_b - 1, \tag{4}$$

we define an associated function $h_k \in A(X \cup K_k)$ as follows:

$$h_k(z) = \begin{cases} F_k(z) & z \in K_k, \\ f(z) & z \in X \setminus K_k. \end{cases}$$

The induction step is then: By Lemma 2, there exists an entire function F_{k+1} such that

$$|F_{k+1}(z) - h_k(z)| < \epsilon(z) \cdot 2^{-(k+1)}, \quad z \in X \cup K_k$$

and

$$F_k^{(j)}(b) = h_k^{(j)}(b), \quad b \in B \cap K_{k+1}, \ j = 0, \cdots, m_b - 1.$$

Then, (2), (3) and (4) hold with k replaced by $k+1$, and hence, by induction, for all $k = 1, 2, \cdots$.

By (3), the sequence $\{F_k\}$ converges to an entire function F. The function F performs the required approximation by (2), and performs the required interpolation by (4). □

In the following proposition, we shall apply Lemma 3 in order to approximate and interpolate the constant function 1, while imposing zeros on the approximating function. A version of this proposition was presented in [11], but therein the interpolation of 1 was only at finitely many points and the interpolation was not with multiplicity.

Proposition 1. *Let X be a chaplet in $\mathbb{C}$. Let B be a discrete subset of X and for each $b \in B$, let m_b be a natural number, with the restriction that $m_b = 1$ if $b \in \partial X$. Let A be a discrete set which is disjoint from X. Suppose X, B, and A are all symmetric with respect to the point $1/2$ and the real axis, and m_b also, that is, $m_b = m_{\overline{b}} = m_{1-b}$. Then, for every strictly positive continuous function $\epsilon : X \to \mathbb{R}$, there exists a function ν holomorphic on $\mathbb{C}$, symmetric with respect to the point $1/2$ and the real axis, and whose zeros are precisely the points of A. Moreover, ν simultaneously approximates and interpolates 1, in that $|\nu(z) - 1| < \epsilon(z)$, for each $z \in X$, and $\nu - 1$ has a zero of multiplicity (at least) m_b, at each point $b \in B$.*

Proof. We merely sketch the proof, since it is similar to that of Proposition 4 in [11]. We may assume that ϵ is symmetric with respect to the point $1/2$ and the real axis. Let g be an entire function, whose zero set is precisely A and let G be a branch of $\log g$ on X. By Lemma 3, there is an entire function h such that $|h - G| < \epsilon/12$

on X and $h-G$ has a zero of multiplicity (at least) m_b, at each point $b \in B$. The function

$$\nu(s) = \frac{g(s)\overline{g(\overline{s})}g(1-s)\overline{g(1-\overline{s})}}{exp\left(h(s)+\overline{h(\overline{s})}+h(1-s)+\overline{h(1-\overline{s})}\right)}$$

has all of the required properties. The additional property that we must verify, that was not proved in [11] is that ν not only interpolates the constant 1 at the points $b \in B$, but in fact interpolates up to multiplicity m_b. That is, we must verify that $\nu^{(j)}(b) = 0$, for $j = 1, \cdots, m_b - 1$. The proof of this is the same as in the proof of Theorem 1, since each of the four factors of ν has this property. □

The next theorem asserts that we may approximate a function $f \in M_Q$ extremely well by one which satisfies the same functional equation and fails radically to satisfy the analogue of the grand Riemann hypothesis.

Theorem 2. *Let f be a function in M_Q. Let A and B be disjoint discrete sets, symmetric with respect to the point $1/2$ and the real axis and suppose B contains the zeros and poles of f. For each $b \in B$, let m_b be a natural number, also symmetric with respect to the point $1/2$ and the real axis; that is, $m_b = m_{\overline{b}} = m_{1-b}$. Let ϵ and η be strictly positive continuous functions on $\mathbb{C}$ and $[0, +\infty)$ respectively. Then, there exists a function $g \in M_Q$, $g \neq f$, and a closed set $X \subset \mathbb{C}$, containing the real and critical axes, such that:*

(i) $\Lambda_Q(g, a) = 0$, for $a \in A$ while $\Lambda_Q(g, \cdot)$ and $\Lambda_Q(f, \cdot)$ have exactly the same zeros with same multiplicities on $\mathbb{C} \setminus A$;

(ii) f and g have the same poles and at each point $b \in B$, the first m_b terms of the Laurent series of f and g coincide;

(iii) $|g(z) - f(z)| < \epsilon(z)$, for $z \in X$;

(iv) area$\{z : z \notin X, |z| > r\} < \eta(r)$, for $r \in [0, +\infty)$.

Proof. We begin by constructing a chaplet X. Let $\{\alpha_j\}$ and $\{\beta_j\}$ be sequences of positive numbers strictly increasing to infinity, such that $\alpha_j < \beta_j < \alpha_{j+1}$. Denote by $\mathcal{A}_j$ the annulus

$$\mathcal{A}_j = \{z : \alpha_j < |z - 1/2| < \beta_j\},$$

and by $\mathcal{A}$ the union of these annuli. We may choose the sequences $\{\alpha_j\}$ and $\{\beta_j\}$ so that $\mathcal{A}_j$ contains A, that is, such that for all $a \in A$, there is a j with $a \in \mathcal{A}_j$.

Choose a point $1/2 + p$, with $p = \alpha_1 e^{i\theta}, 0 < \theta < \pi/2$, lying on the circle ($|z - 1/2| = \alpha_1$), such that the ray $R = \{z = 1/2 + tp : 1 \leq t < +\infty\}$ is disjoint from the set B. Let

$$S_p = \cup_{t \geq 1}\{z : |z - (1/2 + tp)| < \delta(t)\}$$

be a neighborhood of the ray R, where δ is a positive continuous function which decreases so rapidly that the strip S_p is disjoint from the critical and real axes and from the set B. Denote by $\tilde{S}_p$ the reflection of S_p with respect to the real axis and denote by $1 - S_p$ and $1 - \tilde{S}_p$ respectively the reflection of S_p and $\tilde{S}_p$ with respect to the point $z = 1/2$. Denote by S the union of these four strips. For each $b \in B$, let Q_b be a closed disc centered at b, such that the radii are symmetric with respect to the real axis and the point $1/2$. Set $\mathcal{B} = \cup_{b \in B} Q_b$. Finally, set

$$X = (\mathbb{C} \setminus (\mathcal{A} \cup S)) \cup \mathcal{B} \cup \{z : \Im z = 0\} \cup \{z : \Re z = 1/2\}.$$

Then, if the Q_b are sufficiently small, X is a chaplet and X, B, m_b and A satisfy the hypotheses of Proposition 1. Moreover, we may construct X so that the complement is as small as we please. In fact, we may construct X so that condition (iv) is satisfied.

For each pole b, let K_b be a closed disc centered at b and contained in X^0 such that the discs K_b are disjoint and form a locally finite family. Now, let $\epsilon_1(z)$ be a strictly positive continuous function on X such that, for every $z \in X$,

$$\epsilon_1(z) < \begin{cases} \epsilon(z)/|f(z)|, & z \in X \setminus \cup_p K_b; \\ (\min_{K_b} \epsilon)/(\max_{\partial K_b} |f|), & z \in K_b. \end{cases}$$

Let ν be an entire function corresponding to ϵ_1, A, B, m_b in Proposition 1. If b is a pole of f, we may and shall assume that m_b is the multiplicity of the pole. Set $g = \nu f$. Since ν is entire and assumes the value 1 at each pole of f, and with the same multiplicity as that of the pole, the function g has the same poles as f, and with the same principal parts. Thus, $f - g$ is an entire function. Now, $|f(z) - g(z)| < \epsilon(z)$, for $z \in X \setminus \cup_p K_b$. For $z \in K_b$, we have, by the maximum principle,

$$|f(z) - g(z)| \leq \max_{\partial K_b} |f(z) - g(z)| = \max_{\partial K_b} (|1 - \nu(z)| \cdot |f(z)|) < \epsilon(z).$$

We have shown that $|f(z) - g(z)| < \epsilon(z)$, for all $z \in X$. The function g satisfies conditions (i)-(iv).

In case f is the Riemann zeta-function and the Riemann hypothesis is false, then, of course, f is itself a (perfect) approximation of itself failing to satisfy the Riemann hypothesis. However, in the present theorem, it is easy to assure that $g \neq f$, by simply choosing the set A such as to contain a point not among the zeros of f. Thus, the theorem is non-trivial, even in case the Riemann hypothesis fails. □

By a continuous perturbation of a function f in M_Q, we understand a continuous curve in M_Q:

$$[0, 1) \to M_Q$$
$$t \mapsto f_t,$$

such that $f_0 = f$.

Corollary 1. *If f is an L-function, satisfying a functional equation (1), then there is a continuous perturbation, $f_t, 0 \leq t < 1, f_0 = f$, such that each f_t, for $0 < t < 1$, satisfies the same functional equation but fails to satisfy the analogue of the Riemann hypothesis.*

Proof. Let M_Q be the class of meromorphic functions associated to the functional equation for f. Let B be a discrete set, symmetric with respect to the point $1/2$ and the real axis and containing the zeros and poles of f. Let $\{a_n\}$, with $n < |a_n| < n+1$, be a sequence distinct from the zeros of f, and disjoint from the real axis, the critical axis and B. Set $A_n = \{a_k : k > n\}$ and $\epsilon_n = 1/n$. From Theorem 2, we obtain a function $g_n \in M_Q$ and a closed set X_n such that

$$|f(z) - g_n(z)| < 1/n, \quad z \in X_n$$

and

$$g_n(a_k) = 0, \quad k > n.$$

As in the proof of Theorem 2, we see that X_n may be so chosen that it contains the closed disc, centered at the origin and of radius n.

For each n and for $n \leq t \leq n+1$, set

$$g_t = (1-t+n)g_n + (t-n)g_{n+1}.$$

Then, $g_t \in M_Q$,

$$|f(z) - g_t(z)| < 1/n, \quad |z| \leq n, \quad n \leq t \leq n+1,$$

and

$$g_t(a_k) = 0, \quad k > n+1.$$

Thus, $t : [1, +\infty) \to g_t$ is a continuous path of meromorphic functions in M_Q, each of which has infinitely many zeros different from the zeros of f and such that $g_t \to f$.

To conclude the proof, we have only to reparametrize by setting $f_0 = f$ and $f_t = g_{1/t}, 0 < t < 1$. □

3. Approximation by functions satisfying the analogue of the grand Riemann hypothesis

Having approximated zeta-functions by functions which do *not* satisfy the analogue of the Riemann hypothesis, we now consider the possibility of approximating by functions which *do* satisfy the analogue of the Riemann hypothesis. Of course, we cannot have our cake and eat it too. If we *could* approximate the Riemann zeta-function uniformly on compacta by functions which satisfy the analogue of the Riemann hypothesis, this would prove the Riemann hypothesis. Although we do not know how to approximate zeta-functions *uniformly on compacta* by functions satisfying the analogue of the Riemann hypothesis, the next theorem shows that we can, nevertheless, approximate the Riemann zeta-function *extremely well* by functions all of whose non-trivial zeros lie on the critical axis and hence satisfy the analogue of the Riemann hypothesis.

First we need a lemma on approximating zero-free functions by entire zero-free functions.

Lemma 4. *Let X be a chaplet in $\mathbb{C}$. Let $f \in A(X)$ and let ϵ be a strictly positive continuous function on X. Suppose f is zero-free on X. Then, there is a zero-free entire function e^h such that*

$$|e^{h(z)} - f(z)| < \epsilon(z), \quad \textit{for } z \in X.$$

Moreover, if B is a discrete subset of $\mathbb{C}$ lying in the interior of X and for each $b \in B$, m_b is a positive integer, we may find such an h so that e^h interpolates f to order m_b at each point $b \in B$. Moreover, if X, f and B are symmetric with respect to $1/2$, then we may take h symmetric with respect to $1/2$.

Proof. For the function $f \in A(X)$, a function $F \in A(X)$ is needed such that $e^F = f$. In the article [10], it is stated in Lemma 2 that this is the case if X is a set of uniform approximation. Since X is a chaplet it is indeed a set of uniform approximation. Moreover, it is easy to see that $F \in A(X)$.

Since the exponential function is uniformly continuous on compacta, it follows that for any positive continuous function η, there is a positive continuous function δ such that, if w is any continuous function on X such that $|w| < \delta$, then $e^{|w|} - 1 < \eta$. Now set $\eta = \epsilon/|e^F|$. Since X is a chaplet, it follows from Lemma 3 there is an entire function h such that $|h(z) - F(z)| < \delta(z)$ for $z \in X$.

$$|e^h - f| = |e^h - e^F| = |e^{h-F} - 1||e^F| \leq (e^{|h-F|} - 1)|e^F| < \epsilon.$$

Moreover, we may choose h so that e^h interpolates f at the points of B. Indeed, we may choose h so as to interpolate F to order m_b at each point b. Thus, $h - F$ has a zero of order m_b at b. Since the exponential function is locally biholomorphic, the function e^{h-F} assumes the value 1 with multiplicity m_b at b. Since, e^F is different from zero at the point b, the product $(e^{h-F} - 1)e^F$ also has a zero of order m_b at b. That is, $e^h - f = (e^{h-F} - 1)e^F$ has a zero of order m_b at b. So e^h interpolates f to order m_b at the point m.

Suppose, X, f and B are symmetric with respect to 1/2. Since $f(z) = f(1 - z)$ and $e^F = f$, we have that

$$g(z) := \frac{F(z) - F(1-z)}{2\pi i} \in \mathbb{Z}, \quad \text{for all } z \in X.$$

The function g is continuous on X, therefore constant on each component of X. The components of X come in pairs. Each member of a pair is the reflection of the other with respect to 1/2. Let X_c be a collection of components, one from each pair, indexed by a parameter c. We may denote the component paired with X_c by $1 - X_c$. The function g is constant on X_c and so there is an integer k_c such that $F(z) = F(1 - z) + k_c 2\pi i$, for $z \in X_c$. Note that if X_c is the same as its paired component, then on X_c, we have $F(z) = F(1 - z) + k_c 2\pi i = F(z) + k_c 2\pi i + k_c 2\pi i$ and so $k_c = 0$. Now, we had some freedom in choosing the function F. Let us modify F by leaving F as it is on $1 - X_c$ but replacing it on X_c by $F - k2\pi i$. With this new definition of F, we have that $g \equiv 0$ on each X_c. Thus, F is symmetric with respect to 1/2. Now, we may assume that h is also symmetric with respect to 1/2 by replacing $h(z)$ by $(h(z) + h(1 - z))/2$. This proves the lemma. □

In Theorem 2 we approximated a function $f \in M_Q$ by a function which satisfies the same functional equation and fails to satisfy the grand Riemann hypothesis. The following theorem asserts that we may also approximate by one which *does* satisfy the analogue of the grand Riemann hypothesis.

Theorem 3. *Let f be a function in M_Q and suppose (as for the Riemann zeta-function) that no pole of f is a zero of $\Lambda_Q(f, \cdot)$. Let ϵ and η be strictly positive continuous functions on $\mathbb{C}$ and $[0, +\infty)$ respectively. Then, there exists a function $g \in M_Q$, $g \neq f$, and a closed set $X \subset \mathbb{C}$, containing the real and critical axes, such that:*

(i) on the critical axis, $\Lambda_Q(g, \cdot)$ has the same zeros with the same multiplicities as $\Lambda_Q(f, \cdot)$;

(ii) $\Lambda_Q(g, \cdot)$ has no zeros off the critical axis;

(iii) $\Lambda_Q(g, \cdot)$ has the same poles with the same multiplicities as $\Lambda_Q(f, \cdot)$;

(iv) $|g(z) - f(z)| < \epsilon(z)$, for $z \in X$;

(v) area$\{z : z \notin X, |z| > r\} < \eta(r)$, for $r \in [0, +\infty)$.

Proof. As in the proof of Theorem 2, we construct a chaplet X, symmetric with respect to 1/2 and the real axis, containing the real and critical axes and satisfying condition (v). Moreover, by slightly deforming the annuli in the construction of X, we may so construct X that it excludes all of the zeros of $\Lambda_Q(f, \cdot)$ off the critical axis, while retaining the condition that X contains all of the poles of f in its interior.

Let f_o be an entire function, symmetric with respect to 1/2, whose zeros are precisely the zeros of $\Lambda_Q(f, \cdot)$ off the critical axis and with the same multiplicities.

Let B be the set of poles of f and their reflections with respect to the point 1/2. As in the proof of Theorem 2, for each $b \in B$, let K_b be a closed disc centered at

b such that: K_b is contained in X^0; the function f has no pole in K_b other than possibly b ; the discs K_b are disjoint and form a locally finite family. Let M_b be the minimum of $\epsilon(z)$, for $z \in K_b$ and set

$$N_b = \frac{\max_{z\in\partial K_b} |f(z)|}{\min_{z\in K_b} |f_o(z)|}.$$

Note that f_o has no zeros on X and in particular on K_b. Now, let $\epsilon_1(z)$ be a strictly positive continuous function on X such that, for every $z \in X$,

$$\epsilon_1(z) < \begin{cases} \epsilon(z)|f_o(z)|/|f(z)|, & z \in X \setminus \cup_b K_b; \\ M_b/N_b, & z \in K_b. \end{cases}$$

By Lemma 4, there is a zero-free entire function e^h symmetric with respect to 1/2 such that

$$|e^h - f_o| < \epsilon_1 \text{ on } X$$

and, if b is any pole of f and the order of the pole is m_b, then e^h interpolates f_o to order m_b . Hence, $e^h f - f_o f$ is in $A(X)$ and so by the maximum principle, as in the proof of Theorem 2,

$$|e^h f - f_o f| < \epsilon |f_o| \text{ on } X.$$

Thus, if we put $g = (e^h/f_o)f$, then

$$|g - f| < \epsilon \text{ on } X.$$

As we remarked earlier, because g is the product of a function e^h/f_o symmetric with respect to 1/2 and a function f in M_Q, it follows that g is also in M_Q. That is, g satisfies the functional equation. Moreover, $\Lambda_Q(g,\cdot)$ has the same zeros as $\Lambda_Q(f,\cdot)$ with the same multiplicities on the critical axis and no other zeros and $\Lambda_Q(g,\cdot)$ has the same poles as $\Lambda_Q(f,\cdot)$ with the same multiplicities.

We have approximated f by a function g which satisfies the analogue of the grand Riemann hypothesis. In case f is the Riemann zeta-function and the Riemann hypothesis holds, the Riemann zeta-function is, of course, a (perfect) approximation of itself satisfying the Riemann hypothesis. However, it is easy in our theorem to make sure that $g \neq f$. For example, the function h is highly non-unique and so not all of the corresponding functions $g = (e^h/f_o)f$ can be f. Thus, even in case the Riemann hypothesis holds, Theorem 3 is non-trivial for the Riemann zeta-function. □

As a corollary (of the proof of the preceding theorem) we have the following product decomposition for the Riemann zeta-function.

Corollary 2. *The Riemann zeta-function can be written as a product $\zeta = b\zeta_+$, where b has all of the non-trivial zeros of ζ off the critical axis and ζ_+ is an "ideal" zeta function in the sense that it has no non-trivial zeros off the critical axis, has a single simple pole at $z = 1$, satisfies the functional equation and approximates the Riemann zeta-function extremely well in the sense of the previous theorem.*

Proof. In the proof of the previous theorem, we have $g = (e^h/f_o)f$. Take $f = \zeta$ and set $b = f_o e^{-h}$ and $\zeta_+ = g$. □

The Riemann hypothesis would of course follow, if one could show that $b = 1$ in this decompostion. But the product decomposition in Corollary 2 is not unique, since the approximating function g in Theorem 3 is not unique. An improved version of Theorem 3 is therefore desirable, in which the class of approximating functions g is reduced to functions satisfying even more properties of the Riemann zeta function. In Section 4 we shall discuss the possibility of representing the approximating functions in Theorem 2 and Theorem 3 by Dirichlet series.

Philosophical remarks. The above approximations are so strong that the approximator "cannot be distinguished" from the approximatee. Suppose a mischievous angel (a devil?) were to hand us functions ζ_+ and ζ_-, which approximate the Riemann zeta-function as in our theorems and satisfy the corresponding functional equation, where ζ_+ satisfies the analogue of the Riemann hypothesis and ζ_- does not . If the angel claimed that these functions were in fact the Riemann zeta-function, there is no way we could prove it wrong. That is, we could not distinguish these functions from each other or from the Riemann zeta-function. Indeed, we can choose the speed of approximation ϵ so small that on X these functions differ by each other by less than the diameter of an electron and so no (present or future) scientific instrument could distinguish their values on X. While it is true that on $\mathbb{C}\setminus X$, these functions may differ greatly, this set $\mathbb{C}\setminus X$ is so small, that if the fastest conceivable computer (or finite team of computers) where to pick points of $\mathbb{C}$ successively at random, it is likely that the sun would grow cold before the computer would fall upon a point of $\mathbb{C}\setminus X$. We thus have functions ζ_+ and ζ_- satisfying the functional equation, "indistinguishable" from the Riemann zeta-function and from each other, such that ζ_+ has no non-trivial zeros off the critical axis while ζ_- does.

4. Dirichlet series

Let ζ_+ and ζ_- be functions as in the above theorems, which approximate the Riemann zeta function, are practically indistinguishable and respectively satisfy and do not satisfy the analogue of the Riemann hypothesis.

We recall that both functions ζ_+ and ζ_- approximate the Riemann zeta function ζ exceedingly well; the function ζ_+ has no non-trivial zeros off the critical axis, whereas the function ζ_- can be chosen to have a multitude of non-trivial zeros off the critical axis. We should like to say something concerning the representations of these functions by Dirichlet series. To this end we shall employ a result of Frédéric Bayart [1] establishing the existence of a universal Dirichlet series.

Definition. A compact subset K of $\mathbb{C}$ is said to be *admissible* if $\mathbb{C}\setminus K$ is connected, and if K can be written $K = K_1 \cup \cdots \cup K_d$, each K_i being contained in a strip $S_i = \{z : a_i \le \Re(z) \le b_i\}$, with $b_i - a_i < 1/2$, the strips S_i being disjoint.

Theorem 4. [1, Remark 5] *There exists a Dirichlet series* $S(z) = \sum_{n\ge1} a_n n^{-z}$ *absolutely convergent in the open right half-plane* $\{z : \Re(z) > 1\}$, *which is universal in the following sense. For each admissible compact set* K *in the closed left half-plane* $\{z : \Re(z) \le 1\}$ *and each function* g *continuous on* K *and holomorphic on the interior of* K, *there is a sequence of partial sums of* S *which converges to* g *uniformly. The set of such Dirichlet series is dense in the space of Dirichlet series absolutely convergent in* $\{z : \Re(z) > 1\}$.

Bayart states his theorem for the right half-plane $\Re s > 0$, but the substitution $s = z - 1$ yields the above version. From this, we can obtain the following. For a real number σ, denote by $H_{>\sigma}$ the open right-half plane $\{z : \Re(z) > \sigma\}$ and by $H_{\le\sigma}$ the closed left-half plane $\{z : \Re(z) \le \sigma\}$.

Corollary 3. *There exists a Dirichlet series* $S(z) = \sum_{n\ge1} a_n n^{-z}$ *absolutely convergent in the open right half-plane* $H_{>1}$, *which has the following properties. For* *each pair of functions* ζ_+ *and* ζ_- *as in the above theorems, there is a sequence of partial sums of* S *which converges to* ζ *on* $H_{\leq 1}$*, uniformly on admissible compact subsets of* $H_{\leq 1} \setminus \{1\}$*; there is a sequence of partial sums of* S *which converges to* ζ_+ *on* $H_{\leq 1}$*, uniformly on admissible compact subsets of* $H_{\leq 1} \setminus \{1\}$*; and there is a sequence of partial sums of* S *which converges to* ζ_- *on* $H_{\leq 1}$*, uniformly on admissible compact subsets of* $H_{\leq 1} \setminus \{1\}$*. Moreover, for each* $\delta > 0$*, there exists such a Dirichlet series with the further property that for* $\Re(z) \geq 1 + \delta$*,*

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^z} - \sum_{n=1}^{\infty} \frac{1}{n^z} \right| < \delta. \tag{5}$$

Proof. As in [1], let $K_1, K_2, \cdots$ be a sequence of admissible compacta in $H_{\leq 1} \setminus \{1\}$ such that each admissible compact subset of $H_{\leq 1} \setminus \{1\}$ is contained in some K_k. Set $\zeta_k = \zeta$ on K_k and $\zeta_k(1) = k$. Let S be a universal Dirichlet series as in Theorem 4. By Theorem 4, for each k, there is a sequences of partial sums of S:

$$S^{k1}, S^{k2}, \cdots \to \zeta_k, \quad \text{uniformly on } K_k \cup \{1\}.$$

A diagonal sequence of partial sums of S converges to ζ on $H_{\leq 1}$ and uniformly on admissible compact subsets of $H_{\leq 1} \setminus \{1\}$. We do the same for ζ_+ and ζ_-.

By Theorem 4, the set of Dirichlet series such as S is dense in the space of Dirichlet series absolutely convergent in $H_{>1}$. In particular, there is such a Dirichlet series S which satisfies (5). □

A special case of the preceding corollary is the following.

Corollary 4. *Let* E *be a discrete subset of the open left half-plane* $H_{<1/2}$*. For each* $\delta > 0$*, there exists a Dirichlet series* $S(z) = \sum_{n \geq 1} a_n n^{-z}$ *absolutely convergent in the open right half-plane* $H_{>1}$ *such that on the half-plane* $H_{\geq 1+\delta}$*,*

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^z} - \sum_{n=1}^{\infty} \frac{1}{n^z} \right| < \delta.$$

Moreover, a sequence of partial sums of S *converges to* ζ *uniformly on compact subsets of the critical strip* $\{z : 1/2 \leq \Re(z) < 1\}$ *and to zero on* E*.*

Proof. Arrange the set E in a sequence $\{z_n\}$ and set

$$K_n = \{z : 1/2 \leq \Re(z) \leq n/(n+1)\} \cup \{z_1, z_2, \cdots, z_n\}.$$

We define g on K_n by setting $g = \zeta$ on $\{z : 1/2 \leq \Re(z) \leq n/(n+1)\}$ and $g = 0$ on $\{z_1, z_2, \cdots, z_n\}$. Then, K_n is admissible and g is continuous on K_n and holomorphic on the interior. By Theorem 4, there is a Dirichlet series S absolutely convergent in the right half-plane $H_{>1}$ and there is a sequence $n_1 < n_2 < \cdots$ such that, for each k, $|S^{n_k} - g| < 1/k$ on K_k. By Theorem 4, the set of such Dirichlet series is dense in the space of Dirichlet series absolutely convergent in $H_{>1}$. In particular, there is such a Dirichlet series S which satisfies (5). □

Similar results can be proved for L-functions.

5. Examples of L-functions

Let us describe some examples of L-functions in number theory verifying the conditions we ask for.

First examples of L-functions are the Dirichlet L-functions associated to a Dirichlet character χ (mod d) (see for example [3]), that is, a function on the integers which is not identically zero and verifies $\chi(n \cdot m) = \chi(n)\chi(m)$, $\chi(n+d) = \chi(n)$ and $\chi(n) = 0$ if $(n, d) > 1$. They are defined in the half plane $\Re(s) > 1$ by the series

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If χ is a principal character (that is, $\chi(n) = 1$ or 0 for all n), we obtain "essentially" the usual Riemann zeta-function. In all other cases, the function χ can be extended holomorphicly to the whole complex plane. If the character χ is primitive and we define

$$Q(s) := \left(\frac{d}{\pi}\right)^{(s+\delta)/2} \Gamma\left(\frac{s+\delta}{2}\right), \quad \text{where } \delta := \frac{1-\chi(-1)}{2}, \tag{6}$$

then the function $\Lambda_Q(\chi, s) := Q(s)L(\chi, s)$ verifies a functional equation [15] of the form

$$\Lambda_Q(\chi, s) := W(\chi)\Lambda_Q(\overline{\chi}, 1-s),$$

where $W(\chi)$ is an algebraic complex number with absolute value 1, called the root number. This equation is equivalent to the functional equation

$$\Lambda_Q(\chi, s) := W(\chi)\overline{\Lambda_Q}(\chi, 1-s),$$

where, for a complex function Λ, $\overline{\Lambda}(s)$ is the complex conjugate of Λ, defined as $\overline{\Lambda}(s) := \overline{\Lambda(\overline{s})}$. To see this equivalence, it is sufficient to notice that it holds for positive real s.

So, if χ is primitive and is real valued (that is $\chi(n) = \pm 1$ or 0), and $W(\chi) = 1$, then the function $\Lambda_Q(\chi, s) := Q(s)L(\chi, s)$, with Q given by (6), satisfies the functional equation (1). Note that Gauss proved something equivalent to the hypothesis $W(\chi) = 1$ always being satisfied for such real-valued characters.

More general examples are furnished by Artin L-functions associated to group representations $\rho : \mathrm{Gal}(N/K) \to \mathrm{GL}_n(\mathbb{C})$, where N/K is a Galois extension of fields, each of them finite extensions of $\mathbb{Q}$. In this case the conditions we need are that ρ be real valued (meaning its image is in $\mathrm{GL}_n(\mathbb{R})$), and that the root number $W(\rho) = 1$. The fact that ρ is real valued implies directly that the root number $W(\rho) = \pm 1$, so the condition is verified automatically if $L(\rho, 1/2) \neq 0$. On the other hand, by the Fröhlich-Queyrut theorem [6], the so-called real representations (also called real orthogonal) have root number always equal to 1 .

Easier examples of L-functions that *do* satisfy the Grand Riemann Hypothesis are the L-functions associated to a (smooth projective) curve over a finite field. In this case the functions are always of the form $L(s) = P(q^{-s})$, where q is a power of a prime number and $P(t)$ is a polynomial with integer coefficients of even degree $2g$ verifying the conditions, firstly, that $P(\frac{1}{qt}) = t^{2g}q^{-g}P(t)$ and secondly, that all (complex) roots α have absolute value $|\alpha| = 1/\sqrt{q}$. The first condition implies the functional equation, taking $Q(s) := q^{-gs}$, and the second that the zeros of $\Lambda_Q(s) := q^{-gs}P(q^{-s})$ are of the form $\log(\alpha)/\log(q)$, and so have real part equal to 1/2.

Some examples of other L-functions that are known to have a meromorphic extension and a functional equation are the L-functions associated to cuspidal modular forms f (and, more generally, to cuspidal automorphic form π). In these cases we obtain again a functional equation of the form

$$\Lambda_Q(f,s) := Q(s)L(f,s) = \epsilon\overline{\Lambda_Q}(f,1-s),$$

where $Q(s)$ is a function of the form $K^s\prod_{j=1}^n \Gamma(\lambda_j s + \mu_j)$, n is a natural number, K and the λ_j's are positive real numbers and the μ_j's are complex numbers with non-negative real part. So the conditions we need are verified if f is a so-called real modular form (or with real coefficients) and $\epsilon = 1$. This last condition is again automatic if $L(f,1/2) \neq 0$, since ϵ must be ± 1 if the form is real.

This last case includes some L-functions of more geometric origin: the L-functions associated to elliptic curves defined over $\mathbb{Q}$. After the work of Andrew Wiles and others (see [20] and [2]) it is known that the L-function of such a curve is equal to the L-function of a (real) modular form (of weight 2). From the work of Victor A. Kolyvagin [14] it is also known that the condition $L(f,1/2) \neq 0$ is equivalent to the condition the curve have only finitely many points over $\mathbb{Q}$. On the other hand, the condition that the root number be equal to 1 should be equivalent, according to the famous Birch and Swinnerton-Dyer conjecture (see for example the Bourbaki talk [19]), to the rank of the group of rational points of the curve being even. Note that we are considering a modified version of the usual L-function, which, in this case, has a functional equation relating s with $2-s$.

Note that all the cases presented are Dirichlet series having an Euler product. Recall that a complex function f has an Euler product if for $\Re(s) > 1$ it can be expressed as a Dirichlet series and an infinite product varying on the primes p

$$f(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s} = \prod_p f_p(s)$$

such that

$$f_p(s) = \prod_{i=1}^{d_p} (1 - \alpha_i(p) p^{-s})^{-1}$$

where $\alpha_i(p)$ are complex numbers such that $|\alpha_i(p)| < p$, $\lambda(n)$ are also complex numbers, and $\lambda(1) = 1$ and $d_p \geq 1$ is a natural number, which in the classical cases is independent of p (see for example [13], chapter 5). It would be very interesting to show that we can approximate by functions having some of these properties.

Remark 1. *The results in this paper can be modified to cover some of the other cases of functional equations for L-functions as explained in the examples above. For example, one can consider L-functions with functional equation of the form*

$$\Lambda_Q(f,s) := Q(s)f(s) = -\Lambda_Q(f,1-s)$$

by considering functions antisymmetric with respect to the point $s = 1/2$*, i.e* $f(1-s) = -f(s)$*, for all* $s \in \mathbb{C}$*. Such functions must have a zero for* $s = 1/2$*.*

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