Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Formal completions of Néron models for algebraic tori

Oleg Demchenko, Alexander Gurevich and Xavier Xarles

Abstract

We calculate the formal group law which represents the completion of the Néron model for an algebraic torus over $\mathbb{Q}$ split in a tamely ramified abelian extension. To that end, we introduce an analogue of the fixed part of a formal group law with respect to a group action and give a method to compute its Honda type.

Introduction

An explicit description of a formal group law which represents the completion of the Néron model for an elliptic curve over $\mathbb{Q}$ was given by Honda with the aid of the L-series of the curve considered (see [**Ho1**]). He also found such a description for some one dimensional tori over $\mathbb{Q}$. The first attempt to construct such a formal group law for algebraic tori of higher dimension was made by Deninger and Nart [**DN**]. They consider a torus over $\mathbb{Q}$ which is split over an abelian tamely ramified extension K of $\mathbb{Q}$ and define a formal group law in terms of the Galois representation corresponding to this torus. Employing Honda's isomorphism criterion for formal group laws (see [**Ho2**]) they show that after inverting some primes, the formal group obtained becomes isomorphic to the completion of the Néron model for the torus. There is also an older paper by N. Yui [**Yu**], as well as two papers by N. Childress and D. Grant [**CG**] and by D. Grant [**Ga**], where some explicit constructions of formal group laws associated to some special tori are used in order to show some explicit higher reciprocity laws.

In our article, we work in the same setting as Deninger and Nart and calculate a formal group law which represents the completion of the Néron model. Our main tools are Edixhoven's interpretation of the Néron model as a maximal fixed subscheme [**Ed**], Honda's theory of formal group laws [**Ho2**] and a theory of universal fixed pairs for a formal group law with a group action which is introduced within the scope of this note.

The outline of the paper is as follows. In Section 1, we review the notion of the formal completion of a group scheme and introduce related notation. We proceed with the definition and basic properties of formal group laws accompanied with the main results of Honda's theory (Section 2). In Section 3, formal group laws supplied with a group action are studied. We introduce the notion of a universal fixed pair widely employed in subsequent constructions and find universal fixed pairs for certain formal group laws over the ring of integers in an unramfied extension of $\mathbb{Q}_p$ (Theorem 1). Moreover we establish a sufficient condition for a formal group law over $\mathbb{Z}$ to appear in a universal fixed pair (Proposition 3.5). The construction of the Weil restriction of a formal group law is considered in Section 4. We present an explicit expression for the logarithm of the Weil restriction of the multiplicative formal group law (Proposition 4.3)

2000 *Mathematics Subject Classification* 14L05 (primary), 14G25, 11G35(secondary).

The first author wishes to thank JSPS for financial support and the Mathematical Institute of Tohoku University for its hospitality. The second author was partially supported by the Minerva Foundation through the Emmi Noether Institute, the Ministry of Absorption, Israel, and the grant 3-3578 from the Ministry of Science, Israel, and the Russian Foundation for Basic Research. The third author was partially supported by the grant MTM 2006-11391 from DGI.

and further compute its Honda's type. Another Honda's type of the same formal group law is computed in [**Ib**]. An advantage of our type is that its coefficients belong to $\mathbb{Z}$, and thus it can be used for finding a universal fixed pair in the global case.

Section 5 is devoted to the definition of Néron model and its main properties. Our reference for this topic is [**BLR**]. We also recall Edixhoven's result (see [**Ed**]) which asserts that if T is a torus over k and K/k is a tamely ramified Galois extension, then the Néron model for T is isomorphic to the maximal fixed subscheme in the Weil restriction of the Néron model for T_K with respect to the natural action of $\mathrm{Gal}(K/k)$.

In Section 6, we consider a d-dimensional torus T over k split over a Galois extension K of k and prove that the cotangent space of the Weil restriction of the Néron model for T_K is isomorphic to

$$(\mathcal{X} \otimes_{\mathbb{Z}} \mathcal{O}_k) \otimes_{\mathcal{O}_k} \mathrm{Hom}_{\mathcal{O}_k}(\mathcal{O}_K, \mathcal{O}_k)$$

as $\mathrm{Gal}(K/k)$-module, where $\mathcal{X}$ is the Galois module of K-characters corresponding to T (Proposition 6.3). Further we show that if K/k is tamely ramified, then the Weil restriction Φ of the direct sum of d copies of the multiplicative formal group law provided with the Galois action whose linear part is given in Proposition 6.3 admits a universal fixed pair. Since the formal completion commutes with taking the maximal fixed subscheme and Weil restriction, we conclude that the formal group law which appears in this universal fixed pair represents the completion of the Néron model for T (Theorem 2). Further we consider several special cases and apply the techniques of Sections 3 and 4 in order to calculate the formal group law representing the completion of the Néron model explicitly.

The case where $k = \mathbb{Q}_p$ and K is an abelian tamely ramified extension of $\mathbb{Q}_p$ is treated in Section 7. We prove that the formal completion is isomorphic to the direct sum of a p-divisible group whose dimension is equal to that of the maximal subtorus of T split over the maximal unramified subextension of K, and several copies of the additive formal group schemes (Theorem 3). If $K/\mathbb{Q}_p$ is unramified, we get the same answer as in [**DN**]. In another particular case, when $K/\mathbb{Q}_p$ is totally ramified, the formal completion turns out to be isomorphic to the direct sum of several copies of the multiplicative and additive formal group schemes. A similar result for the reduction of the Néron model for such tori appears in [**NX**]. We apply Theorem 3 along with a result on deformations of p-divisible groups (see [**DG**]) to show that in the case considered, the formal completion of the Néron model is uniquely determined by its reduction (Theorem 4). Moreover we give an example of two tori such that their Néron models have isomorphic completions, but non-isomorphic reductions.

Section 8 is devoted to one-dimensional tori over $k = \mathbb{Q}$ which are split over a tamely ramified quadratic extension K of $\mathbb{Q}$. We construct two formal group laws representing the completion of the Néron model (for one of them this was proven in [**DN**]) and reprove Honda's theorem (see [**Ho1**]) which asserts that the formal group law obtained from the Dirichlet series of $K/\mathbb{Q}$ is strongly isomorphic to the formal group law $F_q(x, y) = x + y + \sqrt{q}xy$, where q is the discriminant of $K/\mathbb{Q}$. The case where $k = \mathbb{Q}$ and K is an abelian tamely ramified extension of $\mathbb{Q}$, is considered in Section 9. Due to the Kronecker-Weber theorem, one can suppose that $K = \mathbb{Q}(\xi)$, where ξ is a q-th primitive root of unity, and q is the product of distinct primes. We express the coefficients of the corresponding formal group law in terms of the images of the Frobenius automorphisms with respect to the Galois representation in the character group of T (Theorem 5). This is our main result. As an application, we show that a torus over $\mathbb{Q}$ split over an abelian tamely ramified extension of $\mathbb{Q}$ is determined up to isomorphism by the completion of its Néron model (Theorem 6). A similar result for Jacobians and elliptic curves was proven in [**Na**].

Throughout the paper, we use the following matrix notation. If $U = \{a_{i,j}\}_{0\le i\le m-1; 0\le j\le n-1}$, $V = \{b_{i',j'}\}_{0\le i'\le m'-1; 0\le j'\le n'-1}$ are matrices, their Kronecker product $U \otimes V$ is a matrix $W = \{c_{k,l}\}_{0\le k\le mm'-1; 0\le l\le nn'-1}$, where $c_{i'm+i, j'n+j} = a_{i,j} b_{i',j'}$ for $0 \le i \le m-1, 0 \le j \le n-1, 0 \le$

$i' \leq m'-1, 0 \leq j' \leq n'-1$. Notice that $(U \otimes V)(U' \otimes V') = (UU' \otimes VV')$ and $(U \otimes V)^t = U^t \otimes V^t$.

To make calculations with Kronecker product easier, we sometimes employ the following "matrix-of-matrices" representation. Let R be a ring and let $U^{(i,j)}, 0 \leq i,j \leq n-1$ be matrices from $\mathrm{M}_m(R)$ and $U^{(i,j)} = \{a^{(i,j)}_{k,l}\}_{0 \leq k,l \leq m-1}$. Then the correspondence

$$\{U^{(i,j)}\} \mapsto \{c_{s,t}\}_{0 \leq s,t \leq mn-1}, \quad c_{im+k,jm+l} = a^{(i,j)}_{k,l}$$

is a bijection between $\mathrm{M}_n(\mathrm{M}_m(R))$ and $\mathrm{M}_{mn}(R)$. Remark that if $U \in \mathrm{M}_m(R)$ and $V = \{b_{i,j}\}_{0 \leq i,j \leq n-1} \in M_n(R)$ then $\{Ub_{i,j}\}_{0 \leq i,j \leq n-1} \mapsto U \otimes V$.

Denote $n \times n$ matrices

$$J_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad J'_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad P_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

As usual, the $n \times n$ identity matrix is denoted by I_n, and the $m \times n$ matrix $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$ is denoted by $I_{m,n}$ $(n \leq m)$. Finally, we write δ^j_i for Kronecker's delta, i.e. $\delta^j_i = 1$, if $i = j$, and $\delta^j_i = 0$ otherwise.

1. *Completion of group schemes along the zero section*

Let A be a ring. Let $\mathcal{C}$ be a pro-nilpotent augmented A-algebra, i.e. an A-algebra with augmentation map $\varepsilon : \mathcal{C} \to A$, $\varepsilon(a) = a$ for any $a \in A$, and decreasing chain of ideals $\mathcal{J}_i$ such that Ker $\varepsilon = \mathcal{J}_1$, and $\mathcal{J}_1/\mathcal{J}_i$ is a nilpotent A-algebra for any i. If $\mathcal{C}$ is complete and Hausdorff in the topology defined by $\mathcal{J}_i$, one can define the functor Spf $\mathcal{C}$ from the category of nilpotent A-algebras to the category of sets by Spf $\mathcal{C}(N) = \varinjlim \mathrm{Hom}^*_A(\mathcal{C}/\mathcal{J}_i, A \oplus N)$, where Hom^* is the set of homomorphisms of augmented algebras. A functor from the category of nilpotent A-algebras to the category of sets is called a *formal scheme* over A, if it is isomorphic to Spf $\mathcal{C}$ for some complete Hausdorff pro-nilpotent augmented A-algebra $\mathcal{C}$. Evidently, any formal scheme can be extended to a functor defined on the category of complete Hausdorff pro-nilpotent augmented A-algebras. A continuous homomorphism $h : \mathcal{C}' \to \mathcal{C}$ of complete Hausdorff pro-nilpotent augmented A-algebras defines a morphism Spf h : Spf $\mathcal{C} \to$ Spf $\mathcal{C}'$ in the following way: if $s \in$ Spf $(\mathcal{C})(N)$, then Spf $h(s) = s \circ h \in$ Spf $(\mathcal{C}')(N)$.

Yoneda lemma for formal schemes. *Let $\mathcal{C}$, $\mathcal{C}'$ be complete Hausdorff pro-nilpotent augmented A-algebras, and η : Spf $\mathcal{C} \to$ Spf $\mathcal{C}'$ be a morphism of functors. Then there is a unique continuous homomorphism $h : \mathcal{C}' \to \mathcal{C}$ such that $\eta =$ Spf h.*

Let $\mathcal{H}$ be an A-algebra and $\mathcal{J} \subset \mathcal{H}$ be an ideal in $\mathcal{H}$ such that $\mathcal{H}/\mathcal{J} \cong A$ and $\cap \mathcal{J}^i = 0$. Then $\hat{\mathcal{H}}_{\mathcal{J}} = \varprojlim \mathcal{H}/\mathcal{J}^i$ with the augmentation map $\hat{\mathcal{H}}_{\mathcal{J}} \to \hat{\mathcal{H}}_{\mathcal{J}}/\mathcal{J}_1$ and the chain of ideals $\mathcal{J}_i = \mathrm{Ker}(\hat{\mathcal{H}}_{\mathcal{J}} \to \mathcal{H}_{\mathcal{J}}/\mathcal{J}^i)$ is a complete Hausdorff pro-nilpotent augmented A-algebra. In this case, the formal A-scheme Spf $\hat{\mathcal{H}}_{\mathcal{J}}$ is called the *formal completion of $S =$ Sp $\mathcal{H}$ along $\mathcal{J}$* and is denoted by $\hat{S}_{\mathcal{J}}$. Remark that if i is such that $N^i = 0$, then $\mathrm{Hom}^*_A(\mathcal{H}/\mathcal{J}^i, A \oplus N) = \mathrm{Hom}^*_A(\mathcal{H}, A \oplus N)$. Hence $\hat{S}_{\mathcal{J}}(N)$ can be identified with the subset of $S(A \oplus N) = \mathrm{Hom}_A(\mathcal{H}, A \oplus N)$ consisting of the homomorphisms which map $\mathcal{J}$ in N.

A functor from the category of nilpotent A-algebras to the category of groups is called a *formal group scheme* over A if its composition with the forgetful functor is a formal scheme over A. Due to Yoneda Lemma, giving a formal group scheme structure to a formal scheme Spf $\mathcal{C}$ is equivalent to fixing comultiplication, counit and coinverse in $\mathcal{C}$ which satisfy usual group axioms.

Let G be an affine group scheme over A with Hopf algebra $\mathcal{H}$, and $\mathcal{J}$ denote the augmentation ideal in $\mathcal{H}$. Then $\mathcal{H}/\mathcal{J} \cong A$ and $\cap \mathcal{J}^i = 0$. Thus one can consider the formal completion $\hat{G}_{\mathcal{J}}$ that we denote just by $\hat{G}$. For a nilpotent A-algebra N, the comultiplication in $\mathcal{H}$ induces a group structure on $\hat{G}(N)$ what converts $\hat{G}$ into a formal group scheme. Moreover any morphism $\eta : G \to G'$ of affine group schemes induces a morphism $\hat{\eta} : \hat{G} \to \hat{G}'$ of their formal completions.

2. *Logarithms of formal group laws and their types*

Let A be a ring. We denote by X and Y the sets of variables $x_1, \dots, x_d$ and $y_1, \dots, y_d$, respectively. A d-dimensional *formal group law* over A is a d-tuple of formal power series $F \in A[[X, Y]]^d$ such that

i) $F(X, 0) = 0$;
ii) $F(X, F(Y, Z)) = F(F(X, Y), Z)$;
iii) $F(X, Y) = F(Y, X)$.

Let F and F' be d- and d'-dimensional formal group laws over A. A d'-tuple of formal power series $f \in A[[X]]^{d'}$ is called a *homomorphism* from F to F', if $f(0) = 0$ and $f(F(X, Y)) = F'(f(X), f(Y))$. The matrix $D \in \mathrm{M}_{d',d}(A)$ such that $f(X) = DX \mod \deg 2$ is called the *linear coefficient* of f. Formal group laws are called *strongly isomorphic*, if there exists an isomorphism between them whose linear coefficient is the identity matrix.

If G is a smooth affine group scheme over A with Hopf algebra $\mathcal{H}$ and augmentation ideal $\mathcal{J}$, then any set of elements $x_1, ..., x_d \in \mathcal{J}$ such that $x_1 + \mathcal{J}^2, ..., x_d + \mathcal{J}^2$ form a free A-basis of $\mathcal{J}/\mathcal{J}^2$, gives rise to an isomorphism between $\hat{\mathcal{H}}_{\mathcal{J}}$ and $A[[X]]$ which provides a formal group scheme structure to Spf $A[[X]]$. The images of $x_1, ..., x_d \in A[[X]]$ with respect to the comultiplication form a d-tuple of elements of $A[[X, Y]]$ which is a d-dimensional formal group law over A. Thus, for example, the element $x - 1$ of the augmentation ideal in the Hopf algebra $\mathbb{Z}[x, x']/(1 - xx')$ of the multiplicative group scheme $\mathbb{G}_m$ over $\mathbb{Z}$, induces the multiplicative formal group law $\mathbb{F}_m(x, y) = x + y + xy$.

For a morphism $\eta\colon G \to G'$ of smooth affine group schemes over A, denote by C the matrix of the A-module homomorphism from $\mathcal{J}'/\mathcal{J}'^2$ to $\mathcal{J}/\mathcal{J}^2$ in the bases $x'_1 + \mathcal{J}'^2, \dots, x'_{d'} + \mathcal{J}'^2$ and $x_1 + \mathcal{J}^2, \dots, x_d + \mathcal{J}^2$. Then the linear coefficient of the formal group law homomorphism corresponding to $\hat{\eta}$ in the same bases is C^t, i.e. the transposed of C.

If $\lambda \in A[[X]]^d$ and $\lambda(X) \equiv X \mod \deg 2$, then there exists a unique inverse under composition λ^{-1}. In this case, $F_\lambda(X, Y) = \lambda^{-1}(\lambda(X) + \lambda(Y))$ is a d-dimensional formal group law over A, and $\lambda \in \mathrm{Hom}_A(F_\lambda, (\mathbb{F}_a)^d_A)$, where $\mathbb{F}_a(x, y) = x + y$ is the additive formal group law over $\mathbb{Z}$.

Let $\lambda \in A[[X]]^d$, $\lambda(X) \equiv X \mod \deg 2$ and $\lambda' \in A[[X']]^{d'}$, $\lambda'(X') \equiv X' \mod \deg 2$, where X' is the set of variables $x'_1, \dots, x'_{d'}$. If $D \in \mathrm{M}_{d',d}(A)$, then $\lambda'^{-1} \circ D\lambda \in A[[X]]^{d'}$ is a homomorphism from F_λ to $F_{\lambda'}$.

Proposition 2.1 ([**Ho2**, Theorem 1]). *For any d-dimensional formal group law F over a $\mathbb{Q}$-algebra $\mathcal{A}$, there exists a unique $\lambda \in \mathrm{Hom}_{\mathcal{A}}(F, (\mathbb{F}_a)^d_{\mathcal{A}})$ such that $\lambda(X) \equiv X \mod \deg 2$.*

Let A be a ring of characteristic 0. Then $\mathcal{A} = A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-algebra. If F is a formal group law over A, then applying Proposition 2.1 to $F_{\mathcal{A}}$ we obtain $\lambda \in \mathcal{A}[[X]]^d$ which is called the *logarithm* of F. The logarithm of $\mathbb{F}_m$ is $\mathbb{L}_m(x) = \sum_{i=1}^{\infty} (-1)^{i+1} x^i / i$.

Proposition 2.2 ([**Ho2**, Proposition 1.6]). *Let F and F' be d- and d'-dimensional formal group laws over A with logarithms λ and λ', respectively, and $f \in \mathrm{Hom}_A(F, F')$, then $f = \lambda'^{-1} \circ D\lambda$, where $D \in \mathrm{M}_{d',d}(A)$ is the linear coefficient of f.*

Let k be a finite unramified extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}_k$ and Frobenius automorphism σ. Denote $\mathcal{K}_d = k[[X]]$ and let $\blacktriangle : \mathcal{K}_d \to \mathcal{K}_d$ be a $\mathbb{Q}_p$-algebra map defined by $\blacktriangle(x_i) = x_i^p$ and $\blacktriangle(a) = \sigma(a)$, where $a \in k$. Let $\mathcal{E} = \mathcal{O}_k[[\blacktriangle]]$ be a noncommutative $\mathbb{Q}_p$-algebra with multiplication rule $\blacktriangle a = \sigma(a)\blacktriangle$, $a \in \mathcal{O}_k$. Then $\mathcal{K}_d$ has a left $\mathcal{E}$-module structure which induces a left $\mathrm{M}_d(\mathcal{E})$-module structure on $\mathcal{K}_d^d$.

If $u \in \mathrm{M}_d(\mathcal{E})$, $u \equiv pI_d \mod \blacktriangle$ and $\lambda \in \mathcal{K}_d^d$ are such that $u\lambda \equiv 0 \mod p$, we say that λ is *of type u*. Clearly, if $u \in \mathrm{M}_d(\mathcal{E})$, $u \equiv pI_d \mod \blacktriangle$, then $(u^{-1}p)(\mathrm{id}) \in \mathcal{K}_d^d$ is of type u, and $((u^{-1}p)(\mathrm{id})) \equiv X \mod \deg 2$. Remark also that $\mathbb{L}_m$ is of type $p - \blacktriangle$.

Proposition 2.3 ([**Ho2**, Theorem 2, Proposition 3.3, Theorem 3]).
(i) *If $\lambda \in \mathcal{K}_d^d$ is of type u, then λ is the logarithm of a formal group law over $\mathcal{O}_k$.*
(ii) *For any formal group law F over $\mathcal{O}_k$ with logarithm $\lambda \in \mathcal{K}_d^d$ there exists $u \in \mathrm{M}_d(\mathcal{E})$ such that λ is of type u.*
(iii) *Let F, F' be d- and d'-dimensional formal group laws over $\mathcal{O}_k$ with logarithms λ, λ' of type u, u', respectively, and $D \in \mathrm{M}_{d',d}(\mathcal{O}_k)$. Then $\lambda'^{-1} \circ D\lambda \in \mathrm{Hom}_{\mathcal{O}_k}(F, F')$ if and only if there exists $w \in \mathrm{M}_{d',d}(\mathcal{E})$ such that $u'D = wu$.*

We will also use the following technical result.

Lemma 2.4 ([**Ho2**, Lemma 2.3]). *If $v \in \mathcal{E}$, $\lambda \in \mathcal{K}_1$ is of type $u \in \mathcal{E}$ and $\varphi \in O_k[[X]]$, then $v(\lambda \circ \varphi) \equiv (v\lambda) \circ \varphi \mod p$.*

If $\lambda \in \mathbb{Q}[[X]]^d$ and p is a prime, we call a type of λ considered as an element of $\mathbb{Q}_p[[X]]^d$ its *p-type*. To avoid confusion in this case, instead of $\blacktriangle$ and $\mathcal{E}$, we write $\blacktriangle_p$ and $\mathcal{E}_p$.

Let Ξ be a map from the set of prime numbers to $\mathrm{M}_d(\mathbb{Z})$ such that any two matrices from its image commute. If $p_1, \ldots, p_m$ are distinct primes and $l = \prod_{i=1}^m p_i^{j_i}$, put $A_l = \prod_{i=1}^m \Xi(p_i)^{j_i}$ and $\lambda_\Xi = \sum_{l=1}^\infty A_l X^l / l \in \mathbb{Q}[[X]]$. Finally, define $F_\Xi(X, Y) = \lambda_\Xi^{-1}(\lambda_\Xi(X) + \lambda_\Xi(Y))$.

Proposition 2.5 ([**Ho2**, Theorem 8]). *The formal power series λ_Ξ is of p-type $pI_d - \Xi(p)\blacktriangle_p \in \mathrm{M}_d(\mathcal{E}_p)$, and F_Ξ is a formal group law over $\mathbb{Z}$.*

Proof. For any prime p, we have

$$(pI_d - \Xi(p)\blacktriangle_p)\lambda_\Xi = (pI_d - A_p\blacktriangle_p)\left(\sum_{(l,p)=1}\sum_{i=0}^{\infty} \frac{A_l A_p^i}{lp^i} X^{lp^i}\right)$$

$$= \sum_{(l,p)=1} \frac{pA_l}{l} X^l \equiv 0 \mod p.$$

Thus λ_Ξ is of the required p-type. By Proposition 2.3 (i), F_Ξ is a formal group law over $\mathbb{Z}_p$. Since it is true for any prime p, it is defined over $\mathbb{Z}$. $\square$

3. *Universal fixed pairs for group actions on formal group laws*

The action of a group $\mathcal{G}$ on a formal group law Φ over a ring A is given by a homomorphism $\mathcal{G} \to \mathrm{Aut}_A(\Phi)$. By abuse of notation, we denote the image of $\sigma \in \mathcal{G}$ under this homomorphism also by σ.

Let F be a formal group law over A and $f \in \mathrm{Hom}_A(F, \Phi)$. A pair (F, f) is called *fixed* for $(\Phi, \mathcal{G})$ if $\sigma \circ f = f$ for any $\sigma \in \mathcal{G}$. Yoneda Lemma implies that a pair (F, f) is fixed if and only if for any nilpotent A-algebra N, we have $\mathrm{Im}\ f(N) \subset \Phi(N)^{\mathcal{G}}$.

A fixed pair (F, f) is called *universal* if for any fixed pair (F', f'), there exists a unique $g \in \mathrm{Hom}_A(F', F)$ such that $f' = f \circ g$. Clearly, if (F, f) and $(\tilde{F}, \tilde{f})$ are universal fixed pairs, then $g \in \mathrm{Hom}_A(\tilde{F}, F)$ satisfying $\tilde{f} = f \circ g$ is an isomorphism.

Proposition 3.1. *If (F, f) is a fixed pair such that for any nilpotent A-algebra N the map $f(N) : F(N) \to \Phi(N)^{\mathcal{G}}$ is bijective, then it is universal.*

Proof. We notice that for any nilpotent A-algebra N and any fixed pair (F', f'), there exists a unique homomorphism $g(N) : F'(N) \to F(N)$ such that $f(N) \circ g(N) = f'(N)$, and that $g(N)$ is functorial in N. Then the statement follows from Yoneda Lemma. □

Not every universal fixed pair satisfies the condition of Proposition 3.1. However the universal fixed pairs which appear in the context of the present paper do so (see Theorem 2).

Proposition 3.2. *Let $\mathcal{G}$ act on formal group laws Φ and Φ', and $\varphi \in \mathrm{Hom}_A(\Phi, \Phi')$ commute with the actions of $\mathcal{G}$. If (F, f) and (F', f') are universal fixed pairs for $(\Phi, \mathcal{G})$ and $(\Phi', \mathcal{G})$, respectively, then there exists a unique $\tilde{\varphi} \in \mathrm{Hom}_A(F, F')$ such that $\varphi \circ f = f' \circ \tilde{\varphi}$.*

Proof. For any $\sigma \in \mathcal{G}$, we have $\sigma \circ \varphi \circ f = \varphi \circ \sigma \circ f = \varphi \circ f$. Hence, $(F, \varphi \circ f)$ is a fixed pair for $(\Phi', \mathcal{G})$. Therefore, there exists a unique $\tilde{\varphi} \in \mathrm{Hom}_A(F, F')$ such that $\varphi \circ f = f' \circ \tilde{\varphi}$. □

Let $\mathcal{G}_1, \mathcal{G}_2$ act on Φ and $\sigma_1\sigma_2 = \sigma_2\sigma_1$ for any $\sigma_1 \in \mathcal{G}_1, \sigma_2 \in \mathcal{G}_2$. If (F_1, f_1) is a universal fixed pair for $(\Phi, \mathcal{G}_1)$ then for any $\sigma_2 \in \mathcal{G}_2$ we have $\sigma_1 \circ (\sigma_2 \circ f_1) = \sigma_2 \circ (\sigma_1 \circ f_1) = \sigma_2 \circ f_1$ for any $\sigma_1 \in \mathcal{G}_1$. Thus there exists a unique $\sigma_2' \in \mathrm{Aut}_A(F_1)$ such that $f_1 \circ \sigma_2' = \sigma_2 \circ f_1$. It induces an action of $\mathcal{G}_2$ on F_1.

Proposition 3.3. *Let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ act on a formal group law Φ. If (F_1, f_1) is a universal fixed pair for $(\Phi, \mathcal{G}_1)$ and (F_2, f_2) is a universal fixed pair for $(F_1, \mathcal{G}_2)$ with respect to the induced action of $\mathcal{G}_2$ on F_1, then $(F_2, f_1 \circ f_2)$ is a universal fixed pair for $(\Phi, \mathcal{G})$.*

Proof. Let (F', f') be a fixed pair for $(\Phi, \mathcal{G})$, i.e. $\sigma \circ f' = f'$ for any $\sigma \in \mathcal{G}$. It implies that (F', f') is a fixed pair for $(\Phi, \mathcal{G}_1)$ thus giving $g_1 \in \mathrm{Hom}_A(F', F_1)$ such that $f' = f_1 \circ g_1$. Then for any $\sigma_2 \in \mathcal{G}_2$, we have $f_1 \circ (\sigma_2' \circ g_1) = \sigma_2 \circ f_1 \circ g_1 = \sigma_2 \circ f' = f' = f_1 \circ g_1$ which implies $\sigma_2' \circ g_1 = g_1$ by universality of (F_1, f_1). It gives $g_2 \in \mathrm{Hom}_A(F', F_2)$ such that $g_1 = f_2 \circ g_2$. Thus $(f_1 \circ f_2) \circ g_2 = f_1 \circ g_1 = f'$.

The uniqueness of g_2 can be easily checked in a similar way. □

In the rest of this section, equivalent sufficient conditions for a formal group law with a group action to have a universal fixed pair are given. We do it in the special case of a formal group law over an unramified extension of $\mathbb{Z}_p$ (equivalently, the ring of Witt vectors over a finite field) employing the technique of Honda mentioned in the previous section.

Let R be a ring and $D \in \mathrm{M}_{m,n}(R)$. By abuse of notation, $\mathrm{Ker}(D)$ stands for the free sub-R-module of R^n given by all $x \in R^n$ such that $Dx^t = 0$.

Let k be a finite unramified extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}_k$ and residue field $\bar{k}$.

LEMMA 3.4. *Let $\mathcal{G}'$ be a set of matrices over $\mathcal{O}_k$ with the same number of columns n such that $\cap_{D' \in \mathcal{G}'} \mathrm{Ker}\left(D' \otimes \bar{k}\right) = \{0\}$. Then there exist $D'_i \in \mathcal{G}'$, $D'_i \in \mathrm{M}_{n_i,n}(\mathcal{O}_k)$ and $C_i \in \mathrm{M}_{n,n_i}(\mathcal{O}_k)$ for $1 \leq i \leq m$ such that $\sum_{i=1}^m C_i D'_i \in \mathrm{GL}_n(\mathcal{O}_k)$.*

Proof. It is enough to prove that if $\overline{\mathcal{G}}$ is a set of matrices over $\bar{k}$ with the same number of columns n such that $\cap_{\overline{D} \in \overline{\mathcal{G}}} \mathrm{Ker}\ \overline{D} = \{0\}$, there exist $\overline{D}_i \in \overline{\mathcal{G}}$, $\overline{D}_i \in \mathrm{M}_{n_i,n}(\bar{k})$, $1 \leq i \leq m$, and $\overline{C}_i \in \mathrm{M}_{n,n_i}(\bar{k})$, $1 \leq i \leq m$, such that $\sum_{i=1}^m \overline{C}_i \overline{D}_i \in \mathrm{GL}_n(\bar{k})$. Since we are in a finte dimensional vector space, there exists a finite set of matrices $\overline{D}_1, \ldots, \overline{D}_m \in \overline{\mathcal{G}}$ whose kernels have zero intersection. Construct a matrix D^* with the rows of these matrices. Since $\cap_{i=1}^m \mathrm{Ker}\ \overline{D}_i = \{0\}$, the rank of D^* is equal to n. Then there is a matrix C^* with n rows such that $C^* D^* = I_n$. The required matrices $\overline{C}_1, \ldots, \overline{C}_m$ are formed by the corresponding columns of C^*. $\square$

THEOREM 1. I. *Let $\mathcal{G}^* \subset \mathrm{M}_r(\mathcal{O}_k)$. The following conditions are equivalent*
(i) $\mathrm{rk}_{\mathcal{O}_k} \cap_{D \in \mathcal{G}^*} \mathrm{Ker}\ D = \dim_{\bar{k}} \cap_{D \in \mathcal{G}^*} \mathrm{Ker}\left(D \otimes \bar{k}\right)$;
(ii) *If $x \in \mathcal{O}_k^r$ is such that for any $D \in \mathcal{G}^*$ there is $y_D \in \mathcal{O}_k^r$ satisfying $Dx = py_D$, then there exists $x' \in \mathcal{O}_k^r$ such that $Dx' = y_D$ for any $D \in \mathcal{G}^*$;*
(iii) *There exist $0 \leq e \leq r$ and $Q \in \mathrm{GL}_r(\mathcal{O}_k)$ such that for any $D \in \mathcal{G}^*$*

$$Q^{-1} D Q = \begin{pmatrix} 0 & \hat{D} \\ 0 & \tilde{D} \end{pmatrix}, \qquad \hat{D} \in \mathrm{M}_{e,r-e}(\mathcal{O}_k),\ \tilde{D} \in \mathrm{M}_{r-e}(\mathcal{O}_k),$$

and there are $D_i \in \mathcal{G}^*$, $\hat{C}_i \in \mathrm{M}_{r-e,e}(\mathcal{O}_k)$ and $\tilde{C}_i \in \mathrm{M}_{r-e}(\mathcal{O}_k)$, $1 \leq i \leq m$, such that $\sum_{i=1}^m \hat{C}_i \hat{D}_i + \tilde{C}_i \tilde{D}_i \in \mathrm{GL}_{r-e}(\mathcal{O}_k)$.

II. *Let Φ be an r-dimensional formal group law over $\mathcal{O}_k$ with logarithm Λ. Let a group $\mathcal{G}$ act on Φ and $\mathcal{G}^* = \{D \in \mathrm{M}_r(\mathcal{O}_k) : \Lambda^{-1} \circ (D + I_r)\Lambda \in \mathcal{G}\}$. If $\mathcal{G}^*$ satisfies one of the above equivalent conditions then $(\Phi, \mathcal{G})$ has a universal fixed pair. Moreover, if Λ is of type v, then a universal fixed pair (F, f) can be taken in such a way that the linear coefficient of f is $QI_{r,e}$, and the logarithm of F is of type u, where u is the upper-left $e \times e$-submatrix of $\tilde{v} = Q^{-1} v Q$.*

Proof. I. Remark that $\cap_{D \in \mathcal{G}^*} \mathrm{Ker}\ D$ is a p-divisible $\mathcal{O}_k$-module. Since the reduction of $\cap_{D \in \mathcal{G}^*} \mathrm{Ker}\ D$ is a subspace of $\cap_{D \in \mathcal{G}^*} \mathrm{Ker}\left(D \otimes \bar{k}\right)$, condition (i) is equivalent to the fact that they coincide. Obviously, the latter is equivalent to condition (ii).

Now suppose that for any $D \in \mathcal{G}^*$

$$\begin{pmatrix} 0 & \hat{D} \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} p\hat{y}_D \\ p\tilde{y}_D \end{pmatrix}, \qquad \hat{x}, \hat{y}_D \in \mathcal{O}_k^e,\ \tilde{x}, \tilde{y}_D \in \mathcal{O}_k^{r-e}.$$

Then $\hat{D}\tilde{x} = p\hat{y}_D$, $\tilde{D}\tilde{x} = p\tilde{y}_D$ and for $\tilde{C}_i, \hat{C}_i$ provided by condition (iii) we have $(\sum_{i=1}^m \hat{C}_i \hat{D}_i + \tilde{C}_i \tilde{D}_i)\tilde{x} = p \sum_{i=1}^m \hat{C}_i \hat{y}_{D_i} + \tilde{C}_i \tilde{y}_{D_i}$ which implies that $\tilde{x} = p\tilde{x}'$ for some $\tilde{x}' \in \mathcal{O}_k^{r-e}$. It gives for any $D \in \mathcal{G}^*$

$$\begin{pmatrix} 0 & \hat{D} \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{x}' \end{pmatrix} = \begin{pmatrix} \hat{y}_D \\ \tilde{y}_D \end{pmatrix}$$

as required.

For the reverse implication, take $e = \mathrm{rk}_{\mathcal{O}_k} \cap_{D\in\mathcal{G}^*} \mathrm{Ker}\ D$ and let Q be the transition matrix from a basis of $\mathcal{O}_k^r$ with first e vectors from $\cap_{D\in\mathcal{G}^*} \mathrm{Ker}\ D$ to the standard basis. Then for any $D \in \mathcal{G}^*$, we have

$$Q^{-1}DQ = \begin{pmatrix} 0 & \hat{D} \\ 0 & \tilde{D} \end{pmatrix}, \qquad \hat{D} \in \mathrm{M}_{e,r-e}(\mathcal{O}_k),\ \tilde{D} \in \mathrm{M}_{r-e}(\mathcal{O}_k).$$

Denote $\mathcal{G}' = \{\tilde{D} \in \mathrm{M}_{r-e}(\mathcal{O}_k) \colon D \in \mathcal{G}^*\} \cup \{\hat{D} \in \mathrm{M}_{e,r-e}(\mathcal{O}_k) \colon D \in \mathcal{G}^*\}$.

If $\tilde{x} \in \bar{k}^{r-e}$ and $\tilde{x} \in \cap_{D'\in\mathcal{G}'} \mathrm{Ker}\left(D' \otimes \bar{k}\right)$ then for any $D \in \mathcal{G}^*$

$$\begin{pmatrix} 0 & \hat{D} \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies that

$$\overline{x} = Q \begin{pmatrix} 0 \\ \tilde{x} \end{pmatrix} \in \cap_{D\in\mathcal{G}^*} \mathrm{Ker}\left(D \otimes \bar{k}\right).$$

Then $\overline{x}$ is the reduction of an element from $\cap_{D\in\mathcal{G}^*} \mathrm{Ker}\ D$ by condition (i). It gives $\tilde{x} = 0$, i.e. $\cap_{D'\in\mathcal{G}'} \mathrm{Ker}\left(D' \otimes \bar{k}\right) = \{0\}$. It remains to apply Lemma 3.4.

II. If $\varphi(X) = QX \in \mathcal{K}_r^r$ then $\tilde{\Lambda} = \varphi^{-1} \circ \Lambda \circ \varphi$ is of type $\tilde{v}$. By Proposition 2.3 (i), $\tilde{\Lambda}$ is the logarithm of a formal group law $\tilde{\Phi}$ over $\mathcal{O}_k$, and $\varphi \in \mathrm{Hom}_{\mathcal{O}_k}(\tilde{\Phi}, \Phi)$. Furthermore, $\varphi^{-1} \circ \sigma \circ \varphi \in \mathrm{Aut}_{\mathcal{O}_k} \tilde{\Phi}$ for any $\sigma \in \mathcal{G}$ which defines an action of $\mathcal{G}$ on $\tilde{\Phi}$. If $D + I_r$ is the linear coefficient of σ, then the linear coefficient of $\varphi^{-1} \circ \sigma \circ \varphi$ is equal to $(Q^{-1}DQ + I_r)$. Therefore Proposition 2.3 (iii) implies the existence of $w \in \mathrm{M}_r(\mathcal{E})$ such that $\tilde{v}Q^{-1}DQ = w\tilde{v}$. For $1 \leq i \leq m$, let w_i correspond to D_i and

$$\tilde{v} = \begin{pmatrix} u & * \\ \tilde{u} & * \end{pmatrix}, \quad w_i = \begin{pmatrix} \hat{z}_i & \hat{w}_i \\ \tilde{z}_i & \tilde{w}_i \end{pmatrix},$$

where $u, \hat{z}_i \in \mathrm{M}_e(\mathcal{E})$, $\tilde{u}, \tilde{z}_i \in \mathrm{M}_{r-e,e}(\mathcal{E})$, $\hat{w}_i \in \mathrm{M}_{e,r-e}(\mathcal{E})$, $\tilde{w}_i \in \mathrm{M}_{r-e}(\mathcal{E})$. Then

$$\begin{aligned} \sum_{i=1}^{m} \begin{pmatrix} \hat{C}_i & \tilde{C}_i \end{pmatrix} w_i \tilde{v} &= \sum_{i=1}^{m} \begin{pmatrix} \hat{C}_i & \tilde{C}_i \end{pmatrix} \begin{pmatrix} \hat{z}_i & \hat{w}_i \\ \tilde{z}_i & \tilde{w}_i \end{pmatrix} \begin{pmatrix} u & * \\ \tilde{u} & * \end{pmatrix} \\ &= \sum_{i=1}^{m} \begin{pmatrix} \hat{C}_i & \tilde{C}_i \end{pmatrix} \begin{pmatrix} u & * \\ \tilde{u} & * \end{pmatrix} \begin{pmatrix} 0 & \hat{D}_i \\ 0 & \tilde{D}_i \end{pmatrix} = \begin{pmatrix} 0 & * \end{pmatrix}. \end{aligned}$$

It gives $\sum_{i=1}^m (\hat{C}_i\hat{z}_i + \tilde{C}_i\tilde{z}_i)u + \sum_{i=1}^m (\hat{C}_i\hat{w}_i + \tilde{C}_i\tilde{w}_i)\tilde{u} = 0$. Since $\hat{w}_i \equiv \hat{D}_i \mod \blacktriangle$ and $\tilde{w}_i \equiv \tilde{D}_i \mod \blacktriangle$, we obtain that $\sum_{i=1}^m \hat{C}_i\hat{w}_i + \tilde{C}_i\tilde{w}_i \in \mathrm{M}_{r-e}(\mathcal{E})$ is invertible, and hence, $\tilde{u} = zu$ for some $z \in \mathrm{M}_{r-e,e}(\mathcal{E})$.

Let $\lambda \in \mathcal{K}_e^e$ be of type u and $\lambda(X) \equiv X \mod \deg 2$. Then by Proposition 2.3 (i), λ is the logarithm of a formal group law F over $\mathcal{O}_k$. Since

$$\tilde{v}I_{r,e} = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} I_e \\ z \end{pmatrix} u,$$

Proposition 2.3 (iii) implies that $\tilde{f} = \tilde{\Lambda}^{-1} \circ I_{r,e}\lambda \in \mathrm{Hom}_{\mathcal{O}_k}(F, \tilde{\Phi})$. Moreover, $(Q^{-1}DQ + I_r)I_{r,e} = I_{r,e}$ for any $D \in \mathcal{G}^*$, and hence, by Proposition 2.2, we get $(\varphi^{-1} \circ \sigma \circ \varphi) \circ \tilde{f} = \tilde{f}$ for any $\sigma \in \mathcal{G}$. Thus $(F, \tilde{f})$ is a fixed pair for $(\tilde{\Phi}, \mathcal{G})$.

Now let (F', f') be another fixed pair for $(\tilde{\Phi}, \mathcal{G})$ and the linear coefficient of f' be

$$Z = \begin{pmatrix} \tilde{Z} \\ \hat{Z} \end{pmatrix} \in \mathrm{M}_{r,e'}(\mathcal{O}_k), \qquad \text{where} \quad \tilde{Z} \in \mathrm{M}_{e,e'}(\mathcal{O}_k),\ \hat{Z} \in \mathrm{M}_{r-e,e'}(\mathcal{O}_k).$$

Then $(Q^{-1}DQ + I_r)Z = Z$, and therefore, $\hat{D}_i\hat{Z} = 0$ and $\tilde{D}_i\hat{Z} = 0$, $1 \leq i \leq m$. Since $\sum_{i=1}^m \hat{C}_i\hat{D}_i + \tilde{C}_i\tilde{D}_i$ is invertible, $\hat{Z} = 0$, and hence, $Z = I_{r,e}\tilde{Z}$. According to Proposition 2.3 (ii), there exists $u' \in \mathrm{M}_{e'}(\mathcal{E})$ such that the logarithm λ' of F' is of type u'. By Proposition 2.3

(iii), we have $\tilde{v}Z = w'u'$ for some

$$w' = \begin{pmatrix} \tilde{w}' \\ * \end{pmatrix} \in \mathrm{M}_{r,e'}(\mathcal{E}), \qquad \text{where} \quad \tilde{w}' \in \mathrm{M}_{e,e'}(\mathcal{E}).$$

Then $u\tilde{Z} = \tilde{w}'u'$, and Proposition 2.3 (iii) implies that $g = \lambda^{-1} \circ \tilde{Z}\lambda' \in \mathrm{Hom}_{\mathcal{O}_k}(F', F)$. Besides, by Proposition 2.2, we have $f' = \tilde{f} \circ g$.

If $g' \in \mathrm{Hom}_{\mathcal{O}_k}(F', F)$ is such that $f' = \tilde{f} \circ g'$, and $Z' \in M_{e,e'}(\mathcal{O}_k)$ is the linear coefficient of g', then $I_{r,e}Z' = Z = I_{r,e}\tilde{Z}$, and hence, $Z' = \tilde{Z}$. Then Proposition 2.2 implies that $g' = g$, and thus $(F, \tilde{f})$ is a universal fixed pair for $(\tilde{\Phi}, \mathcal{G})$.

Finally, let $f = \varphi \circ \tilde{f}$. Then $f \in \mathrm{Hom}_{\mathcal{O}_k}(F, \Phi)$, the linear coefficient of f is $QI_{r,e}$, and (F, f) is a universal fixed pair for $(\Phi, \mathcal{G})$. □

Consider a simple example illustrating Theorem 1. Let Φ be a 2-dimensional formal group law over $\mathbb{Z}_p$, $p \neq 2$, with the logarithm of type $v = pI_2 - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \blacktriangle$. Let the action of the group $\mathbb{Z}/2\mathbb{Z}$ on Φ be defined by the condition that the linear coefficient of the non-identity element is equal to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D + I_2$. Then $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ satisfies condition (iii) of Theorem 1, and thus $(\Phi, \mathbb{Z}/2\mathbb{Z})$ admits a universal fixed pair (F, f) such that the linear coefficient of f is $\begin{pmatrix} 1 & 1 \end{pmatrix}^t$, and the logarithm of F is of type $u = p - 2\blacktriangle$.

Alternatively, if the action of the group $\mathbb{Z}/2\mathbb{Z}$ on Φ is given by the condition that the linear coefficient of of the non-identity element is equal to the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = D' + I_2$, then condition (iii) of Theorem 1 is satisfied for $Q' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Here $(\Phi, \mathbb{Z}/2\mathbb{Z})$ admits a universal fixed pair (F', f') such that the linear coefficient of f' is $\begin{pmatrix} 1 & -1 \end{pmatrix}^t$, and the logarithm of F' is of type $u' = p$, i.e. F' is isomorphic to $\mathbb{F}_a$.

Proposition 3.5. *Let Φ be a formal group law over $\mathbb{Z}$ provided with an action of a group $\mathcal{G}$, and for any prime p, the logarithm of Φ be of p-type v_p. If there exist $Q \in \mathrm{GL}_r(\mathbb{Z})$ and a formal group law F over $\mathbb{Z}$ such that for any prime p, Q satisfies condition* (iii) *of Theorem 1,*I *for $k = \mathbb{Q}_p$, and the logarithm of F is of p-type u_p, where u_p is the upper-left $e \times e$-submatrix of $Q^{-1}v_pQ$, then there exists $f \in \mathrm{Hom}_{\mathbb{Z}}(F, \Phi)$ such that the linear coefficient of f is equal to $QI_{r,e}$ and (F, f) is a universal fixed pair for $(\Phi, \mathcal{G})$.*

Proof. By Theorem 1 for any prime p, there exists a universal fixed pair (F_p, f_p) for $(\Phi, \mathcal{G})$ such that the logarithm λ_p of F_p is of type u_p, and the linear coefficient of f_p is $QI_{r,e}$. Proposition 2.3 (iii) implies that $v_pQI_{r,e} = w_pu_p$ for some $w_p \in \mathrm{M}_{r,e}(\mathcal{E}_p)$, and then $f = \Lambda^{-1} \circ QI_{r,e}\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(F, \Phi)$, where Λ and λ denote the logarithms of Φ and F, respectively. Since it is true for any prime p, we have $f \in \mathrm{Hom}_{\mathbb{Z}}(F, \Phi)$. Moreover by Proposition 2.2, we get $f = f_p \circ (\lambda_p^{-1} \circ \lambda)$, and hence, $\sigma \circ f = \sigma \circ f_p \circ (\lambda_p^{-1} \circ \lambda) = f$ for any $\sigma \in \mathcal{G}$. Thus (F, f) is a fixed pair for $(\Phi, \mathcal{G})$.

Let (F', f') be another fixed pair for $(\Phi, \mathcal{G})$. For every prime p, there exists $g_p \in \mathrm{Hom}_{\mathbb{Z}_p}(F', F_p)$ such that $f' = f_p \circ g_p$. Denote the linear coefficients of f' and g_p by $Z \in \mathrm{M}_{r,e'}(\mathbb{Z})$ and $Z_p \in \mathrm{M}_{e,e'}(\mathbb{Z}_p)$, respectively, and let

$$Q^{-1}Z = \begin{pmatrix} \tilde{Z} \\ \hat{Z} \end{pmatrix} \in \mathrm{M}_{r,e'}(\mathbb{Z}), \qquad \text{where} \quad \tilde{Z} \in \mathrm{M}_{e,e'}(\mathbb{Z}),\ \hat{Z} \in \mathrm{M}_{r-e,e'}(\mathbb{Z}).$$

Then $Z = QI_{r,e}Z_p$ implies $\hat{Z} = 0$ and $\tilde{Z} = Z_p$, in particular, the entries of Z_p are in $\mathbb{Z}$. According to Proposition 2.3 (ii), for every prime p, there exists $u'_p \in \mathrm{M}_{e'}(\mathcal{E}_p)$ such that the logarithm λ' of F' is of type u'_p. By Proposition 2.3 (iii), we have $u_p\tilde{Z} = w'_p u'_p$ for some $w'_p \in \mathrm{M}_{e,e'}(\mathcal{E}_p)$, and then $g = \lambda^{-1} \circ \tilde{Z}\lambda' \in \mathrm{Hom}_{\mathbb{Z}_p}(F', F)$. Since it is true for any prime p, we get $g \in \mathrm{Hom}_{\mathbb{Z}}(F', F)$. Besides, by Proposition 2.2, we have $f' = f \circ g$.

If $g' \in \mathrm{Hom}_{\mathbb{Z}}(F', F)$ is such that $f' = f \circ g'$, and $Z' \in \mathrm{M}_{e,e'}(\mathbb{Z})$ is the linear coefficient of g', then $QI_{r,e}Z' = Z = QI_{r,e}\tilde{Z}$, and hence, $Z' = \tilde{Z}$. Then Proposition 2.2 implies that $g' = g$, and thus (F, f) is a universal fixed pair for $(\Phi, \mathcal{G})$. □

4. *Weil restriction for formal group laws*

Let A be a ring, B be an A-algebra which is a free A-module of finite rank with basis $e_0, \dots, e_{n-1}$ that we fix throughout this section. For a positive integer d, denote $B_d = B[[x_1, \dots, x_d]]$. Define the *Weil restriction* functor $\mathcal{R}_{B/A}(\mathrm{Spf}\ B_d)$ from the category of nilpotent A-algebras to the category of sets by $\mathcal{R}_{B/A}(\mathrm{Spf}\ B_d)(N) = \mathrm{Spf}\ B_d(N \otimes_A B)$. Denote $A_d = A[[z_1, \dots, z_{nd}]]$. For any nilpotent A-algebra N, the map

$$\rho_d(N) : \mathrm{Spf}\ A_d(N) \to \mathcal{R}_{B/A}(\mathrm{Spf}\ B_d)(N)$$

defined by $\Big(\rho_d(N)(s)\Big)(x_l) = \sum_{i=0}^{n-1} s(z_{id+l}) \otimes e_i$, $1 \le l \le d$, is a bijection. Therefore, it gives an isomorphism $\rho_d : \mathrm{Spf}\ A_d \to \mathcal{R}_{B/A}(\mathrm{Spf}\ B_d)$ that depends on the fixed basis. It implies, in particular, that $\mathcal{R}_{B/A}(\mathrm{Spf}\ B_d)$ is a formal scheme over A. Let $f : B_{d'} \to B_d$ be a continuous homomorphism of B-algebras. Denote by $\mathcal{R}_{B/A}(f) : A_{d'} \to A_d$ the unique continuous homomorphism of A-algebras such that

$$\sum_{j=0}^{n-1} \mathcal{R}_{B/A}(f)_{jd'+l'}(z_1, \dots, z_{nd})e_j = f_{l'}\left(\sum_{i=0}^{n-1} z_{id+1}e_i, \dots, \sum_{i=0}^{n-1} z_{id+d}e_i\right) \in B \otimes A_d$$

for any $l' = 1, \dots, d'$, where $f_m \in B_d$ and $\mathcal{R}_{B/A}(f)_m \in A_d$ are the images of z_m with respect to f and $\mathcal{R}_{B/A}(f)$, respectively. Certainly, $\mathcal{R}_{B/A}(f)$ also depends on the chosen basis.

Proposition 4.1. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{Spf}\ A_d & \xrightarrow{\mathrm{Spf}\ \mathcal{R}_{B/A}(f)} & \mathrm{Spf}\ A_{d'} \\ \big\downarrow{\scriptstyle \rho_d} & & \big\downarrow{\scriptstyle \rho_{d'}} \\ \mathcal{R}_{B/A}(\mathrm{Spf}\ B_d) & \xrightarrow{\mathcal{R}_{B/A}(\mathrm{Spf}\ f)} & \mathcal{R}_{B/A}(\mathrm{Spf}\ B_{d'}) \end{array}$$

Proof. Let N be a nilpotent A-algebra and $s \in \mathrm{Hom}_A(A_d, N)$. Then

$$\bigg((\mathrm{Spf}\ f)\Big(\rho_d(N)(s)\Big)\bigg)(x_{l'}) = \Big(\rho_d(N)(s)\Big)(f(x_{l'})) = f_{l'}\Big(\rho_d(N)(s)(x_1), \dots, \rho_d(N)(s)(x_d)\Big)$$

$$= f_{l'}\left(\sum_{i=0}^{n-1} s(z_{id+1}) \otimes e_i, \dots, \sum_{i=0}^{n-1} s(z_{id+d}) \otimes e_i\right).$$

for any $l' = 1, \dots, d'$. On the other hand,

$$\bigg(\rho_{d'}(N)\Big((\mathrm{Spf}\ \mathcal{R}_{B/A}(f))(s)\Big)\bigg)(x_{l'}) = \sum_{j=0}^{n-1}\Big((\mathrm{Spf}\ \mathcal{R}_{B/A}(f))(s)\Big)(z_{jd'+l'}) \otimes e_j$$

$$= \sum_{j=0}^{n-1} s\Big(\mathcal{R}_{B/A}(f)(z_{jd'+l'})\Big) \otimes e_j = \sum_{j=0}^{n-1} \mathcal{R}_{B/A}(f)_{jd'+l'}\Big(s(z_1), \dots, s(z_{nd})\Big)e_j.$$

Let F be a d-dimensional formal group law over B. Then F provides a formal group scheme structure to Spf B_d, and hence, also to $\mathcal{R}_{B/A}(\text{Spf } B_d)$. Further, ρ_d allows one to define a formal group scheme structure on Spf A_d, which gives an nd-dimensional formal group law depending on the chosen basis. We denote this formal group law by $\mathcal{R}_{B/A}(F)$.

Proposition 4.2. *If F and F' are formal group laws over B, and $f \in \mathrm{Hom}_B(F, F')$, then $\mathcal{R}_{B/A}(f) \in \mathrm{Hom}_A(\mathcal{R}_{B/A}(F), \mathcal{R}_{B/A}(F'))$.*

Proof. Denote by d, d' the dimensions of F, F', respectively. Let N be a nilpotent A-algebra and $s_1, s_2 \in \mathrm{Hom}_A(A_d, N)$. Then Proposition 4.1 yields

$$\rho_{d'}(N)\Big(\mathcal{R}_{B/A}(f)\big(\mathcal{R}_{B/A}(F)(s_1, s_2)\big)\Big) = f\Big(F\big(\rho_d(N)(s_1), \rho_d(N)(s_2)\big)\Big)$$

$$= F'\Big(f\big(\rho_d(N)(s_1)\big), f\big(\rho_d(N)(s_2)\big)\Big) = \rho_{d'}(N)\Big(\mathcal{R}_{B/A}(F')\big(\mathcal{R}_{B/A}(f)(s_1), \mathcal{R}_{B/A}(f)(s_2)\big)\Big).$$

Suppose that A is of characteristic 0. Then $\mathcal{A} = A \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathcal{B} = B \otimes_{\mathbb{Z}} \mathbb{Q}$ are $\mathbb{Q}$-algebras, and $\mathcal{B}$ is an $\mathcal{A}$-algebra of rank n. We consider $\mathcal{B}$ as a free $\mathcal{A}$-module with the same fixed basis $e_0, \ldots, e_{n-1}$. Denote by λ the logarithm of F.

Proposition 4.3. *The formal series $\mathcal{R}_{\mathcal{B}/\mathcal{A}}(\lambda)$ is the logarithm of $\mathcal{R}_{B/A}(F)$.*

Proof. Denote $\mathcal{B}_d = \mathcal{B}[[x_1, ..., x_d]]$ and $\mathcal{A}_d = \mathcal{A}[[z_1, ..., z_{nd}]]$. Consider the morphism $\rho_d^* :$ Spf $\mathcal{A}_d \to \mathcal{R}_{\mathcal{B}/\mathcal{A}}(\text{Spf } \mathcal{B}_d)$ defined similarly to ρ_d. Since $\mathcal{B} = \mathcal{A} \otimes_A B$, the formal group schemes $\mathcal{R}_{\mathcal{B}/\mathcal{A}}(\text{Spf } \mathcal{B}_d)$ and $\mathcal{R}_{B/A}(\text{Spf } B_d)_{\mathcal{A}}$ coincide as well as the maps ρ_d^* and $(\rho_d)_{\mathcal{A}}$. Hence, $\mathcal{R}_{\mathcal{B}/\mathcal{A}}(F_{\mathcal{B}}) = \mathcal{R}_{B/A}(F)_{\mathcal{A}}$. Besides, it is clear that $\mathcal{R}_{\mathcal{B}/\mathcal{A}}((\mathbb{F}_a)^d_{\mathcal{B}}) = (\mathbb{F}_a)^{nd}_{\mathcal{A}}$. According to Proposition 4.2, we get $\mathcal{R}_{\mathcal{B}/\mathcal{A}}(\lambda) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{R}_{B/A}(F)_{\mathcal{A}}, (\mathbb{F}_a)^{nd}_{\mathcal{A}})$. By definition of $\mathcal{R}_{\mathcal{B}/\mathcal{A}}$, we have $\sum_{j=1}^{n-1} \mathcal{R}_{\mathcal{B}/\mathcal{A}}(\lambda)_{jd+l} e_j \equiv \sum_{i=1}^{n-1} z_{id+l} e_i \mod \deg 2$, and hence, $\mathcal{R}_{\mathcal{B}/\mathcal{A}}(\lambda)_{id+l} \equiv z_{id+l} \mod \deg 2$ for any $l = 1, \ldots, d$ and $i = 0, \ldots, n-1$. Thus $\mathcal{R}_{\mathcal{B}/\mathcal{A}}(\lambda)$ is the logarithm of $\mathcal{R}_{B/A}(F)$. $\square$

The next proposition shows a relation between two Weil restrictions of the same formal group law defined with the aid of two distinct bases.

Proposition 4.4. *Let $e_0, \ldots, e_{n-1}$ and $e'_0, \ldots, e'_{n-1}$ be two free A-bases of B and F be a formal group law over B. If $\rho_d :$ Spf $A_d \to \mathcal{R}_{B/A}(\text{Spf } B_d)$ and $\rho'_d :$ Spf $A'_d \to \mathcal{R}_{B/A}(\text{Spf } B_d)$ are the corresponding maps, $\mathcal{R}_{B/A}(F)$ and $\mathcal{R}'_{B/A}(F)$ are the corresponding formal group laws over A, then there exists $f \in \mathrm{Hom}_A(\mathcal{R}'_{B/A}(F), \mathcal{R}_{B/A}(F))$ with linear coefficient $I_d \otimes W$, where W is the transition matrix from $e'_0, \ldots, e'_{n-1}$ to $e_0, \ldots, e_{n-1}$.*

Proof. Suppose that $W = \{w_{i,j}\}_{0 \le i,j \le n-1} \in \mathrm{GL}_n(A)$, i.e. $e'_j = \sum_{j=0}^{n-1} w_{i,j} e_i$. Define the continuous homomorphism $g : A_d \to A'_d$ as follows: $g(z_{id+l}) = \sum_{j=0}^{n-1} w_{i,j} z'_{jd+l}$ for any $l = 1, \ldots, d$. Let N be a nilpotent A-algebra and $s \in \mathrm{Hom}_A(A'_d, N)$. Then

$$\rho_d(N)(\text{Spf } g(s))(x_l) = \sum_{i=0}^{n-1} s(g(z_{id+l})) \otimes e_i = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} w_{i,j} s(z'_{jd+l}) \otimes e_i.$$

for any $l = 1, \ldots, d$. On the other hand

$$\rho'_d(N)(s)(x_l) = \sum_{j=0}^{n-1} s(z'_{jd+l}) \otimes e'_j = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} s(z'_{jd+l}) \otimes w_{i,j} e_i.$$

Thus $\rho_d \circ (\mathrm{Spf}\ g) = \rho'_d$ and therefore Spf g induces a homomorphism from $\mathcal{R}'_{B/A}(F)$ to $\mathcal{R}_{B/A}(F)$ whose linear coefficient is equal to $I_d \otimes W$. □

5. *Galois action on the Weil restriction of split tori*

Let A be a ring, B be an A-algebra which is a free A-module of finite rank. Let S be a smooth separated scheme over B of finite type. Define *Weil restriction* functor $\mathcal{R}_{B/A}(S)$ from the category of A-algebras to the category of sets by $\mathcal{R}_{B/A}(S)(R) = S(R \otimes_A B)$.

Proposition 5.1 ([**BLR**, Section 7.6, Theorem 4, Proposition 5]). *The functor $\mathcal{R}_{B/A}(S)$ is represented by a separated smooth scheme over A.*

If G is a group scheme over A such that its underlying scheme S is smooth, separated and of finite type, then $\mathcal{R}_{B/A}(S)(R)$ is a group, which defines a group scheme denoted by $\mathcal{R}_{B/A}(G)$.

Let K/k be a finite Galois extension of fields, and T be a torus over k represented by the Hopf algebra H. It is easy to see that if a field K' splits T, then KK' splits $\mathcal{R}_{K/k}(T_K)$, and in particular, the latter scheme is a torus. Define a right action of $\mathrm{Gal}(K/k)$ on $\mathcal{R}_{K/k}(T_K)$ as follows: if $\sigma \in \mathrm{Gal}(K/k)$, R is an k-algebra and $s \in \mathcal{R}_{K/k}(T_K)(R) = \mathrm{Hom}_K(H \otimes_k K, R \otimes_k K)$, then $s\sigma = \hat{\sigma}^{-1} \circ s \circ \tilde{\sigma}$, where $\tilde{\sigma} : H \otimes_k K \to H \otimes_k K$ and $\hat{\sigma} : R \otimes_k K \to R \otimes_k K$ are induced by σ. For every k-algebra R, the map

$$\Omega(R) : T(R) \to \mathcal{R}_{K/k}(T_K)(R)$$

defined by $\Omega(R)(g)(a \otimes v) = g(a)(1 \otimes v)$, $a \in H$, $v \in K$, is a homomorphism. Hence, it gives a morphism $\Omega : T \to \mathcal{R}_{K/k}(T_K)$.

Suppose that k is a local or global field and denote its ring of integers by $\mathcal{O}_k$. Let S be a smooth separated scheme over k. A smooth separated scheme $\mathcal{Y}$ over $\mathcal{O}_k$ is called a *Néron model* of S, if it is a model of S, i.e. $\mathcal{Y}_k = S$, and satisfies the following universal property: for any smooth scheme $\mathcal{Z}$ over $\mathcal{O}_k$ and any k-morphism $g : \mathcal{Z}_k \to \mathcal{Y}_k = S$ there exists a unique $\mathcal{O}_k$-morphism $h : \mathcal{Z} \to \mathcal{Y}$ such that $h_k = g$ (see [**BLR**], Section 1.2, Definition 1). Evidently, if a scheme S admits a Néron model, then it is unique up to isomorphism.

Proposition 5.2 ([**BLR**, Section 1.2, Proposition 6]). *Let G be a smooth separated group scheme over k such that its underlying scheme S admits a Néron model $\mathcal{Y}$ over $\mathcal{O}_k$. Then $\mathcal{Y}$ admits a unique group scheme structure which induces the original group scheme structure on $\mathcal{Y}_k = S$.*

Proposition 5.3 ([**BLR**, Section 10.1, Proposition 6]). *Any torus over k admits a Néron model over $\mathcal{O}_k$.*

By the above Proposition, the schemes T and $\mathcal{R}_{K/k}(T_K)$ admit Néron models which we denote by $\mathcal{T}$ and $\mathcal{U}$, respectively. Due to the universal property of Néron models, the right action of $\mathrm{Gal}(K/k)$ on $\mathcal{R}_{K/k}(T_K)$ can be extended to the right action on $\mathcal{U}$, and Ω can be extended to the morphism $\omega : \mathcal{T} \to \mathcal{U}$.

Let S be a separated scheme over a ring A provided with an action of a finite group $\mathcal{G}$. Define the functor $S^{\mathcal{G}}$ of fixed points by $S^{\mathcal{G}}(R) = S(R)^{\mathcal{G}}$.

PROPOSITION 5.4 ([**Ed**, Proposition 3.1]). *The functor $S^{\mathcal{G}}$ is represented by a closed subscheme of S.*

PROPOSITION 5.5 ([**Ed**, Theorem 4.2]). *If K/k is tamely ramified, then $\omega : \mathcal{T} \to \mathcal{U}$ is a closed immersion which induces an isomorphism $\mathcal{T} \to \mathcal{U}^{\mathrm{Gal}(K/k)}$.*

For a group scheme G, we denote by G_0 the connected component of the unit in G.
The scheme $\mathcal{U}_0$ is invariant with respect to the right action of $\mathrm{Gal}(K/k)$ on $\mathcal{U}$, so we obtain an action on $\mathcal{U}_0$.

PROPOSITION 5.6. *The natural morphism $((\mathcal{U}_0)^{\mathrm{Gal}(K/k)})_0 \to (\mathcal{U}^{\mathrm{Gal}(K/k)})_0$ is an isomorphism.*

Proof. We describe how to construct the inverse morphism. There is a natural morphism from $(\mathcal{U}^{\mathrm{Gal}(K/k)})_0$ to $\mathcal{U}_0$, and its image is a subscheme invariant with respect to the action of $\mathrm{Gal}(K/k)$. That gives a morphism from $(\mathcal{U}^{\mathrm{Gal}(K/k)})_0$ to $(\mathcal{U}_0)^{\mathrm{Gal}(K/k)}$, and the image of this morphism is a connected subscheme. Thus we obtain a morphism from $(\mathcal{U}^{\mathrm{Gal}(K/k)})_0$ to $((\mathcal{U}_0)^{\mathrm{Gal}(K/k)})_0$. $\square$

Let $\mathcal{O}_K$ denote the ring of integers of K.

PROPOSITION 5.7 ([**NX**, Lemma 3.1]). *There exists a natural isomorphism from $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})$ to the connected component of the unit in the Néron model for $\mathcal{R}_{K/k}((\mathbb{G}_m)^d_K)$.*

Suppose that T is split over K, and its dimension is d. Denote by $\mathcal{X}$ the group of group-like elements in $H \otimes_k K$. Then $\mathcal{X}$ is a free $\mathbb{Z}$-module of rank d. We fix a free $\mathbb{Z}$-basis $x_1, \dots, x_d$ in $\mathcal{X}$, and denote by X the set of variables $x_1, \dots, x_d$. Then $\mathcal{X}$ is identified with $\mathbb{Z}^d$, $H \otimes_k K$ is identified with the Hopf algebra $K[X, X^{-1}]$, T_K is identified with $(\mathbb{G}_m)^d_K$, $\mathcal{U}$ is identified with the Néron model for $\mathcal{R}_{K/k}((\mathbb{G}_m)^d_K)$, and according to Proposition 5.7, $\mathcal{U}_0$ is identified with $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})$. Moreover we obtain a right action of $\mathrm{Gal}(K/k)$ on $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})$. If $\sigma \in \mathrm{Gal}(K/k)$, R is an $\mathcal{O}_k$-algebra and

$$s \in \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})(R) = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[X, X^{-1}], R \otimes_{\mathcal{O}_k} \mathcal{O}_K)$$

then $s\sigma = \hat{\sigma}'^{-1} \circ s \circ \tilde{\sigma}'$, where $\tilde{\sigma}'$ is the restriction of $\tilde{\sigma}$ to $\mathcal{O}_K[X, X^{-1}]$, and $\hat{\sigma}'$ is induced by σ.
Let T' be another torus over k split over K with Hopf algebra H' and $\eta \colon T \to T'$ be a morphism. Denote by $\mathcal{T}'$ and $\mathcal{U}'$ the Néron models for T' and $\mathcal{R}_{K/k}(T'_K)$. Then η induces a morphism $\mathcal{U}_0 \to \mathcal{U}'_0$ which commutes with the actions of $\mathrm{Gal}(K/k)$ on $\mathcal{U}_0$ and $\mathcal{U}'_0$. If a $\mathbb{Z}$-basis $x'_1, \dots, x'_{d'}$ in the group $\mathcal{X}'$ of group-like elements in $H' \otimes_k K$ is fixed, then η induces a morphism

$$\tilde{\eta} \colon \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K}) \to \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^{d'}_{\mathcal{O}_K})$$

which also commutes with the actions of $\mathrm{Gal}(K/k)$.

Suppose that η corresponds to the homomorphism $\mathcal{X}' \to \mathcal{X}$ given by the matrix $C = \{c_{l,l'}\}_{1\le l\le d;\, 1\le l'\le d'} \in M_{d,d'}(\mathbb{Z})$ in the bases $x'_1, \dots, x'_{d'}$ and $x_1, \dots, x_d$. If R is an $\mathcal{O}_k$-algebra and

$$s \in \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})(R) = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[X, X^{-1}], R \otimes_{\mathcal{O}_k} \mathcal{O}_K),$$

then $\tilde{\eta}(R)(s)(x'_{l'}) = \sum_{l=1}^{d} c_{l,l'} s(x_l)$ for any $l' = 1, \dots, d'$.

6. *Main construction*

We keep the notations of the previous section. Suppose that the characteristic of k is equal to 0. Denote the degree of K/k by n and fix a free $\mathcal{O}_k$-basis $e_0, \dots, e_{n-1}$ of $\mathcal{O}_K$ such that $e_0 = 1$. Further, denote

$$\mathcal{H} = \mathcal{O}_k[z_{i,l}, \tilde{z}_l]_{\substack{0\le i\le n-1\\ 1\le l\le d}} \Bigg/ \left(1 - \tilde{z}_l \mathcal{N}_{\mathcal{O}_K/\mathcal{O}_k}\left(\sum_{0\le i\le n-1} z_{i,l} e_i\right)\right),$$

where $\mathcal{N}_{\mathcal{O}_K/\mathcal{O}_k}$ is the norm map from $\mathcal{O}_K$ to $\mathcal{O}_k$. For any $\mathcal{O}_k$-algebra R, the map

$$\nu(R) : \mathrm{Sp}\ \mathcal{H}(R) \to \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}(\mathrm{Sp}\ \mathcal{O}_K[X, X^{-1}])(R)$$

defined by $\nu(R)(s)(x_l) = \sum_{i=0}^{n-1} s(z_{i,l}) \otimes e_i$, $1 \le l \le d$, is a bijection. Therefore, it gives an isomorphism $\nu : \mathrm{Sp}\ \mathcal{H} \to \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})$ which allows one to define a group scheme structure on $\mathrm{Sp}\ \mathcal{H}$.

The augmentation ideal $\mathcal{J}$ of the Hopf algebra $\mathcal{H}$ is generated by the elements $z_1, \dots, z_{nd}$, where $z_l = z_{0,l} - 1$, $z_{id+l} = z_{i,l}$, for $1 \le l \le d$, $0 < i \le n-1$. Thus the elements $z_1, \dots, z_{nd}$ provide a coordinate system on $\widehat{\mathrm{Sp}\ \mathcal{H}} = \mathrm{Spf}\ \hat{\mathcal{H}}_{\mathcal{J}}$ and give rise to an nd-dimensional formal group law Φ with logarithm Λ.

PROPOSITION 6.1. *We have $\Phi = \mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{F}_m)^d_{\mathcal{O}_K})$, where $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}$ stays for the Weil restriction of a formal grroup law over $\mathcal{O}_K$ with respect to the basis $e_0, \dots, e_{n-1}$.*

Proof. Let N be a nilpotent $\mathcal{O}_k$-algebra. Denote $N' = N \otimes_{\mathcal{O}_k} \mathcal{O}_K$. The set $\widehat{\mathrm{Sp}\ \mathcal{H}}(N)$ is the subset of $\mathrm{Sp}\ \mathcal{H}(\mathcal{O}_k \oplus N)$ which consists of the elements $s \in \mathrm{Hom}_{\mathcal{O}_k}(\mathcal{H}, \mathcal{O}_k \oplus N)$ such that $s(z_{0,l}) - 1, s(z_{i,l}) \in N$, for $1 \le l \le d$ and $0 < i \le n-1$. On the other hand, the set $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{F}_m)^d_{\mathcal{O}_K})(N) = (\mathbb{F}_m)^d_{\mathcal{O}_K}(N')$ is the subset of $(\mathbb{G}_m)^d_{\mathcal{O}_K}(\mathcal{O}_K \oplus N')$ which consists of the elements $s' \in \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[X, X^{-1}], \mathcal{O}_K \oplus N')$ such that $s'(x_l) - 1 \in N'$, for $1 \le l \le d$. Hence, $\nu(\mathcal{O}_k \oplus N)$ provides a bijection from $\widehat{\mathrm{Sp}\ \mathcal{H}}(N)$ to $(\mathbb{F}_m)^d_{\mathcal{O}_K}(N')$. Moreover, we have $\hat{\mathcal{H}}_{\mathcal{J}} = \mathcal{A}_d$ and $\nu(\mathcal{O}_k \oplus N)$ restricted to $\widehat{\mathrm{Sp}\ \mathcal{H}}(N)$ coincides with $\rho_d(N)$. The formal group laws Φ and $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{F}_m)^d_{\mathcal{O}_K})$ come from the group structure on $(\mathbb{F}_m)^d_{\mathcal{O}_K}(N')$ with the aid of the bijections $\nu(\mathcal{O}_k \oplus N)$ and $\rho_d(N)$, respectively. Hence, they coincide. □

COROLLARY. *For any $l = 1, \dots, d$, we have*

$$\sum_{j=0}^{n-1} \Lambda_{jd+l}(z_1, \dots, z_{nd}) e_j = \mathbb{L}_m\left(\sum_{i=0}^{n-1} z_{id+l} e_i\right) \in K[[z_1, \dots, z_{nd}]].$$

Proof. It follows from Propositions 4.3 and 6.1. □

Let $\mathcal{H}'$, ν', $\mathcal{J}'$, Φ' be defined for the torus T' similar to $\mathcal{H}$, ν, $\mathcal{J}$, Φ. The morphism $\eta\colon T \to T'$ induces a homomorphism $\Phi \to \Phi'$ which corresponds to the morphism $\nu'^{-1} \circ \tilde{\eta} \circ \nu\colon \operatorname{Sp}\ \mathcal{H} \to \operatorname{Sp}\ \mathcal{H}'$ in the bases $z_1, \ldots, z_{nd}$ and $z'_1, \ldots, z'_{nd'}$.

Proposition 6.2. *The linear coefficient of the homomorphism $\Phi \to \Phi'$ of formal group laws induced by η is $C^t \otimes I_n$.*

Proof. Let R be a $\mathcal{O}_k$-algebra and $s \in \operatorname{Hom}_{\mathcal{O}_k}(\mathcal{H}, R)$. Then, for any $l' = 1, \ldots, d'$,

$$(\tilde{\eta} \circ \nu)(R)(s)(x'_{l'}) = \sum_{l=1}^{d} c_{l,l'} \nu(R)(s)(x_l) = \sum_{i=0}^{n-1} \sum_{l=1}^{d} c_{l,l'} s(z_{i,l}) \otimes e_i.$$

On the other hand,

$$(\tilde{\eta} \circ \nu)(R)(s)(x'_{l'}) = (\nu' \circ \nu'^{-1} \circ \tilde{\eta} \circ \nu)(R)(s)(x'_{l'}) = \sum_{i=0}^{n-1} (\nu'^{-1} \circ \tilde{\eta} \circ \nu)(R)(s)(z'_{i,l'}) \otimes e_i.$$

Therefore $(\nu'^{-1} \circ \tilde{\eta} \circ \nu)(z'_{i,l'}) = \sum_{l=1}^{d} c_{l,l'} z_{i,l}$ for any $l' = 1, \ldots, d'$ and $i = 0, \ldots, n-1$. Hence, the homomorphism $\mathcal{J}'/\mathcal{J}'^2 \to \mathcal{J}/\mathcal{J}^2$ induced by $\nu'^{-1} \circ \tilde{\eta} \circ \nu$ maps $z'_{id'+l'}$ to $\sum_{l=1}^{d} c_{l,l'} z_{id+l}$, i.e. the matrix of this homomorphism in the bases $z'_1, \ldots, z'_{nd'}$ and $z_1, \ldots, z_{nd}$ is $C \otimes I_n$. $\square$

The isomorphism ν allows one to define a right action of $\operatorname{Gal}(K/k)$ on $\operatorname{Sp}\ \mathcal{H}$, and hence, an action on $\mathcal{H}$ and on $\mathcal{J}/\mathcal{J}^2$. In the basis $z_1, \ldots, z_{nd}$, the action on $\mathcal{J}/\mathcal{J}^2$ gives the representation $\theta : \operatorname{Gal}(K/k) \to \operatorname{GL}_{nd}(\mathcal{O}_k)$.

The group $\mathcal{X}$ is invariant with respect to the action of the group $\operatorname{Gal}(K/k)$ on $H \otimes_k K$ which is defined by the intrinsic action on K. Thus in the basis $x_1, \ldots, x_d$, it provides the representation $\chi : \operatorname{Gal}(K/k) \to \operatorname{GL}_d(\mathbb{Z})$.

There is a unique action of $\operatorname{Gal}(K/k)$ on $\tilde{\mathcal{O}}_K := \operatorname{Hom}_{\mathcal{O}_k}(\mathcal{O}_K, \mathcal{O}_k)$ such that $\sigma\tilde{a}(\sigma a) = \tilde{a}(a)$ for any $a \in \mathcal{O}_K$, $\tilde{a} \in \tilde{\mathcal{O}}_K$, $\sigma \in \operatorname{Gal}(K/k)$. The elements $\tilde{e}_0, \ldots, \tilde{e}_{n-1}$ defined by $\tilde{e}_i(e_j) = \delta_i^j$ form a free $\mathcal{O}_k$-basis of $\tilde{\mathcal{O}}_K$ which gives the representation $\psi : \operatorname{Gal}(K/k) \to \operatorname{GL}_n(\mathcal{O}_k)$.

Proposition 6.3. *For any $\sigma \in \operatorname{Gal}(K/k)$, we have $\theta(\sigma) = \chi(\sigma) \otimes \psi(\sigma)$.*

Proof. Suppose that $\chi(\sigma) = \{a_{l,l'}\}_{1 \le l,l' \le d}$, $\psi(\sigma) = \{b_{i,i'}\}_{0 \le i,i' \le n-1}$. Let R be an $\mathcal{O}_k$-algebra and $s \in \operatorname{Hom}_{\mathcal{O}_k}(\mathcal{H}, R)$. Then

$$\nu(R)(s) \in \operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[X, X^{-1}], R \otimes_{\mathcal{O}_k} \mathcal{O}_K).$$

Besides, we have the natural mapping $(R \otimes_{\mathcal{O}_k} \mathcal{O}_K) \times \tilde{\mathcal{O}}_K \to R$. Then for any $l' = 1, \ldots, d$ and $i' = 0, \ldots, n-1$, we get

$$\begin{aligned}(-1 + ((\nu(R)(s))\sigma)(x_{l'}))\tilde{e}_{i'} &= (-1 + (\nu(R)(s))(\tilde{\sigma} x_{l'}))\sigma\tilde{e}_{i'} \\ &= \left(-1 + (\nu(R)(s))\left(\prod_{l=1}^{d} x_l^{a_{l,l'}}\right)\right) \sum_{i=0}^{n-1} b_{i,i'} \tilde{e}_i \\ &= \left(-1 + \prod_{l=1}^{d}\left(1 + \sum_{j=0}^{n-1} s(z_{jd+l}) \otimes e_j\right)^{a_{l,l'}}\right) \sum_{i=0}^{n-1} b_{i,i'} \tilde{e}_i\end{aligned}$$

$$\equiv \left(\sum_{l=1}^{d} a_{l,l'} \sum_{j=0}^{n-1} s(z_{jd+l}) \otimes e_j \right) \sum_{i=0}^{n-1} b_{i,i'} \tilde{e}_i$$

$$= \sum_{l=1}^{d} \sum_{i=0}^{n-1} a_{l,l'} b_{i,i'} s(z_{id+l}) \mod s(\mathcal{J}^2).$$

On the other hand,

$$(-1 + (\nu(R)(s\sigma))(x_{l'}))\tilde{e}_{i'} = \left(\sum_{j=0}^{n-1} (s\sigma)(z_{jd+l'}) \otimes e_j \right) \tilde{e}_{i'} = (s\sigma)(z_{i'd+l'}).$$

Thus $\sigma(z_{i'd+l'}) = \sum_{l=1}^{d} \sum_{i=0}^{n-1} a_{l,l'} b_{i,i'} z_{id+l}$. □

COROLLARY. *The $\mathcal{O}_k$-linear map $\Theta\colon (\mathcal{X} \otimes_{\mathbb{Z}} \mathcal{O}_k) \otimes_{\mathcal{O}_k} \tilde{\mathcal{O}}_K \to \mathcal{J}/\mathcal{J}^2$ defined by $\Theta(x_l \otimes \tilde{e}_i) = z_{id+l}$, $1 \le l \le d$, $0 \le i \le n-1$, is $\mathrm{Gal}(K/k)$-equivariant.*

It is known that the $\mathrm{Gal}(K/k)$-module of characters for the torus $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}(T_K)$ is isomorphic to the tensor product of the $\mathrm{Gal}(K/k)$-module $\mathcal{X}$ of characters for T by the group algebra $\mathbb{Z}[\mathrm{Gal}(K/k)]$ considered as $\mathrm{Gal}(K/k)$-module. Corollary of Proposition 6.3 can be, in fact, deduced from this result by descend to the rings of integers.

The right action of $\mathrm{Gal}(K/k)$ on $\mathrm{Sp}\ \mathcal{H}$ induces a right action on Φ. In order to be able to apply the results of Section 3 to Φ, we consider the opposite group $\mathrm{Gal}(K/k)^\circ$, i.e. the group with the same underlying set as $\mathrm{Gal}(K/k)$ and the opposite order of the group operation. Then we obtain an action of $\mathrm{Gal}(K/k)^\circ$ on Φ. Clearly, the linear coefficient of the endomorphism $\sigma \in \mathrm{Gal}(K/k)^\circ$ of Φ is $\theta(\sigma)^t$.

THEOREM 2. *If K/k is tamely ramified, then there exists a universal fixed pair (F, f) for $(\Phi, \mathrm{Gal}(K/k)^\circ)$ such that F represents $\hat{\mathcal{T}}$.*

Proof. According to Proposition 5.5, the restriction of ω to $\mathcal{T}_0$ is an isomorphism from $\mathcal{T}_0$ to $(\mathcal{U}^{\mathrm{Gal}(K/k)})_0$. Proposition 5.6 implies that $\mathcal{T}_0$ is isomorphic to $((\mathcal{U}_0)^{\mathrm{Gal}(K/k)})_0$. Since $\mathcal{U}_0$ is identified with $\mathcal{R}_{\mathcal{O}_K/\mathcal{O}_k}((\mathbb{G}_m)^d_{\mathcal{O}_K})$, and ν is an isomorphism, we conclude that $\mathcal{T}_0$ is isomorphic to $((\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)})_0$. Therefore $\hat{\mathcal{T}}$ is isomorphic to $\widehat{(\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)}}$. Moreover, $(\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)}$ is represented by the algebra $\mathcal{H}/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by elements $a - \sigma a$, $a \in \mathcal{H}$, $\sigma \in \mathrm{Gal}(K/k)$. Consider a formal group law F which represents $\widehat{(\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)}}$. The morphism $\iota : (\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)} \to (\mathrm{Sp}\ \mathcal{H})$ induces a homomorphism $f = \hat{\iota} : F \to \Phi$. Evidently, (F, f) is a fixed pair for $(\Phi, \mathrm{Gal}(K/k)^\circ)$. For any nilpotent $\mathcal{O}_k$-algebra N, $\widehat{(\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)}}(N) = \mathrm{Hom}^*_A(\mathcal{H}/\mathcal{I}, A \oplus N)$ consists of elements $s \in \mathrm{Hom}^*_A(\mathcal{H}, A \oplus N)$ such that $sa - \sigma a = 0$ for any $a \in \mathcal{H}$, $\sigma \in \mathrm{Gal}(K/k)$, and hence, it coincides with $\widehat{\mathrm{Sp}\ \mathcal{H}}(N)^{\mathrm{Gal}(K/k)}$. Therefore the map $f(N) : \widehat{(\mathrm{Sp}\ \mathcal{H})^{\mathrm{Gal}(K/k)}}(N) \to \widehat{\mathrm{Sp}\ \mathcal{H}}(N)^{\mathrm{Gal}(K/k)}$ is bijective, and by Proposition 3.1, (F, f) is a universal fixed pair for $(\Phi, \mathrm{Gal}(K/k)^\circ)$. □

7. *Tori split over a tamely ramified abelian extension of $\mathbb{Q}_p$*

We keep the notations of Section 6. Suppose that $k = \mathbb{Q}_p$, and $K/\mathbb{Q}_p$ is a tamely ramified abelian extension. In this case $K = K_1 K_2$, where $K_1/\mathbb{Q}_p$ is an unramified extension of degree n_1, and $K_2/\mathbb{Q}_p$ is a totally ramified extension of degree n_2, $n = n_1 n_2$. Then $K_1 = \mathbb{Q}_p(\xi)$, where ξ is a primitive $(p^{n_1} - 1)$-th root of unity; $K_2 = \mathbb{Q}_p(\pi)$, where $\pi^{n_2} = p\varepsilon$, $\varepsilon \in \mathbb{Z}_p^*$ and $n_2 \mid p - 1$.

The group $\mathrm{Gal}(K/K_2)$ is isomorphic to $\mathrm{Gal}(K_1/\mathbb{Q}_p) \cong \mathbb{Z}/n_1\mathbb{Z}$. Take $\sigma_1 \in \mathrm{Gal}(K/K_2)$ such that $\sigma_1|_{K_1}$ is the Frobenius automorphism. Then σ_1 generates $\mathrm{Gal}(K/K_2)$. The inertia subgroup $\mathrm{Gal}(K/K_1)$ of $\mathrm{Gal}(K/\mathbb{Q}_p)$ is isomorphic to $\mathrm{Gal}(K_2/\mathbb{Q}_p) \cong \mathbb{Z}/n_2\mathbb{Z}$. Let ζ be a primitive n_2-th root of unity. Take $\sigma_2 \in \mathrm{Gal}(K/K_1)$ such that $\sigma_2(\pi) = \zeta\pi$. Then σ_2 generates $\mathrm{Gal}(K/K_1)$. Consider the following $\mathbb{Z}_p$-basis of $\mathcal{O}_K$: $e_{i_1n_2+i_2} = \pi^{i_2}\xi^{p^{i_1}}$, $0 \le i_1 \le n_1 - 1$, $0 \le i_2 \le n_2 - 1$.

If $\kappa \in K[[t_1, \dots, t_n]]$ then for any $0 \le i_2 \le n_2 - 1$ there exist unique $\kappa^{(i_2)} \in K_1[[t_1, \dots, t_n]]$ such that $\kappa = \sum_{i_2=0}^{n_2-1} \kappa^{(i_2)}\pi^{i_2}$.

LEMMA 7.1 ([**De**, Proposition 1.7, Proposition 1.8]). *Let $\lambda = \sum_{i=1}^{\infty} c_i t^i \in K[[t]]$ be such that $p^s c_{rp^s} \in \mathcal{O}_K$ for any non-negative integers r, s. If $\varphi \in \mathcal{O}_K[[t_1, \dots, t_n]]$, then*
(i) $(\lambda \circ \varphi)^{(0)} \equiv \lambda^{(0)} \circ \varphi^{(0)} \mod p$;
(ii) $(\lambda \circ \varphi)^{(i_2)} \equiv \lambda^{(i_2)} \circ \varphi^{(0)} \mod \mathcal{O}_K$ *for* $0 < i_2 \le n_2 - 1$.

PROPOSITION 7.2. *The formal group law Λ is of type $v = pI_{nd} - I_d \otimes J_{n_2} \otimes P_{n_1}\blacktriangle$.*

Proof. For any $1 \le l \le d$, consider

$$\varphi_l = \sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1} z_{(i_1n_2+i_2)d+l}e_{i_1n_2+i_2} \in \mathcal{O}_K[[z_l, z_{d+l}, \dots, z_{(n-1)d+l}]].$$

By Corollary from Proposition 6.1, we have

$$\sum_{i_2=0}^{n_2-1}\sum_{i_1=0}^{n_1-1} \Lambda_{(i_1n_2+i_2)d+l}e_{i_1n_2+i_2} = \mathbb{L}_m \circ \varphi_l$$

which implies $\sum_{i_1=0}^{n_1-1} \Lambda_{(i_1n_2+i_2)d+l}\xi^{p^{i_1}} = (\mathbb{L}_m \circ \varphi_l)^{(i_2)}$ for any $0 \le i_2 \le n_2 - 1$.

If $i_2 \ne 0$ then by Lemma 7.1 we have

$$\sum_{i_1=0}^{n_1-1} \Lambda_{(i_1n_2+i_2)d+l}\xi^{p^{i_1}} \equiv \mathbb{L}_m^{(i_2)} \circ \varphi_l^{(0)} = 0 \mod \mathcal{O}_K,$$

hence $p\Lambda_{(i_1n_2+i_2)d+l} \equiv 0 \mod p$.

If $i_2 = 0$, then Lemma 7.1 implies $\Lambda_{i_1n_2d+l}\xi^{p^{i_1}} \equiv \mathbb{L}_m \circ \varphi_l^{(0)} \mod p$.

We consider the action of $\blacktriangle$ on $K_1[[z_1, \dots, z_{nd}]]$ introduced in Section 2. Then applying Lemma 2.4, we get

$$(p - \blacktriangle)\sum_{i_1=0}^{n_1-1} \Lambda_{i_1n_2d+l}\xi^{p^{i_1}} \equiv (p - \blacktriangle)(\mathbb{L}_m \circ \varphi_l^{(0)}) \equiv ((p - \blacktriangle)\mathbb{L}_m) \circ \varphi_l^{(0)} \equiv 0 \mod p.$$

Therefore

$$p\sum_{i_1=0}^{n_1-1} \Lambda_{i_1n_2d+l}\xi^{p^{i_1}} \equiv \sum_{i_1=0}^{n_1-1} (\blacktriangle\Lambda_{i_1n_2d+l})\xi^{p^{i_1+1}} \mod p.$$

Hence $p\Lambda_{i_1n_2d+l} - \blacktriangle\Lambda_{(i_1-1)n_2d+l} \equiv 0 \mod p$ for $1 \le i_1 \le n_1 - 1$ and $p\Lambda_l - \blacktriangle\Lambda_{(n_1-1)n_2d+l} \equiv 0 \mod p$.

Thus $p\Lambda \equiv W\blacktriangle\Lambda \mod p$ for some matrix $W = \{w_{\alpha,\beta}\}_{1\le\alpha,\beta\le nd}$, where $w_{i_1n_2+l,(i_1-1)n_2d+l} = 1$ for $1 \le i_1 \le n_1 - 1, 1 \le l \le d$ and $w_{l,(n_1-1)n_2d+l} = 1$ for $1 \le l \le d$, the other entries being equal to 0. Therefore $W = I_d \otimes J_{n_2} \otimes P_{n_1}$ as required. $\square$

For $0 \le i_1 \le n_1 - 2$, $0 \le i_2 \le n_2 - 1$, we have $\sigma_1(e_{i_1 n_2 + i_2}) = e_{(i_1+1)n_2+i_2}$, $\sigma_1(e_{(n_1-1)n_2+i_2}) = e_{i_2}$, i.e. in the basis $e_0, \dots, e_{n-1}$ the automorphism σ_1 is given by the matrix $I_{n_2} \otimes P_{n_1}$. Therefore, $\psi(\sigma_1) = ((I_{n_2} \otimes P_{n_1})^{-1})^t = I_{n_2} \otimes P_{n_1}$. Denote $U_1 = \chi(\sigma_1)^t$. Then by Proposition 6.3, $\theta(\sigma_1)^t = U_1 \otimes I_{n_2} \otimes P_{n_1}^t$.

LEMMA 7.3. *If*

$$\hat{C} = \begin{pmatrix} U_1^{-1} \otimes I_{n_2} \\ U_1^{-2} \otimes I_{n_2} \\ \vdots \\ U_1^{-n_1+1} \otimes I_{n_2} \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} I_{n_2 d} & 0 \\ \hat{C} & I_{nd-n_2 d} \end{pmatrix},$$

and $D = U_1 \otimes I_{n_2} \otimes P_{n_1}^t$, *then* $Q_1^{-1} D Q_1 - I_{nd} = \begin{pmatrix} 0 & \hat{D} \\ 0 & \tilde{D} \end{pmatrix}$, *where* $\hat{D} \in \mathrm{M}_{n_2 d, nd - n_2 d}(\mathbb{Z})$, $\tilde{D} \in \mathrm{M}_{nd-n_2 d}(\mathbb{Z})$, *and* $\hat{C}\hat{D} + \tilde{D}$ *is invertible.*

Proof. First, notice that

$$U_1 \otimes I_{n_2} \otimes P_{n_1}^t = \begin{pmatrix} 0 & U_1 \otimes I_{n_2} & 0 & \cdots & 0 \\ 0 & 0 & U_1 \otimes I_{n_2} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & U_1 \otimes I_{n_2} \\ U_1 \otimes I_{n_2} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then one can directly compute that $Q_1^{-1} D Q_1 - I_{nd}$ is of the required form with $\hat{D} = \begin{pmatrix} U_1 \otimes I_{n_2} & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{M}_{n_2 d, nd-n_2 d}(\mathbb{Z})$, and

$$\tilde{D} = \begin{pmatrix} -2I_{n_2 d} & U_1 \otimes I_{n_2} & 0 & \cdots & 0 & 0 \\ -U_1^{-1} \otimes I_{n_2} & -I_{n_2 d} & U_1 \otimes I_{n_2} & & 0 & 0 \\ -U_1^{-2} \otimes I_{n_2} & 0 & -I_{n_2 d} & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ -U_1^{-n_1+3} \otimes I_{n_2} & 0 & 0 & & -I_{n_2 d} & U_1 \otimes I_{n_2} \\ -U_1^{-n_1+2} \otimes I_{n_2} & 0 & 0 & \cdots & 0 & -I_{n_2 d} \end{pmatrix} \in \mathrm{M}_{nd-n_2 d}(\mathbb{Z}).$$

The last assertion is obvious now. □

LEMMA 7.4 ([**Ed**, Lemma 3.3]). *Let* $\mathcal{G} = \mathbb{Z}/n_2\mathbb{Z}$, $(n_2, p) = 1$, *and* $\bar{M}$ *be a finitely generated* $\mathbb{F}_p[\mathcal{G}]$*-module. Then there exists a unique up to isomorphism* $\mathbb{Z}_p[\mathcal{G}]$*-module* M *which is free as a* $\mathbb{Z}_p$*-module and such that* $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong \bar{M}$.

PROPOSITION 7.5. *Let* $\mathcal{G} = \mathbb{Z}/n_2\mathbb{Z}$, $n_2 \mid p-1$, σ *be a generator of* $\mathcal{G}$ *and* M *be a finitely generated* $\mathbb{Z}_p[\mathcal{G}]$*-module which is free as a* $\mathbb{Z}_p$*-module. Then there exists a free* $\mathbb{Z}_p$*-basis of* M *consisting of eigenvectors of* σ*. The multiplicity of the eigenvalue 1 is equal to* $\mathrm{rk}_{\mathbb{Z}_p} M^{\mathcal{G}}$.

Proof. Since $(n_2, p) = 1$, the algebra $\mathbb{F}_p[\mathcal{G}]$ is semisimple by Mashke's Theorem, and the $\mathbb{F}_p[\mathcal{G}]$-module $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is isomorphic to the direct sum of irreducible $\mathbb{F}_p[\mathcal{G}]$-modules. Let $\bar{M}_0$ be an irreducible $\mathbb{F}_p[\mathcal{G}]$-module, and $x \in \bar{M}_0$, $x \ne 0$. Then $0 = (\sigma^{n_2} - 1)x = \prod_{j=1}^{n_2} (\sigma - \zeta_j)x$, where $\zeta_1, \dots, \zeta_{n_2}$ are the n_2-th roots of unity. Therefore, there exist $y \in M$, $y \ne 0$, and an index $1 \le j \le n_2$ such that $(\sigma - \zeta_j)y = 0$. Obviously $\mathbb{F}_p y$ is an $\mathbb{F}_p[\mathcal{G}]$-submodule of $\bar{M}_0$. Since $\bar{M}_0$ is irreducible and $y \ne 0$, we get $\mathbb{F}_p y = M$, i.e. every irreducible $\mathbb{F}_p[\mathcal{G}]$-module is one-dimensional.

Thus there exists a free $\mathbb{F}_p$-basis $m_1, \dots, m_d$ of $M \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ consisting of eigenvectors of σ. Let $\bar{\alpha}_l \in \mathbb{F}_p$ be the eigenvalue of the vector m_l, $1 \le l \le d$. Define a $\mathbb{Z}_p[\mathcal{G}]$-module structure on $M' = \mathbb{Z}_p^d$ by $\sigma(a_1, \dots, a_d) = (\alpha_1 a_1, \dots, \alpha_d a_d)$, where $\alpha_l \in \mathbb{Z}_p$ is the multiplicative representative of $\bar{\alpha}_l$, $1 \le l \le d$. Then $M' \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong M \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Therefore, Lemma 7.4 implies that $M \cong M'$. □

For $0 \le i_1 \le n_1 - 1$, $0 \le i_2 \le n_2 - 1$, we have $\sigma_2(e_{i_1 n_2 + n_2}) = \zeta^{i_2} e_{i_1 n_2 + i_2}$ and $\sigma_2(\tilde{e}_{i_1 n_2 + i_2}) = \zeta^{-i_2} \tilde{e}_{i_1 n_2 + i_2}$, i.e. $\psi(\sigma_2) = V \otimes I_{n_1}$, where V is the diagonal matrix with i_2-th diagonal entry equal to ζ^{-i_2+1}.

Denote $d_0 = \mathrm{rk}_{\mathbb{Z}_p}(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{<\sigma_2>}$. Then by Proposition 7.5, there exists a free $\mathbb{Z}_p$-basis $y_1, \dots, y_d$ of $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ consisting of eigenvectors of σ_2 such that the eigenvalues of $y_1, \dots, y_{d_0}$ are equal to 1, and the eigenvalues of $y_{d_0+1}, \dots, y_d$ are not equal to 1. Let $\mu\colon \{1, \dots, d\} \to \{0, \dots, n_2 - 1\}$ be a unique function such that the eigenvalue of y_l is $\zeta^{\mu(l)}$ for any $l = 1, \dots, d$. Clearly, $\mu(l) = 0$ if and only if $l \le d_0$.

Let $W \in \mathrm{GL}_d(\mathbb{Z}_p)$ be the transition matrix from $y_1, \dots, y_d$ to $x_1, \dots, x_d$. Then $W^{-1}\chi(\sigma_2)W$ is the diagonal matrix with l-th diagonal entry equal to $\zeta^{\mu(l)}$.

Denote $U_2 = \chi(\sigma_2)^t$. Then by Proposition 6.3, $\theta(\sigma_2)^t = U_2 \otimes V \otimes I_{n_1}$.

Notice that $(W^t U_2 (W^t)^{-1}) \otimes V$ is the diagonal matrix with $(i_2 d + l)$-th diagonal entry equal to $\zeta^{\mu(l) - i_2}$ for $1 \le l \le d$, $0 \le i_2 \le n_2 - 1$. Thus among these diagonal entries there are precisely d which are equal to 1. In order to obtain a matrix with upper-left $d \times d$-submatrix being I_d, we have to conjugate this matrix by a matrix of a permutation. Namely, define a permutation τ of the set $\{1, \dots, n_2 d\}$ in the following way: for $0 \le i_2 \le n_2 - 1, 1 \le l \le d$, put $\tau(i_2 d + l) = i_2' d + l$, where i_2' is the reminder of the division of $i_2 + \mu(l)$ by n_2. Then $\tau(l) = l$ for any $l = 1, \dots, d_0$, and $\tau(l) > d$ for any $l = d_0 + 1, \dots, d$.

LEMMA 7.6. *If $L_\tau \in \mathrm{M}_{n_2 d}(\mathbb{Z})$ is the matrix of τ, i.e. $L_\tau = \{\delta_\alpha^{\tau(\beta)}\}_{1 \le \alpha, \beta \le n_2 d}$, $Q_2 = ((W^t) \otimes I_{n_2})^{-1} L_\tau$ and $D = U_2 \otimes V$, then $Q_2^{-1} D Q_2 - I_{n_2 d} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} \end{pmatrix}$, where $\tilde{D} \in \mathrm{M}_{n_2 d - d}(\mathbb{Z}_p)$ is invertible.*

Proof. As we showed above, $(W^t \otimes I_{n_2}) D (W^t \otimes I_{n_2})^{-1} = (W^{-1}\chi(\sigma_2)W)^t \otimes V$ is a diagonal matrix with the $(i_2 d + l)$-th diagonal entry equal to $\zeta^{\mu(l) - i_2}$ for $0 \le i_2 \le n_2 - 1, 1 \le l \le d$. According to the definition of τ, we obtain that $L_\tau^{-1}(W^t \otimes I_{n_2}) D (W^t \otimes I_{n_2})^{-1} L_\tau$ is a diagonal matrix with first d diagonal entries equal to 1, and the rest of the diagonal entries are n_2-th roots of unity different from 1. Hence, $Q_2^{-1} D Q_2 - I_{n_2 d}$ is of the required form. Finally, since $\zeta^{i_1} - 1$ is invertible for any $i_1 = 1, \dots, n_2 - 1$, we get that $\tilde{D} \in \mathrm{M}_{n_2 d - d}(\mathbb{Z}_p)$ is invertible. □

The character group of the maximal subtorus T_s (resp. T_a) of T which is split (resp. anisotropic) over K_1 is canonically isomorphic to $\mathcal{X}/\operatorname{Ker}\rho_s$ (resp. $\mathcal{X}/\operatorname{Ker}\rho_a$), where $\rho_s\colon \mathcal{X} \to \mathcal{X}$ and $\rho_a\colon \mathcal{X} \to \mathcal{X}$ are defined by

$$\rho_s(x) = \sum_{\sigma \in \mathrm{Gal}(K/K_1)} \sigma x = x + \sigma_2 x + \dots + \sigma_2^{n_2 - 1} x, \quad \rho_a(x) = \sigma_2(x) - x.$$

Denote $d_s = \dim T_s$, $d_a = \dim T_a$. Then $d_s + d_a = d$ (see [**Wa**, Section 7.4]). Let $\tilde{x}_1, \dots, \tilde{x}_{d_s}$ and $\hat{x}_1, \dots, \hat{x}_{d_a}$ be free $\mathbb{Z}$-bases of $\mathcal{X}/\operatorname{Ker}\rho_s$ and $\mathcal{X}/\operatorname{Ker}\rho_a$, respectively, and let $\tilde{\chi} : \mathrm{Gal}(K_1/\mathbb{Q}_p) \to \mathrm{GL}_{d_s}(\mathbb{Z})$ be the representation corresponding to $\tilde{x}_1, \dots, \tilde{x}_{d_s}$. Denote $\tilde{U}_1 = \tilde{\chi}(\sigma_1|_{K_1})^t$.

Let T_s', T_a', ρ_s', ρ_a', d_s', d_a' be defined for the torus T' similar to T_s, T_a, ρ_s, ρ_a, d_s, d_a. Let $\tilde{x}_1', \dots, \tilde{x}_{d_s'}'$ and $\hat{x}_1', \dots, \hat{x}_{d_a'}'$ be free $\mathbb{Z}$-bases of $\mathcal{X}'/\operatorname{Ker}\rho_s'$ and $\mathcal{X}'/\operatorname{Ker}\rho_a'$, respectively. The morphism $\eta\colon T \to T'$ induces the morphisms $\eta_s : T_s \to T_s'$ and $\eta_a : T_a \to T_a'$ which correspond to homomorphisms $\mathcal{X}'/\operatorname{Ker}\rho_s' \to \mathcal{X}/\operatorname{Ker}\rho_s$ and $\mathcal{X}'/\operatorname{Ker}\rho_a' \to \mathcal{X}/\operatorname{Ker}\rho_a$. Denote the matrices of these homomorphisms in the chosen bases by $C_s \in \mathrm{M}_{d_s, d_s'}(\mathbb{Z})$ and $C_a \in \mathrm{M}_{d_a, d_a'}(\mathbb{Z})$, respectively.

Theorem 3. *Let T, T' be tori over $\mathbb{Q}_p$ which are split over an abelian tamely ramified extension K of $\mathbb{Q}_p$, $\mathcal{T}$, $\mathcal{T}'$ be their Néron models, and $\eta\colon T \to T'$ be a morphism. Then*

I. *A formal group law with logarithm of type*

$$pI_d - \begin{pmatrix} \tilde{U}_1 & 0 \\ 0 & 0 \end{pmatrix} \blacktriangle$$

represents $\hat{\mathcal{T}}$.

II. *For two formal groups laws associated to $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ in the sense of part* I, *the morphism η induces their homomorphism with linear coefficient equal to*

$$\begin{pmatrix} C_s^t & 0 \\ 0 & C_a^t \end{pmatrix}.$$

Proof. I. According to Theorem 2, a formal group law which appears in a universal fixed pair for $(\Phi, \mathrm{Gal}(K/\mathbb{Q}_p))$ represents $\hat{\mathcal{T}}$. By Theorem 1 we can take a universal fixed pair (F_1, f_1) for $(\Phi, \langle\sigma_1\rangle)$ such that the linear coefficient of f_1 is $Q_1 I_{nd,n_2d}$, and the logarithm of F_1 is of type u_1, where u_1 is the upper-left $n_2d \times n_2d$-submatrix of $Q_1^{-1} v Q_1$ for v and Q_1 being from Proposition 7.2 and Lemma 7.3, respectively, i.e.

$$v = pI_{nd} - \begin{pmatrix} 0 & 0 & \cdots & 0 & I_d \otimes J_{n_2} \\ I_d \otimes J_{n_2} & 0 & & 0 & 0 \\ 0 & I_d \otimes J_{n_2} & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & I_d \otimes J_{n_2} & 0 \end{pmatrix} \blacktriangle,$$

$$Q_1 = \begin{pmatrix} I_{n_2d} & 0 & \cdots & 0 & 0 \\ U_1^{-1} \otimes I_{n_2} & I_{n_2d} & & 0 & 0 \\ U_1^{-2} \otimes I_{n_2} & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ U_1^{-n_1+1} \otimes I_{n_2} & 0 & \cdots & 0 & I_{n_2d} \end{pmatrix}.$$

An easy calculation shows that $u_1 = pI_{n_2d} - U_1 \otimes J_{n_2}\blacktriangle$. The linear coefficient of f_1 is equal to $Q_1 I_{nd,n_2d}$. The action of $\langle\sigma_2\rangle$ on Φ induces the action on F_1 by condition $f_1 \circ \sigma_2' = \sigma_2 \circ f_1$ for $\sigma_2' \in \mathrm{Aut}_{\mathbb{Z}_p}(F_1)$. If $Z \in \mathrm{M}_{n_2d}(\mathbb{Z}_p)$ is its linear coefficient, we have $I_{nd,n_2d} Z = Q_1^{-1}(U_2 \otimes V \otimes I_{n_1}) Q_1 I_{nd,n_2d}$, which implies that Z is the upper-left $n_2d \times n_2d$-submatrix of $Q_1^{-1}(U_2 \otimes V \otimes I_{n_1}) Q_1$. An easy calculation gives $Z = U_2 \otimes V$.

By Theorem 1 we can take a universal fixed pair (F_2, f_2) for $(F_1, \langle\sigma_2'\rangle)$ such that the linear coefficient of f_2 is $Q_2 I_{n_2d,d}$, and the logarithm of F_2 is of type u_2, where u_2 is the upper-left $d \times d$-submatrix of

$$Q_2^{-1} u_1 Q_2 = pI_{n_2d} - Q_2^{-1}(U_1 \otimes J_{n_2}) Q_2 \blacktriangle = pI_{n_2d} - L_\tau^{-1}(W^{-1}\chi(\sigma_1) W \otimes J_{n_2})^t L_\tau \blacktriangle$$

for Q_2 being from Lemma 7.6. Properties of the permutation τ imply that the upper-left $d \times d$-submatrix of $L_\tau^{-1}(W^{-1}\chi(\sigma_1) W \otimes J_{n_2})^t L_\tau$ is equal to $\begin{pmatrix} \hat{U}_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_d(\mathbb{Z}_p)$, where $\hat{U}_1$ is the upper-left $d_0 \times d_0$-submatrix of $(W^{-1}\chi(\sigma_1) W)^t$.

If we extend ρ_s and ρ_a on $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, then we get $\rho_s(y_l) = n_2 y_l$ and $\rho_a(y_l) = 0$ for $1 \le l \le d_0$, $\rho_s(y_l) = 0$ and $\rho_a(y_l) = (\zeta^{\mu(l)} - 1) y_l$ for $d_0 + 1 \le l \le d$, and consequently, $\mathrm{Ker}\,\rho_s = \langle y_{d_0+1}, \ldots, y_d\rangle$, $\mathrm{Ker}\,\rho_a = \langle y_1, \ldots, y_{d_0}\rangle$. Moreover $y_1 + \mathrm{Ker}\,\rho_s, \ldots, y_{d_0} + \mathrm{Ker}\,\rho_s$ and $y_{d_0+1} + \mathrm{Ker}\,\rho_a, \ldots, y_d + \mathrm{Ker}\,\rho_a$ are free $\mathbb{Z}_p$-bases of $(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p)/\mathrm{Ker}\,\rho_s$ and $(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p)/\mathrm{Ker}\,\rho_a$, respectively. Clearly, $\tilde{x}_1, \ldots, \tilde{x}_{d_s}$ is a free $\mathbb{Z}_p$-basis of $(\mathcal{X}/\mathrm{Ker}\,\rho_s) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p)/\mathrm{Ker}\,\rho_s$, and $\hat{x}_1, \ldots, \hat{x}_{d_a}$ is a free $\mathbb{Z}_p$-basis of $(\mathcal{X}/\mathrm{Ker}\,\rho_a) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}_p)/\mathrm{Ker}\,\rho_a$. It yields in particular $d_0 = d_s$. Denote by $W_s \in \mathrm{GL}_{d_s}(\mathbb{Z}_p)$ the transition matrix from $y_1 + \mathrm{Ker}\,\rho_s, \ldots, y_{d_s} + \mathrm{Ker}\,\rho_s$ to

$\tilde{x}_1, \ldots, \tilde{x}_{d_s}$ and by $W_a \in \mathrm{GL}_{d_a}(\mathbb{Z}_p)$ the transition matrix from $y_{d_s+1} + \operatorname{Ker} \rho_a, \ldots, y_d + \operatorname{Ker} \rho_a$ to $\hat{x}_1, \ldots, \hat{x}_{d_a}$.

The matrices $W^{-1}\chi(\sigma_1)W$ and $W^{-1}\chi(\sigma_2)W$ commute, therefore the former is a block diagonal matrix with two blocks of size $d_s \times d_s$ and $d_a \times d_a$. One can easily see that its upper-left block coincides with $W_s^{-1}\tilde{\chi}(\sigma_1)W_s$. It implies $\hat{U}_1 = W_s^t \tilde{U}_1 (W_s^t)^{-1}$. Thus

$$u_2 = pI_{n_2 d} - \begin{pmatrix} W_s^t \tilde{U}_1 (W_s^t)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \blacktriangle.$$

Now $(F_2, f_1 \circ f_2)$ is a universal fixed pair for $(\Phi, \mathrm{Gal}(K/\mathbb{Q}_p))$ by Proposition 3.3.

According to Proposition 2.3 (i),(iii), there exists a formal group law F_3 whose logarithm is of type

$$\begin{pmatrix} W_s^t & 0 \\ 0 & W_a^t \end{pmatrix}^{-1} u_2 \begin{pmatrix} W_s^t & 0 \\ 0 & W_a^t \end{pmatrix} = pI_d - \begin{pmatrix} \tilde{U}_1 & 0 \\ 0 & 0 \end{pmatrix} \blacktriangle,$$

and

$$f_3(X) = \begin{pmatrix} W_s^t & 0 \\ 0 & W_a^t \end{pmatrix} X$$

is an isomorphism from F_3 to F_2. Thus, $(F_3, f_1 \circ f_2 \circ f_3)$ is a universal fixed pair for $(\Phi, \mathrm{Gal}(K/\mathbb{Q}_p))$ with F_3 as required.

II. The homomorphism $\Phi \to \Phi'$ induced by η commutes with the actions of $\mathrm{Gal}(K/k)$ on Φ and Φ'. According to Proposition 6.2, its linear coefficient is equal to $C^t \otimes I_n$. Let W', F_1', Q_1', F_2', Q_2', τ', μ', W_s', W_a' be defined for Φ' similar to W, F_1, Q_1, F_2, Q_2, τ, μ, W_s, W_a. Then by Proposition 3.2, we get a homomorphism $F_1 \to F_1'$ whose linear coefficient Z_1 satisfies $Q_1' I_{nd', n_2 d'} Z_1 = (C^t \otimes I_n) Q_1 I_{nd, n_2 d}$. It implies that Z_1 is the upper-left $n_2 d' \times n_2 d$-submatrix of ${Q_1'}^{-1}(C^t \otimes I_n)Q_1$. An easy calculation shows that $Z_1 = C^t \otimes I_{n_2}$. The homomorphism $F_1 \to F_1'$ commutes with the actions of $\langle \sigma_2 \rangle$ on F_1 and F_1'. Then Proposition 3.2 yields a homomorphism $F_2 \to F_2'$ whose linear coefficient Z_2 satisfies $Q_2' I_{n_2 d', d'} Z_2 = (C^t \otimes I_{n_2}) Q_2 I_{n_2 d, d}$. Hence, Z_2 is the upper-left $d' \times d$-submatrix of ${Q_2'}^{-1}(C^t \otimes I_{n_2})Q_2 = L_{\tau'}^{-1}((W^{-1}CW')^t \otimes I_{n_2})L_\tau$.

Let $W^{-1}CW' = \{\hat{c}_{l,l'}\}_{1 \le l \le d; 1 \le l' \le d'}$, $Z_2 = \{\tilde{c}_{l',l}\}_{1 \le l' \le d'; 1 \le l \le d}$. Then direct computation shows that $\tilde{c}_{l',l} = \hat{c}_{l,l'} \delta_{\mu(l)}^{\mu'(l')}$. Since $(W^{-1}CW')(W'^{-1}\chi'(\sigma_2)W') = (W^{-1}\chi(\sigma_2)W)(W^{-1}CW')$, we obtain that if $\mu(l) \neq \mu'(l')$, then $\hat{c}_{l,l'} = 0$. It yields $Z_2 = (W^{-1}CW')^t$. Since $\mu(l) = \mu'(l') = 0$ for $1 \le l \le d_s$, $1 \le l' \le d_s'$ and $\mu(l) \neq 0$, $\mu'(l') \neq 0$, for $d_s + 1 \le l \le d$, $d_s' + 1 \le l' \le d'$, we obtain that Z_2 is a block-diagonal matrix with two blocks of size $d_s' \times d_s$ and $d_a' \times d_a$. One can easily see that the upper-left block is $(W_s^{-1} C_s W_s')^t$ and the lower-right block is $(W_a^{-1} C_a W_a')^t$. Finally, the linear coefficient of the composition $F_3 \xrightarrow{f_3} F_2 \longrightarrow F_2' \xrightarrow{(f_3')^{-1}} F_3'$, is equal to

$$\begin{pmatrix} (W_s')^t & 0 \\ 0 & (W_a')^t \end{pmatrix}^{-1} Z_2 \begin{pmatrix} W_s^t & 0 \\ 0 & W_a^t \end{pmatrix} = \begin{pmatrix} C_s^t & 0 \\ 0 & C_a^t \end{pmatrix}$$

as required. □

Corollary. *Suppose that the assumptions of Theorem 3 are satisfied.*

I. *$\hat{\mathcal{T}}$ is isomorphic to the direct sum of a p-divisible group and d_a copies of the additive formal group scheme.*

II. *If $K/\mathbb{Q}_p$ is unramified, then $\hat{\mathcal{T}}$ is represented by a formal group law whose logarithm is of type $pI_d - U_1 \blacktriangle$.*

III. *If $K/\mathbb{Q}_p$ is totally tamely ramified, then $\hat{\mathcal{T}}$ is isomorphic to the direct sum of d_s copies of the multiplicative formal group scheme and d_a copies of the additive formal group scheme.*

Part II of the above Corollary follows from Theorem 1.5 of [**DN**]. Besides, Theorems 0.1 and 1.3 of [**NX**] imply that part III holds for the reduction of the formal group scheme $\hat{\mathcal{T}}$.

A *deformation* over $\mathbb{Z}_p$ of a formal group law $\bar{F}$ over $\mathbb{F}_p$ is any formal group law over $\mathbb{Z}_p$ with reduction equal to $\bar{F}$. Two such deformations are called $\star$-*isomorphic* if there is an isomorphism between them with identity reduction.

PROPOSITION 7.7 ([**DG**, Proposition 9, Corollary of Theorem 2]). *Let $\bar{F}$ be a formal group law over $\mathbb{F}_p$ of dimension d and height h. Then there exists a formal group law Γ over $\mathbb{Z}_p[[t_1,\dots,t_{d(h-d)}]]$ such that for any deformation F of $\bar{F}$ over $\mathbb{Z}_p$ there is a unique formal group law over $\mathbb{Z}_p$ obtained from Γ by specialization of the variables $t_1,\dots,t_{d(h-d)}$ to $\mathbb{Z}_p$ which is $\star$-isomorphic to F. In particular, if $h=d$, then Γ is defined over $\mathbb{Z}_p$, and any deformation of $\bar{F}$ over $\mathbb{Z}_p$ is $\star$-isomorphic to Γ.*

LEMMA 7.8. *Let F, F' be d-dimensional formal group laws over $\mathbb{Z}_p$ with logarithms of types u, u', respectively, such that their reductions are isomorphic. If*

$$u = pI_d - \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \blacktriangle, \quad u' = pI_d - \begin{pmatrix} S' & 0 \\ 0 & 0 \end{pmatrix} \blacktriangle,$$

with matrices $S \in GL_{d_1}(\mathbb{Z}_p)$, $S' \in GL_{d_1'}(\mathbb{Z}_p)$, then F and F' are isomorphic.

Proof. By Theorem 6 of [**Ho2**], there exist $w, z \in \mathrm{GL}_d(\mathcal{E})$ such that $u'z = wu$. In particular, it implies $d_1' = d_1$. Denote by $\tilde{u}$, $\tilde{u}'$, $\tilde{z}$, $\tilde{w}$ the upper-left $d_1 \times d_1$-submatrices of u, u', z, w, respectively. Then we get $\tilde{u}'\tilde{z} = \tilde{w}\tilde{u}$. Let λ, λ' be of types $\tilde{u}$, $\tilde{u}'$, respectively. By Proposition 2.3 (i), λ and λ' are the logarithms of formal group laws $\tilde{F}$ and $\tilde{F}'$. Obviously, F is isomorphic to the direct sum of $\tilde{F}$ and $d-d_1$ copies of the additive formal group law, and F' is isomorphic to the direct sum of $\tilde{F}'$ and $d-d_1$ copies of the additive formal group law. Thus it remains to show that $\tilde{F}$ is isomorphic to $\tilde{F}'$. Let

$$z = \begin{pmatrix} \tilde{z} & \hat{z} \\ * & * \end{pmatrix},$$

with $\hat{z} \in \mathrm{M}_{d_1, d-d_1}(\mathcal{E})$. Since

$$u'z = wu \equiv \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \mod p,$$

we get $\tilde{u}'\hat{z} \equiv 0 \mod p$. Therefore $\hat{z} \equiv 0 \mod p$, and since z is invertible, $\tilde{z}$ is also invertible. Take $D \in \mathrm{GL}_{d_1}(\mathbb{Z}_p)$ such that $\tilde{z} = D \mod \blacktriangle$. Then $\lambda'' = D^{-1}\tilde{z}\lambda$ is of type $\tilde{u}\tilde{z}^{-1}D$, and hence, by Proposition 2.3 (i),(iii), the formal power series λ'' is the logarithm of a formal group law $\tilde{F}''$ over $\mathbb{Z}_p$ which is isomorphic to $\tilde{F}'$. Since $(\lambda^{-1} \circ \tilde{z}^{-1}D\lambda'')(X) = X$, Theorem 5 (ii) of [**Ho2**] implies that the reduction of $\tilde{F}$ coincides with that of $\tilde{F}''$. Since $\tilde{u} = pI_{d_1} - S\blacktriangle$ and S is invertible, $\tilde{F}$ is a p-divisible group of height d_1. Then by Proposition 7.7, any deformation over $\mathbb{Z}_p$ of the reduction of $\tilde{F}$ is isomorphic to $\tilde{F}$. In particular, $\tilde{F}''$ is isomorphic to $\tilde{F}$. □

THEOREM 4. *Let T, T' be tori over $\mathbb{Q}_p$ which are split over an abelian tamely ramified extension of $\mathbb{Q}_p$, and $\mathcal{T}$, $\mathcal{T}'$ be their Néron models. If the reductions of $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic, then $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are isomorphic.*

Proof. It is clear that the reductions of $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are isomorphic. Then by Theorem 3, $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are represented by formal group laws which satisfy the conditions of Lemma 7.8. Therefore $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are isomorphic. □

Remark. It is easy to give an example of tori T, T' such that $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are isomorphic, but the reductions of $\mathcal{T}$ and $\mathcal{T}'$ are not. Take K to be the unramified extension of $\mathbb{Q}_p$ of degree 2, and define $\chi, \chi'\colon \operatorname{Gal}(K/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{Z})$ as follows:

$$\chi(\Delta_p) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \chi'(\Delta_p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let T, T' be tori over $\mathbb{Q}_p$ corresponding to the representations χ, χ'. Then the connected components of the reductions of $\mathcal{T}$, $\mathcal{T}'$ are tori over $\mathbb{F}_p$ which correspond to representations $\bar{\chi}, \bar{\chi}'\colon \operatorname{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{Z})$ defined by

$$\bar{\chi}(\Delta_p) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{\chi}'(\Delta_p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, these representations are not isomorphic, and hence, the reductions of $\mathcal{T}$ and $\mathcal{T}'$ are not isomorphic. On the other hand, by Theorem 3, $\hat{\mathcal{T}}$, $\hat{\mathcal{T}}'$ can be represented by formal group laws with logarithms of types $u = pI_2 - \chi(\Delta_p)\blacktriangle$, $u' = pI_2 - \chi'(\Delta_p)\blacktriangle$, respectively. Since

$$u \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} u',$$

Proposition 2.3 (iii) implies that for $p \neq 2$, $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are isomorphic.

8. *One-dimensional tori over* $\mathbb{Q}$

We keep the notations of Section 6. Suppose that T is a non-split one-dimensional torus over $k = \mathbb{Q}$ which is split over a tamely ramified quadratic extension K of $\mathbb{Q}$. Then $d = 1$, $n = 2$, $\operatorname{Gal}(K/\mathbb{Q}) = \{\mathrm{id}_{\mathbb{Q}}, \sigma\}$ and the discriminant of $\mathcal{O}_K$ is odd, i.e. there exists $\xi \in \mathcal{O}_K$ such that $1, \xi$ form a free $\mathbb{Z}$-basis of $\mathcal{O}_K$ and $q = (\sigma(\xi) - \xi)^2$ is odd. Let $x^2 - rx + s \in \mathbb{Z}[X]$ be the minimal polynomial for ξ. Then $q = r^2 - 4s$. Finally, put $e_0 = 1$, $e_1 = \xi$.

Proposition 8.1. *The formal group law Λ is of p-type $v_p = pI_2 - V_p\blacktriangle_p$, where*

$$V_p = \begin{cases} I_2, & \text{if } \left(\frac{q}{p}\right) = 1 \\ \begin{pmatrix} 1 & r \\ 0 & -1 \end{pmatrix}, & \text{if } \left(\frac{q}{p}\right) = -1 \\ \begin{pmatrix} 1 & r/2 \\ 0 & 0 \end{pmatrix}, & \text{if } p \mid q \end{cases}.$$

Proof. Let $(p, q) = 1$. Then $\mathbb{Q}_p(\xi)/\mathbb{Q}_p$ is a trivial extension if $\left(\frac{q}{p}\right) = 1$, and an unramified extension of degree 2, if $\left(\frac{q}{p}\right) = -1$. Indeed, for $p \neq 2$, this follows from the definition of the Legendre symbol, and for $p = 2$, this is true, since $q \equiv 1 \mod 4$.

Denote $\varphi(z_1, z_2) = z_1 + \xi z_2$. By the Corollary from Proposition 6.1, we have $\Lambda_1 + \Lambda_2\xi = \mathbb{L}_m \circ \varphi$.

We consider the action of $\blacktriangle_p$ on $\mathbb{Q}_p(\xi)[[z_1, z_2]]$ introduced in Section 2. Then, according to Lemma 2.4,

$$(p - \blacktriangle_p)(\Lambda_1 + \Lambda_2\xi) \equiv ((p - \blacktriangle_p)\mathbb{L}_m) \circ \varphi \equiv 0 \mod p.$$

If $\left(\frac{q}{p}\right) = 1$, then $\blacktriangle_p(\xi) = \xi$, and we get $p\Lambda_1 - \blacktriangle_p\Lambda_1 \equiv 0 \mod p$ and $p\Lambda_2 - \blacktriangle_p\Lambda_2 \equiv 0 \mod p$. If $\left(\frac{q}{p}\right) = -1$, then $\blacktriangle_p(\xi) = r - \xi$, and we obtain $p\Lambda_1 - \blacktriangle_p(\Lambda_1 + r\Lambda_2) \equiv 0 \mod p$ and $p\Lambda_2 + \blacktriangle_p\Lambda_2 \equiv 0 \mod p$. Thus in each case, Λ is of the required p-type.

We proceed with the case $p \mid q$. Since $1, \xi$ is a free $\mathbb{Z}$-basis in $\mathcal{O}_K$, $p^2 \nmid q$. Let $\pi = r - 2\xi$. We have $\pi^2 = q$. Therefore, $\mathbb{Q}_p(\xi)/\mathbb{Q}_p$ is a totally ramified extension of degree 2, and e_0, e_1

form a free $\mathbb{Z}_p$-basis of $\mathbb{Z}_p[\xi]$. Then Proposition 6.1 implies that $\Phi = \mathcal{R}_{\mathbb{Z}_p[\xi]/\mathbb{Z}_p}((\mathbb{F}_m)^d_{\mathbb{Z}_p[\xi]})$. Take $e'_0 = 1$, $e'_1 = \pi$, and denote $\Phi' = \mathcal{R}'_{\mathbb{Z}_p[\xi]/\mathbb{Z}_p}((\mathbb{F}_m)^d_{\mathbb{Z}_p[\xi]})$. Let Λ' be the logarithm of Φ'. Then by Proposition 4.4, we get $\Lambda^{-1} \circ (I_d \otimes W)\Lambda' \in \mathrm{Hom}_{\mathbb{Z}_p}(\Phi', \Phi)$, where

$$W = \begin{pmatrix} 1 & r \\ 0 & -2 \end{pmatrix}.$$

Applying Proposition 7.2 for the data $d = 1$, $n_1 = 1$, $n_2 = 2$, we deduce that Λ' is of p-type $pI_2 - J_2\blacktriangle_p$. Hence, Proposition 2.3 (iii) implies that Λ is of p-type $pI_2 - WJ_2W^{-1}\blacktriangle_p$ just as required. □

Since $\sigma(\xi) = r - \xi$, in the basis e_0, e_1 the automorphism σ is given by the matrix $\begin{pmatrix} 1 & r \\ 0 & -1 \end{pmatrix}$. Therefore, $\psi(\sigma) = \begin{pmatrix} 1 & 0 \\ r & -1 \end{pmatrix}$. Since T is non-split, $\chi(\sigma) = -1$. Then $\theta(\sigma)^t = \begin{pmatrix} -1 & -r \\ 0 & 1 \end{pmatrix}$ by Proposition 6.3.

For a prime number p, let $\Xi(p) = \left(\frac{q}{p}\right)$, if $(p, q) = 1$; $\Xi(p) = 0$, if $p \mid q$. Further, let F_Ξ be defined as at the end of Section 2. Due to Proposition 2.5, F_Ξ is a formal group law over $\mathbb{Z}$. Finally, denote the formal group laws $F_{r,s}(x, y) = (x + y + rxy)(1 - sxy)^{-1}$ and $F_q(x, y) = x + y + \sqrt{q}xy$.

Proposition 8.2. *Let T be a non-split one-dimensional torus over $\mathbb{Q}$ which is split over a tamely ramified quadratic extension of $\mathbb{Q}$, and $\mathcal{T}$ be its Néron model. Then*

I. *The formal group law F_Ξ represents $\hat{\mathcal{T}}$;*

II. *The formal group law F_Ξ is strongly isomorphic to $F_{r,s}$ over $\mathbb{Z}$;*

III. *The formal group law F_Ξ is strongly isomorphic to F_q over $\mathbb{Z}[\xi]$.*

Proof. According to Theorem 2, a formal group law from a universal fixed pair for $(\Phi, \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}))$ represents $\hat{\mathcal{T}}$. Take

$$Q = \begin{pmatrix} -r & (r+1)/2 \\ 2 & -1 \end{pmatrix}.$$

Then condition (iii) of Theorem 1,I is obviously satisfied for $k = \mathbb{Q}_p$ and any prime p. Further, applying Proposition 8.1 one can directly check that the upper-left entry of $Q^{-1}v_pQ$ is equal to $pI_d - \Xi(p)\blacktriangle_p$. On the other hand, Proposition 2.5 implies that the logarithm of F_Ξ is of p-type $pI_d - \Xi(p)\blacktriangle_p$. Then by Proposition 3.5, there exists $f_\Xi \in \mathrm{Hom}_{\mathbb{Z}}(F_\Xi, \Phi)$ such that (F_Ξ, f_Ξ) is a universal fixed pair for $(\Phi, \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}))$. Thus F_Ξ represents $\hat{\mathcal{T}}$.

Proposition 6.1 implies that $\Phi = (\Phi_1, \Phi_2)$, where $\Phi_1(x_1, x_2; y_1, y_2) = x_1 + y_1 + x_1y_1 - sx_2y_2$, $\Phi_2(x_1, x_2; y_1, y_2) = x_2 + y_2 + x_1y_2 + x_2y_1 + rx_2y_2$. Besides, one can check that $g(x_1, x_2) = x_2(1 + x_1)^{-1}$ belongs to $\mathrm{Hom}_{\mathbb{Z}}(\Phi, F_{r,s})$ and its linear coefficient is $(0, 1) \in \mathrm{M}_{1,2}(\mathbb{Z})$. According to Proposition 2.3 (ii), for every prime p, there exists $u'_p \in \mathcal{E}_p$ such that the logarithm $\lambda_{r,s}$ of $F_{r,s}$ is of p-type u'_p. By Proposition 2.3 (iii), we have $u'_p(0, 1) = (\hat{w}_p, \tilde{w}_p)v_p$ for some $\hat{w}_p, \tilde{w}_p \in \mathcal{E}_p$. Applying Proposition 8.1 we obtain $\hat{w}_p = 0$ and $u'_p = \tilde{w}_p(p - \Xi(p)\blacktriangle_p)$. Therefore, $\tilde{w}_p \equiv 1 \mod \blacktriangle_p$, and $\lambda_{r,s}$ is of p-type $\tilde{w}_p^{-1}u'_p = p - \Xi(p)\blacktriangle_p$. Further, Proposition 2.3 (iii) implies that $\lambda_{r,s}^{-1} \circ \lambda_\Xi \in \mathrm{Hom}_{\mathbb{Z}_p}(F_\Xi, F_{r,s})$. Since this is true for any prime p, $\lambda_{r,s}^{-1} \circ \lambda_\Xi \in \mathrm{Hom}_{\mathbb{Z}}(F_\Xi, F_{r,s})$, i.e. F_Ξ and $F_{r,s}$ are strongly isomorphic over $\mathbb{Z}$.

Finally, if $\sqrt{q} = r - 2\xi$, then $x(1 + \xi x)^{-1} \in \mathrm{Hom}_{\mathbb{Z}[\xi]}(F_{r,s}, F_q)$. □

Part I of the above Proposition is a special case of Theorem 1.5 of [**DN**], and part III is precisely Theorem 4 of [**Ho1**].

9. *Tori split over a tamely ramified abelian extension of* $\mathbb{Q}$

We keep the notation of Section 6. Suppose that $k=\mathbb{Q}$ and $K/\mathbb{Q}$ is a tamely ramified abelian extension. Due to the Kronecker-Weber Theorem, one can replace K by a larger field so that $K=\mathbb{Q}(\xi)$, where ξ is a primitive q-th root of unity and $q=p_1\cdots p_m$ is the product of distinct primes p_i. Then K is the composite of the extensions $\mathbb{Q}(\xi_i)/\mathbb{Q}$, $1\le i\le m$, where $\xi_i=\xi^{q/p_i}$, and the degree of $K/\mathbb{Q}$ is $n=(p_1-1)\cdots(p_m-1)$. Denote $K_i=\mathbb{Q}_{p_i}(\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_m)$.

For any $1\le i\le m$, fix $s_i\in\mathbb{Z}$ such that s_i is a multiplicative generator modulo p_i, i.e. $s_i\not\equiv 0 \mod p_i$ and $s_i^\alpha\not\equiv 1 \mod p_i$ for any $1\le\alpha\le p_i-2$. Further, for any integer l relatively prime to p_i, take $0\le r_i(l)\le p_i-2$ such that $l\equiv s_i^{r_i(l)} \mod p_i$. Denote also $n_i=\prod_{j=1}^{i-1}(p_j-1)$ for $1\le i\le m$. Consider the bijection

$$\gamma\colon \prod_{i=1}^{m}\{0,\dots,p_i-2\}\to\{0,\dots,n-1\}$$

given by $\gamma(\alpha_1,\dots,\alpha_m)=\sum_{i=1}^m\alpha_i n_i$. Notice that for any matrices $U^{(i)}=\{a^{(i)}_{\alpha_i,\beta_i}\}_{0\le\alpha_i,\beta_i\le p_i-2}$, $1\le i\le m$, and for $U^{(1)}\otimes\cdots\otimes U^{(m)}=\{a_{s,t}\}_{0\le s,t\le n-1}$, we have $a_{\gamma(\alpha_1,\dots,\alpha_m),\gamma(\beta_1,\dots,\beta_m)}=a^{(1)}_{\alpha_1,\beta_1}\cdots a^{(m)}_{\alpha_m,\beta_m}$.

For $0\le\alpha_i\le p_i-2$, $1\le i\le m$, put $e_{\gamma(\alpha_1,\dots,\alpha_m)}=\prod_{i=1}^m\xi_i^{s_i^{\alpha_i}}$. Obviously, $e_0,\dots,e_{n-1}$ is a free $\mathbb{Z}$-basis of $\mathcal{O}_K=\mathbb{Z}[\xi]$.

Proposition 9.1. *The formal group law Λ is of p-type $v_p=pI_{nd}-I_d\otimes V_p\blacktriangle_p$, where $V_p=P^{r_1(p)}_{p_1-1}\otimes\cdots\otimes P^{r_m(p)}_{p_m-1}$ if $p\ne p_i$ for any $i=1,\dots,m$ and*

$$V_{p_i}=P^{r_1(p_i)}_{p_1-1}\otimes\cdots\otimes P^{r_{i-1}(p_i)}_{p_{i-1}-1}\otimes(p_iI_{p_i-1}-J'_{p_i-1})\otimes P^{r_{i+1}(p_i)}_{p_{i+1}-1}\otimes\cdots\otimes P^{r_m(p_i)}_{p_m-1}.$$

Proof. Let $p\ne p_i$ for any $1\le i\le m$. Then $\mathbb{Q}_p(\xi)/\mathbb{Q}_p$ is unramified. For any $1\le l\le d$, consider $\varphi_l=\sum_{i=0}^{n-1}z_{id+l}e_i$. By Corollary from Proposition 6.1, we have $\sum_{j=0}^{n-1}\Lambda_{jd+l}e_j=\mathbb{L}_m\circ\varphi_l$ for any $l=1,\dots,d$.

We consider the action of $\blacktriangle_p$ on $\mathbb{Q}_p(\xi)[[z_1,\dots,z_{nd}]]$ introduced in Section 2. Then applying Lemma 2.4 we get

$$(p-\blacktriangle_p)\sum_{j=0}^{n-1}\Lambda_{jd+l}e_j\equiv((p-\blacktriangle_p)\mathbb{L}_m)\circ\varphi_l\equiv 0 \mod p.$$

Hence, $\sum_{j=0}^{n-1}p\Lambda_{jd+l}e_j\equiv\sum_{j=0}^{n-1}(\blacktriangle_p\Lambda_{jd+l})e_j^p \mod p$. If $0\le\alpha_i,\alpha_i'\le p_i-2$, $1\le i\le m$, are such that $\alpha_i'\equiv\alpha_i+r_i(p) \mod p_i-1$, then $e^p_{\gamma(\alpha_1,\dots,\alpha_m)}=e_{\gamma(\alpha_1',\dots,\alpha_m')}$. Therefore, we obtain

$$p\Lambda_{\gamma(\alpha_1',\dots,\alpha_m')d+l}-\blacktriangle_p\Lambda_{\gamma(\alpha_1,\dots,\alpha_m)d+l}\equiv 0 \mod p.$$

Thus Λ is of the required p-type.

We proceed with the case $p=p_i$. Since $e_0,\dots,e_{n-1}$ is a free $\mathbb{Z}_{p_i}$-basis of $\mathbb{Z}_{p_i}[\xi]$, Proposition 6.1 implies that $\Phi=\mathcal{R}_{\mathbb{Z}_{p_i}[\xi]/\mathbb{Z}_{p_i}}((\mathbb{F}_m)^d_{\mathbb{Z}_{p_i}[\xi]})$. Take π_i such that $\pi_i^{p_i-1}=-p_i$. Then, by the Lubin–Tate theory, $\mathbb{Q}_{p_i}(\xi_i)=\mathbb{Q}_{p_i}(\pi_i)$ and there exists $W=\{w_{\alpha,\beta}\}_{0\le\alpha,\beta\le p_i-2}\in\mathrm{GL}_{p_i-1}(\mathbb{Z}_{p_i})$ such that $\pi_i^\beta=\sum_{\alpha=0}^{p_i-2}w_{\alpha,\beta}\xi_i^{s_i^\alpha}$ for any $0\le\beta\le p_i-2$.

Take $e_0',\dots,e_{n-1}'$, another free $\mathbb{Z}_{p_i}$-basis of $\mathbb{Z}_{p_i}[\xi]$, where

$$e'_{\gamma(\alpha_1,\dots,\alpha_m)}=\pi_i^{\alpha_i}\prod_{1\le j\le m,\, j\ne i}\xi_j^{s_j^{\alpha_j}},\qquad 0\le\alpha_i\le p_i-2, 1\le i\le m.$$

Denote for this basis $\Phi' = \mathcal{R}'_{\mathbb{Z}_p[\xi]/\mathbb{Z}_p}((\mathbb{F}_m)^d_{\mathbb{Z}_p[\xi]})$. Let Λ' be the logarithm of Φ', then Proposition 4.4 implies that $\Lambda^{-1} \circ (I_d \otimes \hat{W})\Lambda' \in \mathrm{Hom}_{\mathbb{Z}_p}(\Phi', \Phi)$, where

$$\hat{W} = I_{p_1-1} \otimes \cdots \otimes I_{p_{i-1}-1} \otimes W \otimes I_{p_{i+1}-1} \otimes \cdots \otimes I_{p_m-1}.$$

For any $1 \le l \le d$, consider $\varphi'_l = \sum_{0\le i\le n-1} z_{id+l}e'_i$. By Corollary from Proposition 6.1, we have $\sum_{0\le i\le n-1} \Lambda'_{id+l}e'_i = \mathbb{L}_m \circ \varphi'_l$.

Remind that for any $\kappa \in \mathbb{Q}_{p_i}(\xi)[[t_1, \dots, t_n]]$, there exist unique $\kappa^{(j)} \in K_i[[t_1, \dots, t_n]]$, for any $0 \le j \le p_i - 2$, such that $\kappa = \sum_{j=0}^{p_i-2} \kappa^{(j)}\pi_i^j$. Thus for any $0 \le \alpha_i \le p_i - 2$,

$$\sum_{\substack{0\le\alpha_j\le p_j-2\\ j\ne i}} \Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \prod_{j\ne i} \xi_j^{s_j^{\alpha_j}} = (\mathbb{L}_m \circ \varphi'_l)^{(\alpha_i)}.$$

If $\alpha_i \ne 0$, then by Lemma 7.1

$$\sum_{\substack{0\le\alpha_j\le p_j-2\\ j\ne i}} \Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \prod_{j\ne i} \xi_j^{s_j^{\alpha_j}} \equiv \mathbb{L}_m^{(\alpha_i)} \circ \varphi_l'^{(0)} = 0 \mod \mathbb{Z}_{p_i}[\xi],$$

whence $p_i\Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \equiv 0 \mod p_i$.

If $\alpha_i = 0$, then Lemma 7.1 implies

$$\sum_{\substack{0\le\alpha_j\le p_j-2\\ j\ne i}} \Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \prod_{j\ne i} \xi_j^{s_j^{\alpha_j}} \equiv \mathbb{L}_m \circ \varphi_l'^{(0)} \mod p_i.$$

We consider the action of $\blacktriangle_{p_i}$ on $K_i[[z_1, \dots, z_{nd}]]$ introduced in Section 2. Then applying Lemma 2.4 we get

$$(p_i - \blacktriangle_{p_i}) \sum_{\substack{0\le\alpha_j\le p_j-2\\ j\ne i}} \Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \prod_{j\ne i} \xi_j^{s_j^{\alpha_j}} \equiv (p_i - \blacktriangle_{p_i}) \left(\mathbb{L}_m \circ \varphi_l'^{(0)}\right) \equiv$$

$$((p_i - \blacktriangle_{p_i})\mathbb{L}_m) \circ \varphi_l'^{(0)} \equiv 0 \mod p_i.$$

Therefore,

$$p_i \sum_{\substack{0\le\alpha_j\le p_j-2\\ j\ne i}} \Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \prod_{j\ne i} \xi_j^{s_j^{\alpha_j}} \equiv \sum_{\substack{0\le\alpha_j\le p_j-2\\ j\ne i}} \blacktriangle_{p_i}\Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \prod_{j\ne i} \xi_j^{s_j^{\alpha_j+r_j(p_i)}} \mod p_i.$$

As before, choose $0 \le \alpha'_j \le p_j - 2$ for $j = 1, \dots, i-1, i+1, \dots, m$ such that $\alpha'_j \equiv \alpha_j + r_j(p_i) \mod p_j - 1$ and put $\alpha_i = 0$. It gives

$$p_i\Lambda'_{\gamma(\alpha'_1,\dots,\alpha'_m)d+l} \equiv \blacktriangle_{p_i}\Lambda'_{\gamma(\alpha_1,\dots,\alpha_m)d+l} \mod p_i.$$

Thus Λ' is of a p_i-type $v' = p_iI_{nd} - I_d \otimes V'\blacktriangle_{p_i}$, where

$$V' = P_{p_1-1}^{r_1(p_i)} \otimes \cdots \otimes P_{p_{i-1}-1}^{r_{i-1}(p_i)} \otimes J_{p_i-1} \otimes P_{p_{i+1}-1}^{r_{i+1}(p_i)} \otimes \cdots \otimes P_{p_m-1}^{r_m(p_i)}.$$

Hence, by Proposition 2.3 (iii), Λ is of p_i-type $v = (I_d \otimes \hat{W})v'(I_d \otimes \hat{W})^{-1} = p_iI_{nd} - I_d \otimes V\blacktriangle_{p_i}$, where $V = \hat{W}V'\hat{W}^{-1}$.

Since $1 = \sum_{\alpha=0}^{p_i-2}(-\xi_i^{s_i^\alpha})$, we get $w_{\alpha,0} = -1$ for any $0 \le \alpha \le p_i - 2$, i.e. all entries of the first column of W are equal to -1. Further, let $W^{-1} = \{w'_{\alpha,\beta}\}_{0\le\alpha,\beta\le p_i-2}$. For any $0 \le \beta \le p_i - 2$, there exists $\tau_\beta \in \mathrm{Gal}(\mathbb{Q}_{p_i}(\xi_i)/\mathbb{Q}_{p_i})$ such that $\tau_\beta(\xi_i) = \xi_i^{s_i^\beta}$. Denote $\zeta_\beta = \tau_\beta(\pi_i)/\pi_i$. Then $\zeta_\beta^{p_i-1} = 1$, $\zeta_\beta \in \mathbb{Z}_{p_i}$, and hence,

$$\tau_\beta(\xi_i) = \sum_{\alpha=0}^{p_i-2} w'_{\alpha,0}\tau_\beta(\pi_i)^\alpha = \sum_{\alpha=0}^{p_i-2} w'_{\alpha,0}\zeta_\beta^\alpha\pi_i^\alpha$$

which implies $w'_{\alpha,\beta} = \zeta_\beta^\alpha w'_{\alpha,0}$. Since $\sum_{\beta=0}^{p_i-2} \zeta_\beta^\alpha = 0$ for any $1 \le \alpha \le p_i - 2$, we get

$$-1 = \sum_{\beta=0}^{p_i-2} \xi_i^{s_i^\beta} = \sum_{\alpha=0}^{p_i-2} \sum_{\beta=0}^{p_i-2} \zeta_\beta^\alpha w'_{\alpha,0} \pi_i^\alpha = (p_i - 1) w'_{0,0},$$

and then $w'_{0,\beta} = w'_{0,0} = -(p_i - 1)^{-1}$, i.e. all entries of the first row of W^{-1} are equal to $-(p_i - 1)^{-1}$. Therefore, $W J_{p_i-1} W^{-1} = (p_i - 1)^{-1} J'_{p_i-1}$, and thus

$$V = P_{p_1-1}^{r_1(p_i)} \otimes \cdots \otimes P_{p_{i-1}-1}^{r_{i-1}(p_i)} \otimes (p_i - 1)^{-1} J'_{p_i-1} \otimes P_{p_{i+1}-1}^{r_{i+1}(p_i)} \otimes \cdots \otimes P_{p_m-1}^{r_m(p_i)}.$$

Clearly, Λ is also of p_i-type $(I_{nd} - I_d \otimes W' \blacktriangle_{p_i})v$, where

$$W' = P_{p_1-1}^{r_1(p_i)} \otimes \cdots \otimes P_{p_{i-1}-1}^{r_{i-1}(p_i)} \otimes \left(I_{p_i-1} - (p_i - 1)^{-1} J'_{p_i-1}\right) \otimes P_{p_{i+1}-1}^{r_{i+1}(p_i)} \otimes \cdots \otimes P_{p_m-1}^{r_m(p_i)}.$$

Finally, since $\left(I_{p_i-1} - (p_i - 1)^{-1} J'_{p_i-1}\right) J'_{p_i-1} = 0$, we obtain that Λ is of the required p_i-type. $\square$

Let $\sigma_i \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}), 1 \le i \le m$, be such that $\sigma_i(\xi_i) = \xi_i^{s_i}$, $\sigma_i(\xi_j) = \xi_j$, for $j \ne i$. They generate $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ and, in the basis $e_0, \ldots, e_{n-1}$, are given by the matrices

$$\hat{P}_i = I_{p_1-1} \otimes \cdots \otimes I_{p_{i-1}-1} \otimes P_{p_i-1} \otimes I_{p_{i+1}-1} \otimes \cdots \otimes I_{p_m-1}.$$

Therefore, $\psi(\sigma_i) = (\hat{P}_i^{-1})^t = \hat{P}_i$. Denote $U_i = \chi(\sigma_i)^t$. Then $U_i U_j = U_j U_i$ and by Proposition 6.3, $\theta(\sigma_i)^t = U_i \otimes \hat{P}_i^t$.

From now on, we employ the "matrix-of-matrices" representation described in Introduction. Denote $D_i = U_i \otimes \hat{P}_i^t - I_{nd}$. Then, under our convention, $D_i = \{d_{s,t}^{(i)}\} \in \mathrm{M}_n(\mathrm{M}_d(\mathbb{Z}))$, where $d_{s,t}^{(i)} = \delta_s^{t-n_i} U_i - \delta_s^t I_d$ for $n_i = \prod_{j=1}^{i-1}(p_j - 1) = \gamma(\varepsilon_i)$, and ε_i stands for the m-tuple $(0, \ldots, 1, \ldots, 0)$ with unity on the i-th position.

Lemma 9.2. *Denote $U^{\langle\alpha\rangle} = U_1^{-\alpha_1} \cdots U_m^{-\alpha_m}$ for $0 \le \alpha_i \le p_i - 2$ and consider the matrix $Q = \{q_{s,t}\} \in \mathrm{GL}_n(\mathrm{M}_d(\mathbb{Z}))$, where*

$$q_{\gamma(\alpha),\gamma(\beta)} = \begin{cases} U^{\langle\alpha\rangle}, & \text{if } \gamma(\beta) = 0 \text{ and } \gamma(\alpha) \ne 0 \\ \delta_{\gamma(\alpha)}^{\gamma(\beta)} I_d, & \text{otherwise} \end{cases}.$$

Then, for any $1 \le i \le m$,

$$Q^{-1} D_i Q = \begin{pmatrix} 0 & \hat{D}_i \\ 0 & \tilde{D}_i \end{pmatrix},$$

where $\hat{D}_i \in \mathrm{M}_{1,n-1}(\mathrm{M}_d(\mathbb{Z}))$ and $\tilde{D}_i \in \mathrm{M}_{n-1}(\mathrm{M}_d(\mathbb{Z}))$.

Moreover, for any prime p there exist $\hat{C}_i^{(p)} \in \mathrm{M}_{n-1,1}(\mathrm{M}_d(\mathbb{Z}_p))$ and $\tilde{C}_i^{(p)} \in \mathrm{M}_{n-1}(\mathrm{M}_d(\mathbb{Z}_p))$ such that $\sum_{i=1}^m \hat{C}_i^{(p)} \hat{D}_i + \tilde{C}_i^{(p)} \tilde{D}_i \in \mathrm{GL}_{n-1}(\mathrm{M}_d(\mathbb{Z}_p))$.

Proof. Observe first that $Q^{-1} = \{q'_{s,t}\} \in \mathrm{GL}_n(\mathrm{M}_d(\mathbb{Z}))$ is the matrix given by $q'_{s,t} = -q_{s,t}$ if $s = 0$ and $t \ne 0$, and $q'_{s,t} = q_{s,t}$ otherwise.

We have $Q^{-1} D_i Q = \{a_{s,t}^{(i)}\}_{0 \le s,t \le n-1}$, where

$$a_{s,t}^{(i)} = \sum_{s',t'=0}^{n-1} q'_{s,s'} d_{s',t'}^{(i)} q_{t',t}.$$

Direct calculation shows that

$$a_{0,0}^{(i)} = -I_d + U_i U^{\langle\varepsilon_i\rangle} = -I_d + U_i U_i^{-1} = 0$$

and

$$a^{(i)}_{\gamma(\alpha),0} = U^{\langle\alpha\rangle} - U^{\langle\alpha\rangle}U_iU^{\langle\varepsilon_i\rangle} = 0$$

if $\gamma(\alpha) \neq 0$, proving the first assertion.

In order to prove the existence of $\hat{C}^{(p)}_i$ and $\tilde{C}^{(p)}_i$, we need to compute explicitly the matrices $\hat{D}_i = \{\hat{a}^{(i)}_t\}_{1\le t\le n-1}$ and $\tilde{D}_i = \{a^{(i)}_{s,t}\}_{1\le s,t\le n-1}$. Here we get

$$\hat{a}^{(i)}_t = a^{(i)}_{0,t} = \delta^t_{n_i}U_i$$

and

$$a^{(i)}_{\gamma(\alpha),\gamma(\beta)} = -\delta^{\gamma(\beta)}_{n_i}U_iU^{\langle\alpha\rangle} + \delta^{\gamma(\beta)}_{\gamma(\alpha)+n_i}U_i - \delta^{\gamma(\beta)}_{\gamma(\alpha)}I_d$$

for $\gamma(\alpha) \neq 0$ and $\gamma(\beta) \neq 0$.

By Lemma 3.4, we only need to prove that, for any prime p

$$\left(\cap^m_{i=1} \mathrm{Ker}\ (\hat{D}_i \otimes \mathbb{F}_p)\right) \cap \left(\cap^m_{i=1} \mathrm{Ker}\ (\tilde{D}_i \otimes \mathbb{F}_p)\right) = \{0\}.$$

Let $x_s \in \mathbb{F}^d_p$, $1 \le s \le n-1$ be such that $(x_1,\dots,x_{n-1})$ lies in the intersection of the kernels. Denote $\overline{U}_i = U_i \otimes \mathbb{F}_p \in \mathrm{GL}_d(\mathbb{F}_p)$, $\overline{U}^{\langle\alpha\rangle} = U^{\langle\alpha\rangle} \otimes \mathbb{F}_p \in \mathrm{GL}_d(\mathbb{F}_p)$. Then, for any $1 \le i \le m$, we have

$$\begin{cases} \overline{U}_i x_{n_i} = 0 \\ -\overline{U}_i\overline{U}^{\langle\alpha\rangle}x_{n_i} + \overline{U}_i x_{\gamma(\alpha)+n_i} - x_{\gamma(\alpha)} = 0 \end{cases}$$

which gives

$$\begin{cases} x_{n_i} = 0 \\ \overline{U}_i x_{\gamma(\alpha)+n_i} = x_{\gamma(\alpha)} \end{cases}.$$

This implies $x_s = 0$ for any $1 \le s \le n-1$ as required. □

For a prime number p, define $\Xi(p) \in M_d(\mathbb{Z})$ in the following way: if $p \neq p_i$ for any $1 \le i \le m$, put $\Xi(p) = \prod^m_{i=1} U^{r_i(p)}_i = \chi(\Delta_p)^t$, where $\Delta_p \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ is the Frobenius automorphism corresponding to p; if $p = p_i$, put

$$\Xi(p_i) = \left(p_iI_d - \sum^{q-2}_{j=0} U^j_i\right) \prod_{1\le j\le m,\, j\neq i} U^{r_j(p_i)}_j = \left(p_iI_d - \sum_{\tau\in G_i} \chi(\tau)^t\right)\chi(\Delta_{p_i})^t,$$

where $G_i = \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}(\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_m))$, and $\Delta_{p_i} \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}(\xi_i))$ is the Frobenius automorphism corresponding to p_i. Further, let F_Ξ be defined as at the end of Section 2. Due to Proposition 2.5, F_Ξ is a formal group law over $\mathbb{Z}$.

Let F'_Ξ be defined for the torus T' similar to F_Ξ.

THEOREM 5. *Let T, T' be tori over $\mathbb{Q}$ which are split over an abelian tamely ramified extension of $\mathbb{Q}$, $\mathcal{T}$, $\mathcal{T}'$ be their Néron models, and $\eta\colon T \to T'$ be a morphism. Then*

I. *The formal group law F_Ξ represents $\hat{\mathcal{T}}$.*

II. *The linear coefficient of the homomorphism $F_\Xi \to F'_\Xi$ induced by $\eta\colon T \to T'$ is C^t.*

Proof. I. According to Theorem 2, a formal group law from a universal fixed pair for $(\Phi, \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}))$ represents $\hat{\mathcal{T}}$. Take $Q \in \mathrm{GL}_{nd}(\mathbb{Z})$ as in Lemma 9.2. Then condition (iii) of Theorem 1,I is satisfied for $k = \mathbb{Q}_p$ and any prime p. Further, we calculate the upper-left $d \times d$-submatrix u_p of $Q^{-1}v_pQ$ with the aid of Proposition 9.1.

If $p \neq p_i$ for $1 \leq i \leq m$, define $r(p) = (r_1(p), \ldots, r_m(p))$. Then

$$u_p = \sum_{\alpha,\beta} \delta_0^{\gamma(\beta)} I_d \left(p\delta_{\gamma(\beta)}^{\gamma(\alpha)} I_d - \delta_{\gamma(\beta - r(p))}^{\gamma(\alpha)} I_d \blacktriangle_p \right) U^{\langle\alpha\rangle} = pI_d - U_1^{r_1(p)} \cdots U_m^{r_m(p)} \blacktriangle_p.$$

If $p = p_i$, denote

$$\delta(p_i, \alpha, \beta) = \delta_{\beta_1 - r_1(p_i)}^{\alpha_1} \cdots \delta_{\beta_{i-1} - r_{i-1}(p_i)}^{\alpha_{i-1}} \delta_{\beta_{i+1} - r_{i+1}(p_i)}^{\alpha_{i+1}} \cdots \delta_{\beta_m - r_m(p_i)}^{\alpha_m}.$$

Then

$$u_{p_i} = \sum_{\alpha,\beta} \delta_0^{\gamma(\beta)} I_d \left(p_i \delta_{\gamma(\beta)}^{\gamma(\alpha)} I_d - (p_i \delta_{\beta_i}^{\alpha_i} - 1) \delta(p_i, \alpha, \beta) I_d \blacktriangle_{p_i} \right) U^{\langle\alpha\rangle}$$

$$= p_i I_d - U_1^{r_1(p_i)} \cdots U_{i-1}^{r_{i-1}(p_i)} U_{i+1}^{r_{i+1}(p_i)} \cdots U_m^{r_m(p_i)} \left(p_i I_d - \sum_{s=0}^{p_i - 2} U_i^s \right) \blacktriangle_{p_i}.$$

Obviously, in both cases $u_p = pI_d - \Xi(p)\blacktriangle_p$.

On the other hand, Proposition 2.5 implies that the logarithm of F_Ξ is of p-type $pI_d - \Xi(p)\blacktriangle_p$. Then by Proposition 3.5, there exists $f_\Xi \in \mathrm{Hom}_{\mathbb{Z}}(F_\Xi, \Phi)$ such that the linear coefficient of f_Ξ is $QI_{nd,d}$, and (F_Ξ, f_Ξ) is a universal fixed pair for $(\Phi, \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}))$. Thus F_Ξ represents $\hat{\mathcal{T}}$.

II. The homomorphism $\Phi \to \Phi'$ induced by η commutes with the actions of $\mathrm{Gal}(K/k)$ on Φ and Φ'. According to Proposition 6.2, the linear coefficient of this homomorphism is equal to $C^t \otimes I_n$. Let (F_Ξ, f_Ξ) and (F'_Ξ, f'_Ξ) be as above. Then by Proposition 3.2, we get a homomorphism $F_\Xi \to F'_\Xi$ whose linear coefficient Z satisfies $Q' I_{nd',d'} Z = (C^t \otimes I_n) Q I_{nd,d}$. This implies that Z is the upper-left $d' \times d$-submatrix of ${Q'}^{-1}(C^t \otimes I_n)Q$. Thus we obtain

$$Z = \sum_{\alpha,\beta} \delta_0^{\gamma}(\beta) I_{d'} \delta_{\gamma(\alpha)}^{\gamma(\beta)} U^{\langle\beta\rangle} = C^t.$$

Theorem 6. *Let T, T' be tori over $\mathbb{Q}$ which are split over an abelian tamely ramified extension of $\mathbb{Q}$, and $\mathcal{T}$, $\mathcal{T}'$ be their Néron models. Then the natural homomorphism $\mathrm{Hom}_{\mathbb{Q}}(T, T') \to \mathrm{Hom}_{\mathbb{Z}}(F_\Xi, F'_\Xi)$ is an isomorphism.*

Proof. Theorem 5 implies the injectivity. Take a homomorphism from F_Ξ to F'_Ξ and denote its linear coefficient by E. If $p \neq p_i$ for any $1 \leq i \leq m$, then F_Ξ and $F_{\Xi'}$ are of p-types $pI_d - \chi(\Delta_p)^t \blacktriangle_p$ and $pI_d - \chi'(\Delta_p)^t \blacktriangle_p$, respectively, where $\Delta_p \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ is the p-Frobenius. By Proposition 2.3 (iii), we get $\chi'(\Delta_p)^t E = E\chi(\Delta_p)^t$. Since any element of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ is the p-Frobenius for some prime p, the matrix $E^t \in \mathrm{M}_{d,d'}(\mathbb{Z})$ defines a homomorphism of $\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$-modules from $\mathcal{X}'$ to $\mathcal{X}$ which gives rise to a morphism from T to T'. According to Theorem 5, this morphism is the inverse image of the homomorphism $F_\Xi \to F'_\Xi$ taken in the beginning of the proof. This proves the surjectivity. □

Corollary. *Suppose that the assumptions of Theorem 6 are satisfied. If $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ are isomorphic, then T and T' are isomorphic.*

Comparing the above Corollary with Theorem 4 and the subsequent remark we see that while in the local case, the completion of the Néron model for a torus contains even less information than its reduction, in the global case the completion of the Néron model determines the torus uniquely up to isomorphism.

References

BLR S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*, Springer-Verlag, New-York Heidelberg Berlin, 1990.

CG N. Childress and D. Grant, Formal groups of twisted multiplicative groups and L-series, *Proc. Symp. Pure Math.* 58, Part 2, Amer. Math. Soc., Providence, RI, 1995, pp. 89-102.

De O. Demchenko, Formal groups over p-adic rings of integers with small ramification and distinguished isogenies, *St.Petersburg Math. J.* 14 (2003) 405–428.

DG O. Demchenko and A. Gurevich, On the moduli space of deformations of a p-divisible group, to appear.

DN C. Deninger and E. Nart, Formal groups and L-series, *Comment. Math. Helvetici* 65 (1990) 318–333.

Ed B. Edixhoven, Néron models and tame ramification, *Comp. Math.* 81 (1992) 291–306.

Ga D. Grant, Geometric proofs of reciprocity laws, *J. reine angew. Math.* 586 (2005), 91-124.

Ho1 T. Honda, Formal groups and zeta-functions, *Osaka J. of Math.* 5 (1968) 199–213.

Ho2 T. Honda, On the theory of commutative formal groups, *J. of Math. Soc. Japan* 22 (1970) 213–246.

Ib T. Ibukiyama, Formal groups and L-functions, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 21 (1974) 249–262.

Na E. Nart, Formal group laws for certain formal groups arising from modular curves, *Comp. Math.* 85 (1993) 109–119.

NX E. Nart and X. Xarles, Additive reduction of algebraic tori, *Arch. Math.* 57 (1991) 460–466.

Yu N. Yui, A note on formal groups and zeta functions, *J. reine angew. Math.* 303/304 (1978), 159-169

Wa W. Waterhouse, *Introduction to affine group schemes*, Graduate texts in mathematics, Springer-Verlag, New-York Heidelberg Berlin, 1979.

Oleg Demchenko
Department of Mathematics and Mechanics
St.Petersburg State University
Universitetski pr. 28, Staryj Petergof
198504, St.Petersburg, Russia

vasja@eu.spb.ru

Alexander Gurevich
Department of Mathematics
Bar-Ilan University
52900, Ramat-Gan, Israel

gurevia1@macs.biu.ac.il

Xavier Xarles
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193, Bellaterra, Barcelona, Catalonia

xarles@mat.uab.cat