ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

## T-DUALITY THROUGH GENERALIZED GEOMETRY

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#### Abstract

We start by giving a short review of some aspects of the subject of generalized geometry, as introduced by Nigel Hitchin, which provides a common framework to study complex and symplectic geometry. This type of structures arise naturally in the context of theoretical physics, especially in quantum field theory and string theory. Following the work of Gil R Cavalcanti and Marco Gualtieri, we present how generalized geometry can be related to T-duality, a topological relation between torus bundles over a common base manifold. In this context we establish the duality as an isomorphism of Courant algebroids. Thanks to this isomorphism we are able to transport additional geometrical structures between the spaces, such as generalized metrics, generalized complex structures and generalized Kähler structures. Moreover, we give another interpretation of T-duality in terms of a Courant reduction. Finally, we explain the physics background of the subject to offer some context and also show some current applications.


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## 1 Introduction

Several geometrical structures can be built on manifolds, such as complex structures, symplectic structures and Riemannian structures. Generalized geometry was firstly introduced by Nigel Hitchin Hit03, Hit10] as a common framework to study complex and symplectic structures. In this formalism, one studies the geometry of a so-called Courant algebroid over a manifold. This object is the Whitney sum of the tangent and cotangent bundle of a manifold equipped with an analogue of the Lie bracket of vector fields, which depends on the choice of some closed 3-form, together with a symmetric pairing, which arises from the duality of the tangent and cotangent bundles together. On this particular bundle we can consider generalized complex structures. These generalize the notion of complex and symplectic structures, and present them as extremal cases. We will see how these constructions provide the language to understand precisely the concept of T-duality and how it works.

T-duality is the term used to name an equivalence of two different quantum field theories or string theories, which arose in the early 1980's [GSB82, KY84, SS86]. The particular case which interests us comes from the relation between type IIA and IIB string theories when we consider a compactification on a circle. Buscher Bus87] presented a relationship between Riemannian structures in the T-dual case, and derived some rules to transport the structure, known as Buscher rules.

A geometrical description of the T-duality picture can be obtained by studying how a closed 3 -form with integral periods interacts with the topology of the space where the theory is constructed. Gualtieri and Cavalcanti [CG04, CG11] based their work on the topological description given by Bouwknegt, Evslin and Mathai in BEM04 to develop their study of the geometry of T-duality. More concretely, T-duality is presented as a commutative diagram consisting of a base manifold, two different principal $T^{k}$-bundles and the correspondence space which can also be seen as a principal $T^{k}$-bundle over the two bundles. Another requirement is imposed on the closed 3 -forms that characterize the Courant algebroids over the principal $T^{k}$-bundles. The goal of this project is to give a detailed description of the work done by Gualtieri and Cavalcanti, providing a sufficient background to make it readable and, hopefully, understandable for any graduate student. Having this in mind, we have devoted the first chapters to lay the foundations of all the concepts and results needed to understand the established results about the geometry of T-duality.

Chapter 2 is devoted to motivate and introduce the framework of generalized geometry. In the first section we give a short review of the most important aspects for our topic of a graduate course on topology and geometry of manifolds.

In Section 2.2 we recall some aspects of the linear structures we are interested in, which are metrics, complex structures and symplectic structures.

In Section 2.3 we introduce the linear algebra tools needed to approach generalized geometry,
by considering the direct sum of a real finite-dimensional vector space and its dual, $V \oplus V^{*}$. In this context, we can characterize complex and symplectic structures in terms of maximally isotropic subspaces, given the natural pairing we obtain on $V \oplus V^{*}$ by contracting the vectors with the 1-forms. These subspaces play the role of describing complex and symplectic structures as particular cases. Moreover, we also give a description of the different symmetries we have with respect to this pairing, especially the B-fields.

In Section 2.4 we give another description of maximally isotropic subspaces in terms of annihilators of forms in the exterior algebra of $V^{*}$. These annihilators are taken with respect to an action of $V \oplus V^{*}$ on this exterior algebra. At the end of the section we explain how we can relate this action with the product in the Clifford algebra of $V \oplus V^{*}$.

In Section 2.5 we present the concept of linear generalized complex structures and show how they naturally generalize complex and symplectic structures. We also use the description of maximally isotropic subspaces in terms of annihilators of forms to give a characterization of linear generalized complex structures in terms of forms.

Finally, in Section 2.6 we present the notion of a Courant algebroid over a manifold $M$. Following the linear case, we show how we can proceed analogously to give the notion of a generalized structure on a manifold which encompasses complex and symplectic structures.

In Chapter 3 we aim to present the topological technical aspects that will be continually used once we start talking about T-duality: principal bundles, connections and Chern classes. In the first section we define vector bundles, which encompass the tangent and the cotangent bundle, and principal bundles. We explain the relationship between them and how under some circumstances they can be equivalent ways to approach the same problem. Taking into account this fact, we define for both cases what connections and curvatures are.

In Section 3.2 we discuss how to completely characterize fibre bundles in terms of some cohomology classes which do not depend on the local information. We explain that one possible way of doing this is by means of characteristic classes, and in particular with Chern classes for the case we are interested in. The goal is to have a clear understanding of these classes since they are used to define T-duality and to work out some major examples.

In Chapter 4 we finally introduce the main ideas of the project. In the first section we expand some notions related to Courant algebroids and define what exact Courant algebroids are. The generalized tangent bundle $T M \oplus T^{*} M$ falls in this definition, and we shall see that in fact all exact Courant algebroids are of this form. The difference between them will be a closed 3 -form $H$, whose cohomology class is called Sěvera class. Moreover, we study the automorphism group of a Courant algebroid, where B-fields play a major role.

In Section 4.2 we state precisely the definition of T-duality, which, as we shall see, is a relationship between circle bundles, and we will give a straightforward generalization to torus bundles. At the end of this section we present a theorem which ensures that T-duality is an isomorphism of differential complexes.

In Section 4.3 we use the theorem mentioned above to construct an isomorphism of Courant algebroids. This isomorphism will be the main tool to transport geometrical structures between the torus bundles.

In Section 4.4 we give a series of results showing how we can transport geometrical structures such as generalized Kähler structures, generalized complex structures and generalized metrics under the assumption of T-duality by means of the isomorphism of Courant algebroids it provides.

To conclude, in Section 4.5 we give an alternative interpretation of T-duality in terms of a reduction of Courant algebroids.

In Chapter 5 we give a short description of the physics framework where T-duality was discovered. We start by motivating the idea of a duality in physics and then proceed to describe how the equivalence of two theories with different ambient space exhibit the nature of T-duality for circle bundles. We end by explaining the process physicists use to find the Buscher rules.

In Chapter 6 we give a short review of the main results of the project and present different lines of research that can be taken from here, some of them moving towards K-theory while the others focus on studying other geometrical structures that can be transported between T-dual bundles.

## 2 Generalized geometry

Generalized geometry is an approach to study some of the classical features that manifolds can be endowed with, such as Riemannian metrics, complex structures, symplectic structures... Before getting to explain this notion, we provide a summary of the concepts and results from topology and geometry of manifolds that will be used later in this project, as it will also serve us to fix some notation. If the reader wants to get a more detailed presentation of the aspects covered in Section 2.1, he can check [War13], Part III of [HN11], Chapter 1 of [BT13], or [Fra11] for a more physics oriented discussion. For the subsequent sections, the reader may check Gua04, Rub and of course Hit10].

### 2.1 Topology and geometry of manifolds

We start by defining the basic object for this project. An $n$-dimensional manifold is a topological space $M$ satisfying three properties: it is second-countable, which means that its topology admits a countable base; Hausdorff, and is locally homeomorphic to $\mathbb{R}^{n}$. Manifolds come with charts, which are pairs $(U, \varphi)$ such that $U \subseteq M$ is an open subset and $\varphi: U \rightarrow \mathbb{R}^{n}$ is continuous map which induces a homeomorphism $U \xrightarrow{\simeq} \varphi(U)$.

These charts can provide additional structure to the manifold when they meet some compatibility condition. We are interested in smooth structures, so we are only going to define that case. A smooth atlas on $M$ is a collection of charts $\mathcal{A}=\left\{\left(U_{i}, \varphi_{i}\right)\right\}_{i \in I}$ where $M=\bigcup_{i \in I} U_{i}$ and for any $i, j \in I$ the map $\varphi_{i}\left(U_{i} \cap U_{j}\right) \rightarrow \varphi_{j}\left(U_{i} \cap U_{j}\right)$ is smooth; so a pair $(M, \mathcal{A})$ where $M$ is a manifold and $\mathcal{A}$ is a smooth atlas for $M$ is called a smooth manifold. For the rest of the project we will be abusing notation and just word it as manifold when the smooth structure is clearly assumed. As it is always convenient, we can list some examples of manifolds:

- $\mathbb{R}^{n}$ for $n \in \mathbb{N}$,
- $S^{n}=\left\{x \in \mathbb{R}^{n+1}:\|x\|=1\right\}$ for $n \in \mathbb{N}$,
- some matrix groups as $\mathrm{GL}(k, \mathbb{R}), \mathrm{O}(k, \mathbb{R}), \ldots$

We say that $f: M \rightarrow \mathbb{R}$ is a smooth function if $f_{i}=f \circ \varphi_{i}^{-1}$ is smooth for some chart $\left(U_{i}, \varphi_{i}\right)$ of its atlas, and we denote the set of smooth functions on $M$ by $\mathcal{C}^{\infty}(M)$. Given a manifold $M$ and a point $p \in M$ we can define the tangent space of $M$ at $p$, which we denote by $T_{p} M$. This tangent space can be defined by different means. One way is to consider equivalence classes of curves: let $\gamma_{1}, \gamma_{2}:(-1,1) \rightarrow M$ such that $\gamma_{1}, \gamma_{2}(0)=p$ and they are smooth in the usual sense. Then $\gamma_{1}$ and $\gamma_{2}$ are equivalent if and only if

$$
\left.\frac{d}{d t} \varphi \circ \gamma_{1}(t)\right|_{t=0}=\left.\frac{d}{d t} \varphi \circ \gamma_{2}(t)\right|_{t=0}
$$

where $(U, \varphi)$ is some chart covering $\operatorname{Im}\left(\gamma_{1}\right)$ and $\operatorname{Im}\left(\gamma_{2}\right)$.
Geometrically speaking, they are equivalent if they define the same tangent vector at $p$. We denote the equivalence class of $\gamma$ by $\gamma^{\prime}(0)$. Hence, we define the $T_{p} M$ as the collection of all tangent vectors to $M$ at $p$. This tangent space is moreover a vector space, of the same dimensions as $M$. To ensure this, we take a chart $\varphi: U \rightarrow \mathbb{R}^{n}$ and consider the differential map at $p$ defined by $d \varphi_{p}: T_{p} M \rightarrow \mathbb{R}^{n}$, where $d \varphi_{p}\left(\gamma^{\prime}(0)\right)=\left.\frac{d}{d t}[(\varphi \circ \gamma)(t)]\right|_{t=0}$, with $\gamma \in \gamma^{\prime}(0)$. Since this map is bijective, we can use it to transfer the vector space structure of $\mathbb{R}^{n}$ back to $T_{p} M$.

Another way to define the tangent space at a point is using derivations. A derivation on a manifold $M$ at a point $p$ is an $\mathbb{R}$-linear map $D: \mathcal{C}^{\infty}(M) \rightarrow \mathbb{R}$ satisfying the Leibniz rule, which means that for $f, g \in \mathcal{C}^{\infty}(M), D(f g)(p)=D f g(p)+f(p) D g$. We then define $T_{p} M$ to be the set of all derivations on $M$ at $p$. Under this point of view is easier to note that $T_{p} M$ is a vector space, as we can set $\left(D_{1}+D_{2}\right) f=D_{1} f+D_{2} f$ and $(\lambda D) f=\lambda D f$. Moreover, using a chart $(U, \varphi)$ with $p \in U$, we can take a basis for this vector space, namely $\left(\frac{\partial}{\partial x_{i}}\right)_{p}$ for $i=1, \ldots, n$, where

$$
\left(\frac{\partial}{\partial x_{i}}\right)_{p}(f)=\frac{\partial\left(f \circ \varphi^{-1}\right)}{\partial x_{i}}(\varphi(p)) .
$$

Together with the tangent space at $p$ comes naturally the cotangent space at $p$, which we denote by $T_{p}^{*} M$. This is the space of $\mathbb{R}$-linear maps from $T_{p} M$ to $\mathbb{R}$, or in other words, the space of 1-forms. Given a smooth function $f: M \rightarrow \mathbb{R}$, we define its differential at $p$ as the map

$$
\begin{array}{rllc}
(d f)_{p}: & T_{p} M & \longrightarrow & \mathbb{R} \\
D & \longmapsto & (d f)_{p} D=D f
\end{array}
$$

As we have done with the tangent space, we can take a chart $(U, \varphi)$ with $p \in U$ to find a basis for the cotangent space, $\left(d x_{i}\right)_{p}$ for $i=1, \ldots, n$, where

$$
\left(d x_{i}\right)_{p}\left(\frac{\partial}{\partial x_{j}}\right)_{p}=\left(\frac{\partial x_{i}}{\partial x_{j}}\right)_{p}=\delta_{i j}
$$

where $\delta_{i j}$ is the usual Kronecker delta.
So far we have built up two structures at every point of the manifold, the tangent and the cotangent space. These will be one of the key points to develop the main aspects of this project, but for now let us focus on the maps between manifolds. Let $M, N$ be two manifolds. A map $F: M \rightarrow N$ is smooth if for for any charts $(U, \varphi)$ with $p \in U$ and $(V, \psi)$ with $\psi(p) \in V$, the map $\psi \circ F \circ \varphi^{-1}: \varphi(U) \rightarrow \psi(V)$ is smooth in the usual sense. Given a smooth map $F: M \rightarrow N$, we can define two maps, one between the tangent spaces $T_{p} M$ and $T_{F(p)} N$ and another one between the cotangent spaces $T_{F(p)}^{*} N$ and $T_{p}^{*} M$. The first one is

$$
\begin{aligned}
D F_{p}: \quad T_{p} M & \longrightarrow
\end{aligned} T_{F(p)} N
$$

and the second one is

$$
\begin{array}{cccc}
\left(D F_{p}\right)^{*}: & T_{F(p)}^{*} N & \longrightarrow & T_{p}^{*} M \\
& (d h)_{F(p)} & \longmapsto d(h \circ F)_{p}=d\left(F^{*} h\right)_{p} .
\end{array}
$$

An important type of map between manifolds is a diffeomorphism, which we can define as a smooth map $F: M \rightarrow N$ such that there exists a smooth map $G: N \rightarrow M$ such that $F \circ G=\operatorname{id}_{N}$ and $G \circ F=\mathrm{id}_{M}$. In this case, note that both $D F_{p}$ and $\left(D F_{p}\right)^{*}$ are isomorphisms for every point $p \in M$.

One could wonder whether it is possible to find objects that encode all the tangent spaces and cotangent spaces of a manifold and, thankfully, there are. Without giving the full details, we define the so-called tangent and cotangent bundles, which are particular cases of vector bundles and this type of objects will be studied in the next section. On one hand we have the tangent bundle $\left(T M, \pi_{T}\right)$ where

$$
T M=\bigsqcup_{p \in M} T_{p} M=\left\{(p, v): p \in M, v \in T_{p} M\right\}
$$

and $\pi_{T}: T M \rightarrow M, \pi_{T}(p, v)=p$ satisfies that for all $p \in M, \pi_{t}^{-1}(p)=T_{p} M$ has a vector space structure. On the other hand we have the cotangent bundle ( $T^{*} M, \pi_{T^{*}}$ ) where

$$
T^{*} M=\bigsqcup_{p \in M} T_{p}^{*} M=\left\{(p, \alpha): p \in M, \alpha \in T_{p}^{*} M\right\}
$$

and $\pi_{T^{*}}: T^{*} M \rightarrow M, \pi_{T^{*}}(p, \alpha)=p$ satisfies that for all $p \in M, \pi_{T^{*}}^{-1}(p)=T_{p}^{*} M$ has a vector space structure. For both bundles we have sections (right inverses of the projections), which we call vector fields and 1-forms, and denote them by $\Gamma(T M)$ and $\Omega^{1}(M)$ respectively.

We now continue by delving into the study of differential forms since they are the basic element to define the cohomology of a manifold. In order to define a differential form of arbitrary degree we must first recall the algebraic notions of tensor algebra and exterior algebra. Given two vector spaces $V$ and $W$ over the same field, we define their tensor product $V \otimes W$ to be a vector space through which any bilinear map $f: V \times W \rightarrow U$ factorizes,


The tensor product is associative and compatible with direct sums. Thanks to this we can define the tensor algebra of a vector space $V$. Let $r \in \mathbb{N}$, and $V^{\otimes r}=V \otimes \cdots \otimes V$, where $V^{\otimes 1}=V$ and $V^{\otimes 0}=k$. The tensor algebra is defined as $T V=\bigoplus_{r \geq 0} V^{\otimes r}$.

To define the exterior algebra, we consider the two-sided ideal $I \subseteq T V$ generated by all the terms of the form $v \otimes w+w \otimes v$ with $v, w \in V$. Then by taking the quotient of the tensor algebra with respect to this ideal we obtain the exterior algebra $\Lambda^{\bullet} V=T V / I$. The class of $v_{1} \otimes \cdots \otimes v_{k} \in V^{\otimes k}$ is denoted by $v_{1} \wedge \cdots \wedge v_{k}$. Both algebras are graded, the tensor algebra right from the definition and the exterior algebra as the quotient is compatible with the direct sum decomposition of the tensor algebra; however, the tensor algebra is noncommutative while the exterior algebra is graded commutative. This means that given $v \in \bigwedge^{p} V, w \in \bigwedge^{q} V$, it holds $v \wedge w=(-1)^{p q} w \wedge v$.

Using this definitions we can now define the space of $k$-forms on a manifold $M$ :

$$
\Omega^{k}(M)=\mathcal{C}^{\infty}\left(\bigwedge^{k} T^{*} M\right)=\left\{\text { smooth sections } \bigwedge^{k} T^{*} M \rightarrow M\right\}
$$

Moreover, we can also define the wedge product of forms $\wedge: \Omega^{p}(M) \times \Omega^{q}(M) \rightarrow \Omega^{p+q}(M)$, which allows us to define an algebra structure $\Omega^{\bullet}(M)=\bigoplus_{p \geq 0} \Omega^{p}(M)$, which is also graded commutative.

Before getting to the definition of the de Rham cohomology it is convenient to present some of the relations between the vector fields and the differential forms together with some of their properties. First of all we shall that we have a natural pairing $\langle\cdot, \cdot\rangle: \Gamma(T M) \times \Omega^{1}(M) \rightarrow \mathcal{C}^{\infty}(M)$, induced by the pairing at each point, $\langle\cdot, \cdot\rangle: T_{p} M \times T_{p}^{*} M \rightarrow \mathbb{R}$. We also have a differential map $d: \mathcal{C}^{\infty}(M) \rightarrow \Omega^{1}(M)$ sending $f$ to $d f$, and an action of the vector fields on the smooth functions on $M, \Gamma(T M) \times \mathcal{C}^{\infty}(M) \rightarrow \mathcal{C}^{\infty}(M)$ given by $(X, f) \mapsto X f$. Given a smooth map $\varphi: M \rightarrow N$ and a $k$-differential form $\alpha$ on $N$, we can define the pullback of $\alpha$,

$$
\left(\varphi^{*} \alpha\right)\left(X_{1}, \ldots, X_{k}\right)(p)=\alpha\left(D \varphi_{p} X_{1}, \ldots, D \varphi_{p} X_{k}\right)(\varphi(p))
$$

One remarkable property of the pullback is its compatibility with the wedge product, as $\varphi^{*}$ ( $\alpha \wedge$ $\beta)=\varphi^{*} \alpha \wedge \varphi^{*} \beta$. Going back to the algebra of differential forms, we have another map $d$, aside from the wedge product, called the exterior derivative which increases the degree of the form by one and it is characterized by four properties:

- $d$ is $\mathbb{R}$-linear,
- $d$ satisfies the Leibniz rule in the graded sense, i.e. if $\alpha \in \Omega^{p}(M), \beta \in \Omega^{q}(M)$ then $d(\alpha \wedge \beta)=d \alpha \wedge \beta+(-1)^{p} \alpha \wedge d \beta$,
- $d$ commutes with pullbacks,
- $d$ extends the derivative map $d: \mathcal{C}^{\infty}(M) \rightarrow \Omega^{1}(M)$.

On top of that, it has the nice property $d^{2}=0$. We also have an map which decreases the degree of a differential form by one, but in this case we need to choose a vector field
$X \in \Gamma(T M)$. This map $i_{X}: \Omega^{k}(M) \rightarrow \Omega^{k-1}(M)$ is defined as $i_{X} \alpha=\alpha(X, \cdot)$ for any $\alpha \in \Omega^{k}(M)$. Finally, we can define the Lie derivatives of vector fields and differential forms with respect to a given vector field $X \in \Gamma(T M)$. In the case of vector fields, we define the Lie derivative of $Y \in \Gamma(T M)$ with respect to $X$ as $\mathcal{L}_{X} Y=[X, Y]$, where $[X, Y]$ is the Lie bracket of vector fields so $[X, Y](f)=X(Y f)-Y(X f)$. For a differential forms $\omega$ we shall use Cartan's formula to define it as $\mathcal{L}_{X} \omega=d i_{X} \omega+i_{X} d \omega$.

What remains now is to define the de Rham cohomology of a manifold $M$. For this, we shall note that we have a cochain complex

$$
0 \longrightarrow \Omega^{0}(M) \xrightarrow{d} \Omega^{1}(M) \xrightarrow{d} \Omega^{2}(M) \xrightarrow{d} \Omega^{3}(M) \xrightarrow{d} \cdots,
$$

that is, a sequence of vector spaces such that $d \circ d=0$ at any level. This implies that $\operatorname{im}\left(\Omega^{k-1}(M) \rightarrow \Omega^{k}(M)\right) \subseteq \operatorname{ker}\left(\Omega^{k}(M) \rightarrow \Omega^{k+1}(M)\right)$ for $k \geq 0$. By defining the $k$-cocycles $Z^{k}(M)=\operatorname{ker}\left(\Omega^{k}(M) \rightarrow \Omega^{k+1}(M)\right)$ and the $k$-coboundaries $B^{k}(M)=\operatorname{im}\left(\Omega^{k-1}(M) \rightarrow \Omega^{k}(M)\right)$ we can construct the $k$-th cohomology group of $M$,

$$
H_{d R}^{k}(M)=Z^{k}(M) / B^{k}(M)
$$

### 2.2 Linear structures

After refreshing the most important facts about manifolds for this project, we now devote some time to get in touch with some structures that can be defined on vector spaces and motivate the notion of generalized geometry.

The first one we can define is one of the most familiar, a linear riemannian metric. For this, let $V$ be a vector space over $\mathbb{R}$ and consider a positive-definite symmetric nondegenerate bilinear map

$$
g: V \times V \rightarrow \mathbb{R}
$$

We call $g$ a riemannian metric on $V$. It allows us to have a notion of orthogonality: two vectors $u, v \in V$ are orthogonal if $g(u, v)=0$. We can go further and define orthogonal subspaces. Given a subspace $U \subseteq V$, we define its orthogonal complement as

$$
U^{\perp}=\{v \in V: g(v, U)=0\} .
$$

This provides a decomposition of the total space $V$ as $U \oplus U^{\perp}=V$. Similarly to this we have linear symplectic structures, given by a skew-symmetric nondegenerate bilinear map

$$
\omega: V \times V \rightarrow k,
$$

where now $V$ is a $k$-vector space with $k=\mathbb{R}, \mathbb{C}$. We can define the symplectic complement in an analogous way to the orthogonal complement,

$$
U^{\omega}=\{u \in V: \omega(u, U)=0\} .
$$

The symplectic complement has some nice properties, almost as the orthogonal complement. For example, $\operatorname{dim} U+\operatorname{dim} U^{\omega}=\operatorname{dim} V$ and $\left(U^{\omega}\right)^{\omega}=U$. However, for the direct sum decomposition we require the subspace to be symplectic, that is, $\omega_{\mid U}$ is symplectic form. In this case, we have $U \oplus U^{\omega}=V$. The most remarkable difference with the riemannian metrics is that symplectic structures require an extra condition on the vector space. A similar process to the Gram-Schmidt orthogonalization shows that the dimension of $V$ must be even.

We introduce one last type of structures: complex structures over real vector spaces, which are $J \in \operatorname{End}(V)$ such that $J^{2}=-\mathrm{id}$. As we are dealing with an endomorphism now, the situation varies a bit. For a subspace $U$ to be stable under the complex structure we require that $J(U) \subseteq U$, since it will imply that $J(U)=U$, and denote the corresponding complex structure by $J_{\mid U}$. It can be shown that the dimension of $V$, as in the case of symplectic structures, must be even for it to admit a complex structure. We can link complex structures with the complexification of a real vector space. First of all, we define the complexification of $V$ as

$$
V_{\mathbb{C}}=V \times V=\{u+i v: u, v \in V\},
$$

where the $i$ is a formal element and should not be confused with the complex number. By complexifying $V$, we extend the scalars from $\mathbb{R}$ to $\mathbb{C}$, by defining

$$
(a+i b)(u+i v)=(a u-b v)+i(b u+a v)
$$

We have a natural conjugation on $V_{\mathbb{C}}, \overline{u+i v}=u-i v$, which resembles the conjugation on $\mathbb{C}$. Moreover, given a complex structure $J$ on $V$, we can extend it to a complex structure on $V_{\mathbb{C}}$ simply by acting linearly in both components, $J(u+i v)=J(u)+i J(v)$. The main difference with the original complex structure is that now, as we have a complex vector space, we can diagonalize it for the roots of the minimal polynomial of $J$ are $\pm i$. The diagonalization gives the $+i$-eigenspace and the $-i$-eigenspace, which we denote by $V^{1,0}$ and $V^{0,1}$ respectively. Both eigenspaces can be described in a neat way, as eigenvector actually come in pairs,

$$
\begin{aligned}
& V^{1,0}=\{v-i J(v): v \in V\}, \\
& V^{0,1}=\{v+i J(v): v \in V\} .
\end{aligned}
$$

Looking at the dimensions, we have $\operatorname{dim}_{\mathbb{R}} V=\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}$ and $\operatorname{dim}_{\mathbb{R}} V=\operatorname{dim}_{\mathbb{R}} V^{1,0}=\operatorname{dim}_{\mathbb{R}} V^{0,1}$. This allows us to state the following isomorphisms

$$
(V, J) \cong\left(V^{1,0}, J_{\mid V^{1,0}}\right)=\left(V^{1,0}, i\right), \quad(V, J) \cong\left(V^{0,1}, J_{\mid V^{0,1}}\right)=\left(V^{0,1},-i\right),
$$

given that the map $f \circ V \rightarrow V^{1,0}, v \mapsto v-i J(v)$ satisfies $f: J=J_{\mid V^{1,0}} \circ f$. The situation is analogous for $V^{0,1}$.

This idea of decomposing the space in two subspaces, one being the conjugate of the other, is key to describe linear complex structures. A given complex subspace $L \subseteq V_{\mathbb{C}}$ such that
$L \cap \bar{L}=\{0\}$ and $\operatorname{dim}_{\mathbb{R}} L=\operatorname{dim}_{\mathbb{R}} V$ determines a decomposition $V_{\mathbb{C}}=L \oplus \bar{L}$ and $V=\{l+$ $\bar{l}: l \in L\}$. Moreover, considering $L$ with the same hypothesis, there exists a unique linear complex structure $J$ such that $L=V^{1,0}$. To see this, we can consider an endomorphism $J$ on $V=\{l+\bar{l}: l \in L\}$ given by $l+\bar{l} \mapsto i l-i \bar{l}$. This is by construction a linear complex structure on $V$. Uniqueness follows from the fact that the condition $L=V^{1,0}$ completely determines the complexification of any such $J$. It must be the map $l^{\prime}+\bar{l} \mapsto i l^{\prime}-i \bar{l}$, but $J$ is precisely the restriction of this map to $V$.

### 2.3 Generalized linear algebra

We have seen three major examples of structures that vector spaces can be endowed with and some similarities between them. It would be desirable to have a common framework for these types of structures to study their properties and find some general characterization. For this purpose we have the setting of what we can call generalized linear algebra.

Given a vector space $V$, we can consider the formal sum $V \oplus V^{*}$, and denote its elements by $X+\xi$ with $X \in V, \xi \in V^{*}$. This construction comes with a canonical pairing,

$$
\langle X+\xi, Y+\eta\rangle=\frac{1}{2}\left(i_{Y} \xi+i_{X} \eta\right)
$$

Thanks to this pairing, for a subspace $U \subseteq V \oplus V^{*}$ we can define its orthogonal,

$$
U^{\perp}=\left\{v \in V \oplus V^{*}:\langle v, U\rangle=0\right\} .
$$

We shall remark the fact that the pairing has split signature $(n, n)$. To see this, let $\left\{e_{i}\right\}$ be a basis of $V$ and $\left\{e^{i}\right\}$ its dual basis. We have the following relations:

$$
\left\langle e_{i}, e_{j}\right\rangle=0, \quad\left\langle e_{i}, e^{j}\right\rangle=\delta_{i j}, \quad\left\langle e^{i}, e^{j}\right\rangle=0
$$

Then, we can consider the ordered basis ( $e_{1}+e^{1}, \ldots, e_{n}+e^{n}, e_{1}-e^{1}, \ldots, e_{n}-e^{n}$ ) of $V \oplus V^{*}$ to realize that the matrix representation of the pairing is

$$
\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right),
$$

where all entries are $n \times n$ blocks.
A subspace $U \subseteq V \oplus V^{*}$ is said to be isotropic if $U \subseteq U^{\perp}$, and maximally isotropic if it is isotropic and is not strictly contained in an isotropic subspace. This subspaces can be characterized by the property $U=U^{\perp}$.

Lemma 2.1. Let $V$ be a vector space and $L \subseteq V \oplus V^{*}$ be a maximally isotropic space. Then $\operatorname{dim} L=\operatorname{dim} V$.

Proof. Let $L$ be maximally isotropic, and let $S \subseteq V \oplus V^{*}$ be a complement of $L$ in $L^{\perp}$, that is, $L^{\perp}=L \oplus S$. If there are $u, v \in S$ such that $\langle u, u\rangle<0$ and $\langle v, v\rangle>0$, there exists $\lambda \in k$ such that $\langle u+\lambda v, u+\lambda v\rangle=0$. Then $L \subseteq L \oplus \operatorname{span}(u+\lambda v) \subseteq L^{\perp}$, and by maximality of $L$ this cannot be. Hence, $L^{\perp}$ is either positive semidefinite or negative semidefinite. Without loss of generality, we assume that $L^{\perp}$ is positive semidefinite. Let $\left(e_{i}\right)$ be a basis of $V$, and ( $e^{i}$ ) the corresponding dual basis for $V^{*}$, and define $N=\operatorname{span}\left(e_{i}-e^{i}\right.$ ) (if $L^{\perp}$ happens to be negative semidefinite, we would consider $\left.P=\operatorname{span}\left(e_{i}+e^{i}\right)\right)$. Then it follows that $L^{\perp} \cap N=\{0\}$ and

$$
\operatorname{dim} L^{\perp}=\operatorname{dim}\left(L^{\perp} \oplus N\right)-\operatorname{dim} N \leq 2 n-n=n .
$$

But, as the pairing is nondegenerate, we have $\operatorname{dim} L^{\perp}=2 n-\operatorname{dim} L$, so $\operatorname{dim} L \geq n$, and hence $\operatorname{dim} L=n$.

In particular, this lemma implies that maximally isotropic subspaces $L \subseteq V \oplus V^{*}$ are characterized by $L=L^{\perp}$.

As usual, when dealing with spaces with pairings, it is useful to understand the automorphisms which preserve such pairing. We define this subgroup of $\mathrm{GL}\left(V \oplus V^{*}\right)$ as

$$
O\left(V \oplus V^{*}\right)=\left\{f \in \operatorname{GL}\left(V \oplus V^{*}\right):\langle f \cdot, f \cdot\rangle=\langle\cdot, \cdot\rangle\right\} .
$$

Taking block-matrix decomposition, we can represent any element $f \in \mathrm{GL}\left(V \oplus V^{*}\right)$ as

$$
f \equiv\left(\begin{array}{ll}
A & C \\
B & D
\end{array}\right)
$$

where $A: V \rightarrow V, B: V \rightarrow V^{*}, C: V^{*} \rightarrow V$ and $D: V^{*} \rightarrow V^{*}$. An automorphism $f \in$ $\mathrm{GL}\left(V \oplus V^{*}\right)$ belongs to $O\left(V \oplus V^{*}\right)$ if and only if $i_{A X+C \xi}(B X+D \xi)=i_{X} \xi$. We now give three examples of these symmetries of the pairing, the last one being of central importance as shown later:

Taking $B, C=0$ and $A \in G L(V)$, we have $\left(\begin{array}{cc}A & 0 \\ 0 & \left(A^{-1}\right)^{*}\end{array}\right)$.
Taking $A=D=1, B=0$ and $\beta \in \Lambda^{2} V$, we have $\left(\begin{array}{ll}1 & \beta \\ 0 & 1\end{array}\right)$.
Taking $A=D=1, C=0$ and $B \in \bigwedge^{2} V^{*}$, we have $\left(\begin{array}{ll}1 & 0 \\ B & 1\end{array}\right)$. These type of symmetries are called B-fields.

We can now give a general description of the maximally isotropic subspaces $L \subseteq V \oplus V^{*}$. Let $\pi_{V}: V \oplus V^{*} \rightarrow V$ be the projection on $V$, and define $E=\pi_{V}(L)$. Consider now $\operatorname{Ann}(E)$. We are going to show that $\operatorname{Ann}(E) \subseteq L$. For any $\eta \in \operatorname{Ann}(E)$ and $X+\xi \in L$ we have that

$$
\langle\eta, X+\xi\rangle=\frac{1}{2} i_{X} \eta=0
$$

Thus, $\operatorname{Ann}(E) \subseteq L^{\perp}$, but that means that $L \subseteq \operatorname{span}(L, \operatorname{Ann}(E))$ so by maximality of $L$, $\operatorname{Ann}(E) \subseteq L$. As a consequence, $\operatorname{Ann}(E)=L \cap V^{*}$. A question arises now: for $X \in E$, which are the $\xi \in V^{*}$ such that $X+\xi \in L$ ? Taking the difference $(X+\xi)-(X+\eta)=\xi-\eta$, we see that $\xi-\eta \in \operatorname{Ann}(E)$ by the previous observation. Hence, we can define the following map:

$$
\begin{array}{ll}
\varepsilon: & E \longrightarrow V^{*} / \operatorname{Ann}(E) \\
X & \longmapsto E^{*} \\
& \longmapsto \xi+\operatorname{Ann}(E) \\
\longmapsto \xi_{\mid E},
\end{array}
$$

where $X+\xi \in L . \varepsilon$ is in fact an element of $\bigwedge^{2} E^{*}$ because of the isotropy of $L$. This allows us to describe maximally isotropic subspaces as

$$
L(E, \varepsilon)=\left\{X+\xi: X \in E, \xi_{\mid E}=\varepsilon(X)\right\} .
$$

For any choice of $E$ and $\varepsilon$ we obtain a maximally isotropic space. We shall remark that the image of a maximally isotropic subspace by a symmetry of the canonical pairing is again a maximally isotropic subspace, as it is isotropic and has the right dimension. For instance, given a B-field $B \in \bigwedge^{2} V^{*}$ we have

$$
\left(\begin{array}{ll}
1 & 0 \\
B & 1
\end{array}\right) L(E, \varepsilon)=L\left(E, \varepsilon+i^{*} B\right),
$$

where $i^{*}: \bigwedge^{2} V^{*} \rightarrow \bigwedge^{2} E^{*}$ is the map induced by the inclusion $i: E \rightarrow V$.

### 2.4 Clifford algebra and Clifford action

We have finished the previous section with a characterization of maximally isotropic subspaces in terms of a subspace of $E$ and a map $\varepsilon \in \bigwedge^{2} E^{*}$. There is another description of such subspaces in terms of annihilators which is also useful.

Given an element $X+\xi \in V \oplus V^{*}$ we can define an action on $\varphi \in \Lambda^{\bullet} V^{*}$ given by

$$
(X+\xi) \cdot \varphi=i_{X} \varphi+\xi \wedge \varphi
$$

We can define the annihilator of $\varphi$ with respect to this action,

$$
\operatorname{Ann}(\varphi)=\left\{X+\xi \in V \oplus V^{*}:(X+\xi) \cdot \varphi=0\right\}
$$

A nice property about this action is the following: for any $X+\xi \in V \oplus V^{*}$

$$
(X+\xi)^{2} \cdot \varphi=(X+\xi) \cdot\left(i_{X} \varphi+\xi \wedge \varphi\right)=i_{X}(\xi \wedge \varphi)+\xi \wedge i_{X} \varphi=i_{X} \alpha \cdot \varphi=\langle X+\xi, X+\xi\rangle \cdot \varphi
$$

Hence, any annihilator of a form is an isotropic space. Some examples of maximally isotropic spaces described by annihilators are $V=\operatorname{Ann}(1)$ and $V^{*}=\operatorname{Ann}\left(\operatorname{vol}_{V}\right)$, where $\operatorname{vol}_{V} \in \bigwedge^{\operatorname{dim} V} V^{*}$ is a volume form. We can also describe some more intricate cases: let $\omega \in \bigwedge^{2} V$ and consider
the maximally isotropic subspace $\operatorname{gr}(\omega)=\left\{X+i_{X} \omega: X \in V\right\}$. We can describe it as the annihilator of

$$
\varphi=e^{-\omega}=\sum_{k=0}^{\lfloor\operatorname{dim} V / 2\rfloor}(-1)^{k} \frac{\omega^{k}}{k!} .
$$

We can use this example to introduce the action of B-fields on annihilators. Considering $B \in \bigwedge^{2} V^{*}$, we can write

$$
e^{B}=\left(\begin{array}{ll}
1 & 0 \\
B & 1
\end{array}\right),
$$

which we can recognize as a symmetry of the canonical pairing.
Lemma 2.2. Let $\varphi \in \Lambda^{\bullet} V^{*}$ and consider the isotropic space $\operatorname{Ann}(\varphi)$. If $B \in \Lambda^{2} V^{*}$ then

$$
e^{B} \operatorname{Ann}(\varphi)=\operatorname{Ann}\left(e^{-B} \wedge \varphi\right) .
$$

Proof. Let $X+\xi \in \operatorname{Ann}(\varphi)$. We can compute

$$
\left(e^{B}(X+\xi)\right) \cdot \sum_{k \geq 0} \frac{(-1)^{k}}{k!} B^{k} \wedge \varphi=\sum_{k \geq 1} \frac{(-1)^{k}}{(k-1)!} i_{X} B \wedge B^{k-1} \wedge \varphi+\sum_{k \geq 0} \frac{(-1)^{k}}{k!} i_{X} B \wedge B^{k} \wedge \varphi=0
$$

which shows that $e^{B} \operatorname{Ann}(\varphi) \subseteq \operatorname{Ann}\left(e^{-B} \wedge \varphi\right)$. Conversely, let $X+\xi \in \operatorname{Ann}\left(e^{-B} \wedge \varphi\right)$. Then, by writting it as $X+\left(\xi-i_{X} B\right)+i_{X} B$, we can see that $X+\left(\xi-i_{X} B\right) \in \operatorname{Ann}(\varphi)$.

Proposition 2.3. Let $\varphi \in \Lambda^{\bullet} V^{*} . \operatorname{Ann}(\varphi)$ is maximally isotropic if and only if

$$
\varphi=\lambda e^{B} \wedge\left(\theta_{1} \wedge \ldots \wedge \theta_{r}\right)
$$

for $\lambda \in \mathbb{R} \backslash\{0\}, B \in \bigwedge^{2} V^{*}$, and some linearly independent $\theta_{1}, \ldots, \theta_{r} \in V^{*}$.
Proof. We have already seen that maximally isotropic subspaces can be described as $L(E, \varepsilon)$ for $E \in V$ and $\varepsilon \in \bigwedge^{2} E^{*}$. Given the inclusion $i: E \rightarrow V$, we have the surjective induced map $i^{*}: \bigwedge^{2} V^{*} \rightarrow \bigwedge^{2} E^{*}$. Thus there exists $B \in \bigwedge^{2} V^{*}$ such that $i^{*} B=\varepsilon$, which implies that $L(E, \varepsilon)=e^{B} L(E, 0)$. Now, $L(E, 0)=E+\operatorname{Ann}(E) \subseteq \operatorname{Ann}\left(\operatorname{vol}_{\operatorname{Ann}(E)}\right)$. But $E+\operatorname{Ann}(E)$ is a maximally isotropic space, so equality must hold, which translates to

$$
L(E, \varepsilon)=e^{B} L(E, 0)=e^{B} \operatorname{Ann}\left(\operatorname{vol}_{\operatorname{Ann}(E)}\right)=\operatorname{Ann}\left(e^{-B} \wedge \operatorname{vol}_{\operatorname{Ann}(E)}\right)
$$

Since volume forms are decomposable forms, the result follows with the exception of the parameter $\lambda$, as it does not affect the annihilator, so it can be included without any problem.

Remark 2.4. If an element $\varphi \in \Lambda^{\bullet} V^{*}$ can be expressed as

$$
\varphi=\lambda e^{B} \wedge\left(\theta_{1} \wedge \ldots \wedge \theta_{r}\right)
$$

for $\lambda \in \mathbb{R} \backslash\{0\}, B \in \bigwedge^{2} V^{*}$ and some linearly independent $\theta_{1}, \ldots, \theta_{r} \in V^{*}$ we say it is a pure spinor.

We have another point of view to understand the action of $V \oplus V^{*}$ on $\Lambda^{\bullet} V^{*}$ using the Clifford algebra of $V \oplus V^{*}$. As we have pointed, $V \oplus V^{*}$ comes with a canonical pairing. If we denote by $Q$ the quadratic form associated to this pairing, we define

$$
\mathrm{Cl}\left(V \oplus V^{*}, Q\right)=T\left(V+V^{*}\right) / \operatorname{gen}\left(v \otimes v-Q(v): v \in V \oplus V^{*}\right),
$$

where $T\left(V+V^{*}\right)$ is the tensor algebra of $V+V^{*}$, and $\operatorname{gen}\left(v \otimes v-Q(v): v \in V \oplus V^{*}\right)$ is an ideal in $T\left(V+V^{*}\right)$.

We shall denote the class $\left[v_{1} \otimes \ldots \otimes v_{r}\right]$ of $v_{1} \otimes \ldots \otimes v_{r} \in T\left(V+V^{*}\right)$ by $v_{1} \ldots v_{r}$. The grading on $T\left(V+V^{*}\right)$ provides also $\mathrm{Cl}\left(V \oplus V^{*}, Q\right)$ with a grading, but in this case is a $\mathbb{Z}_{2}$-grading, as it only distinguishes the parity, since the ideal we are quotientig by, even though it is not homogeneous, it only contains elements of even degree. Hence, we can decompose the Clifford algebra as $\mathrm{Cl}\left(V \oplus V^{*}, Q\right)=\mathrm{Cl}_{0}\left(V \oplus V^{*}, Q\right) \oplus \mathrm{Cl}_{1}\left(V \oplus V^{*}, Q\right)$, where $\mathrm{Cl}_{0}\left(V \oplus V^{*}, Q\right)=\left[\left(V \oplus V^{*}\right)^{\otimes \text { even }}\right]$ and $\mathrm{Cl}_{1}\left(V \oplus V^{*}, Q\right)=\left[\left(V \oplus V^{*}\right)^{\otimes \text { ood }}\right]$. Using this fact, we can write any element of $\mathrm{Cl}\left(V \oplus V^{*}, Q\right)$ as $\alpha=\alpha_{0}+\alpha_{1}$. Henceforth, we omit writing the quadratic form $Q$, as it will be easy understood, to ease the notation.

The way to relate the Clifford algebra $\mathrm{Cl}\left(V \oplus V^{*}\right)$ with the action $\left(V \oplus V^{*}\right) \times \Lambda^{\bullet} V^{*} \rightarrow \Lambda^{\bullet} V^{*}$ is to realize we can include $\Lambda^{\bullet} V^{*}$ inside the Clifford algebra. Indeed, as $V^{*}$ is isotropic, $\Lambda^{\bullet} V^{*}=$ $\mathrm{Cl}\left(V^{*}\right)$ and this is a subalgebra of $\mathrm{Cl}\left(V \oplus V^{*}\right)$. Then, we can rewrite the action as

$$
\mathrm{Cl}\left(V \oplus V^{*}\right) \otimes \mathrm{Cl}\left(V^{*}\right) \rightarrow \mathrm{Cl}\left(V^{*}\right) .
$$

To understand how this action works, it is better to take a basis $\left\{e_{i}\right\}$ for $V$, with dual basis $\left\{e^{i}\right\}$, so that $\left\{e_{i}\right\} \cup\left\{e^{i}\right\}$ is a basis of $V \oplus V^{*}$. Then, under product in the Clifford algebra, we have

$$
e_{i}^{2}=0, \quad\left(e^{i}\right)^{2}=0, \quad e_{i} e^{i}=1-e^{i} e_{i}, \quad e_{i} e^{j}=-e^{j} e_{i} .
$$

Let $e \in V$ and $1 \in \Lambda^{\bullet} V^{*}$. On one hand, the action we have gives us $i_{e} 1=0$, while the Clifford product gives us $e 1=e$. Although it may seem an incompatible fact, we can understand $\Lambda^{\bullet} V^{*}$ inside $\mathrm{Cl}\left(V \oplus V^{*}\right)$ in a compatible way, as $\mathrm{Cl}^{*}(V) \cdot \operatorname{det} V$, where $\operatorname{det} V$ is generated by $e_{1} \ldots e_{n}$. Under this correspondence, 1 is mapped to $e_{1} \ldots e_{n}$, so now $e_{1}\left(e_{1} \ldots e_{n}\right)=0\left(e_{2} \ldots e_{n}\right)=0$. And, in fact, this is the correct way to understand the action using the Clifford algebra:

$$
(X+\xi) \cdot \varphi \wedge \operatorname{det} V=(X+\xi) \varphi \operatorname{det} V
$$

As a final comment, we can define two special subgroups of $\mathrm{Cl}\left(V \oplus V^{*}\right)$, the Pin and Spin subgroups, which are double covers of $O\left(V \oplus V^{*}\right)$ and $S O\left(V \oplus V^{*}\right)$ respectively.

$$
\begin{gathered}
\operatorname{Pin}\left(V \oplus V^{*}\right)=\left\{g=v_{1} \ldots v_{r}: v_{i} \in V \oplus V^{*}, Q\left(v_{i}\right)= \pm 1\right\} \\
\operatorname{Spin}\left(V \oplus V^{*}\right)=\left\{g=v_{1} \ldots v_{2 r}: v_{i} \in V \oplus V^{*}, Q\left(v_{i}\right)= \pm 1\right\}=\operatorname{Pin}\left(V \oplus V^{*}\right) \cap \mathrm{Cl}_{0}\left(V \oplus V^{*}\right)
\end{gathered}
$$

### 2.5 Linear generalized complex structures

Now that we have a better understanding of maximally isotropic subspaces and how we can describe them, it is time to return to the starting point and care about what we can say about riemanninan metrics, symplectic and complex structures...

Definition 2.5. A linear generalized complex structure $\mathcal{J}$ is an endomorphism of $V \oplus V^{*}$ such that $\mathcal{J}^{-1}=-\mathrm{id}$ and $\langle\mathcal{J} u, \mathcal{J} v\rangle=\langle u, v\rangle$ for any $u, v \in V \oplus V^{*}$.

It is rather obvious that this definition is just the generalized analogue of linear complex structures with the additional requirement of the preservation of the pairing. However, we gain more than a mere generalization.

Examples 2.6. Let $J$ be a linear complex structure on $V$ and $\omega$ a linear symplectic structure on $V$. By considering the endomorphisms

$$
\mathcal{J}_{J}=\left(\begin{array}{cc}
-J & 0 \\
0 & J^{*}
\end{array}\right), \quad \mathcal{J}_{\omega}=\left(\begin{array}{cc}
0 & -\omega^{-1} \\
\omega & 0
\end{array}\right),
$$

we obtain two linear generalized complex structures, so with this definition we gather all the structures we were interested in the first place as particular examples.

As a consequence of the pairing preserving condition, we get a orthogonality property:

$$
\langle\mathcal{J} v, v\rangle=\left\langle\mathcal{J}^{2} v, \mathcal{J} v\right\rangle=\langle-v, \mathcal{J} v\rangle=-\langle\mathcal{J} v, v\rangle \Longrightarrow\langle\mathcal{J} v, v\rangle=0 .
$$

This is the key observation to prove the following proposition about the dimension required to admit such structures.

Proposition 2.7. A vector space $V$ admits a linear generalized complex structure if and only if the dimension of $V$ is even.

Proof. We first assume that $V$ admits a linear generalized complex structure $\mathcal{J}$, and take a nonzero $v_{1} \in V \oplus V^{*}$ such that $\left\langle v_{1}, v_{1}\right\rangle=0$. Then, by the orthogonality of $\mathcal{J}$, we have that $\left\langle\mathcal{J} v_{1}, \mathcal{J} v_{1}\right\rangle=0$. The vectors $v_{1}$ and $\mathcal{J} v_{1}$ are linearly independent as it can be easily checked, and by the computation we have done before, we know that they are orthogonal. Hence, $S_{1}=\operatorname{span}\left(v_{1}, \mathcal{J} v_{1}\right)$ is an isotropic subspace. If it is not maximal, then consider $v_{2} \in$ $S_{1}^{\perp} \backslash S_{1}$ such that $\left\langle v_{2}, v_{2}\right\rangle=0$. By the same argument as before, $\operatorname{span}\left(v_{2}, \mathcal{J} v_{2}\right)$ is an isotropic subspace and so is $S_{2}=\operatorname{span}\left(v_{1}, \mathcal{J} v_{1}, v_{2}, \mathcal{J} v_{2}\right)$, given that it can be easily checked that the four vectors $v_{1}, \mathcal{J} v_{1}, v_{2}, \mathcal{J} v_{2}$ are linearly independent. We can repeat this process until we obtain a maximally isotropic subspace $S_{m}=\operatorname{span}\left(v_{1}, \mathcal{J} v_{1}, \ldots, v_{m}, \mathcal{J} v_{m}\right)$, which is clearly even. Now, recalling that the dimension of a maximally isotropic subspace of $V \oplus V^{*}$ is $\operatorname{dim} V$, we have that $\operatorname{dim} V$ must be even.

For the converse, as we know that $V$ admits, for instance, a linear symplectic structure if $\operatorname{dim} V$ is even, then by the example we know we can find a linear generalized complex structure from such a symplectic structure.

By complexifying $V \oplus V^{*}$, we can again find the $+i$-eigenspace of $\mathcal{J}$, which brings us again the descriptions of the maximally isotropic spaces we had before. For example, taking again $\mathcal{J}_{J}$ and $\mathcal{J}_{\omega}$ as in the example we have

$$
L_{J}=V^{0,1} \oplus\left(V^{1,0}\right)^{*}, \quad L_{\omega}=\left\{X-i_{\omega} X: v \in V_{\mathbb{C}}\right\} .
$$

These subspaces of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ satisfy $L_{J} \cap \bar{L}_{J}=\{0\}=L_{\omega} \cap \bar{L}_{\omega}$. Moreover, we have that $L_{J}=L_{J}^{\perp}$ and $L_{\omega}=L_{\omega}^{\perp}$, so they are maximally isotropic. More generally, what we have is the following result.

Lemma 2.8. The $+i$-eigenspace $L$ of a linear generalized complex structure is a maximally isotropic subspace of $\left(V \oplus V^{*}\right)_{\mathbb{C}}$.

Proof. Let $v \in L$, so we have $\mathcal{J} v=i v$. In $\left(V \oplus V^{*}\right)_{\mathbb{C}}$ we have the $\mathbb{C}$-linear extension of the canonical pairing in $V \oplus V^{*}$. Thus

$$
\langle v, v\rangle=\langle\mathcal{J} v, \mathcal{J} v\rangle=\langle i v, i v\rangle=-\langle v, v\rangle .
$$

Using the polarization identity, we obtain that $L$ is isotropic. Finally, the fact that $\operatorname{dim} V=$ $\operatorname{dim}_{\mathbb{C}} L$ concludes the proof.

Furthermore, we can check that a linear complex structure can be expressed as a maximally isotropic subspace $L \subset\left(V \oplus V^{*}\right)_{\mathbb{C}}$ such that $L \cap \bar{L}=\{0\}$. Using the lemma, we already know that $L$ is maximally isotropic, and that $\mathcal{J}$ acts as the multiplication by $i$ on $L$. Consequently, it acts as the multiplication by $-i$ on $\bar{L}$. Hence, what remains to be check to be able to ensure that the endomorphism $\mathcal{J}$ defined in this way, is its orthogonality. Since any element of $V+V^{*}$ can be written as $l+\bar{l}$ for $l \in L$, we have

$$
\begin{aligned}
\langle\mathcal{J}(l+\bar{l}), \mathcal{J}(l+\bar{l})\rangle & =\langle i l-i \bar{l}, i l-i \bar{l}\rangle=-\langle l, l\rangle+2\langle l, \bar{l}\rangle-\langle\bar{l}, \bar{l}\rangle \\
& =2\langle l, \bar{l}\rangle=\langle l, l\rangle+2\langle l, \bar{l}\rangle+\langle\bar{l}, \bar{l}\rangle=\langle l+\bar{l}, l+\bar{l}\rangle
\end{aligned}
$$

where we have used the fact that both $L$ and $\bar{L}$ are isotropic so $\langle l, l\rangle=0\langle\bar{l}, \bar{l}\rangle$. Hence, $\mathcal{J}$ is orthogonal and is indeed a linear complex strucutre. From here we can derive the description of these structures as annihilators of forms. First of all, we have a generalization of the action of $V \oplus V^{*}$ on $\Lambda^{\bullet} V^{*}$ into

$$
\left(V \oplus V^{*}\right)_{\mathbb{C}} \times \bigwedge^{\bullet} V_{\mathbb{C}}^{*} \longrightarrow \bigwedge^{\bullet} V_{\mathbb{C}}^{*} .
$$

Thus, we can derive straightforwardly the description of $L_{J}$ and $L_{\omega}$ in terms of aniihilators:

$$
L_{J}=\operatorname{Ann}\left(\operatorname{vol}_{\left(V^{1,0}\right)^{*}}\right), \quad L_{\omega}=\operatorname{Ann}\left(e^{i \omega}\right) .
$$

And, of course, thanks to Proposition 2.3, we have that the forms whose annihilator describes a maximally isotropic subspace are of the form

$$
\varphi=\lambda e^{B+i \omega} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r}
$$

where $\lambda \in \mathbb{C} \backslash\{0\}, B, \omega \in \bigwedge^{2} V^{*}$ and linearly independent $\theta_{i} \in V^{*}$. However this is not all, since so far the condition $L \cap \bar{L}=\{0\}$ has not appeared yet. For this we have to introduce a new pairing in $\Lambda^{\bullet} V^{*}$.

Let ${ }^{T}$ be the operation on $\Lambda^{\bullet} V^{*}$ defined by extending linearly

$$
\left(\alpha_{1} \wedge \ldots \wedge \alpha_{r}\right)^{T}=\alpha_{r} \wedge \ldots \wedge \alpha_{1}
$$

where $\alpha_{i} \in V^{*}$. We define the Chevalley pairing $(\cdot, \cdot): \Lambda^{\bullet} V^{*} \times \Lambda^{\bullet} V^{*} \rightarrow \operatorname{det} V^{*}=\Lambda^{\text {top }} V^{*}$ by

$$
(\varphi, \psi)=\left(\varphi^{T} \wedge \psi\right)_{\mathrm{top}}
$$

where $\alpha_{\text {top }}$ denotes the maximal exterior power component.
Lemma 2.9. Let $v \in V \oplus V^{*}$. Then $(v \cdot \varphi, \psi)=(\varphi, v \cdot \psi)$. Consequently, for $x \in \mathrm{Cl}\left(V \oplus V^{*}\right)$

$$
(x \cdot \varphi, \psi)=\left(\varphi, x^{T} \cdot \psi\right)
$$

and for $g \in \operatorname{Spin}\left(V \oplus V^{*}\right)$

$$
(g \cdot \varphi, g \cdot \psi)= \pm(\varphi, \psi)
$$

Proof. As the pairing is linear in the forms, we can reduce to the case of forms of pure degree. Let $\varphi \in \bigwedge^{s} V^{*}$ and $\psi \in \Lambda^{t} V^{*}$. Then for $X \in V$

$$
(X \cdot \varphi, \psi)=(\varphi, X \cdot \psi)
$$

when $s+t=\operatorname{dim} V \oplus 1$, because $\varphi^{T} \wedge \psi=0$ and $i_{X}\left(\varphi^{T}\right)=(-1)^{s}\left(i_{X} \varphi\right)^{T}$. The other cases are irrelevant as the top component would be immediately 0 . Similarly, for $\xi \in V^{*}$

$$
(\xi \cdot \varphi, \psi)=(\varphi, \xi \cdot \psi)
$$

when $s+t=\operatorname{dim} V-1$ because of the anticommutation of the wedge product. Again, the other cases are discarded because the top component would be zero. The second identity follows immediately and so does the third recalling that $g^{T} g= \pm 1$ since $g \in \operatorname{Spin}\left(V \oplus V^{*}\right)$.

This pairing comes in handy, as it provides a description of the condition $L \cap \bar{L}=\{0\}$ in terms of spinors. The following results show precisely how this is done.

Lemma 2.10. Let $L=\operatorname{Ann}(\varphi)$ be a maximally isotropic subspace. Then $L \cap V=\{0\}$ if and only if $\varphi_{\text {top }} \neq 0$.

Proof. Let us assume that $\varphi_{\text {top }} \neq 0$. Then, for any $X \in V \backslash\{0\}$, we have $i_{X} \varphi_{\text {top }} \neq 0$ and consequently $i_{X} \varphi \neq 0$. Therefore $L \cap V=\{0\}$. For the other implication, we assume $L=$ $L\left(E, i^{*} B\right)$. If $E=0$ then $L(0,0)=V^{*}$ and $\varphi=\operatorname{vol}_{V}^{*} \in \operatorname{det} V^{*}$ so $\varphi_{\mathrm{top}} \neq 0$. Considering the case $E \neq 0$, since $L\left(E, i^{*} B\right) \cap V=\{0\}$, we have that $i^{*} B$ must be nondegenerate and hence $r$ is even. Therefore, for $X \in E$,

$$
i_{X}\left(B^{k} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r}\right)=k i_{X} B \wedge B^{k-1} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r}
$$

so $B^{j} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r} \neq 0$ for $k=1, \ldots,(n-r) / 2$ by induction, and the term $k=(n-r) / 2$ corresponds precisely to $\varphi_{\text {top }}$.

Lemma 2.11. Let $L=\operatorname{Ann}(\varphi)$ be a maximally isotropic subspace. Then $L \cap L\left(E^{\prime}, 0\right)=\{0\}$ if and only if $\left(\varphi, \operatorname{vol}_{\operatorname{Ann}\left(E^{\prime}\right)}\right) \neq 0$.

Proof. Let $L=L\left(E, i^{*} B\right), \operatorname{dim} E=r$ and $\operatorname{dim} E^{\prime}=r^{\prime}$. If $r+r^{\prime}$ is odd then the Chevalley pairing of the forms is zero immediately because the top components have even degree. Otherwise, we have

$$
\left(\varphi, \operatorname{vol}_{\operatorname{Ann}\left(E^{\prime}\right)}\right)= \pm B^{\left(n-r-r^{\prime}\right) / 2} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r} \wedge \tilde{\theta}_{1} \wedge \ldots \wedge \tilde{\theta}_{r^{\prime}}
$$

where $\theta_{1}, \ldots, \theta_{r}$ are linearly independent elements of $V^{*}$ and $\tilde{\theta}_{1} \wedge \ldots \wedge \tilde{\theta}_{r^{\prime}}$ is the volume form of the subspace $\operatorname{Ann}\left(E^{\prime}\right)$.

If $L \cap L\left(E^{\prime}, 0\right) \neq\{0\}$, then we can distinguish two cases:

- $X \in E^{\prime}$ belongs to $E \cap \operatorname{ker} B$, and so $i_{X}\left(\varphi, \operatorname{vol}_{\operatorname{Ann}\left(E^{\prime}\right)}\right)=0$.
- $\xi \in \operatorname{Ann}\left(E^{\prime}\right)$ belongs to $\operatorname{Ann}(E)$, and so $\theta_{1} \wedge \ldots \wedge \theta_{r} \wedge \tilde{\theta}_{1} \wedge \ldots \wedge \tilde{\theta}_{r^{\prime}}=0$.

If $L \cap L\left(E^{\prime}, 0\right)=\{0\}$, then $i^{*} B$ is nondegenerate on $E \cap E^{\prime}$ and $\operatorname{Ann}(E) \cap \operatorname{Ann}\left(E^{\prime}\right)=\{0\}$. Again, we have two cases:

- If $E \cap E^{\prime} \neq\{0\}$ then

$$
i_{X}\left(B^{k} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r} \wedge \tilde{\theta}_{1} \wedge \ldots \wedge \tilde{\theta}_{r^{\prime}}\right)=k i_{X} B \wedge B^{k-1} \wedge \theta_{1} \wedge \ldots \wedge \theta_{r} \wedge \tilde{\theta}_{1} \wedge \ldots \wedge \tilde{\theta}_{r^{\prime}}
$$

for $k=1, \ldots,\left(n-r-r^{\prime}\right) / 2$ by induction and so the pairing is different from zero.

- If $E \cap E^{\prime}=\{0\}$ and $\operatorname{Ann}(E) \cap \operatorname{Ann}\left(E^{\prime}\right)=\{0\}$ then $\left(\varphi, \operatorname{vol}_{\operatorname{Ann}\left(E^{\prime}\right)}\right) \neq 0$.

Lemma 2.12. Let $L=\operatorname{Ann}(\varphi)$ and $L^{\prime}=\operatorname{Ann}(\psi)$ be maximally isotropic subspaces. Then $L \cap L^{\prime}=\{0\}$ if and only if $(\varphi, \psi) \neq 0$.

Proof. Using the action of B-fields on forms, we can rewrite $L^{\prime}=e^{-B} L\left(E^{\prime}, 0\right)$ where $\psi=e^{B} \wedge \psi^{\prime}$. Then $e^{B} L \cap L\left(E^{\prime}, 0\right)=\{0\}$ if and only $\left(e^{-B} \wedge \varphi, \psi^{\prime}\right)=0$. But by Lemma 2.9, this is equivalent to $\left(\varphi, e^{B} \wedge \psi^{\prime}\right)=0$, and up to a sign, $\left(\varphi, e^{B} \wedge \psi^{\prime}\right)=(\varphi, \psi)$, so we have concluded the proof.

Finally, we can gather all these lemmas to state the characterization of the condition $L \cap \bar{L}=0$ in terms of spinors.

Proposition 2.13. A linear generalized complex structure is given by a pure form $\varphi=\lambda e^{B+i \omega} \wedge$ $\theta_{1} \wedge \ldots \wedge \theta_{r} \in \Lambda^{\bullet} V_{\mathbb{C}}^{*}$ such that $(\varphi, \bar{\varphi}) \neq 0$.

As we had for linear complex structures, there is not uniqueness in the form describing the structure. Moreover, we can be more precise about the characterization of linear complex structure from the generalized linear algebra point of view.

Lemma 2.14. Linear complex structures are in bijective correspondence to linear generalized complex structures of diagonal form $\left(\begin{array}{ll}0 \\ 0 & \mathbf{0}\end{array}\right)$.

Proof. This is immediate after noticing that the right-upper block must be a linear complex structure and the left-lower block must be minus its dual, so there is no other possible choice.

Proposition 2.15. The forms in $\Lambda^{\bullet} V_{\mathbb{C}}^{*}$ whose annihilator gives a linear complex structure are those decomposable forms $\Omega=\lambda \theta_{1} \wedge \ldots \wedge \theta_{n / 2}$ such that $\Omega \wedge \bar{\Omega} \neq 0$. Moreover, two forms give the same structure if and only if they are multiples of each other.

Proof. Linear generalized complex structure of the form

$$
\left(\begin{array}{cc}
-J & 0 \\
0 & J^{*}
\end{array}\right)
$$

are given by forms $\varphi=\lambda \theta_{1} \wedge \ldots \wedge \theta_{r}$. As we require that $(\varphi, \bar{\varphi}) \neq 0$, we must have $r=n / 2$ and $\varphi \wedge \bar{\varphi} \neq 0$. Finally, by the previous lemma, the $-i$-eigenspace of the linear complex structure is the projection to $V$ of $\operatorname{Ann}(\varphi)$, so the result is proven.

With this, we have conclude study of characterization of maximally isotropic subspaces and its relation with linear generalized complex structures. Nevertheless, there remains a small fact related to them, as there is a number which determines what type of structure we do have related to the underlying linear structure.

Definition 2.16. We define the type of a linear generalized complex structure as

- Given an automorphism $\mathcal{J} \in \mathrm{O}\left(\left(V \oplus V^{*}\right)_{\mathbb{C}}\right)$ such that $\mathcal{J}^{2}=-\mathrm{id}$,

$$
\operatorname{type}(\mathcal{J})=\frac{1}{2} \operatorname{dim}_{\mathbb{R}}\left(V^{*} \cap \mathcal{J} V^{*}\right)
$$

- Given a maximally isotropic space $L=L(E, \varepsilon)$,

$$
\operatorname{type}(L)=\operatorname{dim}_{\mathbb{C}} V_{\mathbb{C}}-\operatorname{dim}_{\mathbb{C}} E=\operatorname{dim}_{\mathbb{C}} \operatorname{Ann}(E)=\operatorname{dim}_{\mathbb{C}}\left(V_{\mathbb{C}}^{*} \cap L\right)
$$

- Given a maximally isotropic space $L=\operatorname{Ann}(\varphi)$, where $\varphi=\varphi_{0}+\ldots+\varphi_{n}, \varphi_{i} \in \bigwedge^{i} V_{\mathbb{C}}^{*}$,

$$
\operatorname{type}(\varphi)=\min \left\{k: \varphi_{k} \neq 0\right\} .
$$

Proposition 2.17. Linear generalized complex structures of type 0 and $n / 2$ are B-field transforms of, respectively, linear symplectic and linear complex structures.

Proof. Before starting the proof of both cases, we remark a fact that will save us some work. As even forms are commutative in the alternate algebra, we can write $e^{B+i \omega}=e^{B} \wedge e^{i \Omega}$. As we have already seen, $e^{B} \wedge \varphi$ correspond to the B-field transformation $e^{-B} \operatorname{Ann}(\varphi)$. But because B-fields are symmetries of the pairing, we can study all the structures up to the action of real B-fields.

We start with the structures of type 0 . This means that we have $\varphi=e^{B+i \omega}=e^{B} \wedge e^{i \omega}$ such that $(\varphi, \bar{\varphi}) \neq 0$. Thus

$$
(\varphi, \bar{\varphi})=\left(e^{B} \wedge e^{i \omega}, e^{B} \wedge e^{-i \omega}\right)=\left(e^{2 i \omega}, 1\right) \neq 0
$$

if and only if $\omega^{n / 2} \neq 0$. This tells us that $\omega$ is nondegenerate, so type 0 structures are B-field transformations of linear symplectic structures.

For structures of type $n / 2$, these are complex B-field transformations of complex structures $\Omega=\theta_{1} \wedge \ldots \wedge \theta_{n / 2}$. However, we have seen that B-fields are symmetries only for real $B$. We know that $\Omega$ is an $(m, 0)$-form and we can decompose $B+i \omega=\xi^{2,0}+\xi^{1,1}+\xi^{0,2}$ where $\xi^{i, j}$ is an $(i, j)$-form. Taking into account that there cannot be neither $(n / 2+2,0)$ nor ( $n / 2+1,1)$-forms, we only keep $\xi^{0,2}$. By considering another B-field $B^{\prime}=\xi^{0,2}+\overline{\xi^{0,2}}$, we can rewrite the original form as $\varphi=e^{B+i \omega} \wedge \Omega=e^{B^{\prime}} \wedge \Omega$, so $\varphi$ is a real B-field transformation of a linear complex structure.

### 2.6 Integrability conditions and generalized geometry

We now have all the ingredients from linear algebra to translate linear generalized complex structures to the tangent spaces of a manifold, where we can carry such structures simply by building them pointwise. However, this process does not work always, as we have some conditions on the transition functions between charts and they may not be compatible with the additional data transferred to the tangent spaces. For this we have to add some integrability conditions, and we can start by giving these conditions for the case of complex and symplectic structures.

Definition 2.18. A complex structure on a manifold $M$ is a bundle map $J: T M \rightarrow T M$ such that $J^{2}=-\mathrm{id}$ and the $+i$-eigenbundle $L$ of $J$ is involutive with respect to the Lie bracket, $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$.

Definition 2.19. A symplectic structure on a manifold $M$ is a nondegenerate 2 -form $\omega \in$ $\Omega^{2}(M)$ such that $d \omega=0$.

These two structures are related to the tangent and cotangent bundle respectively, but we have developed more machinery to work jointly with the generalized tangent bundl $\rrbracket, T M \oplus$ $T^{*} M$. Following the reasoning from the linear algebra part, we should focus on finding analogous constructions on $T M$ and transport them into $T M \oplus T^{*} M$. Thus, a good starting point could be to find a bracket on the sections of $T M \oplus T^{*} M$, and for this we make use of the notion of vector bundles with well behaved brackets.

Definition 2.20. A Lie algebroid is a vector bundle $E \rightarrow M$ together with a bundle map $\pi: E \rightarrow T M$, called the anchor map, and a Lie bracket on $\Gamma(E)$, such that for $X, Y \in \Gamma(E)$ and $f \in \mathcal{C}^{\infty}(M)$,

$$
[X, f Y]=X(f) Y+f[X, Y]
$$

As a consequence, the anchor map commutes with the bracket, $\pi([X, Y])=[\pi(X), \pi(Y)]$.

According to this definition, we shall search for a bracket in $\Gamma\left(T M \oplus T^{*} M\right)$ in a similar fashion, which has also to take into account the integrability conditions already mentioned. To start with, an almost complex structure $J$ on $M$ (we do not care about the integrability for now) determines the subbundle

$$
L_{J}=T M^{0,1}+\left(T M^{1,0}\right)^{*}
$$

Now, we complexify the generalized tangent bundle and begin to search for a suitable bracket. Because of the anchor map, which in our case is the projection to $T M_{\mathbb{C}}$, the bracket of $[X+$ $\xi, Y+\eta]$ should project to $[X, Y]$. If we now recall the integrability condition of $J$, which is the involutivity of $T M^{0,1}$, it is a good idea to consider a bracket of the form $[X+\xi, Y+\eta]=[X, Y]+P$ where $P$ is some 1-form which involves $X+\xi$ and $Y+\eta$.

If we move to the other extreme case, the symplectic structures, we have the subbundle

$$
L_{\omega}=\{X+\omega(X, \cdot): X \in T M\} .
$$

On one hand, we have $[X+\omega(X, \cdot), Y+\omega(Y, \cdot)]=[X, Y]+P$, and on the other hand we have $d \omega=0$, which comes from the integrability condition. As we want the subbundle to be

[^0]involutive, necessarily $\omega([X, Y])=P$, and this has to agree with $d \omega=0$. Using the known fact that $i_{[X, Y]}=\mathcal{L}_{X} i_{Y}-i_{Y} \mathcal{L}_{X}$, we try to find some other condition on $P$ :
$$
P=\omega([X, Y])=i_{[X, Y]} \omega=\mathcal{L}_{X} i_{Y} \omega-i_{Y} \mathcal{L}_{X} \omega=\mathcal{L}_{X} i_{Y} \omega-i_{Y} d i_{X} \omega-i_{Y} i_{X} d \omega
$$

Because of we are imposing $d \omega=0$, we get that in this case we must have $O=\mathcal{L}_{X} i_{Y} \omega-i_{Y} \mathcal{L}_{X} \omega$. This fact points to general definition of $P$.

Definition 2.21. The Dorfman bracket of sections of $T M \oplus T^{*} M$ is given by

$$
[X+\xi, Y+\eta]=[X, Y]+\mathcal{L}_{X} \eta-i_{Y} d \xi
$$

The first question that arises is whether the Dorfman bracket is a Lie bracket. Linearity is obvious, so we should care about skewsymmetry and the Jacobi identity. For the skewsymmetry we have

$$
[X+\xi, X+\xi]=[X, X]+\mathcal{L}_{X} \xi-i_{X} d \xi=d i_{X} \xi
$$

so regretably, this happens to fail, as this is not zero always. Nevertheless, for the Jacobi identity we have more luck.

Lemma 2.22. The Dorfman bracket satisfies for $u, v, w \in \Gamma\left(T M \oplus T^{*} M\right)$,

$$
[u,[v, w]]=[[u, v], w]+[v,[u, w]] .
$$

Hence, at least the Dorfman bracket is itself a derivation of it. Moreover, the condition from the Lie algebroid is satisfied.

Lemma 2.23. For $u, v \in \Gamma\left(T M \oplus T^{*} M\right)$ and $f \in \mathcal{C}^{\infty}(M)$, we have

$$
[u, f v]=\pi(u)(f) v+f[u, v] .
$$

Proof. Let $u=X+\xi, v=Y+\eta$ with $X, Y \in \Gamma(T M)$ and $\xi, \eta \in \Omega^{1}(M)$. Then

$$
\begin{aligned}
{[X+\xi, f Y+f \eta)] } & =[X, f Y]+\mathcal{L}_{X}(f \eta)-i_{Y} d \eta=X(f) Y+f[X, Y]+X(f) \eta+f \mathcal{L}_{X} \eta-i_{f Y} d \eta \\
& =X(f)(Y+\eta)+f[X, Y]+f \mathcal{L}_{X} \eta-f i_{Y} d \eta \\
& =X(f)(Y+\eta)+f\left([X, Y]+\mathcal{L}_{X} \eta-i_{Y} d \eta\right) \\
& =\pi(u)(f) v+f[u, v] .
\end{aligned}
$$

Aside from this, we may wonder what happens with the canonical pairing we have on $T M \oplus$ $T^{*} M$, which comes from the pointwise canonical pairing. It turns out that the Dorfman bracket acts as a derivation of it.

Proposition 2.24. The Dorfman bracket satisfies, for $u, v, w \in \Gamma\left(T M \oplus T^{*} M\right)$,

$$
\pi(u)\langle v, w\rangle=\langle[u, v], w\rangle+\langle v,[u, w]\rangle .
$$

We can gather all these properties to actually define a new structure over a manifold $M$.
Definition 2.25. A Courant algebroid $(E,\langle\cdot, \cdot\rangle,[\cdot, \cdot], \pi)$ over a manifold $M$ consists of a vector bundle $E \rightarrow M$ together with a nondegenerate symmetric bilinear form $\langle\cdot, \cdot\rangle$, a linear bracket $[\cdot, \cdot]$ on the sections $\Gamma(E)$ and a bundle map $\pi: E \rightarrow T M$ such that the following properties are satisfied for any $u \in \Gamma(E)$ :
(1) $[u, u]=D\langle u, u\rangle$.
(2) The operator $[u, \cdot]$ is a derivation of the bracket.
(3) The operator $[u, \cdot]$ is a derivation of the pairing.
where we define the map $D: \mathcal{C}^{\infty}(M) \rightarrow \Gamma(E)$, for $f \in \mathcal{C}^{\infty}(M)$, by $D f=(2\langle\cdot, \cdot\rangle)^{-1} \pi^{*} d f$. Additionally, as a consequence of these three properties we have that

- The anchor map preserves the bracket, $\pi([u, v])=[\pi(u), \pi(v)]$.
- The bracket satisfies the Leibniz rule, $[u, f v]=\pi(u)(f) v+f[u, v]$.

Clearly, as we can recognize, we have that $\left(T M \oplus T^{*} M,\langle\cdot, \cdot\rangle,[\cdot, \cdot], \pi\right)$ is a Courant algebroid over $M$, and this is a sufficiently nicely behaved structure to deal with generalized structures. Hence, we shall start to search for a description of such structures in terms of annihilators. We begin giving the definition of a real Dirac structure, since the complex case which are generalized complex structure can be work analogously.

Definition 2.26. A Dirac structure is a maximally isotropic subbundle $L \subseteq T M \oplus T^{*} M$ whose sections are involutive with respect to the Dorfman bracket.

Definition 2.27. The canonical bundle of a Dirac structure $L$ is the smooth line bundle $K$ of $\Lambda^{\bullet} V^{*}$ given pointwise by

$$
K_{x}=\left\{\varphi \in \bigwedge^{\bullet} V^{*}: \operatorname{Ann}(\varphi)=L_{x}\right\} \cup\{0\} .
$$

Now that we have such a subbundle, we look for the integrability condition. To state it, we require a few results. We start with an extension of the Lie derivative of forms to an action of $\Gamma\left(T M \oplus T^{*} M\right)$ on $\Omega^{\bullet}(M)$, defined by

$$
\mathcal{L}_{u} \varphi=d(u \cdot \varphi)+u \cdot(d \varphi), \quad u \in \Gamma\left(T M \oplus T^{*} M\right)
$$

Lemma 2.28. For $\varphi \in \Omega^{\bullet}(M)$ and $u, v \in \Gamma\left(T M \oplus T^{*} M\right)$, we have

$$
[u, v] \cdot \varphi=\left[\mathcal{L}_{u}, v \cdot\right] \varphi .
$$

Proposition 2.29. The subbundle $L=\operatorname{Ann}(\varphi)$ is involutive if and only if

$$
u \cdot(v \cdot d \varphi)=0
$$

Proof. To be involutive means that for any $u, v \in \Gamma(L),[u, v] \in \Gamma(L)$. Then, $[u, v] \in \Gamma(L)$ if and only if $[u, v] \cdot \varphi=0$, but

$$
[u, v] \cdot \varphi=\left[\mathcal{L}_{u}, v \cdot\right] \varphi=\mathcal{L}_{u}(v \cdot \varphi)-v \cdot\left(\mathcal{L}_{u} \varphi\right)=-v \cdot(d(u \cdot \varphi)+u \cdot d \varphi)=-v \cdot(u \cdot d \varphi),
$$

so $L$ is involutive if and only if $u \cdot(v \cdot d \varphi)=0$ for any $u, v \in L$.
As useful as this description may seem, it is not like that, as we are describing $\varphi$ using sections of its annihilator. To find something more meaningful, we have the following lemma.

Lemma 2.30. Let $L$ be a maximally isotropic subbundle. The canonical subbundle $K$ of $L$ is the subbundle annihilated by any section of $L$. The subbundle $\left(T M \oplus T^{*} M\right) \cdot K$, that is, the bundle whose sections are exactly $\Gamma\left(T M \oplus T^{*} M\right) \cdot \Gamma(K)$, is the bundle annihilated by exactly any two sections of $L$.

Proof. The first statement follows directly from the pointwise definition of $K$. For the second one, let $u \in \Gamma\left(T M \oplus T^{*} M\right)$ and $l \in \Gamma(L)$. Then

$$
l \cdot(u \cdot \varphi)=-u \cdot(l \cdot \varphi)+2\langle u, l\rangle \varphi=2\langle u, l\rangle \varphi .
$$

This expression is not zero for all $u$ and $l$. However, if we take $l^{\prime}$ another section of $L$, then

$$
l^{\prime} \cdot(l \cdot(u \cdot \varphi))=l^{\prime} \cdot(2\langle u, l\rangle \varphi)=0
$$

so it is exactly annihilated by any two sections of $L$.
Proposition 2.31. A maximally isotropic subbundle $L$ given by $\operatorname{Ann}(\varphi)$ is involutive if and only if there exists $X+\xi \in \Gamma\left(T M \oplus T^{*} M\right)$ such that $d \varphi=(X+\xi) \cdot \varphi$.

This proposition is a consequence of the Lemma 2.30 and Proposition 2.29 and provides a weaker integrability condition than the required, for example, for a symplectic structure, which is $d \varphi=0$.

We are arriving at the end of the section, we may conclude by giving a description of the integrability condition in terms of the bundle map $\mathcal{J}: T M \oplus T^{*} M \rightarrow T M \oplus T^{*} M$ and not its $+i$-eigenspace. We know that the $+i$-eigenspace is $L=\left\{u-i \mathcal{J} u: u \in T M \oplus T^{*} M\right\}$. We know
that the integrability condition in terms of $L$ is that the Dorfman bracket of two sections of $L$ is again a section of $L$. Explicitly, since

$$
[u-i \mathcal{J} u, v-i \mathcal{J} v]=[u, v]-[\mathcal{J} u, \mathcal{J} v]-i([\mathcal{J} u, v]+[u, \mathcal{J} v]),
$$

we require that

$$
[\mathcal{J} u, \mathcal{J} v]=[u, v]+\mathcal{J}([\mathcal{J} u, v]+[u, \mathcal{J} v]) .
$$

Definition 2.32. The Nijenhuis tensor of $\mathcal{J}$ is given by

$$
N_{\mathcal{J}}(u, v)=[\mathcal{J} u, \mathcal{J} v]-[u, v]-\mathcal{J}([\mathcal{J} u, v]+[u, \mathcal{J} v]) .
$$

Hence, we can describe a generalized complex structure on a manifold $M$ in three ways:

- As a bundle map $\mathcal{J}: T M \oplus T^{*} M \rightarrow T M \oplus T^{*} M$ which preserves the canonical pairing, satisfies $\mathcal{J}^{2}=-\mathrm{id}$ and $N_{\mathcal{J}}=0$.
- As a maximally isotropic subbundle $L \subseteq\left(T M \oplus T^{*} M\right)_{\mathbb{C}}$ such that $L \cap \bar{L}=\{0\}$ and $L$ is involutive under the Dorfman bracket.
- A line subbundle $K \subseteq \Lambda^{\bullet} T^{*} M_{\mathbb{C}}$ locally given by a pure form $\varphi$ such that $(\varphi, \bar{\varphi}) \neq 0$ for the Chevalley pairing and $d \varphi=(X+\xi) \cdot \varphi$ for some $X+\xi \in \Gamma\left(T M \oplus T^{*} M\right)$.

To conclude, we can analyze the type of a generalized complex structure, which now is a function from the manifold to the integers. Taking a small neighbourhood $U$, we know that the subbundle $L$ is described by a form $\varphi \in \Omega^{\bullet}(U)$. Decomposing it in components with welldefined degree, $\varphi=\varphi_{0}+\ldots+\varphi_{n}$, we get that the type of the structure is zero everywhere as $\varphi_{0}$ does not vanish, and generically this will be the case. We can define the type-change locus as

$$
\left\{x \in M: \operatorname{type}\left(L_{x}\right) \neq 0\right\} .
$$

This set is locally the zero set of a function $\varphi_{0}$, so it is a closed subset of codimension 2 for generalized complex structures.

## 3 Vector bundles, principal bundles and characteristic classes

We devote this chapter to define both vector and principal bundles over a manifold $M$ and to establish some relations between them, as both constructions play an important role for the T-duality theory. On the one hand, vector bundles appear as the tangent and cotangent bundle, but most importantly as the generalized tangent bundle. On the other hand, principal bundles will appear as we will have some Lie groups acting on the manifolds and we will require the fibres of the bundle to have some compatibility -whatever that means for the moment- with it. Moreover, on both types of bundles we have the notions of connection and curvarture, which will be needed also for the theory of T-duality. For Section 3.1 we have followed mainly Joy07, but also [Mit01, Rub18]. For Section 3.2 we have extracted most of the facts from Zha11] and [Fra11].

### 3.1 Vector bundles and principal bundles

Definition 3.1. Let $M$ be a manifold. A vector bundle $E$ over $M$ is a fibre bundle whose fibres are vector spaces. That is, $E$ is a manifold eqquiped with a smooth projection $\pi: E \rightarrow M$. For each $m \in M$ the fibre $E_{m}=\pi^{-1}(m)$ has the structure of a vector space, and there is an open neighbourhood $U_{m}$ of $m$ such that $\pi^{-1}\left(U_{m}\right) \cong U_{m} \times V$ where $V$ is the fibre of $E$.

We can see that the definitions of the tangent and the cotangent bundle of a manifold fit into this description, and so do the generalized tangent bundle and the subbundles characterizing generalized complex structures.

Definition 3.2. Let $M$ be a manifold and $G$ a Lie group. A principal bundle $P$ over $M$ with structure group $G$ is a manifold $P$ eqquiped with a smooth projection $\pi: P \rightarrow M$, and an action of $G$ on $P$, which we will write as $p \stackrel{g}{\longmapsto} p \cdot g$, for $g \in G$ and $p \in P$. This $G$-action must be smooth,free and transitive, so the orbits of the $G$-action are the fibers and the orbit space $P / G$ is homeomorphic to the base $M$ The fibers have the structure of $G$-torsors, as they are homeomorphic to $G$ but have not a group structure as there is not a preferred choice of an identity element.

Both definitions offer a different points of view to study properties of the base manifold. Moreover, the following constructions show that they can be equivalent ways to study the same problem if some conditions are required.

Definition 3.3. Let $M$ be a manifold, and $E \rightarrow M$ a vector bundle with fibre $\mathbb{R}^{k}$. Define a manifold $F M_{E}$ by

$$
F M_{E}=\left\{\left(m, e_{1}, \ldots, e_{k}\right): m \in M,\left(e_{1}, \ldots, e_{k}\right) \text { is a basis for } E_{m}\right\}
$$

Define $\pi: F M_{E} \rightarrow M$ by $\left(m, e_{1}, \ldots, e_{k}\right) \mapsto m$. For each $A \in \operatorname{GL}(k, \mathbb{R})$ and $\left(m, e_{1}, \ldots, e_{k}\right) \in$ $F M_{E}$, define $\left(m, e_{1}, \ldots, e_{k}\right) \cdot A=\left(m, e_{1}^{\prime}, \ldots, e_{k}^{\prime}\right)$ where $e_{i}^{\prime}=\sum_{j=1}^{k} A_{i j}^{-1} e_{j}$. This gives an action of $\mathrm{GL}(k, \mathbb{R})$ on $F M_{E}$, which makes $F M_{E}$ into a principal bundle over $M$, with structure group $\mathrm{GL}(k, \mathbb{R})$. We call $F M_{E}$ the frame bundle of $E$.

Definition 3.4. Let $M$ be a manifold and $P$ a principal bundle over $M$ with structure group a Lie group $G$. Let $\rho$ be a representation of $G$ on a vector space $V$. Then $G$ acts on the product space $P \times V$ by the principal bundle action on the first factor and $\rho$ on the second. Define $\rho(P)=(P \times V) / G=P \times V /\left\{(p, v) \sim\left(p g, g^{-1} v\right)\right\}$. Now, $P / G=M$, so the obvious $\operatorname{map}(P \times V) / G \rightarrow P / G$ gives a projection from $\rho(P)$ to $M$. Since $G$ acts freely on $P$, this projection has fibre $V$, and it can be checked that $\rho(P)$ is a vector bundle over $M$ with fibre $V$.

When we take $\rho$ to be the canonical representation of $\mathrm{GL}(k, \mathbb{R})$ on $\mathbb{R}^{k}$, then $E \cong \rho\left(F M_{E}\right)$, so in this case we have a bijective correspondence between vector bundles over $M$ with fibre $\mathbb{R}^{k}$ and principal $\mathrm{GL}(k, \mathbb{R}$-bundles over $M$. However, principal bundles are a more general construction than vector bundles, as we can have the structure group $G$ to be any Lie group and not necessary $\mathrm{GL}(k, \mathbb{R})$ for some $k$.

Let $P$ be a principal bundle over $M$ with structure group $G$, and let $\mathfrak{g}$ be the Lie algebra of $G$. Let ad: $G \rightarrow \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of $G$ on $\mathfrak{g}$. Then, by the Definition 3.4 we obtain a natural vector bundle $\operatorname{ad}(P)$ over $M$ with fibre $\mathfrak{g}$, which we call the adjoint bundle.

As a general remark, let $\rho$ be a representation of $G$ on $V$, and $\pi: P \times V \rightarrow \rho(P)$ the natural projection. Then $P \times V$ is a trivial vector bundle over $P$ with fibre $V$. Given a smooth section of $\rho(P)$, we can take its pullback through the projection to get a smooth section of $P \times V$, which will be invariant under the action of $G$ on $P \times V$. This provides us a bijective correspondence between sections of $\rho(P)$ over $M$ and $G$-invariant sections of $P \times V$ over $P$, that is, $G$-invariant maps $P \rightarrow V$.

Now that we have gotten acquainted with these two types of bundles and how to translate one to the other, we can pass to define connections and curvartures on them and see that either on principal or vector bundles, the corresponding objects are equivalent.

Definition 3.5. Let $M$ be a manifold and $E \rightarrow M$ a vector bundle. A connection $\nabla^{E}$ on $E$ is a linear map $\nabla^{E}: \Gamma(E) \rightarrow \Gamma\left(E \otimes T^{*} M\right)$ satisfying

$$
\nabla^{E}(f e)=f \nabla^{E} e+e \otimes d f
$$

where $e \in \Gamma(E)$ and $f \in \mathcal{C}^{\infty}(M)$.
Given a connection $\nabla^{E}$ on $E$ and $v \in \Gamma(T M)$ we can write $\nabla_{v}^{E} e=\nabla^{E} e(v)$, where we contracting the $T M$ part from $v$ with the $T^{*} M$ part from $\nabla^{E} e$. Hence, if $f, g \in \mathcal{C}^{\infty}(M)$ we have

$$
\nabla_{f v}^{E}(g e)=f g \nabla_{v}^{E} e+f\left(\mathcal{L}_{v} g\right) e
$$

Assume $E$ is a vector bundle over $M$ with fibre $\mathbb{R}^{k}$ and let $e_{1}, \ldots, e_{k}$ be sections of $E$ over some open set $U \subseteq M$ such that $\left(e_{1}, \ldots, e_{k}\right)$ is a basis for $E$ at each point of $U$. Then we can write any section over $U$ as a linear combinations of the $e_{i}$ 's. Let $f_{1}, \ldots, f_{k}$ be smooth sections of $E \otimes T^{*} M$ over $U$, and define

$$
\nabla^{E}\left(\sum_{i=1}^{k} \varphi_{i} e_{i}\right)=\sum_{i=1}^{k}\left(\varphi_{i} f_{i}+e_{i} \otimes d \varphi_{i}\right)
$$

for all smooth functions $\varphi_{1}, \ldots, \varphi_{k}$ on $U$. Then $\nabla^{E}$ is a connection on $E$ over $U$ and every such connection can be written in this way.

Given a vector bundle $E$ over $M$, we can consider $\operatorname{End}(E)=E \otimes E^{*}$. The curvature $R\left(\nabla^{E}\right)$ of a connection $\nabla^{E}$ is a section of $\operatorname{End}(E) \otimes \bigwedge^{2} T^{*} M$, defined as follows.

Proposition 3.6. Let $M$ be a manifold, $E$ a vector bundle over $M$, and $\nabla^{E}$ a connection on $E$. Suppose that $v, w \in \Gamma(T M)$ are vector fields and $e \in \Gamma(E)$, and $\varphi, \psi, \eta \in \mathcal{C}^{\infty}(M)$. Then

$$
\left(\nabla_{\varphi v}^{E} \nabla_{\psi w}^{E}-\nabla_{\psi w}^{E} \nabla_{\varphi v}^{E}-\nabla_{[\varphi v, \psi w]}^{E}\right)(\eta e)=\varphi \psi \eta\left(\nabla_{v}^{E} \nabla_{w}^{E}-\nabla_{w}^{E} \nabla_{v}^{E}-\nabla_{[v, w]}^{E}\right) e .
$$

Thus the expression $\nabla_{v}^{E} \nabla_{w}^{E} e-\nabla_{w}^{E} \nabla_{v}^{E} e-\nabla_{[v, w]}^{E} e$ is pointwise linear in $v, w$ and also in $e$. Also, it is clearly antisymmetric in $v$ and $w$. Therefore, there exists a unique section $R\left(\nabla^{E}\right) \in$ $\Gamma\left(\operatorname{End}(E) \otimes \bigwedge^{2} T^{*} M\right)$, called the curvature of $\nabla^{E}$, that satisfies

$$
R\left(\nabla^{E}\right) \cdot(e \otimes v \wedge w)=\nabla_{v}^{E} \nabla_{w}^{E} e-\nabla_{w}^{E} \nabla_{v}^{E} e-\nabla_{[v, w]}^{E} e
$$

for all $v, w \in \Gamma(T M)$ and $e \in \Gamma(E)$.
Now suppose that $\pi: P \rightarrow M$ is a principal bundle over $M$ with structure group $G$. Let $p \in P$, and set $\pi(p)=m$. Then we have $D \pi_{p}: T_{p} P \rightarrow T_{m} M$. Define $V_{p} \subseteq T_{p} P$ to be $\operatorname{ker}\left(D \pi_{p}\right)$. Then the union of these subspaces forms a subbundle of $T P$, which we call the vertical subbundle.

Definition 3.7. Let $M$ be a manifold and $P$ a principal bundle over $M$ with structure group $G$, a Lie group. A connection on $P$ is a vector subbundle $H \subseteq T P$, called the horizontal subbundle, that is invariant under the $G$-action on $P$ and satisfies $T_{p} P=V_{p} \oplus H_{p}$ for each $p \in P$. If $\pi(p)=m$, the $D \pi_{p}$ maps $T_{p} P=V_{p} \oplus H_{p}$ onto $T_{m} M$, and since $V_{p}$ is precisely the kernel of this map, we get an isomorphism between $H_{p}$ and $T_{m} M$.

Remark 3.8. We can be more precise about this. When we say that $H$ is invariant under the $G$-action, we mean that

$$
H_{p g}=d\left(R_{g}\right)_{p}\left(H_{p}\right)
$$

for any $p \in P$ and $g \in G$, where $d\left(R_{g}\right)_{p}$ is the differential of the right action of $G$ on $P$ at $p$. The fundamental vector fields that generate the $G$-action on $P$ give us an isomorphism between $V$ and $P \times \mathfrak{g}$, so we have that $\iota: V_{p} \cong \mathfrak{g}$. If we set $\pi_{V}$ to be the projection $T P \rightarrow V$, then we can understand the connection as a 1-form $\theta \in \Omega^{1}(P ; \mathfrak{g})$ defined by $\theta(p)=\iota \circ \pi_{V}(p)$.

As an immediate consequence of this definition, we have that $H$ is naturally isomorphic to $\pi^{*}(T M)$. So if we have $v \in \Gamma(T M)$ there exists a unique $\lambda(v) \in H \subseteq T P$ such that $D \pi_{p}\left(\lambda(v)_{\mid p}\right)=v_{\mid \pi(p)}$ for each $p \in P$. We call $\lambda(v)$ the horizontal lift of $v$, and it is a $G$-invariant vector field on $P$.

We now define the curvature of such a connection on a principal bundle. For $v, w \in \Gamma(T M)$ and $\varphi, \psi \in \mathcal{C}^{\infty}(M)$, it can be shown that

$$
[\lambda(\varphi v), \lambda(\psi w)]-\lambda([\varphi v, \psi w])=\varphi \psi([\lambda(v), \lambda(w)]-\lambda([v, w])),
$$

where $[\cdot, \cdot]$ is the Lie bracket of vector field, so the expression $[\lambda(v), \lambda(w)]-\lambda([v, w])$ is pointwise tensorial and antisymmetric on $v$ and $w$. Moreover, since $D \pi(\lambda(v))=v$ then

$$
\begin{aligned}
D \pi([\lambda(v), \lambda(w)]) & =D \pi(\lambda(v)(\lambda(w)(\cdot))-\lambda(w)(\lambda(v)(\cdot))) \\
& =D \pi(\lambda(v)(\lambda(w)(\cdot)))-D \pi(\lambda(w)(\lambda(v)(\cdot)))=[v, w]=D \pi(\lambda([v, w])) .
\end{aligned}
$$

All three sections $\lambda(v), \lambda(w), \lambda([v, w])$ are invariant under the $G$-action, so $[\lambda(v), \lambda(w)]-$ $\lambda([v, w])$ is invariant under the action of $G$ on $P \times \mathfrak{g}$, but these sections were in bijective correspondence with sections of $\operatorname{ad}(P)$. We use this fact to define the curvature $R(P, H)$ of a connection $H$ on $P$.

Proposition 3.9. Let $M$ be a manifold, $G$ a Lie group with Lie algebra $\mathfrak{g}, P$ a principal bundle over $M$ with structure group $G$, and $H$ a connection on $P$. Then there exists a unique section $R(P, H)$ of the vector bundle $\operatorname{ad}(P) \otimes \bigwedge^{2} T^{*} M$ called the curvature of $D$, that satisfies

$$
\pi^{*}(R(P, H) \cdot v \wedge w)=[\lambda(v), \lambda(w)]-\lambda([v, w])
$$

for all $v, w \in \Gamma(T M)$. Notice that the left-hand side takes values on $\mathfrak{g}$ while the right hand takes values on $v \subseteq T P$, so we shall use the isomorphism $V_{p} \cong \mathfrak{g}$ to identify both sides.

With these, we conclude the definitions and now we proceed to relate the notion of connection on vector and principal bundles. Let $\rho$ be a representation of $G$ on $V$ and define $E=\rho(P)$. Given a connection $H$ on $P$, we are going to construct a unique connection $\nabla^{E}$ on $E$. Let $e \in \Gamma(E)$. Then $\pi^{*}(e) \in \Gamma(P \times V)$ so we can regard it as a map $P \rightarrow V$, whose derivative is a linear map $D \pi^{*}(e)_{p}: T_{p} P \rightarrow V$ for every $p \in P$. Thus $D \pi^{*}(e) \in \Gamma\left(V \otimes T^{*} P\right)$. For each $p \in P$ we have the isomorphisms,

$$
T_{p} P=V_{p} \oplus H_{p}, \quad V_{p} \cong \mathfrak{g}, \quad H_{p} \cong \pi^{*}\left(T_{\pi(p)} M\right) .
$$

They render us a splitting $V \otimes T^{*} P \cong\left(V \otimes \mathfrak{g}^{*}\right) \oplus\left(V \otimes \pi^{*}\left(T^{*} M\right)\right)$. Let us denote by $\pi_{H}\left(D \pi^{*}(e)\right)$ the component of $D \pi^{*}(e)$ in $\Gamma\left(V \otimes \pi^{*}\left(T^{*} M\right)\right)$. Notice that both $\pi^{*}(e)$ and the splitting are invariant under the $G$-action, so necessarily $\pi_{H}\left(D \pi^{*}(e)\right)$ is also $G$-invariant. Since we have a bijective correspondence between $G$-invariant sections of $V \otimes \pi^{*}\left(T^{*} M\right)$ over $P$ and sections of $E \otimes T^{*} M$ over $M, \pi_{H}\left(D \pi^{*}(e)\right)$ must be the pullback of a unique element of $\Gamma\left(E \otimes T^{*} M\right)$, and this is the fact needed to define the corresponding connection $\nabla^{E}$ on $E$.

Definition 3.10. Let $M$ be a manifold, $P$ a principal bundle over $M$ with structure group $G$, and $H$ a connection on $P$. Let $\rho$ be a representation of $G$ on $V$, and define $E=\rho(P)$. If $e \in \Gamma(E)$, then $\pi_{H}\left(D \pi^{*}(e)\right)$ is a $G$-invariant section of $V \otimes \pi^{*}\left(T^{*} M\right)$ over $P$. Define $\nabla^{E} e \in$ $\Gamma\left(E \otimes T^{*} M\right)$ to be the unique section of $E \otimes T^{*} M$ with pullback $\pi_{H}\left(D \pi^{*}(e)\right)$ under the natural projection $V \otimes \pi^{*}\left(T^{*} M\right) \rightarrow E$. This defines a connection $\nabla^{E}$ on the vector bundle $E$ over $M$.

We can state a final result, which is a straighforward consequence of the the previous definitions, and ensures the equivalence of the curvatures defined on vector and principal bundles.

Proposition 3.11. Let $M$ be a manifold, $G$ a Lie group with Lie algebra $\mathfrak{g}, P$ a principal bundle over $M$ with structure group $G$, and $H$ a connection on $P$ with curvature $R(P, H)$. Let $\rho$ be a representation of $G$ on a vector space $V, E$ the vector bundle $\rho(P)$ over $M$, and $\nabla^{E}$ the connection given in the previous definition, with curvature $R\left(\nabla^{E}\right)$. We have that $\mathfrak{g}$ and $\operatorname{End}(V)$ are representations of $G$ and $\rho$ gives a $G$-equivariant linear map $d \rho: \mathfrak{g} \rightarrow \operatorname{End}(V)$, which induces a map $d \rho: \operatorname{ad}(P) \rightarrow \operatorname{End}(E)$ of the vector bundles $\operatorname{ad}(P)$ and $\operatorname{End}(E)$ over $M$. Let

$$
d \rho \otimes \operatorname{id}: \operatorname{ad}(P) \otimes \bigwedge^{2} T^{*} M \rightarrow \operatorname{End}(V) \otimes \bigwedge^{2} T^{*} M
$$

Then $(d \rho \otimes \mathrm{id})(R(P, h))=R\left(\nabla^{E}\right)$.

### 3.2 Invariant polynomials and Chern classes

The curvature form $R$ on the base space for a fibre bundle contains information about how the bundle is twisted. Using this fact, we could try to compare the bundles using these curvature forms. This approach leads to some problems, since depending on the local chart we use to describe it, it could resemble or not other curvature forms. Moreover, a fibre bundle does not admit a unique connection, so there is not a unique curvature. To avoid these problems, we are going to construct an invariant polynomial in terms of $R$ which does not depend on the local chart and some independent information from the connection can be extracted.

Having this goal in mind, we start by discussing invariant polynomials of matrices.
Definition 3.12. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. A $G$-invariant $k$-linear symmetric function $P$ is a map $P: \bigoplus^{k} \mathfrak{g} \rightarrow \mathbb{F}$ such that

- For $c_{1}, c_{2} \in \mathbb{C}$ and $A_{1}, \ldots, A_{k} \in \mathfrak{g}$,

$$
P\left(A_{1}, \ldots, c_{1} A_{i_{1}}+c_{2} A_{i_{2}}, \ldots, A_{k}\right)=c_{1} P\left(A_{1}, \ldots, A_{i_{1}}, \ldots, A_{k}\right)+c_{2} P\left(A_{1}, \ldots, A_{i_{2}}, \ldots, A_{k}\right)
$$

- For $1 \leq i, j \leq k, P\left(A_{1}, \ldots, A_{i}, \ldots, A_{j}, \ldots, A_{k}\right)=P\left(A_{1}, \ldots, A_{j}, \ldots, A_{i}, \ldots, A_{k}\right)$.
- For $g \in G, P\left(\operatorname{Ad}_{g}\left(A_{1}\right), \ldots, \operatorname{Ad}_{g}\left(A_{k}\right)\right)=P\left(A_{1}, \ldots, A_{k}\right)$.

We denote the set of all such maps $I^{k}(G)$.
We can construct a graded algebra by setting $I^{\bullet}(G)=\bigoplus_{k} I^{k}(G)$ and defining the product as

$$
(P \cdot Q)\left(A_{1}, \ldots, A_{p+q}\right)=\frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} P\left(A_{\sigma(1)}, \ldots, A_{\sigma(p)}\right) Q\left(A_{\sigma(p+1)}, \ldots, A_{\sigma(p+q)}\right),
$$

where $P \in I^{p}(G), Q \in I^{q}(G)$ and $A_{i} \in \mathfrak{g}$. This structure allows us to define invariant polynomials.

Definition 3.13. A homogeneous invariant polynomial $P$ of degree $k$ is a map $P: \mathfrak{g} \rightarrow \mathbb{F}$ for which exists $\tilde{P} \in I^{k}(G)$ such that $P(A)=\tilde{P}(A, \ldots, A)$. An invariant polynomial is a finite sum of homogeneous invariant polynomials of different degrees.

Example 3.14. Let $G$ be a Lie group with a $k$-dimensional representation, using the differential of the representation, we define

$$
P(A)=\operatorname{det}\left(I+t \frac{i A}{2 \pi}\right)
$$

for $A \in \mathfrak{g}$. It is easy to check that $P$ is invariant under the adjoint representation, and can be expanded as

$$
P(A)=1+t P_{1}(A)+\cdots+t^{k} P_{k}(A)
$$

where $P_{i}$ is $G$-invariant homogeneous invariant polynomial. As an example, it is easy to check that $P_{1}(A)=\frac{i}{2 \pi} \operatorname{Tr}(A)$ and $P_{k}(A)=\operatorname{det}\left(\frac{i A}{2 \pi}\right)$.

By definition, homogeneous invariant polynomials are defined over a Lie algebra $\mathfrak{g}$, but we can extend the domain to $\mathfrak{g}$-valued differential forms betting

$$
P\left(A_{1} \xi_{1}, \ldots, A_{k} \xi_{k}\right)=\xi_{1} \wedge \ldots \wedge \xi_{k} P\left(A_{1}, \ldots, A_{k}\right), \quad P \in I^{k}(G), A_{i} \in \mathfrak{g}, \xi_{i} \in \Omega^{r_{i}}(M)
$$

and extending it linearly. For a homogeneous invariant polynomial of degree $k$, we have

$$
P(A \xi)=\xi^{k} P(A)
$$

and extending by linearity we get the right definition for invariant polynomials. Thanks to this extension, we have the following results.

Proposition 3.15. Let $X, A_{i} \in \mathfrak{g}$ and $P \in I^{k}(G)$.

$$
\sum_{i=1}^{k} P\left(A_{1}, \ldots,\left[X, A_{i}\right], \ldots, A_{k}\right)=0
$$

In addition, for $A, \Omega_{i} \mathfrak{g}$-valued differential forms of degree $p$ and $p_{i}$ respectively,

$$
\sum_{i=1}^{k}(-1)^{p\left(p_{1}+\cdots+p_{i-1}\right)} P\left(\Omega_{1}, \ldots,\left[A, \Omega_{i}\right], \ldots, \Omega_{k}\right)=0
$$

Proposition 3.16. Let $P \in I^{k}(G)$ and $\Omega_{i}$ be $\mathfrak{g}$-valued differential forms.

$$
d P\left(\Omega_{1}, \ldots, \Omega_{k}\right)=\sum_{i=1}^{k}(-1)^{p_{1}+\cdots+p_{i-1}} P\left(\Omega_{1}, \ldots, d \Omega_{i}, \ldots, \Omega_{k}\right) .
$$

Let $R$ be a local curvature form. On the intersection of two charts $U_{i} \cap U_{j}$, we have the relation $R_{j}=t_{i j}^{-1} R_{i} t_{i j}$, where $t_{i j}$ is the group element of $G$ that gives the transition chart, so it may seem like $P(R)$ is not globally defined. However, we can rewrite the previous relation as $R_{j}=\operatorname{Ad}_{t_{i j}^{-1}} R_{i}$, so we know that $P(R)$ is globally defined. Moreover, it has some nice properties.

Theorem 3.17. Let $R$ be a local curvature form and $P$ an invariant polynomial. Then

- $d P(R)=0$,
- If $R$ and $R^{\prime}$ are the curvatures of two connections over the same fibre bundle, then $P(R)-P\left(R^{\prime}\right)$ is exact.

As a consequence of this theorem, we have can describe an invariant of the manifold in terms of $P(R)$.

Corollary 3.18. Let the basis space $B$ be a $2 m$-dimensional orientable compact manifold without boundary, $E$ a fiber bundle over $B$ and $P_{m}$ a degree $m$ invariant polynomial. Then

$$
\int_{B} P_{m}(R)
$$

is independent of the connection choice of the fibre bundle $E$.

According to the theorem, given a fibre bundle $E$ over $B$ and an invariant polynomial $P$, we can define without any problem a de Rham class $\chi_{E}(P)=[P(R)] \in H^{\bullet}(B)$, as if we consider another curvature form $R^{\prime}, \chi_{E}(P)=\left[P\left(R^{\prime}\right)\right]=[P(R)+d Q]=[P(R)]=\chi_{E}(P)$. We call $\chi_{E}(P)$ a characteristic class. The Chern-Weil homomorphism is precisely this assignment of a cohomology class.

Theorem 3.19 (Chern-Weil homomorphism). Let $E$ be a fibre bundle over $B$.

- $\chi_{E}: I^{\bullet}(G) \rightarrow H^{\bullet}(B)$ is a ring homomorphism.
- Let $f: B \rightarrow B^{\prime}$ be a differential map and $f^{*} E$ be the pullback fibre bundle of $E$. Then $\chi_{f^{*} E}(P)=f^{*} \chi_{E}(P)$.

A direct consequence of the theorem is that if $E$ is a trivial fibre bundle, $\chi_{E}$ sends every invariant polynomial to the zero class. Indeed, for a trivial fibre bundle, we can always find a connection whose curvature is zero, so $\chi_{E}(P)=[P(0)]=0$.

In spirit of Example 3.14, let $G=\mathrm{GL}(k, \mathbb{C})$. For $A \in \mathrm{GL}(k, \mathbb{C})$, we define the invariant polynomials $P_{j}$ to be such that

$$
\operatorname{det}\left(I+\frac{i A}{2 \pi}\right)=\sum_{j=1}^{k} t^{j} P_{j}(A) .
$$

Definition 3.20. Let $\pi: E \rightarrow B$ be a complex vector bundle whose fibre is $\mathbb{C}^{k}$. We define the $j$-th Chern class to be

$$
c_{j}(E)=\left[P_{j}(R)\right] \in H^{2 j}(B),
$$

and the total Chern class as

$$
c(E)=1+c_{1}(E)+\ldots+c_{k}(E) \in H^{\bullet}(B) .
$$

We can describe in a simple way the $P_{j}(R)$ if $P_{j}$ can be written as a polynomial of matrix elements. That is, if given $P_{j} \in I^{j}(\mathrm{GL}(k, \mathbb{C}))$, using the Einstein summation notation, we have

$$
P_{j}(A)=c_{\alpha_{1} \beta_{1} \ldots \alpha_{j} \beta_{j}} A_{\beta_{j}}^{\alpha_{1}} \ldots A_{\beta_{j}}^{\alpha_{j}}, \quad \forall A \in \mathrm{GL}(k, \mathbb{C}),
$$

then for an arbitrary $\mathfrak{g l}(k, \mathbb{C})$-valued 2-form $R$,

$$
P_{j}(R)=c_{\alpha_{1} \beta_{1} \ldots \alpha_{j} \beta_{j}} R_{\beta_{1}}^{\alpha_{1}} \wedge \ldots \wedge R_{\beta_{j}}^{\alpha_{j}} .
$$

Thanks to this fact, we can ensure that

$$
c(E)=\left[\operatorname{det}\left(I+\frac{i R}{2 \pi}\right)\right],
$$

and again recalling Example 3.14, we have $c_{1}=\left[P_{1}(R)\right]=\left[\frac{i}{2 \pi} \operatorname{Tr}(R)\right]$ and $c_{k}=\left[P_{k}(R)\right]=$ $\left[\operatorname{det}\left(\frac{i R}{2 \pi}\right)\right]$. And, of course, this agrees with the fact that

$$
\begin{aligned}
\operatorname{det}\left(I+\frac{i R}{2 \pi}\right) & =\exp \left(\log \left(\operatorname{det}\left(I+\frac{i R}{2 \pi}\right)\right)\right)=\exp \left(\operatorname{Tr}\left(\log \left(I+\frac{i R}{2 \pi}\right)\right)\right) \\
& =\exp \left(-\sum_{j=1}^{\infty}\left(\frac{-i}{2 \pi}\right)^{n} \operatorname{Tr}\left(R^{j}\right)\right)=1+\frac{i}{2 \pi} \operatorname{Tr}(R)+\frac{1}{8 \pi^{2}}\left(\operatorname{Tr}\left(R^{2}\right)-\operatorname{Tr}(R)^{2}\right)+\ldots
\end{aligned}
$$

Moreover, we know that when taking the cohomology class of this expression, the sum is finite, because $c_{j}(E)=0$ for $2 j>\operatorname{dim} B$.

Example 3.21. Let $\pi: E \rightarrow B$ be a complex line bundle, that is, the fibre is complex onedimensional. Then the total Chern class is

$$
c(E)=1+[R],
$$

where the curvature $R$ is a real 2 -form.

As simple as it may look, this example, together with the following definition, is a core element to the statement, and construction, of T-dual bundles.

Definition 3.22. Let $\pi_{E}: E \rightarrow B$ and $\pi_{F}: F \rightarrow B$ be two complex vector bundles. Consider the diagram

where $i: X \rightarrow X \times X$ is the diagonal map $i(x)=(x, x) . E \times F$ is a complex vector bundle over $X \times X$. We define the Whitney sum $E \oplus F$ as the pullback $i^{*}(E \times F)$.

The following result gives the naturality of the Chern classes and the compatibility with the Whitney sums.

Theorem 3.23. Let $\pi_{E}: E \rightarrow B$ be a complex vector bundle.

- Let $f: B \rightarrow B^{\prime}$ be a smooth map. Then

$$
c\left(f^{*} E\right)=f^{*} c(E)
$$

- Let $\pi_{F}: F \rightarrow B$ be another complex vector bundle. Consider the Whitney sum $E \oplus F$. Then

$$
c(E \oplus F)=c(E) \wedge c(F) .
$$

## 4 T-duality

### 4.1 Exact Courant algebroids

We have previously stressed that the main objects to study on generalized geometry are Courant algebroids. Nevertheless, we had only pointed some relevant facts and results about them. From here, and following [BCG07, $\mathrm{D}^{+14}, \mathrm{Dru}$, we will be interested in exact Courant algebroids, and as we shall see, there is actually not that much freedom to choose such an alegbroid. Because of the anchor map $\pi: E \rightarrow T M$, in general, we have the sequence

$$
0 \longrightarrow T^{*} M \longrightarrow E \xrightarrow{\pi} T M \longrightarrow 0 .
$$

Definition 4.1. A Courant algebroid $E$ is exact if for every $x \in M$, the pairing $\langle,\rangle_{x}$ is nondegenerate and $E_{x}$ fits into the sequence

$$
0 \longrightarrow T_{x}^{*} M \longrightarrow E_{x} \longrightarrow T_{x} M \longrightarrow 0 .
$$

Then, we can see that for an exact Courant algebroid, pointwise, what we have is a vector space $E_{x}$ which is isomorphic to $T_{x} M \oplus T_{x}^{*} M$. We can define an isotropic splitting for a Courant $\operatorname{algebroid} E$ as a bundle map

$$
\nabla: T M \rightarrow E
$$

such that for every $x \in M, \nabla_{x}: T_{x} M \rightarrow E_{x}$ is an isotropic splitting. For the case of exact Courant algebroids, the image $\nabla(T M)$ is a maximally isotropic subbundle, and it will be a Dirac structure, i.e. involutive with respect to the bracket if and only if

$$
H(X, Y, Z)=\langle[\nabla X, \nabla Y], \nabla Z\rangle=0, \quad \forall X, Y, Z \in \Gamma(T M)
$$

This $H$ is in fact a closed 3-form on $M$, and its cohomology class $[H] \in H_{d R}^{3}(M)$ does not depend on the splitting. If we were to change the splitting $\nabla$ for $\nabla+B$ with $B \in \Omega^{2}(M)$, then $H$ would change into $H+d B$, and so it would define the same cohomology class. Moreover, any two splittings differ by a 2 -form. This 3 -form $H$ is called the curvature of the splitting $\nabla$, and its cohomology class $[H]$ is called the Ševera class of $E$.

Given an isotropic splitting $\nabla$, we can construct a bundle isomorphism $\Phi_{\nabla}: E \rightarrow T M \oplus T^{*} M$ given by

$$
\Phi_{\nabla}(e)=\pi(e)+s_{\nabla}(e),
$$

where $s_{\nabla}(e) \in T^{*} M$ is such that $\pi^{*} s_{\nabla}(e)=e-\nabla \pi(e)$. Let us check how the bracket of $E$ reads on $T M \oplus T^{*} M$. Given $X+\xi, Y+\eta \in \Gamma\left(T M \oplus T^{*} M\right)$, we have

$$
\Phi_{\nabla}([\nabla X+\xi, \nabla Y+\eta])=[X, Y]+s_{\nabla}([\nabla X, \nabla Y])+s_{\nabla}([\nabla X, \eta])+s_{\nabla}([\xi, \nabla Y]) .
$$

For any $Z \in \Gamma(T M)$, we have that

$$
i_{Z} s_{\nabla}([\nabla X, \nabla Y])=\langle[\nabla X, \nabla Y], \nabla Z\rangle=H(X, Y, Z),
$$

and it is a matter of computation to check that $s_{\nabla}([\nabla X, \eta])=\mathcal{L}_{X} \eta$ and $s_{\nabla}([\xi, \nabla Y])=-i_{Y} d \xi$. Hence,

$$
\Phi_{\nabla}([\nabla X+\xi, \nabla Y+\eta])=[X, Y]+\mathcal{L}_{x} \eta-i_{Y} d \xi+i_{Y} i_{X} H
$$

This looks very familiar to us, as it is a slighlt modification of the Dorfman bracket.
Definition 4.2. Let $H \in \Omega^{3}(M)$ be a closed form. We call the bracket on sections of $T M \oplus T^{*} M$

$$
[X+\xi, Y+\eta]_{H}=[X, Y]+\mathcal{L}_{X} \eta-i_{Y} d \xi+i_{Y} i_{X} H
$$

the $H$-twisted Courant bracket.
It is interesting to study the group $\operatorname{Aut}(E)$ of bundle automorphisms preserving all the structures we have.

Definition 4.3. The automorphism group $\operatorname{Aut}(E)$ of a Courant algebroid is the group of pairs $(\Psi, \psi)$, where $\Psi: E \rightarrow E$ is a bundle automorphism covering $\psi \in \operatorname{Diff}(M)$ such that

- $\psi^{*}\langle\Psi(\cdot), \Psi(\cdot)\rangle=\langle\cdot, \cdot\rangle$,
- $[\Psi(\cdot), \Psi(\cdot)]=\Psi[\cdot, \cdot]$,
- $\pi \circ \Psi=\psi_{*} \circ \pi$.

By restricting to the case $E=\left(T M \oplus T^{*} M \cdot[\cdot, \cdot \cdot]_{H}\right)$, we can find two illustrative examples of such automorphisms. First, let us consider $\psi \in \operatorname{Diff}(M)$, and define $\Psi_{\psi}=\psi_{*}+\left(\psi^{-1}\right)^{*}$, which trivially preserves the pairing. For any two sections $X+\xi, Y+\eta$ of the generalized tangent bundle, we have the following identities

- $\left[\psi_{*} X, \psi_{*} Y\right]=\psi_{*}[X, Y]$,
- $\mathcal{L}_{\psi_{*} X}\left(\psi^{*}\right)^{-1} \eta=\left(\psi^{*}\right)^{-1} \mathcal{L}_{X} \eta$,
- $i_{\psi_{*} Y} d\left(\psi^{*}\right)^{-1} \xi=\left(\psi^{*}\right)^{-1} i_{Y} \xi$,
- $i_{\psi_{*} Y} i_{\psi_{*} X} H=\left(\psi^{*}\right)^{-1} i_{Y} i_{X} \psi^{*} H$.

Then, $\left[\Psi_{\psi}(X+\xi), \Psi_{\psi}(Y+\eta)\right]_{H}=\Psi_{\psi}\left([X+\xi, Y+\eta]_{\psi^{*} H}\right)$, so for every $\psi \in \operatorname{Diff}(M)$ such that $\psi^{*} H=H$ we have that $\Psi_{\psi} \in \operatorname{Aut}(E)$.

Now, let $B \in \Omega^{2}(M)$ and consider the B-field action $e^{B}$ on sections of $E$. We already know that B-field are symmetries of the pairing, so we shall only be interested in the bracket.

$$
\begin{aligned}
{\left[e^{B}(X+\xi), e^{B}(Y+\eta)\right]_{H} } & =[X, Y]+\mathcal{L}_{X}\left(\eta+i_{Y} B\right)-i_{Y} d\left(\xi+i_{X} B\right)+i_{Y} i_{X} H \\
& =[X, Y]+\mathcal{L}_{X} \eta-i_{Y} d \xi+i_{[X, Y]} B+i_{Y} i X(H+d B) \\
& =e^{B}[X+\xi, Y+\eta]_{H+d B}
\end{aligned}
$$

Therefore, if $B$ is a closed 2-form, the $B$-field provides an automorphism of $E$. Moreover, B-fields allow us to describe the autormorphisms of the Courant algebroid in a simple way.

Proposition 4.4. Let $E$ be a Courant algebroid and $\nabla: T M \rightarrow E$ an isotropic splitting. For $(\Psi, \psi) \in \operatorname{Aut}(E)$, there exists $B \in \Omega^{2}(M)$ such that

$$
\Phi_{\nabla} \circ \Psi \circ \Phi_{\nabla}^{-1}=\Psi_{\psi} \circ e^{B}
$$

Let $H$ be the curvature of the splitting $\nabla$. Then $H-\psi^{*} H=d B$ holds so $\psi$ is a diffeomorphism that preserves the Sěvera class $[H]$.

From this result we extract a description of $\operatorname{Aut}(E)$ given a splitting $\nabla$ :

$$
\operatorname{Aut}(E)=\left\{(\psi, B) \in \operatorname{Diff}(M) \times \Omega^{2}(M): H-\psi^{*} H=d B\right\}
$$

As the description is dependent on the choice we have to understand how it changes when we take another splitting $\nabla+B^{\prime}$ with $B^{\prime} \in \Omega^{2}(M)$. If this is the case, then

$$
\Phi_{\nabla+B^{\prime}} \circ \Psi \circ \Phi_{\nabla+B^{\prime}}^{-1}=e^{-B^{\prime}} \circ\left(\Psi_{\psi} \circ e^{B}\right) \circ e^{B^{\prime}}=\Psi_{\psi} \circ e^{B^{\prime}-\psi^{*} B^{\prime}+B}
$$

so the change $\nabla \mapsto \nabla+B^{\prime}$ induces the change $(\psi, B) \mapsto\left(\psi, B^{\prime}-\psi^{*} B^{\prime}+B\right)$. This last result grants us a full understanding of the group $\operatorname{Aut}(E)$. Now we proceed to study its Lie algebra $\operatorname{Der}(E)$. An element of $\operatorname{Der}(E)$ is an infinitessimal symmetry of $E$ and is described by $(A, X)$, where $A: \Gamma(E) \rightarrow \Gamma(E)$ and $X \in \Gamma(T M)$ such that for any sections $e_{1}, e_{2}$ of $E$ and a smooth function $f$ on $M$, the following properties are satisfied:

- $A\left(f e_{1}\right)=f A\left(e_{1}\right)+\left(\mathcal{L}_{X} f\right) e_{1}$,
- $\left\langle A\left(e_{1}\right), e_{2}\right\rangle+\left\langle e_{1}, A\left(e_{2}\right)\right\rangle=\mathcal{L}_{X}\left\langle e_{1}, e_{2}\right\rangle$,
- $A\left(\left[e_{1}, e_{2}\right]\right)=\left[A\left(e_{1}\right), e_{2}\right]+\left[e_{1}, A\left(e_{2}\right)\right]$,
- $\pi\left(A\left(e_{1}\right)\right)=\left[X, \pi\left(e_{1}\right)\right]$.

We would like to remark a special type of objects, called inner symmetries, which are completely determined by some $e \in \Gamma(E)$, as $(A, X)=([e, \cdot], \pi(e))$.

Returning to the case we are mainly interested, let $E=T M \oplus T^{*} M$ and $\nabla: T M \oplus T^{*} M \rightarrow$ $T M$ a splitting with curvature $H$, and consider the $H$-twisted bracket. Given a one-parameter subgroup $\left(\psi_{t}, B_{t}\right) \in \operatorname{Aut}\left(T M \oplus T^{*} M\right)$, we get an element of $\operatorname{Der}\left(T M \oplus T^{*} M\right)$ by differentiating it at $t=0$, rendering $(X, B) \in \operatorname{Der}\left(T M \oplus T^{*} M\right)$. By the proposition, since $H-\psi_{t}^{*} H=d B_{t}$, then by differentiation we get $\mathcal{L}_{X} H=-d B$. By changing the splitting $\nabla \mapsto \nabla+B^{\prime}$ with $B^{\prime} \in \Omega^{2}(M)$, we know that $\left(\psi_{t}, B_{t}\right) \mapsto\left(\psi_{t}, B^{\prime}-\psi_{t}^{*} B^{\prime}+B_{t}\right)$. By differentiating again, we get that $(X, B) \mapsto\left(X, B-\mathcal{L}_{X} B^{\prime}\right)$.

Let us now find which $A: \Gamma\left(T M \oplus T^{*} M\right) \rightarrow \Gamma\left(T M \oplus T^{*} M\right)$ corresponds to (X,B) $\in$ $\operatorname{Der}\left(T M \oplus T^{*} M\right)$. Let $Y+\eta \in \Gamma\left(T M \oplus T^{*} M\right)$, then

$$
\begin{aligned}
\left.\frac{d}{d t}\right|_{t=0} \Psi_{\psi_{-t}} \circ e^{B_{-t}}(Y+\eta) & =\left.\frac{d}{d t}\right|_{t=0}\left[\left(\psi_{-t}\right)_{*}+\left(\psi_{t}\right)^{*}\right] \circ e^{B_{-t}}(Y+\eta)= \\
& =\left.\frac{d}{d t}\right|_{t=0}\left[\left(\psi_{-t}\right)_{*} Y+\left(\psi_{t}\right)^{*}\left(\eta-i_{Y} B_{-t}\right)\right]= \\
& =[X, Y]+\mathcal{L}_{X} \eta-i_{Y} B .
\end{aligned}
$$

Hence, we define $A(Y+\eta)=[X, Y]+\mathcal{L}_{X} \eta-i_{Y} B$. Let us focus now on the inner symmetries. Recall that $[X+\xi, Y+\eta]_{H}=[X, Y]+\mathcal{L}_{X} \eta-i_{Y} d \xi+i_{Y} i_{X} H=[X, Y]+\mathcal{L} X \eta-i Y\left(d \xi-i_{X} H\right)$, so we can identify $(A, X)$ with $\left([X+\xi, \cdot]_{H}, X\right)$, rendering us an inner symmetry. Let us define the map

$$
\begin{aligned}
& \mathrm{ad}: \quad \Gamma\left(T M \oplus T^{*} M\right) \longrightarrow \\
& X+\xi \longmapsto \\
& X+\xi \operatorname{Der}\left(T M \oplus T^{*} M,[\cdot, \cdot]_{H}\right) \\
&\left(X, d \xi-i_{X} H\right) .
\end{aligned}
$$

Given $(X, B) \in \operatorname{Der}\left(T M \oplus T^{*} M\right)$, since $d\left(i_{X} H+B\right)=d i_{X} H+i_{X} d H+d B=\mathcal{L}_{X} H+d B=$ $-d B+d B=0$, we have a cohomology class $\left[i_{X} H+B\right] \in H^{2}(M)$. This class will be zero precisely when there exists $\xi \in \Omega^{1}(M)$ such that $d \xi=i_{X} H+B$, which we can rewrite as $B=d \xi-i_{X} H$. Hence, we can characterize the inner symmeties as the kernel of the map

$$
\begin{array}{rlc}
\chi: \quad \operatorname{Der}\left(T M \oplus T^{*} M\right) & \longrightarrow & H^{2}(M) \\
(X, B) & \longmapsto\left[i_{X} H+B\right] .
\end{array}
$$

In particular, this shows that the map ad is not surjective; nor injective, since if $\operatorname{ad}(X+\xi)=0$ then $[X+\xi, Y+\eta]_{H}=0$ for any $Y+\eta \in \Gamma\left(T M \oplus T^{*} M\right)$. By taking $Y=0$ and $\eta=d f$ with $f \in \mathcal{C}^{\infty}(M)$, we then have $d f(X)=0$. Since this holds for any $f \in \mathcal{C}^{\infty}(M), X=0$. Similarly, choosing $\eta=0, i_{Y} d \xi=0$ implies that $d \xi=0$, as $Y$ is arbitrary.

All this discussion amounts to the statement of this exact sequence,

$$
0 \longrightarrow \Omega_{c l}^{1}(M) \xrightarrow{\pi^{*}} \Gamma\left(T M \oplus T^{*} M\right) \xrightarrow{\text { ad }} \operatorname{Der}\left(T M \oplus T^{*} M\right) \xrightarrow{\chi} H^{2}(M) \longrightarrow 0
$$

### 4.2 Dual principal $S^{1}$-bundles and T-duality

We now introduce the mathematical setting of T-duality, which, according to BHM04, BEM04, Bug19, is a topological relation between principal circle bundles that renders an equivalence of two string theories. Although at first only circle bundles were considered, the T-duality relation can be defined for principal torus bundles. This duality between bundles can be used to transport geometrical structures between them, and can be stated as a relation of Courant algebroids over them.

Let $B$ be a manifold and $M_{1}, M_{2}$ principal $S^{1}$-bundles over $B$. Consider $H_{1} \in \Omega^{3}\left(M_{1}\right)$ an invariant closed integral form on $M_{1}$ and $H_{2} \in \Omega^{3}\left(M_{2}\right)$ an invariant closed integral form on $M_{2}$.

We have the following commutative diagram

where $\pi_{1}: M_{1} \rightarrow B$ and $\pi_{2}: M_{2} \rightarrow B$ are the projections to the base, and $p_{1}: M_{1} \times_{B} M_{2} \rightarrow M_{1}$ and $p_{2}: M_{1} \times_{B} M_{2} \rightarrow M_{2}$ are the projections on the corresponding component from

$$
M_{1} \times_{B} M_{2}=\left\{\left(m_{1}, m_{2}\right) \in M_{1} \times M_{2}: \pi_{1}\left(m_{1}\right)=\pi_{2}\left(m_{2}\right)\right\},
$$

which we call the correspondence space.
Definition 4.5. $\left(M_{1}, H_{1}\right)$ and $\left(M_{2}, H_{2}\right)$ are T-dual if $\left(\pi_{1}\right)_{*} H_{2}=c_{1},\left(\pi_{2}\right)_{*} H_{1}=c_{2}$, where $c_{1}$ is the first Chern class of $M_{1}$ and $c_{2}$ is the first Chern class of $M_{2}$; and $p_{1}^{*} H_{1}-p_{2}^{*} H_{2}=d F$ for some $F \in \Omega^{2}\left(M_{1} \times_{B} M_{2}\right)$.

This is the classical definition for T-dual $S^{1}$-bundles, but it would be meaningless if we could not construct examples of it which are interesting. For this, consider $\pi_{1}: M_{1} \rightarrow B$ a principal $S^{1}$-bundle, $H_{1} \in \Omega^{3}\left(M_{1}\right)$ an invariant closed integral form on $M_{1}$ and $\theta_{1} \in \Omega^{1}\left(M_{1}\right)$ a connection form. In general the connection form would be in $\Omega^{1}\left(M_{1} ; \mathfrak{s}^{1}\right)$, but since $\mathfrak{s}^{1} \cong \mathbb{R}$, we can choose an appropriate isomorphism such that $\left(\pi_{1}\right)_{*} \theta_{1}=1$. Now we construct $M_{2}$ a T-dual $S^{1}$-bundle. Define $c_{2}=\left(\pi_{1}\right)_{*} H_{1} \in \Omega^{2}(B)$ and consider $c_{1}$ the first Chern class of $M_{1}$, i.e. $\pi_{1}^{*} c_{1}=d \theta_{1}$. As for invariant forms we have an isomorphism

$$
\Omega_{S^{1}}^{k}\left(M_{1}\right) \cong \bigoplus_{p+q=k} \Omega^{p}(B) \otimes \bigwedge^{q}\left(\mathfrak{s}^{1}\right)^{*}
$$

we can write $H_{1}=\pi_{1}^{*} c_{2} \wedge \theta_{1}+\pi_{1}^{*} h$ for some $h \in \Omega^{3}(B)$. Notice that $c_{2}$ is a closed form, given that the push-forward $\left(\pi_{1}\right)_{*}$ commutes with the de Rham differential, so $d c_{2}=d\left(\pi_{1}\right)_{*} H_{1}=$ $\left(\pi_{1}\right)_{*} d H_{1}=0$ since $H_{1}$ is closed; and that it is an integral form because

$$
\int_{S} c_{2}=\int_{\pi_{1}^{-1}(S)} H_{1}
$$

for any surface $S \subseteq B$ and $H_{1}$ is integral. Hence, $c_{2}$ can be viewed as a first Chern class of some $S^{1}$-bundle $M_{2}$ of $B$, with connection form $\theta_{2} \in \Omega^{1}\left(M_{2}\right)$ such that $\pi_{2}^{*} c_{2}=d \theta_{2}$. To conclude we need to find some $H_{2} \in \Omega^{3}\left(M_{2}\right)$ which satisfies the required conditions. By choosing
$H_{2}=\pi_{2}^{*} c_{1} \wedge \theta_{2}+\pi_{2}^{*} h \in \Omega^{3}\left(M_{2}\right)$, we can easily compute

$$
\begin{aligned}
p_{1}^{*} H_{1}-p_{2}^{*} H_{2} & =p_{1}^{*}\left(\pi_{1}^{*} c_{2} \wedge \theta_{1}+\pi_{1}^{*} h\right)-p_{2}^{*}\left(\pi_{2}^{*} c_{1} \wedge \theta_{2}-\pi_{2}^{*} h\right) \\
& =p_{1}^{*} \pi_{1}^{*} c_{2} \wedge p_{1}^{*} \theta_{1}-p_{2}^{*} \pi_{2}^{*} c_{1} \wedge p_{2}^{*} \theta_{2} \\
& =p_{2}^{*} \tau_{2}^{*} c_{2} \wedge p_{1}^{*} \theta_{1}-p_{1}^{*} \pi_{2}^{*} c_{1} \wedge p_{2}^{*} \theta_{2} \\
& =p_{2}^{*} d \theta_{2} \wedge p_{1}^{*} \theta_{1}-p_{1}^{*} d \theta_{1} \wedge p_{2}^{*} \theta_{2} \\
& =d\left(p_{2}^{*} \theta_{2}\right) \wedge p_{1}^{*} \theta_{1}-d\left(p_{1}^{*} \theta_{1}\right) \wedge p_{2}^{*} \theta_{2} \\
& =d\left(-p_{1}^{*} \theta_{1} \wedge p_{2}^{*} \theta_{2}\right) .
\end{aligned}
$$

We will generalize these definitions and constructions to the case of torus bundles, as we can split the torus in circles and apply to each circle the previous steps.

Definition 4.6 ([CG04, CG11]). Let $B$ be a manifold and $\pi_{1}: M_{1} \rightarrow B, \pi_{2}: M_{2} \rightarrow B$ two principal $T^{k}$-bundles. Let $H_{1}$ and $H_{2}$ be invariant closed 3 -forms on $M_{1}$ and $M_{2}$ respectively, and consider the following commutative diagram

$M_{1}$ and $M_{2}$ are said to be T-dual if there exists $F$ an $T^{2 k}$-invariant form on $M_{1} \times{ }_{B} M_{2}$ such that $F: \mathfrak{t}_{1}^{k} \otimes \mathfrak{t}_{2}^{k} \rightarrow \mathbb{R}$ is nondegenerate and $p_{1}^{*} H_{1}-p_{2}^{*} H_{2}=d F$.

Notice that in our definition we have dropped the requirement of $H_{1}$ and $H_{2}$ being integral, so we cannot associate a well-defined first Chern class. However, if it was the case, then we would require $H_{1_{i}}$ and $H_{2_{i}}$ to be integral for each circle component of the tori. Another important remark is that $H_{1}(X, Y, \cdot)=0$ for $X, Y \in \mathfrak{t}_{1}^{k} \subseteq T M_{1}$. This comes from the fact that $p_{2}^{*} H_{2}(X, \cdot, \cdot)=0$ for any $X \in \operatorname{ker}\left(p_{2}\right)_{*}$, so in particular for $X \in \mathfrak{t}_{1}^{k}$. As a similar reasoning can be applied to $H_{2}$, we get that $d F(X, Y, \cdot)=0$ for any $X, Y \in \mathfrak{t}_{1}^{k} \oplus \mathfrak{t}_{2}^{k}$.

Given $M_{1}$ a principal $T^{k}$-bundle over a base manifold $B$ with an invariant closed integral 3form $H_{1}$, the procedure to construct a dual $T^{k}$-bundle is analogous to the case of circle bundles. Let $\theta_{1} \in \Omega\left(M_{1}, \mathfrak{t}_{1}^{k}\right)$ be a connection on $M_{1}$ and let $H$ satisfy $H(X, Y, \cdot)=0$ for $X, Y \in \mathfrak{t}_{1}^{k}$. Since we have the isomorphism

$$
\Omega_{T^{k}}^{k}\left(M_{1}\right)=\bigoplus_{p+q=k} \Omega^{p}(B) \otimes \bigwedge^{q}\left(\mathfrak{t}^{k}\right)^{*}
$$

we can decompose $H_{1}=\left\langle\pi_{1}^{*} c_{2}, \theta_{1}\right\rangle+\pi_{1}^{*} h$ with $c_{2} \in \Omega^{2}\left(B ;\left(\mathfrak{t}_{2}^{k}\right)^{*}\right)$ an integral form, $h \in \Omega^{3}(B)$, and $\langle\cdot, \cdot\rangle$ meaning that we contract the part of $\left(\mathfrak{t}_{1}^{k}\right)^{*}$ in $c_{2}$ with the part of $\mathfrak{t}_{1}^{k}$ in $\theta_{1}$. Now $c_{2}$
can be viewed as the first Chern class of the dual bundle $M_{2}$. Then we can take a connection $\theta_{2} \in \Omega^{1}\left(M_{2},\left(\mathfrak{t}_{2}^{k}\right)^{*}\right)$ such that $d \theta_{2}=\pi_{2}^{*} c_{2}$, and define $H_{2}=\left\langle\pi_{2}^{*} c_{1}, \theta_{2}\right\rangle+\pi_{2}^{*} h$ where $c_{1}$ is the corresponding first Chern class of $M_{1}$, so $d \theta_{1}=\pi_{1}^{*} c_{1}$. To conclude, we check that the difference of the pullbacks of $H_{1}$ and $H_{2}$ on the correspondence space is in fact an exact form:

$$
\begin{aligned}
p_{1}^{*} H_{1}-p_{2}^{*} H_{2} & =p_{1}^{*}\left(\left\langle\pi_{1}^{*} c_{2}, \theta_{1}\right\rangle+\pi_{1}^{*} h\right)-p_{2}^{*}\left(\left\langle\pi_{2}^{*} c_{1}, \theta_{2}\right\rangle+\pi_{2}^{*} h\right) \\
& =\left\langle p_{1}^{*} \pi_{1}^{*} c_{2}, p_{1}^{*} \theta_{1}\right\rangle-\left\langle p_{2}^{*} \pi_{2}^{*} c_{1}, p_{2}^{*} \theta_{2}\right\rangle \\
& =\left\langle p_{2}^{*} \pi_{2}^{*} c_{2}, p_{1}^{*} \theta_{1}\right\rangle-\left\langle p_{1}^{*} \pi_{1}^{*} c_{1}, p_{2}^{*} \theta_{2}\right\rangle \\
& =\left\langle p_{2}^{*} d \theta_{2}, p_{1}^{*} \theta\right\rangle-\left\langle p_{1}^{*} d \theta_{1}, p_{2}^{*} \theta_{2}\right\rangle \\
& =d\left\langle-p_{1}^{*} \theta_{1}, p_{2}^{*} \theta_{2}\right\rangle .
\end{aligned}
$$

Thus, $F=-\left\langle p_{1}^{*} \theta_{1}, p_{2}^{*} \theta_{2}\right\rangle$, and it is nondegenerate on $\mathfrak{t}_{1}^{k} \otimes \mathfrak{t}_{2}^{k}$ as it is the wedge product of connection forms.

In the following two examples we show that, unlike the original case of $S^{1}$-bundles, there is not unicity for the dual bundle, either because of dropping the integrality condition on $H$ or because there is in fact some hidden choice when $k>1$. We will omit to write some of the pullbacks to ease the notation when there is no ambiguity to do so.

Example 4.7. We consider two $S^{1}$-bundles over $S^{2}$, the trivial one, $M_{1}=S^{1} \times S^{2}$, and the one given by the Hopf fibration, $M_{2}$. For the first one we consider the 3-form $H_{1}=d \theta_{1} \wedge \theta_{2}$ where $\theta_{1}$ is a connection on $S^{2}$ so $d \theta_{1}$ is a curvature form and can be identified with a Chern class, and $\theta_{2}$ an invariant volume form on $S^{1}$. As $\mathfrak{s}^{1} \cong \mathbb{R}$ is one-dimensional, $H_{1}(X, Y, \cdot)=0$ for $X, Y \in \mathfrak{s}^{1}$ holds straightforwardly. For the Hopf fibration, we just consider the zero 3-form, $H_{2}=0$. Then we can compute

$$
p_{1}^{*} H_{1}-p_{2}^{*} 0=p_{1}^{*}\left(d \theta_{1} \wedge \theta_{2}\right)=d\left(p_{1}^{*} \theta \wedge p_{1}^{*} \theta_{2}\right)
$$

where we have used the fact that $\theta_{2}$ is a volume form, so $d \tilde{\theta}=0$. We obtain $F=p_{1}^{*} \theta \wedge p_{1}^{*} \theta_{2}$ and it is nondegenerate on $\mathfrak{s}^{1} \otimes \mathfrak{s}^{1}$.

Alternatively, for two Hopf fibrations, with the same $H=\theta \wedge d \theta$, where $\theta$ is a connection on $S^{3}$, we have

$$
p^{*} H-\tilde{p}^{*} H=\theta \wedge d \theta-\theta \wedge d \theta=\theta \wedge d \theta+d \theta \wedge \theta=d(\theta \wedge \theta)
$$

so $F=\theta \wedge \theta$. This shows that the Hopf fibration with such 3-form $H$ is self T-dual. Thus, we can use these two examples to show that $M_{1}$ does not determine by itself its T-dual pair but $H_{1}$ has also to be considered.

Example 4.8. Consider a $T^{2}$-bundle $M_{1}$ over $B$, with $H_{1}=0$. As $T^{2} \simeq S^{1} \times S^{1}$, we have $\mathfrak{t}^{2} \simeq \mathbb{R} \times \mathbb{R}$. Consider a connection on $M_{1}$, which we can identify with a pair $\left(\theta_{1}, \theta_{2}\right)$ of connections corresponding to the two circles, and denote by $c_{i}$ the curvature forms $d \theta_{i}$. Now,
we do consider the trivial $T^{2}$-bundle over $B$, namely, $S^{1} \times S^{1} \times B$ with $H_{2}=c_{1} \wedge \theta_{1}+c_{2} \wedge \theta_{2}$. Then for $X, Y \in \mathfrak{t}^{2}$, we have $X=\left(X_{1}, X_{2}\right)$ and $Y=\left(Y_{1}, Y_{2}\right)$ and as in the previous example $H_{2}(X, Y, \cdot)=0$. Now, a simple calculation

$$
p_{1}^{*} H-p_{2}^{*} H_{2}=-c_{1} \wedge \theta_{1}-c_{2} \wedge \theta_{2}=-d \theta_{1} \wedge \theta_{1}-d \theta_{2} \wedge \theta_{2}=d\left(\frac{1}{2}\left(\theta_{1} \wedge \theta_{1}+\theta_{2} \wedge \theta_{2}\right)\right)
$$

shows that $F$ is nondegenerate on $\mathfrak{t}^{2} \otimes \mathfrak{t}^{2}$ as $\theta_{1}, \theta_{2}$ are connections.
Another T-dual principal $T^{2}$-bundle to $M_{1}$ can be obtained by considering the connection $\tilde{\theta}$ such that $d \tilde{\theta}_{1}=c_{1}$ and $d \tilde{\theta}_{2}=-c_{2}$, and the 3 -form $H_{2}=d\left(\tilde{\theta}_{1} \wedge \tilde{\theta}_{2}\right)=c_{1} \wedge \tilde{\theta}_{2}+\tilde{\theta}_{1} \wedge c_{2}$. Hence,

$$
\begin{aligned}
p_{1}^{*} H-p_{2}^{*} \tilde{H} & =-\left(c_{1} \wedge \tilde{\theta}_{2}+c_{2} \wedge \tilde{\theta}_{1}\right)=-\left(d \theta_{1} \wedge \tilde{\theta}_{2}-d \theta_{2} \wedge \tilde{\theta}_{1}\right) \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}\right)-\left(\theta_{1} \wedge d \tilde{\theta}_{2}+\theta_{2} \wedge d \tilde{\theta}_{1}\right) \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}\right)-\left(-\theta_{1} \wedge c_{2}+\theta_{2} \wedge c_{1}\right) \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}\right)-d\left(\theta_{1} \wedge \theta_{2}\right)=-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}+\theta_{1} \wedge \theta_{2}\right) .
\end{aligned}
$$

Taking $F=-\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}+\theta_{1} \wedge \theta_{2}\right)$, we see that it is nondegenerate on $\mathfrak{t}^{2} \otimes \mathfrak{t}^{2}$.
Remark 4.9. We shall stress that in the process to find a T-duality as in Example 4.8, the product of the Chern class and the connection must be done between pairs of dual circles. By this we mean that we do not have much freedom when choosing this combination. To ilustrate this, let us naively generalized the previous example to $T^{k}$-bundles. Let $\left(\theta_{1}, \ldots, \theta_{k}\right)$ be connections of the circles in the $T^{k}$-bundle $M_{1}$ and $\left(\tilde{\theta}_{1}, \ldots, \tilde{\theta}_{k}\right)$ connections of the circles in the $T^{k}$-bundle $M_{2}$. Let us consider $\sigma \in S_{k}$, and set $H_{1}=0$ and $H_{2}=\sum_{i=1}^{k} c_{i} \wedge \tilde{\theta}_{\sigma(i)}$, where $c_{i}=m_{i} d \tilde{\theta}_{i}$. Then, we have

$$
p^{*} H-\tilde{p}^{*} \tilde{H}=-\sum_{i=1}^{k} c_{i} \wedge \tilde{\theta}_{\sigma(i)}=-d\left(\sum_{i=1}^{k} \theta_{i} \wedge \tilde{\theta}_{\sigma(i)}\right)+\sum_{i=1}^{k} m_{\sigma(i)} \theta_{i} \wedge d \theta_{\sigma(i)}
$$

For the $T^{k}$-bundles to be T-dual we require that this quantity is an exact form. But this condition means that the permutation has to decompose as a product of disjoint transpositions. We present two examples, the first one where we clearly fail to establish a T-duality because $\sigma$ does not decompose in disjoint transpositions.

Let $k=3$ and $\sigma=(123)$.

$$
\begin{aligned}
p^{*} H-\tilde{p}^{*} \tilde{H} & =-\left(c_{1} \wedge \tilde{\theta}_{2}+c_{2} \wedge \tilde{\theta}_{3}+c_{3} \wedge \tilde{\theta}_{1}\right)= \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{3}+\theta_{3} \wedge \tilde{\theta}_{1}\right)+\left(m_{2} \theta_{1} \wedge c_{2}+m_{3} \theta_{2} \wedge c_{3}+m_{1} \theta_{3} \wedge c_{1}\right)= \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{3}+\theta_{3} \wedge \tilde{\theta}_{1}\right)+\left(m_{2} \theta_{1} \wedge d \theta_{2}+m_{3} \theta_{2} \wedge d \theta_{3}+m_{1} \theta_{3} \wedge d \theta_{1}\right) .
\end{aligned}
$$

There is no possible way to arrange some terms in the second parenthesis and obtain an exact form. Now we show and example that work.

Taking $\sigma=(12)$,

$$
\begin{aligned}
p^{*} H-\tilde{p}^{*} \tilde{H} & =-\left(c_{1} \wedge \tilde{\theta}_{2}+c_{2} \wedge \tilde{\theta}_{1}+c_{3} \wedge \tilde{\theta}_{3}\right)= \\
& =-\left(d\left(\theta_{1} \wedge \tilde{\theta}_{2}\right)-\theta_{1} \wedge d \tilde{\theta}_{2}+d\left(\theta_{2} \wedge \tilde{\theta}_{1}\right)-\theta_{2} \wedge d \tilde{\theta}_{1}+d\left(\theta_{3} \wedge \tilde{\theta}_{3}\right)-\theta_{3} \wedge d \tilde{\theta}_{3}\right)= \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}+\theta_{3} \wedge \tilde{\theta}_{3}\right)+\left(m_{2} \theta_{1} \wedge c_{2}+m_{1} \theta_{2} \wedge c_{1}+m_{3} \theta_{3} \wedge c_{3}\right)= \\
& =-d\left(\theta_{1} \wedge \tilde{\theta}_{2}+\theta_{2} \wedge \tilde{\theta}_{1}+\theta_{3} \wedge \tilde{\theta}_{3}\right)+\left(m_{2}\left(\theta_{1} \wedge d \theta_{2}+d \theta_{1} \wedge \theta_{2}\right)+\frac{m_{3}}{2} d\left(\theta_{3} \wedge \theta_{3}\right)\right)= \\
& =-d\left(\theta_{1} \wedge \theta_{2}+\theta_{2} \wedge \tilde{\theta}_{1}+\theta_{3} \wedge \tilde{\theta}_{3}-m_{2} \theta_{1} \wedge \theta_{2}-\frac{m_{3}}{2} \theta_{3} \wedge \theta_{3}\right)
\end{aligned}
$$

The proof of the general statement is just a matter of comparing the terms that appear in $\sum_{i=1}^{k} m_{\sigma(i)} \theta_{i} \wedge d \theta_{\sigma(i)}$.

There is a result which gives us a nice consequence of the T-duality between two $T^{k}$-bundles, which we will use later to transport geometrical structures between them.

Theorem 4.10 ( $(\overline{\mathrm{BHM} 04}])$. Let $M_{1}$ and $M_{2}$ be $T^{k}$-principal bundles over $B$, with invariant closed 3 -forms $H_{1}$ and $H_{2}$ respectively. If they are T-dual, so $p_{1}^{*} H_{1}-p_{2}^{*} H_{2}=d F$, then the map

$$
\tau:\left(\Omega_{T^{k}}^{\bullet}\left(M_{1}\right), d_{H_{1}}\right) \longrightarrow\left(\Omega_{T^{k}}^{\bullet}\left(M_{2}\right), d_{H_{2}}\right), \quad \tau(\rho)=\left(p_{2}\right)_{*}\left(e^{F} \wedge p_{1}^{*} \rho\right)
$$

is an isomorphism of twisted differential complexes.

### 4.3 T-duality viewed as a map of Courant algebroids

We have explained the setting of T-duality and have shown some examples to illustrate the behaviour of bundles under this duality, ultimately given by an isomorphism of twisted differential complexes. So far, it has been a purely topological treatment and it is time now to use the tools from generalized geometry to search for some relationship between geometrical structures over T-dual bundles.

Consider the T-dual setting


Now, over $M_{1}$ and $M_{2}$ we consider the Courant algebroids $T M_{1} \oplus T^{*} M_{1}$ and $T M_{2} \oplus T^{*} M_{2}$ with Ševera classes $\left[H_{1}\right]$ and $\left[H_{2}\right]$ respectively. We would like to take the Clifford action of sections of $T M_{1} \oplus T^{*} M_{1}$ and find some relation with the isomorphism $\tau$, so it turns into an isomorphism
of Clifford modules. Since $M_{1}$ and $M_{2}$ are principal bundles and $H_{1}$ and $H_{2}$ are invariant forms, it is natural to consider invariant elements of the Courant algebroids.

Ideally, what we would like to do is to pullback invariant elements of $T M_{1} \oplus T^{*} M_{1}$ to the tangent bundle of the correspondence space and then push them forward to $T M_{2} \oplus T^{*} M_{2}$. However, there are some problems. First of all, if we have $X+\xi \in T M_{1} \oplus T^{*} M_{1} / T^{k}$, when applying the pullback $p_{1}^{*}$ we do not have any problems with $p_{1}^{*} \xi$ as the pullback of forms is welldefined, but that is not the case for vector fields, because if we take a lift $p_{1}^{*} X$ on $T\left(M_{1} \times{ }_{B} M_{2}\right)$ then we can add some component on $\mathfrak{t}_{2}^{k}$ and get another lift of $X$. In any case, let $\hat{X}$ be a lift of $X$. We can apply the B-field transformation given by $-F$ to $\hat{X}+p_{1}^{*} \xi$, so we obtain $\hat{X}+p_{1}^{*} \xi-F(\hat{X}, \cdot)$. If we now try to push it forward, then we have two problems: $\left(p_{2}\right)_{*}(\hat{X})$ will depend on the choice of the lift and we require the form $p_{1}^{*} \xi-F(\hat{X}, \cdot)$ to be basic, that is, it has to be a pullback of a form on $B$, so $\mathcal{L}_{Y}(\xi-F(\hat{X}, \cdot))=0=\xi(Y)-F(\hat{X}, Y)$ for any $Y \in \mathfrak{t}_{1}^{k}$. Nevertheless, since $F$ is nondegenerate on $\mathfrak{t}_{1}^{k} \otimes \mathfrak{t}_{2}^{k}$, there exists only one possible lift of $X$ which satisfies the later condition, so we end up solving the problem of choice. Thus, we get a map

$$
\varphi: T M_{1} \oplus T^{*} M_{1} / T^{k} \longrightarrow T M_{2} \oplus T^{*} M_{2} / T^{k}, \quad X+\xi \longmapsto\left(p_{2}\right)_{*}\left(\hat{X}+p_{1}^{*} \xi-F(\hat{X}, \cdot)\right),
$$

for $\hat{X}$ the unique lift of $X$ such that $\xi(Y)=F(\hat{X}, Y)$ for any $Y \in \mathfrak{t}_{1}^{k}$.
As it is defined, this map satisfies

$$
\tau(v \cdot \rho)=\varphi(v) \cdot \tau(\rho)
$$

where $v \in T M_{1} \oplus T^{*} M_{1} / T^{k}$ and $\rho \in \Omega_{T^{k}}^{\bullet}\left(M_{1}\right)$. Even more, this map is in fact an isomorphism of Courant algebroids.

Theorem 4.11 (CG04, CG11). Let $M_{1}$ and $M_{2}$ be two $T^{k}$-principal bundles over $B$ with invariant closed 3 -forms $H_{1}$ and $H_{2}$ respectively. If they are T-dual then

$$
\begin{aligned}
\left\langle v_{1}, v_{2}\right\rangle & =\left\langle\varphi\left(v_{1}\right), \varphi\left(v_{2}\right)\right\rangle, \\
{\left[v_{1}, v_{2}\right]_{H_{1}} } & =\left[\varphi\left(v_{1}\right), \varphi\left(v_{2}\right)\right]_{H_{2}},
\end{aligned}
$$

this is, $\varphi$ is an isomorphism of Courant algebroids.
An insightful way to understand how this isomorphism acts is to take connections on $M_{1}$ and $M_{2}$, so we can split the invariant generalized bundles as

$$
\begin{aligned}
& \left(T M_{1} \oplus T^{*} M_{1}\right) / T_{1}^{k} \cong T B \oplus \mathfrak{t}_{1}^{k} \oplus T^{*} B \oplus\left(\mathfrak{t}_{1}^{k}\right)^{*}, \\
& \left(T M_{2} \oplus T^{*} M_{2}\right) / T_{2}^{k} \cong T B \oplus \mathfrak{t}_{2}^{k} \oplus T^{*} B \oplus\left(\mathfrak{t}_{2}^{k}\right)^{*}
\end{aligned}
$$

Then $\varphi$ sends elements of $\mathfrak{t}_{1}^{k}$ to elements of $\left(\mathfrak{t}_{2}^{k}\right)^{*}$ and elements of $\left(\mathfrak{t}_{1}^{k}\right)^{*}$ to elements $\mathfrak{t}_{2}^{k}$. To show this, let us consider the case of circle bundles at the begining of Section 4.2. We had
connections $\theta_{1}$ and $\theta_{2}$ on $M_{1}$ and $M_{2}$ respectively, so we have the splittings $T M_{i} \oplus T^{*} M_{i} / S^{1}=$ $T B \oplus\left\langle\partial_{\theta_{i}}\right\rangle \oplus T^{*} B \oplus\left\langle\theta_{i}\right\rangle, i=1,2$. This implies that we can write an element of $T M_{1} \oplus T^{*} M_{1} / T^{k}$ as $X+f \partial_{\theta_{1}}+\xi+g \theta_{1}$, where $X \in T B$ is an invariant horizontal vector field, $\xi \in T^{*} B$ is an invariant 1-form and $f, g$ are smooth functions. After pulling it back to the correspondence space, we have

$$
X+f \partial_{\theta_{1}}+\xi+g \theta+k \partial_{\theta_{2}},
$$

where $k \partial_{\theta_{2}}$ denotes the added component of $\hat{X}$ in $\mathfrak{s}_{2}^{1}$ with respect to $X$. We had that $F=$ $-p_{1}^{*} \theta_{1} \wedge p_{2}^{*} \theta_{2}$, so if we make the B-transformation, we have

$$
X+f \partial_{\theta_{1}}+\xi+g \theta+k \partial_{\theta_{2}}+\theta_{1} \wedge \theta_{2}\left(f \partial_{\theta_{1}}+k \partial_{\theta_{2}}\right) .
$$

Now, since we have $\theta_{1} \wedge \theta_{2}\left(f \partial_{\theta_{1}}+k \partial_{\theta_{2}}\right)=\theta_{1}\left(f \partial_{\theta_{1}}\right) \otimes \theta_{2}-\theta_{1} \otimes \theta_{2}\left(k \partial_{\theta_{2}}\right)=f \theta_{2}-k \theta_{1}$.we require $\xi+g \theta_{1}+f \theta_{2}-k \theta_{1}$ to be basic. $\xi$ vanishes when we evaluate at some element in $\mathfrak{s}_{1}^{1}$, and so does $f \theta_{2}$. Thus, we get that $g-k=0$. Hence, we have that

$$
\varphi\left(X+f \partial_{\theta_{1}}+\xi+g \theta_{1}\right)=X+g \partial_{\theta_{2}}+\xi+f \theta_{2},
$$

so we have mapped $\partial_{\theta_{1}} \mapsto \theta_{2}$ and $\theta_{1} \mapsto \partial_{\theta_{2}}$, exhibiting the permutation we mentioned.

### 4.4 Generalized complex structures and the effect of T-duality

In the previous section we have introduced an isomorphism of Courant algebroids between two T-dual bundles. This isomorphism can further be used to transport geometrical structures between the bundles. To introduce how this works recall that a generalized complex structure can be described at a point as the Clifford annihilator of a line in $\Lambda^{\bullet} T_{\mathbb{C}}^{*} M$. If $L \subseteq T_{\mathbb{C}} M \oplus T_{\mathbb{C}}^{*} M$ is the generalized complex structure as a subbundle, then it is closed under the Courant bracket, isotropic with respect the pairing and satisfies $L \cap \bar{L}=\{0\}$. Consider that $\rho$ is the nonvanishing section of $\Lambda T_{\mathbb{C}}^{*} M$ that describes such $L$, then $\rho=e^{B+i \omega} \wedge \Omega$ where $\Omega$ is a decomposable form and $B, \omega$ are 2 -forms. This is the maximality requirement. For the condition $L \cap \bar{L}=\{0\}$, we have $(\rho, \bar{\rho}) \neq 0$ for the Chevalley pairing; and for the involutivity with respect to the Courant bracket, there must exist a local section $v$ of $T_{\mathbb{C}} M \oplus T_{\mathbb{C}}^{*} M$ such that $d_{H} \rho=v \cdot \rho$, where we incorporate the Sěvera class into the exterior derivative by setting $d_{H} \varphi=d \varphi+H \wedge \varphi$ for any form $\varphi$.

Let $\mathcal{J}$ be a generalized complex structure on $M$. Then it induces a splitting of $\Lambda^{\bullet} T_{\mathbb{C}}^{*} M$ into subbundles $U^{k}$, fulfilling the relation $U^{n-k}=\bigwedge^{k} \bar{L} \cdot U^{n}$. This can be understood as a generalization of the $(p, q)$-decomposition given by a complex structure. By denoting $\mathcal{U}^{k}$ the local sections of $U^{k}$ then the exterior differential has a mixed grading, as $d_{H}: \mathcal{U}^{k} \rightarrow \mathcal{U}^{k-1}+\mathcal{U}^{k+1}$. This allows us to decompose it in two operators $\partial: \mathcal{U}^{k} \rightarrow \mathcal{U}^{k+1}$ and $\bar{\partial}: \mathcal{U}^{k} \rightarrow \mathcal{U}^{k-1}$, which generalize the Dolbeault operators.

To have a broader view, we can introduce two more particular cases, as we already know that generalized complex structures gather complex structures and symplectic structures. The first one is a generalized Kähler structure.

Definition 4.12. A generalized Kähler structure on a manifold $M$ is a pair of commuting generalized complex structures $\mathcal{J}_{A}$ and $\mathcal{J}_{B}$ such that $G=\mathcal{J}_{A} \mathcal{J}_{B}$ is a metric on the generalized bundle $T M \oplus T^{*} M$ fulfilling $\langle G v, v\rangle>0$ if $v \neq 0$.

There is a particular case of generalized Kähler structures which are generalized Calabi-Yau metrics, which are generalized Kähler structures where each generalized complex structure is described by $d_{H}$-closed forms $\rho_{A}$ and $\rho_{B}$ satisfying $\left(\rho_{A}, \bar{\rho}_{A}\right)=\left(\rho_{B}, \bar{\rho}_{B}\right)$. The last structure we introduce are generalized metrics.

Definition 4.13. A generalized metric on a vector space $V$ is an orthogonal self-adjoint map $G: V \oplus V^{*} \rightarrow V \oplus V^{*}$ for which $\langle G v, v\rangle>0$ for $v \neq 0$.

From the fact that $G$ is orthogonal and self-adjoint, we get the identities $G=G^{*}=G^{-1}$ and $G^{2}=$ id. Hence, $V \oplus V^{*}$ splits as a direct sum of $\pm 1$-eigenspaces, $C_{ \pm}$. This fact leads us to recognize that a generalized metric is equivalent to a choice of orthogonal spaces $C_{ \pm}$where the natural pairing is positive and negative definite respectively. Considering $V \subseteq V \oplus V^{*}, C_{+}$ may be expressed as the graph of some element in $\bigoplus^{2} V^{*}=\operatorname{Sym}^{2} V^{*} \oplus \bigwedge^{2} V^{*}$. Let $g \in \operatorname{Sym}^{2} V^{*}$ and $b \in \Lambda^{2} V^{*}$, then

$$
C_{+}=\{X+g(X, \cdot)+b(X, \cdot): X \in V\} .
$$

We should recall that the pairing must be positive definite on $C_{+}$, so we have some restrictions:

$$
\langle X+g(X, \cdot)+b(X, \cdot), X+g(X, \cdot)+b(X, \cdot)\rangle=g(X, X)>0 \quad \text { for } X \neq 0
$$

This shows that $g$ is in fact a metric on $V$. In a similar way, we can describe $C_{-}$in terms of the graph of $g_{-}+b_{-} \in \bigoplus^{2} V^{*}$. Since $\left\langle C_{+}, C_{-}\right\rangle=0$, we have
$0=\left\langle X+g(X, \cdot)+b(X, \cdot), Y+g_{-}(Y, \cdot)+b_{-}(Y, \cdot)\right\rangle=g(X, Y)+b(X, Y)+g_{-}(Y, X)+b_{-}(Y, X)$.
As this must hold for any $X, Y \in V$, it means that $g_{-}=-g$ and $b_{-}=b$, so $C_{-}$is the graph of $-g+b$. This construction can be brought to the generalized tangent bundle in the same way a riemannian metric is built on a manifold.

With these two new structures we proceed now to state the result which allows us to transport the structures between the T-dual bundles.

Theorem 4.14 ([CG04, CG11]). Let $\left(M_{1}, H_{1}\right)$ and $\left(M_{2}, H_{2}\right)$ be T-dual spaces. Any invariant generalized complex structure, generalized Kähler structure or generalized Calabi-Yau metric on $M_{1}$ is transformed in a similar one by the isomorphism of Courant algebroids $\varphi$.

Proof. Let $\varphi$ be the isomorphism of Courant algebroids. If $L \subseteq T_{\mathbb{C}} M_{1} \oplus T_{\mathbb{C}}^{*} M_{1}$ is a generalized complex structure, using Theorem $6.2, \varphi(L)$ is closed under the Courant bracket $[\cdot, \cdot]_{H_{2}}$. Thanks to the orthogonality of $\varphi$ with respect to the canonical pairing, $\varphi(L)$ is still maximal isotropic, and moreover

$$
\varphi(L) \cap \overline{\varphi(L)}=\varphi(L) \cap \varphi(\bar{L})=\varphi(L \cap \bar{L})=\{0\}
$$

as $\varphi$ is in particular real.
Let $\mathcal{J}_{A}$ and $\mathcal{J}_{B}$ be two generalized complex structures on $M_{1}$ defining a generalized Kähler on $M_{1}$. Then on $M_{2}$ we have the generalized complex structures $\tilde{\mathcal{J}}_{A}=\varphi \circ \mathcal{J}_{A} \circ \varphi^{-1}$ and $\tilde{\mathcal{J}}_{B}=\varphi \circ \mathcal{J}_{B} \circ \varphi^{-1}$. It is easy to check that they also commute:

$$
\begin{aligned}
\tilde{\mathcal{J}}_{A} \circ \tilde{\mathcal{J}}_{B} & =\varphi \circ \mathcal{J}_{A} \circ \varphi^{-1} \circ \varphi \mathcal{J}_{B} \circ \varphi^{-1}=\varphi \circ \mathcal{J}_{A} \circ \mathcal{J}_{B} \circ \varphi^{-1} \\
& =\varphi \circ \mathcal{J}_{B} \circ \mathcal{J}_{A} \circ \varphi^{-1}=\varphi \circ \mathcal{J}_{B} \circ \varphi^{-1} \circ \varphi \circ \mathcal{J}_{A} \circ \varphi^{-1}=\tilde{\mathcal{J}}_{B} \circ \tilde{\mathcal{J}}_{A} .
\end{aligned}
$$

Finally, as $\varphi$ is orthogonal with respect to the natural pairing, we have a generalized metric $\tilde{G}=\varphi \circ G \circ \varphi^{-1}=\varphi \circ \mathcal{J}_{A} \circ \mathcal{J}_{B} \circ \varphi^{-1}$ on $M_{2}$, so we end up with a generalized Kähler structure on $M_{2}$.

To finish the proof, let $\rho_{A}$ and $\rho_{B}$ be the $d_{H_{1}}$-closed forms that describe $\mathcal{J}_{A}$ and $\mathcal{J}_{B}$, and let $L_{A}$ and $L_{B}$ be their Clifford annihilator. Taking the isomorphism of differential complexes $\tau$ from Theorem 6.1, we have that $\tau\left(\rho_{A}\right), \tau\left(\rho_{B}\right)$ are $d_{H_{2}}$-closed and $\varphi\left(L_{A}\right), \varphi\left(L_{B}\right)$ have maximal dimension and correspond to the annihilators of $\tau\left(\rho_{A}\right)$ and $\tau\left(\rho_{B}\right)$ respectively. Finally, to show that $\left.\left(\tau\left(\rho_{A}\right), \overline{\tau\left(\rho_{A}\right)}\right)=\left(\tau\left(\rho_{B}\right), \overline{\tau\left(\rho_{B}\right.}\right)\right)$, we use the map $\psi: \bigwedge^{n} T^{*} M_{1} / T_{1}^{k} \rightarrow \bigwedge^{n} T^{*} M_{2} / T_{2}^{k}$ defined by $\psi\left(\theta_{1} \operatorname{vol}_{B}\right)=\theta_{2} \operatorname{vol}_{B}$, where $\operatorname{vol}_{B}$ is the volume form of the base space $B$. It can be checked that $\psi\left(\left(\xi_{1}, \xi_{2}\right)\right)=(-1)^{k}\left(\tau\left(\xi_{1}\right), \tau\left(\xi_{2}\right)\right)$, so

$$
\begin{aligned}
\left(\tau\left(\rho_{A}\right), \overline{\tau\left(\rho_{A}\right)}\right) & =\left(\tau\left(\rho_{A}\right), \tau\left(\bar{\rho}_{A}\right)\right)=(-1)^{k} \psi\left(\left(\rho_{A}, \bar{\rho}_{A}\right)\right) \\
& =(-1)^{k} \psi\left(\left(\rho_{B}, \bar{\rho}_{B}\right)\right)=\left(\tau\left(\rho_{B}\right), \tau\left(\bar{\rho}_{B}\right)\right)=\left(\tau\left(\rho_{B}\right), \overline{\tau\left(\rho_{B}\right)}\right)
\end{aligned}
$$

As a corollary, we have that T-duality preserves the decomposition of $\Lambda^{\bullet} T_{\mathbb{C}}^{*} M$ into subbundles $U_{k}$.

Corollary 4.15 ([CG04, CG11]). Let $\left(M_{1}, \mathcal{J}_{1}\right)$ and $\left(M_{2}, \mathcal{J}_{2}\right)$ be two complex generalized manifolds which are T-dual, then $\tau\left(\mathcal{U}_{M_{1}}^{k}\right)=\mathcal{U}_{M_{2}}^{k}$ and

$$
\tau\left(\partial_{M_{1}} \xi\right)=\partial_{M_{2}} \tau(\xi), \quad \tau\left(\bar{\partial}_{M_{1}} \xi\right)=\bar{\partial}_{M_{2}} \tau(\xi)
$$

Proof. Let $L_{1}$ be the $+i$-eigenspace of the generalized complex structure $\mathcal{J}_{1}$ and $\rho_{1}$ the form whose annihilator is $L_{1}$. Then $L_{2}=\varphi\left(L_{1}\right)$ is the $+i$-eigenspace of the generalized complex structure $\mathcal{J}_{2}$. Since $\varphi$ is real $\bar{L}_{2}=\varphi\left(\bar{L}_{1}\right)$, which implies

$$
\mathcal{U}_{M_{2}}^{n-k}=\Omega^{k}\left(\bar{L}_{2}\right) \cdot \tau\left(\rho_{1}\right)=\tau\left(\Omega^{k}\left(\bar{L}_{1}\right) \cdot \rho_{1}\right)=\tau\left(\mathcal{U}_{M_{1}}^{k}\right) .
$$

What remains to check is the statement about the operators. Let $\xi \in \mathcal{U}_{M_{1}}^{k}$,

$$
\left(\partial_{M_{2}}-\bar{\partial}_{M_{2}}\right) \tau(\xi)=d_{H_{2}} \tau(\xi)=\tau\left(d_{H_{1}} \xi\right)=\tau\left(\left(\partial_{M_{1}}-\bar{\partial}_{M_{1}}\right) \xi\right),
$$

and since $\tau\left(\mathcal{U}_{M_{1}}^{k}\right)=\mathcal{U}_{M_{2}}^{k}$, the equalities about the operators must hold.
Theorem 4.14 in particular shows that T-duality transports generalized complex structures from one manifold to the other. However it is not clear whether the type of the structure is preserved or not under the duality. In the following example we explore this question.

Example 4.16. Recall that the type of a generalized complex structure locally given by a form $\rho=e^{B+i \omega} \wedge \Omega$ is defined to be the degree of $\Omega$. Let us first consider the case where both $M_{1}$ and $M_{2}$ are circle bundles over the base $B$. By Theorem 4.14, $\tau(\rho)$ defines the corresponding generalized complex structure on $M_{2}$ if $\rho$ defines a generalized complex structure on $M_{1}$.

Expanding the exponential on $\tau(\rho)$, we have the terms

$$
\int_{S^{1}}(F+B+i \omega)^{j} \wedge \Omega .
$$

Hence, the type of $\tau(\rho)$ will be the lowest $j$ such that the integral is not zero. Thus, we have

$$
\operatorname{type}(\tau(\rho))=\operatorname{type}(\rho)+2 j-1
$$

In this case, we only have two possible values for $j$. If $\Omega$ is a basic form, that is, a pullback of a form on $B$, then $j=1$, so the type increases by 1 . If $\Omega$ is not basic, then $j=0$, and the type decreases by 1 . Now, let us consider the case where the fibres are $T^{k}$. Let us define

$$
l=\max \left\{j: \bigwedge^{j} T\left(T^{k}\right) \cdot \Omega \neq 0\right\}, \quad r=\operatorname{rank} \omega_{\mid \operatorname{Ann}(\Omega) \cap T\left(T^{k}\right)} .
$$

Then, by a similar reasoning as before, we have

$$
\operatorname{type}(\tau(\rho))=\operatorname{type}(\rho)+k-2 l-r .
$$

Let us restrict to the case where $M_{1}$ and $M_{2}$ have dimension $2 k$, and the fibres over $B$ have complex or symplectic structures, so $k$ is even.

- The first case we consider is that $M_{1}$ has a complex structure and so do the fibres. This implies that $l=k / 2$ and $r=0$. Hence, the type of the structure is preserved, so $M_{2}$ inherits a complex structure with complex structure on the fibres.
- The second case we consider is that $M_{1}$ also has a complex structure but the fibres have a real one, that is, $T\left(T^{k}\right) \cap J\left(T\left(T^{k}\right)\right)=\{0\}$. Now we have $l=k$ but also $r=0$. Thus $\operatorname{type}(\tau(\rho))=\operatorname{type}(\rho)+k-2 k+0=\operatorname{type}(\rho)-k=0$, so $M_{2}$ inherits a symplectic structure, and the fibres have the corresponding structure, that is, a Lagrangian structure.
- The third case we consider is that $M_{1}$ has a symplectic structure and so do the fibres. Then $l=0$ and $r=k$. This means that $\operatorname{type}(\tau(\rho))=\operatorname{type}(\rho)$, so $M_{2}$ inherits the same structure as $M_{1}$.
- The fourth and last case we consider is that $M_{1}$ has a symplectic structure but the fibres are lagrangian, so $l=0$ and also $r=0$. Thus type $(\tau(\rho))=\operatorname{type}(\rho)+k$, so $M_{2}$ inherits a complex structure with a real structure on the fibres.

Another example, which is not included in Theorem 4.13, is the transport of a generalized metric, especially in the case of circle bundles, as it constitutes the basis for the Buscher rules, independently found in theoretical physics.

Example 4.17. Let $\left(M_{1}, H_{1}\right)$ and $\left(M_{2}, H_{2}\right)$ be T-dual circle bundles over the base $B$, with connections $\theta_{1}$ and $\theta_{2}$ respectively. Let us assume that $M_{1}$ is endowed with an invariant generalized metric $G$. We already know that such a $G$ is equivalent to a choice of $g \in \Gamma\left(\operatorname{Sym}^{2} T^{*} M_{1}\right)$ and $b \in \Omega^{2}\left(M_{1}\right)$, so we must find how these two elements transform.

Since the Courant algebroid isomorphism $\varphi$ is orthogonal with respect to the canonical pairing, it suffices to find $\varphi\left(C_{+}\right)$as a graph of the corresponding two forms $\tilde{g}, \tilde{b}$. Let $v=X+f \partial_{\theta_{1}}$ be an invariant section of $T M_{1}$, and note that we can write $g$ and $b$ as

$$
g=g_{0} \theta_{1} \odot \theta_{1}+g_{1} \odot \theta_{1}+g_{2}, \quad b=b_{1} \wedge \theta_{1}+b_{2}
$$

where $\odot$ denotes the symmetric product of forms. Then, an element of $C_{+}$is of the form

$$
X+f \partial_{\theta_{1}}+\left(i_{X} g_{2}+f g_{1}-f b_{1}+i_{X} b_{2}\right)+\left(g_{1}(X)+b_{1}(X)+f g_{0}\right) \theta_{1}
$$

Now we can apply $\varphi$ to find a general element of $\varphi\left(C_{+}\right)$. According to what we had found in the calculations at the end of Section 4.3, we have

$$
X+\left(g_{1}(X)+f g_{0}+b_{1}(X)\right) \partial_{\theta_{2}}+\left(i_{X} g_{2}+i_{X} b_{2}+f g_{1}-f b_{1}\right)+f \theta_{2}
$$

Comparing this to the graph of a general choice $\tilde{g}+\tilde{b}$, we obtain the relations

$$
\begin{gathered}
\tilde{g}=\frac{1}{g_{0}} \theta_{2} \odot \theta_{2}-\frac{1}{g_{0}} b_{1} \odot \theta_{2}+g_{2}+\frac{1}{g_{0}}\left(b_{1} \odot b_{1}-g_{1} \odot g_{1}\right), \\
\tilde{b}=-\frac{1}{g_{0}} g_{1} \wedge \theta_{2}+b_{2}+\frac{1}{g_{0}} g_{1} \wedge b_{1} .
\end{gathered}
$$

This completely determines the corresponding generalized metric on $M_{2}$.
We can use this example to work our way to give a complete description of the transport of a generalized Kähler structure.

Example 4.18. The choice of a generalized metric $(g, b)$ gives an orthogonal decomposition of the generalized tangent bundle $T M \oplus T^{*} M=C_{+} \oplus C_{-}$. If we consider the projections $\pi_{+}: C_{+} \rightarrow T M$ and $\pi_{-}: C_{-} \rightarrow T M$, it is clear that they are isomorphisms. Hence, having an endomorphism $A \in \operatorname{End}(T M)$, we can use $\pi_{+}$and $\pi_{-}$to induce two endomorphisms $A_{+} \in$ $\operatorname{End}\left(C_{+}\right)$and $A_{-} \in \operatorname{End}\left(C_{-}\right)$.

Given $(M, H),(\tilde{M}, \tilde{H})$ two T-dual spaces and a generalized metric $(g, b)$ on $M$, we can transport it via $\varphi$, obtaining


In this setting, we can consider $A_{+}=J_{+}$and $A_{-}=J_{-}$to be the two commuting complex structures defining a generalized Kähler structure on $M$. Since $\varphi$ is orthogonal and so are $\pi_{ \pm}$, the metric properties of $J_{ \pm}$will be preserved when transported via $C_{ \pm}$, so

$$
\tilde{J}_{ \pm}=\left(\tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}\right) \circ J_{ \pm} \circ\left(\tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}\right)^{-1}
$$

give complex structures on $\tilde{M}$. We can give an explicit description of $\tilde{J}_{ \pm}$when choosing $\theta$ to be a metric connection, that is, $\theta=\frac{1}{g\left(\partial_{\theta}, \partial_{\theta}\right)} g\left(\partial_{\theta}, \cdot\right)$. For this, we need to describe $\tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}$. Let $X$ be orthogonal to $\partial_{\theta}$. This in particular implies that $g_{1}(X)=0$, which simplifies the computation

$$
\begin{aligned}
& \tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}(X)=\tilde{\pi}_{\pi} \circ \varphi\left(X+i_{X} g_{2}+b_{1}(X) \theta+i_{X} b_{2}\right) \\
&=\tilde{\pi}_{ \pm}\left(X+b_{1}(X) \partial_{\tilde{\theta}}+i_{X} b_{2} \pm i_{X} g_{2}\right)=X+b_{1}(X) \partial_{\tilde{\theta}} \\
& \tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}\left(\partial_{\theta}\right)=\tilde{\pi}_{ \pm} \circ \varphi\left(\partial_{\theta}+b_{1} \pm\left(g_{1}+g_{0}^{-1} \theta\right)\right)=\tilde{\pi}_{ \pm}\left(g_{0}^{-1} \partial_{\tilde{\theta}}+\tilde{\theta}\right)= \pm \frac{1}{g_{0}} \partial_{\tilde{\theta}} .
\end{aligned}
$$

At this point we shall stress that the T-dual connection to $\theta$ is not the metric connection of the T-dual metric. Indeed, since $\tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}(X)=X+b_{1}(X) \partial_{\tilde{\theta}}$ is perpendicular to $\partial_{\tilde{\theta}}$, if we use the metric connections on both sides, the map $\tilde{\pi}_{ \pm} \circ \varphi \circ \pi_{ \pm}^{-1}$ acts as the identity from the orthogonal complement of $\partial_{\theta}$ to the orthogonal complement of $\partial_{\tilde{\theta}}$. Hence, if we denote $V_{ \pm}$as the orthogonal complement to $\operatorname{span}\left(\partial_{\theta}, J_{ \pm} \partial_{\theta}\right)$, we can describe the T-dual complex structures as

$$
\tilde{J}_{ \pm} v=\left\{\begin{array}{l}
J_{ \pm}, \quad \text { if } v \in V_{ \pm} \\
\pm \frac{1}{g_{0}} J_{ \pm} \partial_{\theta}, \quad \text { if } v=\partial_{\tilde{\theta}} \\
\mp g_{0} \partial_{\tilde{\theta}}, \quad \text { if } J_{ \pm} \partial_{\tilde{\theta}}
\end{array}\right.
$$

Geometrically, we can interpret this in the following way. As $M$ and $\tilde{M}$ are circle bundles, we can identify $\operatorname{span}\left(\partial_{\theta}\right)$ with $\operatorname{span}\left(\partial_{\tilde{\theta}}\right)$ and their orthogonal complements as $T M \cong T B \oplus \mathbb{R} \cong T \tilde{M}$. Then $\tilde{J}_{+}$is equivalent to $J_{+}$by stretching the axes $\partial_{\theta}$ and $J_{+} \partial_{\theta}$ by a factor of $g_{0}$, and $\tilde{J}_{-}$is $J_{-}$ conjugated and stretched in the same axes.

### 4.5 Lifted actions and reduced Courant algebroids

In this section we study T-duality from a different point of view, using the reduction process of Courant algebroids BCG07, Dru]. In order to do this, we need to define some new objects using the last constructions detailed at the end of Section 4.1.

Let $M$ be a manifold and $E$ a Courant algebroid over $M$. Let $G$ be a connected, compact Lie group acting on $M$ by $g \mapsto \psi_{g} \in \operatorname{Diff}(M)$. We obtain the infinitesimal action

$$
\begin{array}{rllc}
\Sigma: & \mathfrak{g} & \longrightarrow \Gamma(T M) \\
u & \longmapsto & u_{M}
\end{array}
$$

Our first goal is to lift the action of $G$ on $M$ to an action of $G$ on $E$. From the infinitesimal perspective, this is equivalent to find a homomorphism $\mathfrak{g} \rightarrow \operatorname{Der}(E)$ that makes the following diagram commutative


In particular, we will focus on actions by inner symmetries, that is, we will consider $\mathfrak{g} \rightarrow$ $\Gamma(E) \rightarrow \operatorname{Der}(E)$. The reason to do this is that considering only $\mathfrak{g} \rightarrow \Gamma(E)$ prevents us to regard $T M$ and $T^{*} M$ on an equal footing, since on one hand $\mathfrak{g}$ has an antisymmetric bracket but the bracket on $\Gamma(E)$ satisfies a more general condition. We introduce a new object that we will use to solve this problem.

Definition 4.19. A Courant algebra over a Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ endowed with a bilinear bracket $[\cdot, \cdot]: \mathfrak{a} \times \mathfrak{a}$ and a map $\rho: \mathfrak{a} \rightarrow \mathfrak{g}$ such that

- $\left[a_{1},\left[a_{2}, a_{3}\right]\right]=\left[\left[a_{1}, a_{2}\right], a_{3}\right]+\left[a_{2},\left[a_{1}, a_{3}\right]\right], a_{1}, a_{2}, a_{3} \in \mathfrak{a} ;$
- $\rho\left(\left[a_{1}, a_{2}\right]\right)=\left[\rho\left(a_{1}\right), \rho\left(a_{2}\right)\right], a_{1}, a_{2} \in \mathfrak{a}$.

A Courant algebra is exact if $\left[a_{1}, a_{2}\right]=0$ for any $a_{1}, a_{2} \in \operatorname{ker}(\rho)$ and $\rho$ is surjective.

The next example shows that we are in the right path, as $\Gamma(E)$ has precisely this structure in the case that we are interested, the exact Courant algebroids.

Example 4.20. Let $M$ be a manifold and $E$ a Courant algebroid over $M$. Consider the extension of the anchor map $p: \Gamma(E) \rightarrow \Gamma(T M)$. Then by definition $\Gamma(E)$ is a Courant algebra over $\Gamma(T M)$. If $E$ happens to be exact, then $\operatorname{ker}(p)=\Gamma\left(T^{*} M\right)$, and clearly $\left[\Gamma\left(T^{*} M\right), \Gamma\left(T^{*} M\right)\right]=0$. Therefore, $\Gamma(E)$ is an exact Courant algebra over $\Gamma(T M)$ if and only if $E$ is an exact Courant algebroid over $M$.

Now we can define extended $G$-actions and extended $\mathfrak{g}$-actions, which are the previous step to the aforementioned lifted actions.

Definition 4.21. Let $M$ be a manifold and $\mathfrak{g}$ a Lie algebra acting infinitesimally on $M$ by $\Sigma: \mathfrak{g} \rightarrow \Gamma(T M)$. Given an exact Courant algebroid $E$ over $M$, an extended $\mathfrak{g}$-action on $E$ is given by an exact Courant algebra $\rho: \mathfrak{a} \rightarrow \mathfrak{g}$ together with a linear map $\chi: \mathfrak{a} \rightarrow \Gamma(E)$ which satisfies the following conditions:

- $\chi$ is bracket preserving,
- $\operatorname{ad} \circ \chi_{\mid \operatorname{ker}(\rho)}=0$,
- $\Sigma \circ \rho=p \circ \chi$.

Recall that we had the exact sequence

$$
0 \longrightarrow \Omega_{c l}^{1}(M) \xrightarrow{\pi^{*}} \Gamma(E) \xrightarrow{\text { ad }} \operatorname{Der}(E) \xrightarrow{\chi} H^{2}(M) \longrightarrow 0
$$

Using it and the fact that ad $\circ \chi_{\mid \operatorname{ker}(\rho)}=0$, we can see that $\operatorname{Im}\left(\operatorname{ker}_{\mid \operatorname{ker}(\rho)}\right) \subseteq \Omega_{c l}^{1}(M)$. By taking the quotient with respect to $\operatorname{ker}(\rho)$, we can descend the map ad $\circ \chi$ to $\mathfrak{a} / \operatorname{ker}(\rho) \cong \mathfrak{g}$, ad $\circ \chi: \mathfrak{g} \rightarrow \operatorname{Der}(E)$. This gives us the following commutative diagram


Definition 4.22. An extended $G$-action is an extended $\mathfrak{g}$-action such that $\widetilde{\text { ad } \circ \chi}: \mathfrak{g} \rightarrow \operatorname{Der}(E)$ integrates to a group homomorphism $G \rightarrow \operatorname{Aut}(E)$.

We will be interested in a particular type of extended actions called isotropic lifted actions, which come with an isotropic subbundle $K_{\mathfrak{g}}$ of $E$. A lifted $\mathfrak{g}$-action is a $\mathfrak{g}$-extended action $\chi: \mathfrak{a} \rightarrow \Gamma(E)$ such that $\mathfrak{a}=\mathfrak{g}$ and $\rho=\mathrm{id}: \mathfrak{g} \rightarrow \mathfrak{g}$. If it integrates to a $G$-action, we call it a lifted $G$-action.

Let $E$ be a Courant algebroid over $M, \Sigma: \mathfrak{g} \rightarrow \Gamma(T M)$ an infinitesimal action of $\mathfrak{g}$ on $M$ and let $\chi: \mathfrak{g} \rightarrow \Gamma(E)$ a lifted $\mathfrak{g}$-action. We define

$$
K_{\mathfrak{g}}=\{\chi(u)(x) \in E:(u, x) \in \mathfrak{g} \times M\} .
$$

It will be a subbundle if $\left\{u_{M}(x): u \in \mathfrak{g}\right\}$ has constant dimension when varying $x \in M$. Henceforward, we assume that $\Sigma: \mathfrak{g} \rightarrow \Gamma(T M)$ integrates to a free action of a connected, compact Lie group $G$ on $M$. Under this assumption, we get that $\left\{u_{M}(x): u \in \mathfrak{g}\right\}$ has constant dimension $\operatorname{dim} \mathfrak{g}$.

Definition 4.23. Let $\chi: \mathfrak{g} \rightarrow \Gamma(E)$ be a lifted $G$-action. We say that an isotropic splitting $\nabla: T M \rightarrow E$ is invariant if $\nabla \circ\left(\psi_{g}\right)_{*}=\Psi_{g} \circ \nabla$ for every $g \in G$.

We can restate the condition infinitesimally as $\nabla\left[u_{M}, X\right]=[\chi(u), \nabla X]$ for every $u \in \mathfrak{g}$, $X \in \Gamma(T M)$. We define $\xi_{u}=\pi_{T^{*} M}\left(\Phi_{\nabla} \circ \chi(u)\right) \in \Omega^{1}(M)$, so we have $\Phi_{\nabla} \circ \chi(u)=u_{M}+\xi_{u}$. These elements allow us to characterize invariant splitting via their image under ad.

Proposition 4.24. A splitting $\nabla$ is invariant if and only if $\operatorname{ad}\left(u_{M}+\xi_{M}\right)=\left(u_{M}, 0\right)$ for every $u \in \mathfrak{g}$. In particular, if $H$ is the curvature of the splitting, $\mathcal{L}_{u_{M}} H=0$.

Proof. Using $\Phi_{\nabla}$ we can identify $E$ with $T M \oplus T^{*} M$. Taking into account that $\operatorname{ad}\left(u_{M}, \xi_{u}\right)=$ $\left(u_{M}, d \xi_{u}-i_{u_{M}} H\right)$, we have to prove that

$$
\left[u_{M}+\xi_{M}, X\right]_{H}=\left[u_{M}, X\right] \Longleftrightarrow d \xi_{u}-i_{u_{M}} H=0
$$

but this is immediate, as

$$
\left[u_{M}+\xi_{M}, X\right]_{H}=\left[u_{M}, X\right]-i_{X}\left(d \xi_{u}-i_{u_{M}} H\right) .
$$

To conclude, $\mathcal{L}_{u_{M}} H=d i_{u_{M}} H+i_{u_{M}} d H=d i_{u_{M}}=d\left(d \xi_{u}\right)=0$.
As a corollary, we have a description for the B -fields such that $\nabla+B$ is again an invariant splitting.

Corollary 4.25. Let $B \in \Omega^{2}(M)$ and $\nabla$ be an invariant splitting. Then $\nabla+B$ is an invariant splitting if and only if $\mathcal{L}_{u_{M}} B=0$.

Proof. By changing the splitting $\nabla$ by $\nabla+B$, we know that the symmetries change by $(X, A) \mapsto$ $\left(X, A-\mathcal{L}_{X} B\right)$, so $\left(u_{M}, d \xi_{u}-i_{u_{M}} H\right) \mapsto\left(u_{M}, d \xi_{u}-i_{u_{M}} H-\mathcal{L}_{u_{M}} B\right)$. Since $\nabla$ is an invariant splitting, $d \xi_{u}-i_{u_{M}} H=0$, so $\nabla+B$ will be invariant if and only if $\mathcal{L}_{u_{M}} B=0$.

We have already mentioned that if the $G$-action $\chi: \mathfrak{g} \rightarrow \Gamma\left(T M \oplus T^{*} M\right)$ is free and proper and is a lift of $\Sigma: \mathfrak{g} \rightarrow \Gamma(T M)$, then the distribution $K_{\mathfrak{g}} \subseteq T M \oplus T^{*} M$ is a smooth subbundle. What we have not said is that both $K_{\mathfrak{g}}$ and $K_{\mathfrak{g}}^{\perp}$ are invariant under the $G$-action. From this fact we can deduce that the $G$-invariant sections of $K_{\mathfrak{g}}^{\perp}$ are closed under the bracket, and the $G$-invariant sections of $K_{\mathfrak{g}} \cap K_{\mathfrak{g}}^{\perp}$ are an ideal of the former, so

$$
E_{\text {red }}=\frac{K_{\mathfrak{g}}^{\perp}}{K_{\mathfrak{g}} \cap K_{\mathfrak{g}}^{\perp}} / G
$$

is a bundle over $M / G$ which inherits a bracket and a nondegenerate pairing. This is a result that follows from the following theorem.

Theorem 4.26 ([BCG07]). Consider a free and proper $G$-action on $M$ and let $\chi: \mathfrak{g} \rightarrow T M \oplus$ $T^{*} M$ be a lift of this action which preserves the splitting. Then the distribution

$$
E_{\text {red }}=\frac{K_{\mathfrak{g}}^{\perp}}{K_{\mathfrak{g}} \cap K_{\mathfrak{g}}^{\perp}} / G
$$

is a bundle over $M / G$ which inherits a bracket and a nondegenerate pairing. Thus, $E_{\text {red }}$ is a Courant algebroid over $M / G$, which is exact if and only if $K_{\mathfrak{g}}$ is isotropic.

We define the bundle $E_{\text {red }}$, together with the inherited bracket and nondegenerate pairing, as the reduced Courant algebroid, and $M_{\text {red }}=M / G$ as the reduced manifold. We present two examples of this construction which will ease the understanding of the particular case of T-duality.

Example 4.27. Let $M$ be a principal $G$-bundle and choose a connection $\theta \in \Omega^{1}(M ; \mathfrak{g})$. We can understand the extended action $\chi: \mathfrak{g} \rightarrow T M \oplus T^{*} M$ as $X+\xi$, where $X \in \Gamma\left(T M ; \mathfrak{g}^{*}\right)$ and $\xi \in \Gamma\left(T^{*} M ; \mathfrak{g}^{*}\right)$. It is important to notice that for the generators of the action $X+\xi, i_{X} \theta=1$ and $\mathcal{L}_{X} \theta=0$. Consider $B=\langle\theta, \xi\rangle+\frac{1}{2}\langle\theta \wedge \theta,\langle\xi, X\rangle\rangle$, and apply the corresponding B-field transformation to $X+\xi$, so we get

$$
X+\xi-i_{X}\langle\theta, \xi\rangle-\frac{1}{2} i_{X}\langle\theta \wedge \theta,\langle\xi, X\rangle\rangle=X+\xi-\xi+\langle\theta,\langle\xi, X\rangle\rangle-\frac{1}{2} 2\langle\theta,\langle\xi, X\rangle\rangle=X,
$$

and the 3 -form curvature transforms to $H_{\text {red }}=H+d B$. As shown in the computation, the extended action after the B-field transformation lies entirely on $T M$, so $K_{\mathfrak{g}}=T M$ and $K_{\mathfrak{g}}^{\perp}=$ $T M+\operatorname{Ann}(\Sigma(\mathfrak{g}))$. Therefore,

$$
E_{\text {red }}=\frac{K_{\mathfrak{g}}^{\perp}}{K_{\mathfrak{g}} \cap K_{\mathfrak{g}}^{\perp}} / G=\frac{K_{\mathfrak{g}}^{\perp}}{K_{\mathfrak{g}}} / G \cong(T M / \Sigma(\mathfrak{g}) \oplus \operatorname{Ann}(\Sigma(\mathfrak{g}))) / G \cong T M / G \oplus T^{*} M / G,
$$

where $M / G$ is endowed with the 3 -form curvature $H_{\text {red }}$.
Example 4.28. Let $\mathcal{G}$ be a $2 k$-dimensional Lie group with Lie algebra $\mathfrak{G}$, and consider a lifted $\mathcal{G}$-action on $M$. Endow $\mathfrak{G}$ with the pairing $\langle\chi(\cdot), \chi(\cdot)\rangle$, which is nondegenerate and with split signature. Let $K_{\mathfrak{G}}$ be the distribution generated by $\mathfrak{G}$. As the pairing is nondegenerate on $K_{\mathfrak{G}}$, we have that $K_{\mathfrak{G}} \cap K_{\mathfrak{G}}^{\perp}=\{0\}$, so $K_{\mathfrak{G}} \nsubseteq K_{\mathfrak{G}}^{\perp}$. Hence, $K_{\mathfrak{G}}$ is not isotropic, which implies that $E_{\text {red }}$ is not exact,

$$
E_{\text {red }}=\frac{K_{\mathfrak{G}}^{\perp}}{K_{\mathfrak{G}} \cap K_{\mathfrak{G}}^{\perp}} / \mathcal{G}=K_{\mathfrak{G}}^{\perp} / \mathcal{G} .
$$

Thanks to the split signature of the pairing on $K_{\mathfrak{G}}$, we can decompose it as $K_{\mathfrak{G}}=K \oplus \tilde{K}$. Then, $K^{\perp}=K_{\mathfrak{G}}^{\perp}+K$ and $\tilde{K}^{\perp}=K_{\mathfrak{G}}^{\perp}+\tilde{K}$, which allows us to describe the vector bundle with symmetric pairing structure of $E_{\text {red }}$ as

$$
E_{r e d} \cong \frac{K^{\perp}}{K} / \mathcal{G} \cong \frac{\tilde{K}^{\perp}}{\tilde{K}} / \mathcal{G}
$$

These are not Courant algebroids because they are not involutive with respect to bracket.
The obstruction to endow $E_{\text {red }}$ with a natural bracket can be solved in the following way. Let us assume that $\mathcal{G}=G^{k} \times \tilde{G}^{k}$, so $\mathfrak{G}=\mathfrak{g} \oplus \tilde{\mathfrak{g}}$, and asume both $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. These conditions allow us to choose a natural isotropic splitting $K_{\mathfrak{G}}=K_{\mathfrak{g}} \oplus K_{\tilde{\mathfrak{g}}}$, and each of the components now are closed under the bracket, so $\left(K_{\mathfrak{g}}^{\perp} / K_{\mathfrak{g}}\right) / \mathcal{G}$ and $\left(K_{\mathfrak{\mathfrak { g }}}^{\perp} / K_{\tilde{\mathfrak{g}}}\right) / \mathcal{G}$ inherit a Courant bracket, and both structure coincide with $K_{\mathfrak{G}}^{\perp} / \mathcal{G}$. This discussion can be summarized in the following commutative diagram


This is precisely the case where T-duality arises from this point of view.
Theorem 4.29 (CG11]). Let $(\mathcal{M}, \mathcal{H})$ be the total space of a principal $T^{k} \times \tilde{T}^{k}$-bundle over $B$ and let $\chi: \mathfrak{t}^{k} \times \tilde{\mathfrak{t}}^{k} \rightarrow \Gamma\left(T \mathcal{M} \oplus T^{*} \mathcal{M}\right)$ be a lift of the action of $T^{k} \times \tilde{T}^{k}$ for which $K_{\mathfrak{t}^{k} \times \tilde{\mathrm{t}}^{k}}$ is of split signature and nondegenerate and such that $K_{\mathrm{t}^{k}}$ and $K_{\mathfrak{\mathrm { t }}^{k}}$ are isotropic. Then, $M=\mathcal{M} / \tilde{T}^{k}$ and $\tilde{M}=\mathcal{M} / T^{k}$ are T-dual. Moreover, any T-dual pair arises in this form.

Proof. As shown in Example 4.27, we can perform a B-field transformation so that the lifted action of $\tilde{T}^{k}$ lies entirely on $T \mathcal{M}$, so without loss of generality, we assume that the action is of this form. Let $\mathcal{H}$ be the 3 -form curvature associated to this splitting. Then, for $M=\mathcal{M} / \tilde{T}^{k}$ we have a 3 -form $H$ such that $\mathcal{H}$ is its pullback.

Analogously, we can take a $T^{k} \times \tilde{T}^{k}$-invariant form $F \in \Omega^{2}(\mathcal{M})$ and perform the B-field transformation so the action of $T^{k}$ lies entirely on $T \mathcal{M}$. Then, on $\tilde{M}=\mathcal{M} / T^{k}$ we have the 3-form $\tilde{H}=\mathcal{H}+d F$. Hence, $\mathcal{M}=M \times_{B} \tilde{M}$ and $H-\tilde{H}=d F$. Since the pairing on $K_{\mathrm{t}^{k} \times \tilde{\mathfrak{f}}^{k}}$ is nondegenerate, it turns out that $F$ is a nondegenerate pairing between $K_{\mathfrak{t}^{k}}$ and $K_{\mathfrak{t}^{k}}$, so $(M, H)$ and $(\tilde{M}, \tilde{H})$ are indeed T-dual.

To prove that any T-dual pair has this form, let $(M, H)$ and $(\tilde{M}, \tilde{H})$ be T-dual spaces. Let $\mathcal{M}=M \times{ }_{B} \tilde{M}$ and consider the lifted $T^{k} \times \tilde{T}^{k}$-action $\chi(t, \tilde{t})=X_{t}-i_{X_{t}} F+X_{\tilde{t}}$. Lifting only the action of $T^{k}$ and $\tilde{T}^{k}$ with this $\chi$ shows that the corresponding actions are isotropic. This, together with the fact that $F$ is nondegenerate, means that the natural pairing restricted to $K_{\mathrm{t}^{k} \times \tilde{\mathfrak{t}^{k}}}$ is nondegenerate and of split signature.

This result gives an intuitive idea of Theorem 4.11, as both Courant algebroids given by the $T^{k}$-invariant sections of $T M \oplus T^{*} M$ and the $\tilde{T}^{k}$-invariant sections of $T \tilde{M} \oplus T^{*} \tilde{M}$ are isomorphic to the reduction of $\mathcal{M}$ by the full action of $T^{k} \times \tilde{T}^{k}$.

## 5 Physics interpretation of T-duality

Following some of the aspects in Bug19, we shall give a mild explanation of the physics interpretation and treatment of T-duality. One way to approach this is to get a simple example of what physicists call a duality. Let us consider the classical electromagnetic theory in a vacuum. We have two vector fields in $\mathbb{R}^{3}, \vec{E}$ and $\vec{B}$, which we call electric and magnetic field respectively, and satisfy the so-called Maxwell's equations,

$$
\nabla \cdot \vec{E}=0, \quad-\nabla \times \vec{E}=\frac{\partial \vec{B}}{\partial t}, \quad \nabla \cdot \vec{B}=0, \quad \nabla \times \vec{B}=\frac{\partial \vec{E}}{\partial t}
$$

where $t$ denotes the time and we have set the speed of light $c=1$. If we consider the transformation $(\vec{E}, \vec{B}) \mapsto(\vec{B},-\vec{E})$, then it is easy to check that the Maxwell's equations are also satisfied. This points out that in reality assigning the label of electric and magnetic field is just a convention, as we can swap their roles using this transformation.

When working in theoretical physics, systems are studied using a functional $S$ called action, which usually is an integral over the coordinates and the time of some function, called the lagrangian, which depends on the coordinates, the derivative of the coordinates with respect to time and the time. Let us see what the action of an electromagnetic system in the vacuum. is. For this, we consider the time as another coordinate, and take the Minkowski space $\mathbb{R}^{1,3}=$ $\{(t, x, y, z): t, x, y, z \in \mathbb{R}\}$, with the metric tensor

$$
\eta=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array}\right)
$$

In this setting, we can express the electric and magnetic fields as 2-forms,

$$
\begin{gathered}
E=E_{x} d t \wedge d x+E_{y} d t \wedge d y+E_{z} d t \wedge d z \\
B=B_{x} d z \wedge d y+B_{y} d x \wedge d z+B_{z} d y \wedge d x
\end{gathered}
$$

We can sum both forms to obtain what is called the field strength tensor,

$$
F=E+B=\left(\begin{array}{cccc}
0 & E_{x} & E_{y} & E_{z} \\
-E_{x} & 0 & -B_{z} & B_{y} \\
-E_{y} & B_{z} & 0 & -B_{x} \\
-E_{z} & -B_{y} & B_{x} & 0
\end{array}\right) .
$$

We can now write the action in terms of $F$,

$$
S=-\frac{1}{4} \int F \wedge \star F
$$

where $\star$ denotes the Hodge star operator with respect to the Minkowski metric $\eta$. It is easy to check that

$$
\star F=\left(\begin{array}{cccc}
0 & -B_{x} & -B_{y} & -B_{z} \\
B_{x} & 0 & -E_{z} & E_{y} \\
B_{y} & E_{z} & 0 & -E_{x} \\
B_{z} & -E_{y} & E_{x} & 0
\end{array}\right),
$$

which agrees with the transformation $(E, B) \mapsto(B,-E)$. Then, by transforming the strength field tensor in this fashion, the action changes by a total minus sign

$$
S=-\frac{1}{4} \int F \wedge \star F \mapsto S^{\prime}=-\frac{1}{4} \int \star F \wedge \star(\star F)=-\frac{1}{4} \int \star F \wedge F=\frac{1}{4} \int F \wedge \star F=-S .
$$

As the dynamics of the system is found by extremizing the action, a global sign does not have any effect, so we say it is a symmetry of the system, and it leaves the action invariant. This kind of duality between the electric and magnetic fields that preserves the action gives a nice intuition of what will happen in T-duality.

It may be insightful to spend some time describing the first appearence of T-duality in phyisics. Let us consider a closed string embedded and moving in a target space $M=\mathbb{R}^{1,24} \times S^{1}$. We can describe its movement using a map

$$
X: \Sigma \longrightarrow M
$$

where $\Sigma$ is a 2-dimensional space, in fact, a $1+1$-dimensional space, as we have 1 component to describe the time, $\tau$, and another one to describe the point on the string, $\sigma$. We can decompose $X$ in coordinates, $X(\tau, \sigma)=\left(X^{0}(\tau, \sigma), \ldots, X^{24}(\tau, \sigma), X^{25}(\tau, \sigma)\right)$. Regarding the structure of $M$, we want to impose an especial condition to the last coordinate:

$$
X^{25}(\tau, \sigma+2 \pi)=X^{25}(\tau, \sigma)+2 \pi m R .
$$

The meaning of this constraint is very easy to understand. As the 25 -th coordinate takes values in $S^{1}$, we require some kind of periodicity. Geometrically speaking, we have to wrap the string around this $S^{1}$, stretching it if needed, so that when we advance by $2 \pi$ on the spatial coordinate, the string wraps around the $S^{1} m$ times. We call this $m \in \mathbb{Z}$ the winding number, and $R$ is the intrinsic radius of the $S^{1}$.

The kinetic energy of the string, using the Einstein summation notation, can be written as

$$
T=h^{\alpha \beta} \sqrt{-h} g_{\mu \nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu},
$$

where $g_{\mu \nu}$ is the metric tensor of $M, h_{\alpha \beta}$ is the intrinsic metric of the worldsheet of the string, that is, the surface it describes as it moves, $h^{\alpha \beta}$ is its inverse, and $h$ is the determinant of the intrinsic metric. We introduce the minus sign as the signature of the metric is $(1,1)$. In this case, the lagrangian is equal to the kinetic energy, so we can write the action as follows:

$$
S=\frac{1}{4 \pi \lambda} \int_{\Sigma}\left(h^{\alpha \beta} \sqrt{-h} g_{\mu \nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu}\right) d \tau d \sigma
$$

where the constant is added for physical purposes we do not care about for the moment. By extremizing the action, we obtain the so-called Euler-Lagrange equations, which are the equations of movement of the string in this case. The relevant term for our case is the solution for the 25 -th coordinate. The solution can be described as the sum of two terms, which can be interpreted as waves propagating to the right and to the left,

$$
\begin{aligned}
& X_{L}^{25}=\frac{1}{2} X^{25}+\frac{1}{2} \lambda\left(\frac{n}{R}+\frac{m R}{\lambda}\right)+\text { oscillator terms } \\
& X_{R}^{25}=\frac{1}{2} X^{25}+\frac{1}{2} \lambda\left(\frac{n}{R}-\frac{m R}{\lambda}\right)+\text { oscillator terms. }
\end{aligned}
$$

Here, $n \in \mathbb{Z}$ is the quantum number associated to momentum in the compact direction $S^{1}$, which can be interpreted as the mode of vibration. Having the solutions for the equations of motion, one can compute the momentum, and then compute the spectrum of the masses that are allowed for the string by means of the equation $E^{2}=M^{2} c^{4}+p^{2} c^{2}$, which relates the energy of the string with its mass and momentum. Since we have set $c=1$, we have

$$
M^{2}=-p_{\mu} p^{\mu}=\frac{n^{2}}{R^{2}}+\frac{m^{2} R^{2}}{\lambda^{2}}+\frac{2}{\alpha}(N+\tilde{N}-2)
$$

We shall focus on the first two terms. In the first term appears $n$, which indicates that it is the momentum contribution to the mass, while in the second term appears $m$, so it must have some relation with the fact that the string wraps around the $S^{1}$. To understand this term, we shall stress that strings have an intrinsic tension $T=\frac{1}{2 \pi \lambda}$, so stretching it will increase its energy. As the string is wrapping $m$ times around an $S^{1}$ with radius $R$, at least its length must be $l=2 \pi m R$. Hence, if the string has no momentum, we have that its minimum mass is $M=l \cdot T=\frac{m R}{\lambda}$. Thus, the second term is precisely the contribution to the energy of the wrapping.

Now is the time to introduce the T-duality. Let us consider the set of transformations

$$
R \mapsto \frac{\lambda}{R}, \quad n \mapsto m, \quad m \mapsto n .
$$

It is an easy computation to check that the mass spectrum remains invariant. This means that if the string were moving on another target space $M^{\prime}=\mathbb{R}^{1,24} \times S^{1}$ where the radius of $S^{1}$ is $\lambda / R$, we would not be able to tell both theories apart, or in other words, the theories are dual. In fact, T-dual, as we shall recognise much better later.

We are now going to derive the so-called Buscher rules. For this, we consider a more general setting than before. Let $X: \Sigma \rightarrow M$ be the map describing the movement of a string, where $\left(\Sigma, h_{\alpha \beta}\right)$ is a 2 -dimensional Lorentzian manifold and $\left(M, g_{\mu \nu}\right)$ is a (pseudo)-Riemannian manifold, which comes equipped with a B-field, that is, a locally defined two-form. In this case,
we have the action

$$
\begin{aligned}
S & =\frac{1}{4 \pi} \int_{\Sigma}\left(h^{\alpha \beta} \sqrt{-h} g_{\mu \nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu}+\varepsilon^{\alpha \beta} B_{\mu \nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu}\right) d \tau d \sigma \\
& =\frac{1}{4} \int_{\Sigma} g_{\mu \nu} d X^{\mu} \wedge \star d X^{\nu}+B_{\mu \nu} d X^{\mu} \wedge d X^{\nu} .
\end{aligned}
$$

Let $v=v^{i} \partial_{i}$ be a vector field on $M$ and $\epsilon$ a constant parameter, and set the transformation $\left(X^{\prime}\right)^{i}=X^{i}+v^{i} \epsilon$. We write this as $\delta_{\epsilon} X=v^{i} \epsilon$. We can compute how this changes the action.

$$
\delta_{\epsilon} S=\int_{\Sigma} \epsilon\left(\mathcal{L}_{v} g\right)_{i j} d X^{i} \wedge \star d X^{j}+\epsilon\left(\mathcal{L}_{v} B\right)_{i j} d X^{i} \wedge d X^{j}
$$

Hence, the action is invariant under this transformation $\left(\delta_{\epsilon} S=0\right)$ if and only if $\mathcal{L}_{v} g=0=\mathcal{L}_{v} B$. We can take benefit from the first fact, as the condition $\mathcal{L}_{v} g=0$ means that $v$ is a Killing vector, so it generates a one-parameter group of isometries. Assuming that the system has at least one continuous isometry, we can perform a change of coordinates such that $\left\{X^{\mu}\right\}=\left\{X^{i}, \theta\right\}$ and $v=\partial_{\theta}$. Hence, the isometry in this case is just a translation $\theta \mapsto \theta+\epsilon$. Let us also assume that $\mathcal{L}_{\partial_{\theta}} B=0$. We have that

$$
\delta_{\epsilon} X^{i}=0, \quad \delta_{\epsilon} \theta=\epsilon .
$$

In order to obtain the Buscher rules, we have to allow $\epsilon$ to depend on the worldsheet coordinates, $\epsilon=\epsilon(\tau, \sigma)$. When we do this, we need to make a small adjustment, known as as gauging, in order to keep the action invariant. Let $\mathcal{A}$ be a field such that $\delta_{\epsilon} \mathcal{A}=d \epsilon$, and set the covariant derivatives as

$$
d X^{i} \mapsto \mathcal{D} X^{i}=d X^{i}, \quad d \theta \mapsto \mathcal{D} \theta=d \theta-\mathcal{A} .
$$

Then, we can rewrite the action as

$$
S=\frac{1}{4 \pi} \int_{\Sigma} g_{\mu \nu} \mathcal{D} X^{\mu} \wedge \star \mathcal{D} X^{\nu}+B_{\mu \nu} \mathcal{D} X^{\mu} \wedge \mathcal{D} X^{\nu}
$$

which preserves the original form. In addition to this, consider also the term

$$
\frac{1}{2 \pi} \int_{\Sigma} \mathcal{F} \hat{\theta}
$$

where $\mathcal{F}=d \mathcal{A}$ and $\hat{\theta}$ is an auxiliary field such that $\delta_{\epsilon} \hat{\theta}=0$. In summary, what we have is a new action

$$
S=\frac{1}{4 \pi} \int_{\Sigma} g_{\mu \nu} \mathcal{D} X^{\mu} \wedge \star \mathcal{D} X^{\nu}+B_{\mu \nu} \mathcal{D} X^{\mu} \wedge \mathcal{D} X^{\nu}+\frac{1}{2 \pi} \int_{\Sigma} \mathcal{F} \hat{\theta}
$$

which is invariant under the local gauge transformation

$$
\delta_{\epsilon} X^{i}=0, \quad \delta_{\epsilon} \theta=\epsilon, \quad \delta_{\epsilon} \mathcal{A}=d \epsilon, \quad \delta_{\epsilon} \hat{\theta}=0 .
$$

If we find the equations of motion for the gauge field $\mathcal{A}$ and substitute it again in $S$, we can rewrite it as

$$
\hat{S}=\frac{1}{4 \pi} \int_{\Sigma} \hat{g}_{\mu \nu} d \hat{X}^{\mu} \wedge \star d \hat{X}^{\nu}+\hat{B}_{\mu \nu} d \hat{X}^{\mu} \wedge d \hat{X}^{\nu}
$$

where $\left\{\hat{X}^{\mu}\right\}=\left\{X^{i}, \hat{\theta}\right\}$, and the fields $\hat{g}_{\mu \nu}, \hat{B}_{\mu \nu}$ satisfy

$$
\begin{gathered}
\hat{g}_{\hat{\theta} \hat{\theta}}=\frac{1}{g_{\theta \theta}}, \quad \hat{g}_{i \hat{\theta}}=\frac{B_{i \theta}}{g_{\theta \theta}}, \quad \hat{g}_{i j}=g_{i j}-\frac{1}{g_{\theta \theta}}\left(g_{i \theta} g_{j \theta}-B_{i \theta} B_{j \theta}\right) ; \\
\hat{B}_{i \hat{\theta}}=\frac{g_{i \theta}}{g_{\theta \theta}}, \quad \hat{B}_{i j}=B_{i j}-\frac{1}{g_{\theta \theta}}\left(B_{i \theta} g_{j \theta}-g_{i \theta} B_{j \theta}\right) .
\end{gathered}
$$

These are the famous Buscher rules, which can be easily compared to the ones we obtained in Example 4.17, and the fundamental example which arises in string theory. As a last remark, if we consider that the target space $M$ is a cylinder with a flat metric, given by the line element

$$
d s^{2}=\sum_{i}\left(d X^{i}\right)^{2}+R^{2} d \theta^{2}
$$

and the zero B-field, we have that $g_{i j}=\delta_{i j}$ and $g_{\theta \theta}=R^{2}$. Then, applying the Buscher rules, we obtain $\hat{g}_{i j}=g_{i j}$ and $\hat{g}_{\hat{\theta} \hat{\theta}}=R^{-2}$, which establishes the relation $R \mapsto \lambda / R$ we found in the previous example, as we have set $\lambda=1$.

## 6 Conclusions and further developments

As we have seen, we have come a long path to understand the concept of T-duality through generalized geometry. We have stated a precise definition of the notion of T-duality and given a characterization of it in terms of an isomorphism between the Courant algebroids over the torus bundles. This isomorphism turns out to be the key element to transport geometrical structures between the bundles, which leads to some interesting new paths to investigate both in theoretical physics and mathematics.

From the physics point of view, some generalizations have been recently developed. As an example, in BEM18, a generalization of T-duality called spherical T-duality is developed. In this case, torus bundles are replaced by $S^{3}$-bundles and the 3 -forms $H$ are now 7 -forms, which are exchanged by the duality with the Euler class. Moreover, the duality provides other nice results, as the fact that the Pontryagin class as well as the second Stiefel-Whitney class are fixed.

Regarding the algebraic topological and geometrical aspects that arise from the notion of T-duality, some K-theory can be used to go further. In [BEM18], a Fourier-Mukai transform between the K-theory of trivial $\mathrm{SU}(2)$-bundles is developed, and some of the related results can be extended to higher dimensions. In a similar fashion, in LSW20 spherical T-duality is treated from the K-theory point of view, showing that any cohomology theory which admits T-duality between $S^{q}$-bundles with $q>1$ has to be rational.

When it comes to the differential geometrical approach, some developments have been done in [MP16], showing that not only geometrical structures are carried by T-duality but also supersymmetry, by exchanging complex supersymmetric systems with symplectic supersymmetric systems. Aside from that, in GFS20, an interplay between the generalized Ricci flow, which is a partial differential equation for a generalized metric, and the T-duality is stated, showing that solutions of gauge-fixed generalized Ricci flow and generalized Einstein pairs are exchanged under the duality.

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[^0]:    ${ }^{1}$ This is the Whitney sum of the bundles $T M$ and $T^{*} M$. The precise definition of this construction is given in Section 3.2

