Introduction to Surface Group Representations and Higgs Bundles
Assignment 8

Weizmann Institute
First Semester 2017-2018

There is no formal submission of the assignments but you must work on them.
One of the students will present a solution and we will discuss alternatives.

Problem 1.

You are not expected to work on this completely independently, which you can of course if you wish, but to find references for them and look at the proof.

- Look for the definition of partitions of unity and use the bump functions of Assignment 3 to show the existence or partitions of unity on a topological or smooth manifold.
- Find how partitions of unity can be used to define a riemannian metric on a manifold, that is, a smoothly varying family of positive definite symmetric inner products $\langle \cdot, \cdot \rangle_p$ on T_pM for each p. (The smoothness can be easily stated as giving a (smooth) section of the bundle $M \to \mathrm{Sym}^2 T^*M$.)
- Find out if there exist partitions for analytic and complex manifolds. Why?

Problem 2.

A complex structure on a real vector space is an endomorphism $J : V \to V$ such that $J^2 = -\,\mathrm{Id}$. An almost complex structure on a manifold M is a smootly varying family of complex structures on T_pM for each $p \in M$.

Let Σ be a compact orientable surface.

- Given a riemannian metric on Σ, find a way to define an almost complex structure on Σ.
- What happens if we try to do it the other way round? Does an almost complex structure determine a metric? If not, how far is it from determining a metric?
- Look for the definition of integrability of almost complex structures and why any almost complex structure on a surface is necessarily integrable, and hence defines a complex structure.