Introduction to Surface Group Representations and Higgs Bundles
Assignment 5

Weizmann Institute
First Semester 2017-2018

There is no formal submission of the assignments but you must work on them.
One of the students will present a solution and we will discuss alternatives.

Problem 1.

Consider a vector or a principal G-bundle over M. Let $\{U_i\}$ be an open cover for which there are trivializations $\{\varphi_i\}$ whose transition maps are $\{g_{ij}\}$.

- Prove that by using a trivialization φ_i over $U_i \subset M$, a section of a vector or a principal G-bundle over M can be seen, on U_i, as a map $s_i : U_i \to V$ or $s_j : U_i \to G$, respectively.
- Let $\{s_i : U_i \to V\}$ or $\{s_i : U_i \to G\}$ be a collection of smooth maps. What do they have to satisfy in order to define a global section of a vector or a principal G-bundle over M?

Problem 2.

Around any point p of a surface S we can choose a neighbourhood U_1 of p and a chart c_1 mapping U_1 to an open ball centered at the origin of $\mathbb{R}^2$, that is $c(p) = 0$. Define $U_2 := S \setminus \{p\}$. The intersection U_{12}, an annulus around p, is mapped by c_1 to the ball in $\mathbb{R}^2$ without the origin. If we see this ball in $\mathbb{C} \cong \mathbb{R}^2$, we have a map $\varphi : U_{12} \to \mathbb{C}^*$. We define a vector bundle with fibre $\mathbb{C} \cong \mathbb{R}^2$ on S by giving a cocycle of transition maps for the open cover $\{U_1, U_2\}$. It is enough to give only one transition map, say, g_{12}. Define $g_{12} : U_{12} \to \mathrm{GL}(\mathbb{C})$ by $g_{12}(x)(z) = \varphi(x)z$, for $x \in U_{12}$ and $z \in \mathbb{C}$, and call the resulting bundle L_p.

- Define a section of L_p that vanishes only at the point p.