Introduction to Surface Group Representations and Higgs Bundles
Assignment 4

Weizmann Institute
First Semester 2017-2018

There is no formal submission of the assignments but you must work on them.
One of the students will present a solution and we will discuss alternatives.

Problem 1. A principal G-bundle for a Lie group G is defined as a fibre bundle $E \to M$ with typical fibre G such that the transition maps are of the form $U_{ij} \to G \subset \mathrm{Diff}(G)$, where G acts on the fibre G by left multiplication. It could have made more sense to define it differently, as a fibre bundle $E \to M$ with typical fibre G, such that the transition maps are maps $U_{ij} \to \mathrm{Aut}(G) \subset \mathrm{Diff}(G)$, where $\mathrm{Aut}(G)$ are the diffeomorphisms of G that are also group homomorphisms.

- Prove that any bundle with the alternative definition such that the group G acts transitively on the fibres of $E$is a trivial bundle.
- Consider the Klein bottle as an S^1-bundle over S^1. Show that it is a bundle with the alternative definition but it is not trivial.

Problem 2.

- Prove that a vector bundle E is trivial if and only if there exist $n = \mathrm{rk}\, E$ global sections $s_1, \dots, s_n$ that are linearly independent (in the sense that for any $x \in M$, the elements $s_1(x), \dots, s_n(x)$ are linearly independent in the vector space E_x).
- Prove that the tangent bundle of S^1 is trivial.
- Invoke the hairy-ball theorem to prove that the tangent bundle of the 2-sphere, $T\mathrm{S}^2$, is not trivial. What about $T\mathrm{S}^3$?

Problem 3. When we passed from vector bundles to frame bundles and from principal G-bundles to associated vector bundles, we only did it in terms of cocycles of transition maps, but we can actually do it in a more tangible way.

Let P be a principal G-bundle and $\rho : G \to GL(V)$ a representation of G in a vector space V. Consider the product manifold $P \times V$ and quotient it by the equivalence relation $(p, v) \sim (p \cdot g, \rho(g^{-1})(v))$ for all $p \in P$, $v \in V$ and $g \in G$. We denote this bundle by $P(V) := P \times F/\sim$.

- Show that $P(V)$ is a vector bundle.
- Show that if $\{(U_{ij}, g_{ij})\}$ is a cocycle of transition maps for P, by choosing the same open sets we find a cocycle of transition maps $\{(U_{ij}, \rho \circ g_{ij})\}$.

Conversely, let E be a rank n vector bundle with local trivializations $\{(U_i, \varphi_i)\}$. For any $x \in M$, consider the set

$$\mathrm{GL}(\mathbb{R}^n, E_x) := \{\varphi : \mathbb{R}^n \to E_x \mid \varphi \text{ is linear and invertible}\}.$$

- Use the fact that $\det : \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}$ is continuous to endow the space of invertible linear maps between two vector spaces, say $L(V, W)$, with the structure of a differentiable manifold. Consequently, $\mathrm{GL}(\mathbb{R}^n, E_x)$ is a manifold.
- Give $\mathrm{GL}(\mathbb{R}^n, E_x)$ the structure of a right $\mathrm{GL}(n)$-torsor.

We now consider all these torsors together as a set:

$$\mathrm{GL}(\mathbb{R}^n, E) := \bigcup_{x \in M} \mathrm{GL}(\mathbb{R}^n, E_x).$$

There is a canonical projection π to M, mapping $v \in \mathrm{GL}(\mathbb{R}^n, E_x)$ to $x \in M$.

- Define a local trivialization for $\pi^{-1}(U_i) = \bigcup_{x \in U_i} \mathrm{GL}(\mathbb{R}^n, E_x)$.
- Use these local trivializations to prove that $\mathrm{GL}(\mathbb{R}^n, E)$ is a principal $GL(n)$-bundle.
- What are the transition functions of $\mathrm{GL}(\mathbb{R}^n, E)$ for the open cover $\{U_i\}$?