Introduction to Surface Group Representations and Higgs Bundles
Assignment 3

Weizmann Institute
First Semester 2017-2018

There is no formal submission of the assignments but you must work on them.

One of the students will present a solution and we will discuss alternatives.

Problem 1. A vector bundle is a fibre bundle $V \to M$ such that the generic fibre F is a vector space and the transition maps between any two local trivializations are linear, i.e., $g_{ij}(x, -) \in \mathrm{GL}(F)$. The fibres of a fibre bundle canonically carry the structure of a vector space.

A principal G-bundle for a Lie group G is defined as a fibre bundle $E \to M$ with fibre G, such that the transition functions are maps $U_{ij} \to G$, where G acts on the fibre G by left multiplication.

- What structure can you define on the fibres of a principal G-bundle?

Problem 2. Consider the real function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

- Is $f(x)$ a smooth function?
- Sketch the graph of $f(x)$.

Define

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}.$$

- Sketch the graph of $g(x)$.

Define

$$h(x) = g(x+2)g(2-x).$$

- Sketch the graph of $h(x)$.

Finally, we look at the properties of $f(x)$:

- Recall or look for Taylor's theorem and write the Taylor series of $f(x)$.
- Recall or look for the concept of real analytic function and decide whether $f(x)$ is real analytic or not.

Extra problem

Please, work on this only if you feel relatively confident about the contents of the course. It is not strictly relevant for the course. We will mention the solution and discuss your ideas.

Let C, T be two topological manifolds living, as topological submanifolds, in the same ambient topological manifold M. An ambient isotopy between C and T is a continuous map

$$\Psi : C \times [0,1] \to M$$

such that $\Psi(x,0) = x \in C \subset M$, $\Psi_{|C\times\{t\}}$ is a homeomorphism onto its image for any t, and $\Psi(C,1) = T \subset M$.

Recall the definition of the cylinder and the 2-twisted cylinder in $\mathbb{R}^4$:

$$\{(e^{i\theta}, r) \mid \theta \in [0,2\pi], r \in (-1,1)\},$$
$$\{(e^{i\theta}, re^{i\theta}) \mid \theta \in [0,2\pi], r \in (-1,1)\}.$$

- Is there an ambiente isotopy between the cylinder and the 2-twisted cylinder in $\mathbb{R}^4$?
- If not, can you regard them in a bigger ambient manifold and define an ambient isotopy?