Introduction to Surface Group Representations and Higgs Bundles
Assignment 2

Weizmann Institute
First Semester 2017-2018

There is no formal submission of the assignments but you must work on them.
One of the students will present a solution and we will discuss alternatives.

Problem 1. Given a cocycle $\{(U_{ij}, g_{ij})\}$ associated to an open cover $\{U_i\}$, not necessarily coming from a fibre bundle, we do the following constructions. We consider all the sets $U_i \times F$ and glue them following g_{ij}: so $(x, v) \sim (x, w)$ when $(x, v) \in U_i \times F$, $(x, w) \in U_j \times F$ and $v = g_{ij}(x, w)$. This means doing a quotient

$$E' = \coprod_i U_i \times F / \sim .$$

- Define charts for E'.
- Define a differentiable and surjective projection $\pi : E' \to M$.
- Show that E' is locally trivial. What are the transition maps?

Assume now that $\{(U_{ij}, g_{ij})\}$ are the transition maps of a fibre bundle E.

- Write a diffeomorphism $f : E \to E'$ without using the local description.

Problem 2. So far we have given simple examples and constructions of bundles. Let us move to a more involved example, the tangent bundle of a manifold. In the mental image we have of a surface, sitting on $\mathbb{R}^3$, we have a clear notion of a tangent space at a point. This strong intuition relies on the actual embedding on $\mathbb{R}^3$ and is not available if we work with abstract manifolds. It is, say, extrinsic, and we want something intrinsic. An intrinsic way to think about a tangent vector on p on a surface would be looking at a curve $\gamma : (-\epsilon, \epsilon) \to M$ passing through p at time 0. The tangent vector, in the extrinsic way, is $\gamma'(0)$. There are many curves giving the same vector. We want to say that two curves γ, σ are equivalent if $\gamma'(0) = \sigma'(0)$... but we cannot do this, it does not make sense for an abstract manifold. Instead, we can take a chart c around p and move this to $\mathbb{R}^n$, where we can take derivatives. Two curves are hence equivalent if $(c \circ \gamma)'(0) = (c \circ \sigma)'(0)$. This clearly defines an equivalence relation and a tangent vector at p is an equivalence class of curves passing through p (for any $(-\epsilon, \epsilon)$. The tangent space at a point $p \in M$ is

$$T_pM = \{\gamma : (-\epsilon, \epsilon) \to M \mid \gamma(0) = p\}/(\gamma \sim \sigma \leftrightarrow (c \circ \gamma)'(0) = (c \circ \sigma)'(0)).$$

- Show that the definition does not depend on the choice of chart.

What kind of structure does T_pM have, if any? In our intuitive image of a tangent space, we get a plane, a vector space.

- Define a vector space structure on T_pM.
- Find and describe a basis for T_pM

This is a very geometrical way of understanding the tangent space to a point, but we also want to see vectors in action. This is done by generalizing the concept of directional derivative. Given $[\gamma] \in T_pM$ and a smooth function $f : U \to \mathbb{R}$ on an open neighbourhood U of p, set

$$[\gamma](f) := (f \circ \gamma)'(0).$$

- Write a formula for $[\gamma](fg)$.

If you feel corageous, consider

$$TM = \cup_{p \in M} T_pM$$

and endow it with a fibre bundle structure.