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# Berger's Holonomy Theorem, and a First Incursion into Lie Algebroid Holonomy

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Universiteit Utrecht
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**Jaime Pedregal Pastor**
*Supervised by* pr. dr. Gil R. Cavalcanti
*Cosupervised by* dr. Roberto Rubio

# Berger's Holonomy Theorem, and a First Incursion into Lie Algebroid Holonomy

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## Abstract

The holonomy group of a connection is very intimately related to the curvature of the connection and to the existence and quantity of parallel sections, thus controlling an important part of the geometry. The Lie groups that can arise as the holonomy group of the Levi-Civita connection of a Riemannian manifold were classified by Berger, resulting in a list of seven possible groups. These holonomies give rise to special geometries, like Kähler, Calabi–Yau or hyperkähler geometries. It was not until fifty years later that Olmos offered a geometric proof of Berger's theorem, as an alternative to Berger's more algebraic proof. The first part of this work is dedicated to presenting Olmos's proof, orderly developing the requisites needed to understand it.

In the second part we introduce the generalization of holonomy to the Lie algebroid setting. Lie algebroids are, in a sense, a generalization of the tangent bundle and, as such, it makes sense to consider Lie algebroid connections and Lie algebroid holonomy. This new concept presents some remarkable new features. The first one is the failure of the Ambrose–Singer theorem: the holonomy algebra is not only determined by curvature, but also by the isotropy of the algebroid. We give a new proof of this Lie algebroid Ambrose–Singer theorem, and provide some original examples of flat Lie algebroid connections with non-discrete holonomy. Secondly, the notion of Lie algebroid holonomy is a leafwise notion, so the holonomy can jump from leaf to leaf. When considering general Lie algebroid connections on vector bundles, this behavior can be quite wild: it can jump either up or down when changing to smaller leaves. We provide as well original examples of such behaviors.

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# Contents

# Introduction

On the trivial vector bundle $E = M \times \mathbb{R}^r$ over a smooth manifold $M$, there is a canonical way of taking derivatives of sections of $E$ along directions in $M$. Indeed, sections of $E$ can be identified with maps in $C^\infty(M, \mathbb{R}^r)$, and the derivative of $\sigma \in C^\infty(M, \mathbb{R}^r)$ in the direction of $v \in T_xM$ is just given by $\sigma_* v$. A section of $E$ whose derivative vanishes in all directions is constant.

On a general vector bundle $E \to M$, though, the above construction is only canonical locally, whereas globally there is no canonical way of taking directional derivatives. A consistent choice of directional derivatives is what we call a connection on $E$. This allows us to talk about "constant" or "parallel" sections: those whose directional derivatives vanish in all possible directions. The existence and quantity of such sections on a given bundle is controlled by the holonomy group of the connection.

Concretely, the holonomy group of a connection at a point $x \in M$ consists of all the possible linear automorphisms of the fiber $E_x$ that arise as parallel transport along loops based at $x$. Parallel transport is a way of connecting fibers of $E$ by means of the connection: if $v \in E_x$ and $\gamma : [0, 1] \to M$ is a smooth curve starting at $x$, then there is a unique section of $E$ which is parallel along $\gamma$, meaning that its derivative in the direction of $\dot\gamma(t)$ vanishes. The value of such a section at time 1 is the parallel transport of $v$ along $\gamma$, and we call it $\tau_\gamma v$. The holonomy group at $x$ is the subgroup of $\mathrm{GL}(E_x)$ given by transformations of the form $\tau_\gamma$ for all loops $\gamma$ based at $x$. As said, this group is very closely related to the space of parallel sections of $E$. This goes under the name of the holonomy principle: every vector in $E_x$ which is invariant under the holonomy group gives rise to a unique parallel section of $E$, and all parallel sections arise in this manner. The reason we are interested in the (non)existence of parallel sections is because this is very intimately related to the geometry of $E$. Indeed, the holonomy group contains essentially the same information as the curvature (actually, a bit more). The curvature of a connection is an obstruction for the connection to define a cochain complex on $E$-valued differential forms on $M$, and it is related to holonomy through the celebrated Ambrose–Singer theorem: the Lie algebra of the holonomy group is spanned by the parallel transport of every curvature endomorphism on $M$.

If $E$ is the tangent bundle of a Riemannian manifold $(M, g)$, then it is well known from Riemannian geometry that there is a unique connection which is compatible with $g$ and moreover torsion-free, called the Levi-Civita connection. The holonomy group of $TM$ at $x \in M$ for the Levi-Civita connection is called the Riemannian holonomy group of $M$, and we denote it by $\mathrm{Hol}_x(M)$. As stated, the group $\mathrm{Hol}_x(M)$ determines part of the geometry of $M$. Indeed, if $\mathrm{Hol}_x(M)$ leaves some tensor over $x$ invariant, then by the holonomy principle there is some global tensor field on $M$ which is nowhere vanishing, parallel and equals the given tensor at $x$. Applying this reasoning to the simplest examples already gives some interesting results:

1. The group $\mathrm{SO}(n)$ is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ preserving the canonical metric on $\mathbb{R}^n$ and the canonical volume form $dx^1 \wedge \cdots \wedge dx^n$, for $(x^j)_j$ coordinates in $\mathbb{R}^n$. Hence, if $\mathrm{Hol}_x(M) \subseteq \mathrm{SO}(n)$, then there is parallel volume form on $M$. In particular, $M$ is orientable.

2. The group $\mathrm{U}(n)$ is the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ preserving the canonical metric on $\mathbb{R}^{2n}$ and the canonical linear complex structure $J$ on $\mathbb{R}^{2n}$, given by $J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}$, for $(x^j, y^j)_j$ coordinates in $\mathbb{R}^{2n}$. Actually, $\mathrm{U}(n) \subseteq \mathrm{SO}(n)$, so that it also preserves the canonical volume form. Hence, if $\mathrm{Hol}_x(M) \subseteq \mathrm{U}(n)$, then there is a parallel almost complex structure and a parallel volume form on $M$, which is equivalent to $M$ being Kähler.

3. The group $\mathrm{SU}(n)$ is the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ preserving the canonical metric on $\mathbb{R}^{2n}$, the canonical linear complex structure on $\mathbb{R}^{2n}$ and the canonical complex volume form $dz^1 \wedge \cdots \wedge dz^n$, for $(z^j = x^j + iy^j)_j$ complex coordinates in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Also, $\mathrm{SU}(n) \subseteq \mathrm{SO}(n)$ as well, so it also preserves the canonical volume form. Hence, if $\mathrm{Hol}_x(M) \subseteq \mathrm{SU}(n)$, then there is a parallel almost complex structure, a parallel complex volume form and a parallel real volume form on $M$. This is the definition of $M$ being Calabi–Yau.

4. The group $\mathrm{Sp}(n)$ is the subgroup of $\mathrm{GL}(4n, \mathbb{R})$ preserving the canonical metric on $\mathbb{R}^{4n}$ and the canonical linear quaternionic structure on $\mathbb{R}^{4n}$, given by two linear complex structures $I$ and $J$ on $\mathbb{R}^{4n}$ defined by $I\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}$ and $I\frac{\partial}{\partial b^j} = \frac{\partial}{\partial a^j}$, and $J\frac{\partial}{\partial x^j} = \frac{\partial}{\partial a^j}$ and $J\frac{\partial}{\partial y^j} = \frac{\partial}{\partial b^j}$, for $(x^j, y^j, a^j, b^j)_j$ coordinates in $\mathbb{R}^{4n}$. Actually, $\mathrm{Sp}(n) \subseteq \mathrm{SU}(n)$, so it also preserves a complex and a real volume form. Hence, if $\mathrm{Hol}_x(M) \subseteq \mathrm{Sp}(n)$, then on $M$ there are two parallel almost complex structures $I$ and $J$ satisfying the quaternionic relations $IJ = -JI$, a parallel complex volume form and a parallel real volume form. This is the definition of $M$ being hyperkähler.

A remarkable result by Berger [Ber55], together with some later refinements by Alekseevskii [Ale68] and Brown and Gray [BG72], states that the four examples just listed plus $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ and the two exceptional cases $G_2$ and $\mathrm{Spin}(7)$ are the only possible groups that a connected and simply connected Riemannian manifold which is not reducible or locally symmetric can have as holonomy. Here reducible means that the action of $\mathrm{Hol}_x(M)$ on $T_xM$ is reducible, and locally symmetric means that every point is a fixed point of a local isometric involution inverting the direction of geodesics. Moreover, each one of these groups can be realized as the holonomy of some manifold and, actually, of some compact manifold [Yau78, Bea83, Gal87, Joy96a, Joy96b, Joy96c].

The first objective of this thesis is to understand the proof of Berger's theorem. The original proof by Berger is algebraic and relies on the classification of Lie groups: it considers the list of closed connected Lie subgroups of $\mathrm{SO}(n)$ which act irreducibly on $\mathbb{R}^n$ and applies two algebraic tests to each one of them. These tests are essentially two symmetry tests having to do with the symmetry properties of the Riemann curvature, and those groups that survive both tests can be holonomy groups.

Seven years later, Simons [Sim62] offered a new proof, still quite algebraic in nature. He showed that if the holonomy group acts irreducibly and in a nonsymmetric manner, then it must act transitively on the unit sphere of $T_xM$. The transitive actions on the sphere had already been classified by Montgomery and Samelson [MS43] and Borel [Bor49], and they are the ones above listed.

Relatively recently, Olmos [Olm05] found a different proof of Simons's theorem, this one geometric in flavor, which relies heavily on the Riemannian theory of submanifolds of Euclidean space. This is the proof we will be following. The proof has two main ingredients: if $G \subseteq \mathrm{SO}(n)$ is a compact connected Lie subgroup acting irreducibly, then

1. for all $g \in G$ and $x \in \mathbb{R}^n$ nonzero, there is a smooth curve from $x$ to $gx$ in the orbit $Gx$ such that the action of $g$ on the normal space to the orbit at $x$ can be realized as the normal parallel transport along the curve;

2. when the action is moreover not transitive on the unit sphere, for every nonzero $x \in \mathbb{R}^n$ there is some vector $\xi$ which is normal to the orbit at $x$ but not a multiple of $x$ such that the normal spaces to the orbits through the points of the curve $x + t\xi$, for $t \in \mathbb{R}$, span all of $\mathbb{R}^n$.

From these two propositions it follows that if the holonomy acts irreducibly on $T_xM$ but not transitively, then it must act symmetrically, meaning that it leaves the Riemann curvature invariant. From this one can deduce that $M$ must be locally symmetric.

The second part of the thesis focuses on the holonomy of Lie algebroids. Lie algebroids are, in a way, a generalization of the tangent bundle of a manifold $M$. They are vector bundles $A \to M$ with two structures that together make it behave like $TM$: a Lie bracket on its space of sections and a way to take derivatives of smooth functions of $M$ in the directions of $A$, i.e., a bundle map $\rho : A \to TM$, called the anchor. Both objects are related by a Leibniz rule. The way to think about Lie algebroids is as a version of the tangent bundle of $M$ tailored for particular geometrical applications. Examples of this are regular foliations, that is, involutive subbundles $F \subseteq TM$, where the "tailored tangent bundle" to look at is $F$; Poisson geometry, where the "tailored tangent bundle" is the cotangent bundle $T^*M$; or manifolds with boundary, where the "tailored vector fields" are those tangent to the boundary.

Every construction on $TM$ using vector fields as derivations and their Lie brackets can be generalized to Lie algebroids, like differential forms and the de Rham differential, or connections, parallel transport and holonomy. Of course, a Lie algebroid connection, or $A$-connection, on a vector bundle $E \to M$ will be a consistent way of taking directional derivatives of sections of $E$ along directions given by $A$. To define parallel transport we already run into a problem: in the classical case, a section $\sigma$ of $E$ along a smooth curve $\gamma : [0,1] \to M$ was said to be parallel if its derivative in the direction of $\dot\gamma(t)$ vanished for all $t$. But following the "tailored tangent bundle" principle, we should substitute $\dot\gamma$, which is a section of $TM$ along $\gamma$, by the "tailored velocity" of $\gamma$, a section of $A$ along $\gamma$. These are called $A$-paths, and it is along them that we can parallel transport. As in the classical case, this leads to the notion of Lie algebroid holonomy: linear automorphisms of the fiber $E_x$ which are parallel transport along $A$-paths whose base paths are closed loops at $x$.

The Lie algebroid holonomy presents some remarkable new features, when compared with the classical holonomy. First of all, the Ambrose–Singer theorem does not hold anymore. This was proven by Fernandes [Fer02], and here we give a new proof of this fact. For Lie algebroid holonomy, the curvature endomorphisms do not span the holonomy algebra, but we have to add new terms coming from the fact that the anchor might not be injective. The classical Ambrose–Singer theorem gives that a flat connection (one whose curvature vanishes identically) must have a discrete holonomy group. The Ambrose–Singer theorem for Lie algebroids gives instead that flat connections can still have non-discrete holonomy, if the anchor is not injective. We give explicit original examples of such behavior.

On the other hand, Lie algebroid holonomy is a leafwise object. Any Lie algebroid comes with an involutive (possibly singular) distribution on the base manifold: the image of the anchor. This integrates to a (possibly singular) foliation on $M$. The smooth curves on $M$ that can be lifted to $A$-paths, i.e., those having a "tailored velocity" in $A$, must stay in a single leaf, and so the Lie algebroid holonomy only sees what is happening at the leaf level. Hence, the holonomy can jump from leaf to leaf. We also give original examples of such behavior.

The thesis is organized as follows. Chapter 1 is devoted to introducing the basic language of connections and holonomy that we will be using throughout the entire text. We start with the general notion of connections, parallel transport and holonomy on vector bundles, as well as curvature. We also prove the Ambrose–Singer theorem. Finally, we particularize to the case of the tangent bundle and consider Riemannian geometry: we introduce torsion, the Levi-Civita connection, the Riemann and related curvatures and geodesics.

Chapter 2 goes into the Riemannian theory of submanifolds that we will need for Olmos's proof of Simons's theorem. Whereas Section 2.1 is absolutely fundamental to follow the proof, Section 2.2 is only necessary for the first step in Olmos's proof (Proposition 3.47). If willing to take some details of the proof of Proposition 3.47 in faith, Section 2.2 can be skipped in a first reading.

In Chapter 3 we finally give the proof of Berger's theorem and study some of its consequences. In Section 3.1 we start by describing reducible spaces and what their holonomy looks like, including the de Rham decomposition theorem. Then we pass on to symmetric spaces, in Section 3.2: we establish some of their basic geometric properties and study their underlying Lie theoretic nature, and from here we conclude what their holonomy is. Lastly, Section 3.2.3 contains again some details for one of the main ingredients for the proof of Proposition 3.47. The other main ingredient in such proof is studied in Section 3.3: the normal holonomy theorem. This is the analog of the pointwise de Rham decomposition theorem for the normal holonomy of a submanifold. Finally, in Section 3.4 we prove Simons's theorem using the previous machinery. In Section 3.5 we deduce Berger's theorem about Riemannian holonomy from Simons's holonomy theorem. Using the classification of transitive actions on the sphere, we recast Berger's theorem in its original fashion: as Berger's list of possible holonomy groups for a Riemannian manifold. As it has already been stated, different holonomy groups give different geometric properties to the manifold. These special geometries (Kähler, Calabi–Yau, hyperkähler and quaternionic Kähler) are explored in Section 3.6 and we give some examples.

In Chapter 4 we turn to Lie algebroid connections and holonomy. We first give the basic definitions, examples and properties of Lie algebroids, including the induced singular foliation. In Section 4.2 we introduce Lie algebroid connections, parallel transport and holonomy. We give a new proof of the Ambrose–Singer-Fernandes theorem, by following the "tailored tangent bundle" principle, adapting the proof of the classical Ambrose–Singer theorem to the Lie algebroid case, replacing $TM$ by $A$. We finally give original examples of flat Lie algebroid connections having non-discrete holonomy and of holonomy jumps between leaves.

In order to keep the flow of the text, we have included in Appendix A the proofs of some results formulated or used in the main body which require uninteresting computations. Also, although in the main text we focus on the view of connections as covariant derivatives, for completeness we have added an introduction to linear Ehresmann connections in Appendix B.

1

# Holonomy

On the trivial rank $r$ vector bundle $E = M \times \mathbb{R}^r$ over a smooth manifold $M$. Sections of $E$ can be identified with $C^\infty(M, \mathbb{R}^r)$. On such a bundle there is a canonical notion of what it means to take the derivative of a section $\sigma \in C^\infty(M, \mathbb{R}^r)$ in the direction of $v \in T_x M$, it is just $\sigma_* v$. This expression is $\mathbb{R}$-linear on $v$ and satisfies the following Leibniz rule in $\sigma$: if $f \in C^\infty(M)$, then

$$(f\sigma)_* v = f(x)\sigma_* v + (vf)\sigma(x).$$

The "constant" or "parallel" sections of the bundle are those whose directional derivatives in all directions vanish. In this case, they are the constant maps.

On a general vector bundle $E \to M$ over a smooth manifold $M$ there is no canonical way of taking directional derivatives. A choice of such directional derivatives is what we call a connection on $E$. The name connection comes from the fact that it allows us to compare different fibers of $E$ through the concept of parallel transport.

In this chapter we review the basic concepts of connection and curvature, we introduce parallel transport and holonomy, we state and prove the Ambrose–Singer theorem and finally we particularize all these constructions to the tangent bundle.

## 1.1. Connections

For a vector bundle $E \to M$, we write $\Omega^k(M, E)$ for the space of $E$-valued $k$-differential forms:

$$\Omega^k(M, E) := \Gamma(\Lambda^k T^* M \otimes E).$$

**Definition 1.1.** A **connection** on a vector bundle $E \to M$ is an $\mathbb{R}$-linear operator $\nabla : \Gamma(E) \to \Omega^1(M, E)$ satisfying the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \quad \text{for } f \in C^\infty(M) \text{ and } \sigma \in \Gamma(E).$$

We denote $\nabla\sigma(u)$ by $\nabla_u\sigma$, for $u \in TM$. A section $\sigma \in \Gamma(E)$ is called **parallel** if $\nabla\sigma = 0$. ♦

Every vector bundle admits a connection, for instance by taking the canonical connection described above on trivializing charts and gluing them together with a partition of unity, see for instance [Tu17, Thm. 10.6]. Also, as a consequence of the Leibniz rule we have that if $\nabla$ and $\nabla'$ are two connections on $E$, then $\nabla - \nabla' : \Gamma(E) \to \Omega^1(M, E)$ is $C^\infty(M)$-linear, so that actually $\nabla - \nabla' \in \Omega^1(M, \operatorname{End} E)$.

A connection on $E$ immediately induces a connection on $E^*$, defined by

$$\nabla_u\lambda(\sigma) := u(\lambda(\sigma)) - \lambda(\nabla_u\sigma), \quad \text{for } \lambda \in \Gamma(E^*),\ \sigma \in \Gamma(E) \text{ and } u \in TM.$$

On the other hand, if $E' \to M$ is another vector bundle with connection $\nabla'$, then both $\nabla$ and $\nabla'$ induce a connection $\overline{\nabla}$ on $E \otimes E'$ given by

$$\overline{\nabla}_u(\sigma \otimes \sigma') := \nabla_u\sigma \otimes \sigma' + \sigma \otimes \nabla'_u\sigma', \quad \text{for } \sigma \in \Gamma(E),\ \sigma' \in \Gamma(E') \text{ and } u \in TM.$$

In particular, they induce a connection $\nabla'$ on every tensor product $E^{\otimes k} \otimes E^{*\otimes l} \otimes E'$: if $T \in \Gamma(E^{\otimes k} \otimes E^{*\otimes l} \otimes E')$, then for $\lambda^i \in \Gamma(E^*)$, $\sigma_i \in \Gamma(E)$ and $u \in TM$, it is given by

$$\begin{aligned}\nabla'_uT(\lambda^1, \ldots, \lambda^k, \sigma_1, \ldots, \sigma_l) = \nabla'_u(T(\lambda^1, \ldots, \lambda^k, \sigma_1, \ldots, \sigma_l))& \\ - \sum_{i=1}^{k} T(\lambda^1, \ldots, \nabla_u\lambda^i, \ldots, \lambda^k, \sigma_1, \ldots, \sigma_l)& \\ - \sum_{i=1}^{l} T(\lambda^1, \ldots, \lambda^k, \sigma_1, \ldots, \nabla_u\sigma_i, \ldots, \sigma_l).&\end{aligned} \tag{1.1}$$

This seems to potentially introduce some ambiguity, since $T$ can be considered as a section of $E^{\otimes k} \otimes E^{*\otimes l} \otimes E'$ or as a section of $E^{\otimes p} \otimes E^{*\otimes q} \otimes (E^{\otimes(k-p)} \otimes E^{*\otimes(l-q)} \otimes E')$, for $0 \le p \le k$ and $0 \le q \le l$, which have two a priori different connections. It is a routine exercise to check that they actually agree, so that $\nabla'_uT$ is perfectly well defined.

A connection can be extended to higher degree forms in a unique manner, by imposing that the Leibniz rule be satisfied.

**Definition 1.2.** On a vector bundle $E \to M$ with a connection $\nabla$, the **covariant differential** is the unique $\mathbb{R}$-linear operator $D : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ satisfying the Leibniz rule

$$D(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^k\alpha \wedge \nabla\sigma, \quad \text{for } \alpha \in \Omega^k(M) \text{ and } \sigma \in \Gamma(E).$$ ♦

Explicitly, it is given by a Koszul-type formula: if $\alpha \in \Omega^k(M, E)$ and $X_i \in \mathfrak{X}(M)$, then

$$\begin{aligned}D\alpha(X_0, \ldots, X_k) = \sum_i (-1)^i \nabla_{X_i}(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k))& \\ + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).&\end{aligned}$$

The space $\Omega^\bullet(M, E)$ has several module structures, and it is interesting to know how these structures behave under the covariant differential. First of all, the wedge product of forms induces a graded $\Omega^\bullet(M)$-module structure on $\Omega^\bullet(M, E)$, by defining a wedge product as

$$\begin{array}{ccc}\Omega^k(M) \times \Omega^l(M, E) & \longrightarrow & \Omega^{k+l}(M, E) \\ (\alpha, \beta \otimes \sigma) & \longmapsto & (\alpha \wedge \beta) \otimes \sigma.\end{array}$$

Also, the composition of endomorphisms turns $\Omega^\bullet(M,\operatorname{End}E)$ into a graded ring, by defining

$$\begin{aligned}\Omega^k(M,\operatorname{End}E)\times\Omega^l(M,\operatorname{End}E) &\longrightarrow \Omega^{k+l}(M,\operatorname{End}E)\\ (\alpha\otimes A,\beta\otimes B) &\longmapsto (\alpha\wedge\beta)\otimes AB,\end{aligned}$$

and this in turn induces a $\Omega^\bullet(M,\operatorname{End}E)$-module structure as well on $\Omega^\bullet(M,E)$ by

$$\begin{aligned}\Omega^k(M,\operatorname{End}E)\times\Omega^l(M,E) &\longrightarrow \Omega^{k+l}(M,E)\\ (\alpha\otimes A,\beta\otimes \sigma) &\longmapsto (\alpha\wedge\beta)\otimes A\sigma.\end{aligned}$$

**Lemma 1.3.** *The covariant differential is a degree 1 derivation of both the $\Omega^\bullet(M)$-module and the $\Omega^\bullet(M,\operatorname{End}E)$-module structures on $\Omega^\bullet(M,E)$, by which we mean that for all $\alpha\in\Omega^k(M)$, $A\in\Omega^l(M,\operatorname{End}E)$ and $\beta\in\Omega^m(M,E)$ we have that the following Leibniz rules hold:*

$$D(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^k\alpha\wedge D\beta \quad \textit{and} \quad D(A\wedge\beta)=DA\wedge\beta+(-1)^lA\wedge D\beta.$$

*Proof.* We will prove the results for decomposable forms. To that effect, let $\alpha\in\Omega^k(M)$, $\beta\in\Omega^m(M)$ and $\sigma\in\Gamma(E)$. Then

$$\begin{aligned}D(\alpha\wedge(\beta\otimes\sigma)) &= D((\alpha\wedge\beta)\otimes\sigma)=d(\alpha\wedge\beta)\otimes\sigma+(-1)^{k+m}\alpha\wedge\beta\wedge\nabla\sigma\\ &= d\alpha\wedge(\beta\otimes\sigma)+(-1)^k\alpha\wedge(d\beta\otimes\sigma+(-1)^m\beta\wedge\nabla\sigma)\\ &= d\alpha\wedge(\beta\otimes\sigma)+(-1)^k\alpha\wedge D(\beta\otimes\sigma).\end{aligned}$$

For the second Leibniz rule, we first prove it for $A\in\Gamma(\operatorname{End}E)$ and $\sigma\in\Gamma(E)$. By the definition of the induced connection on $\operatorname{End}E$, we have that for all $X\in\mathfrak{X}(M)$,

$$\nabla_XA(\sigma)=\nabla_X(A\sigma)-A(\nabla_X\sigma).$$

This exactly means that $DA\wedge\sigma=D(A\wedge\sigma)-A\wedge D\sigma$, as wanted. Let now $\alpha\in\Omega^l(M)$ and $\beta\in\Omega^m(M)$. Then

$$\begin{aligned}D((\alpha\otimes A)\wedge(\beta\otimes\sigma)) &= D((\alpha\wedge\beta)\otimes A\sigma)=d(\alpha\wedge\beta)\otimes A\sigma+(-1)^{l+m}\alpha\wedge\beta\wedge D(A\wedge\sigma)\\ &= (d\alpha\wedge\beta)\otimes A\sigma+(-1)^l(\alpha\wedge d\beta)\otimes A\sigma\\ &\qquad +(-1)^{m+l}\alpha\wedge\beta\wedge(DA\wedge\sigma+A\wedge D\sigma)\\ &= (d\alpha\otimes A+(-1)^l\alpha\wedge DA)\wedge(\beta\otimes\sigma)\\ &\qquad +(-1)^l(\alpha\otimes A)\wedge(d\beta\otimes\sigma+(-1)^m\beta\wedge\nabla\sigma)\\ &= D(\alpha\otimes A)\wedge(\beta\otimes\sigma)+(-1)^l(\alpha\otimes A)\wedge D(\beta\otimes\sigma),\end{aligned}$$

and this ends the proof. □

**Definition 1.4.** The **curvature** $F$ of a connection $\nabla$ on $E$ is the $\mathbb{R}$-linear operator $F:\mathfrak{X}^2(M)\to\Gamma(\operatorname{End}E)$ given by

$$F(X,Y)\sigma=\nabla_X\nabla_Y\sigma-\nabla_Y\nabla_X\sigma-\nabla_{[X,Y]}\sigma, \quad \text{for } X,Y\in\mathfrak{X}(M) \text{ and } \sigma\in\Gamma(E).$$

We say that $(E,\nabla)$ is **flat** if $F=0$. ♦

*Remark* 1.5. It is straightforward to see, using the Leibniz rule for $\nabla$, that actually $F$ is $C^\infty(M)$-linear, so that $F\in\Omega^2(M,\operatorname{End}E)$.

One way to think of $F$ is as the obstruction of $D$ to square to zero, as the following shows.

**Proposition 1.6.** *For all $\alpha\in\Omega^\bullet(M,E)$ we have that $D^2\alpha=F\wedge\alpha$.*

*Proof.* Let $\sigma \in \Gamma(E)$. Then the Koszul formula gives that, if $X, Y \in \mathfrak{X}(M)$,

$$D^2\sigma(X,Y) = \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma = F(X,Y)\sigma = (F \wedge \sigma)(X,Y).$$

If now $\alpha \in \Omega^k(M)$, then by the Leibniz rule and because $F$ is a 2-form,

$$\begin{aligned} D^2(\alpha \otimes \sigma) &= D(d\alpha \otimes \sigma + (-1)^k \alpha \wedge \nabla\sigma) \\ &= d^2\alpha \otimes \sigma + (-1)^{k+1} d\alpha \wedge \nabla\sigma + (-1)^k d\alpha \wedge \nabla\sigma + \alpha \wedge D^2\sigma \\ &= \alpha \wedge F \wedge \sigma = F \wedge (\alpha \otimes \sigma), \end{aligned}$$

as wanted. □

An interesting property of $F$ is that it is always $D$-closed.

**Proposition 1.7** (Second Bianchi identity)**.** $DF = 0$.

*Proof.* By Proposition 1.6 and the Leibniz rule in Lemma 1.3, for any $\sigma \in \Gamma(E)$ we have that

$$D^3\sigma = D(F \wedge \sigma) = DF \wedge \sigma + F \wedge \nabla\sigma = DF \wedge \sigma + D^3\sigma,$$

from where we get $DF = 0$. □

Connections which are compatible with additional structures on $E$ present special features. The ones we will be using are metric connections.

**Definition 1.8.** A **metric** $\langle\cdot,\cdot\rangle$ on the bundle $E$ is a section of the symmetric product $S^2E^*$ which is fiberwise nondegenerate, meaning that for every $x \in M$, if $v \in E_x$ is such that $\langle v, w\rangle = 0$ for all $w \in E_x$, then $v = 0$. If it is not only fiberwise nondegenerate but fiberwise positive-definite, meaning that $\langle v, v\rangle > 0$ for all nonzero $v \in E_x$, then we call it a **positive metric**.

A connection $\nabla$ on $E$ is **metric** (or compatible with the metric) if

$$X\langle\sigma,\nu\rangle = \langle\nabla_X\sigma,\nu\rangle + \langle\sigma,\nabla_X\nu\rangle, \quad \text{for all } \sigma,\nu \in \Gamma(E) \text{ and } X \in \mathfrak{X}(M).$$ ♦

**Proposition 1.9.** *Let $\nabla$ be a metric connection on $E$. Then the curvature is skew-symmetric with respect to the metric, by which we mean that $\langle F \wedge \sigma, \nu\rangle + \langle\sigma, F \wedge \nu\rangle = 0$, for all $\sigma,\nu \in \Gamma(E)$. If $\mathfrak{so}(E)$ is the subbundle of* End $E$ *given by skew-symmetric endomorphisms, then $F \in \Omega^2(M, \mathfrak{so}(E))$.*

*Proof.* Direct computation: for all $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned} \langle F(X,Y)\sigma,\nu\rangle &= \langle\nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma,\nu\rangle \\ &= -\langle\nabla_Y\sigma,\nabla_X\nu\rangle + \langle\nabla_X\sigma,\nabla_Y\nu\rangle + \langle\sigma,\nabla_{[X,Y]}\nu\rangle \\ &\qquad + X\langle\nabla_Y\sigma,\nu\rangle - Y\langle\nabla_X\sigma,\nu\rangle - [X,Y]\langle\sigma,\nu\rangle \\ &= \langle\sigma, F(Y,X)\nu\rangle + X\langle\nabla_Y\sigma,\nu\rangle - Y\langle\nabla_X\sigma,\nu\rangle - [X,Y]\langle\sigma,\nu\rangle \\ &\qquad - Y\langle\sigma,\nabla_X\nu\rangle + X\langle\sigma,\nabla_Y\nu\rangle \\ &= \langle\sigma, F(Y,X)\nu\rangle + (XY - YX - [X,Y])\langle\sigma,\nu\rangle \\ &= -\langle\sigma, F(X,Y)\nu\rangle \end{aligned}$$ □

A metric on $E$ induces an isomorphism $E \cong E^*$ by sending $v \in E$ to $\langle v,\cdot\rangle \in E^*$. This induces a metric on $E^*$ and, hence, on every tensor product $E^{\otimes k} \otimes E^{*\otimes l}$ by

$$\langle v_1 \otimes \cdots \otimes v_k \otimes \xi^1 \otimes \cdots \otimes \xi^l, w_1 \otimes \cdots \otimes w_k \otimes \zeta^1 \otimes \cdots \otimes \zeta^l\rangle := \langle v_1, w_1\rangle \ldots \langle v_k, w_k\rangle\langle\xi^1,\zeta^1\rangle \ldots \langle\xi^l,\zeta^l\rangle.$$

If a connection $\nabla$ on $E$ is metric, then the induced connections on tensor products and duals are also metric.

One last construction that we will use is the induced connection on a pullback bundle.

**Lemma 1.10.** *Let $E \to M$ be a vector bundle with a connection $\nabla$, and $\phi : N \to M$ a smooth map from a smooth manifold $N$. Then $\phi^*E$ has a connection $\phi^*\nabla$, the* ***pullback connection****, given by*

$$(\phi^*\nabla)(\phi^*\sigma) := \phi^*(\nabla\sigma), \quad \text{for } \sigma \in \Gamma(E).$$

*Proof.* We need only check that it is well defined by checking the Leibniz rule for a section $\phi^*(f\sigma)$, for $f \in C^\infty(M)$. Since $\phi^*$ commutes with the differential,

$$(\phi^*\nabla)(\phi^*(f\sigma)) = \phi^*(\nabla(f\sigma)) = \phi^*(df \otimes \sigma + f\nabla\sigma) = d\phi^* f \otimes \phi^*\sigma + \phi^* f \phi^*(\nabla\sigma). \qquad \square$$

Explicitly,

$$(\phi^*\nabla)_u(\phi^*\sigma) = \nabla_{\phi_* u}\sigma, \quad \text{for } u \in TM.$$

Of course, not every section of $\phi^*E$ can be written as $\phi^*\sigma$ for some $\sigma \in \Gamma(E)$, but they can all be written as finite $C^\infty(M)$-linear combinations of such pullback sections.

Not surprisingly, the curvature of the pullback connection is the pullback of the curvature.

**Lemma 1.11.** *Let $E$ be a vector bundle with a connection $\nabla$ and curvature $F$, and $\phi : N \to M$ a smooth map from a smooth manifold $N$. Then the curvature of $\phi^*\nabla$ is $\phi^*F \in \Omega^2(N, \operatorname{End}\phi^*E)$.*

*Proof.* Write $\overline{\nabla}$ for $\phi^*\nabla$, $\bar{D}$ for its covariant differential and $\bar{F}$ for its curvature. Then if $\alpha \in \Omega^k(M)$ and $\sigma \in \Gamma(E)$, we have that

$$\begin{aligned}\bar{D}\phi^*(\alpha \otimes \sigma) &= \bar{D}(\phi^*\alpha \otimes \phi^*\sigma) = d\phi^*\alpha \otimes \phi^*\sigma + (-1)^k \phi^*\alpha \wedge \overline{\nabla}(\phi^*\sigma) \\ &= \phi^*(d\alpha \otimes \sigma + (-1)^k \alpha \wedge \nabla\sigma) = \phi^* D(\alpha \otimes \sigma).\end{aligned}$$

Then, by Proposition 1.6, we have that

$$\bar{F} \wedge \phi^*\sigma = \bar{D}^2\phi^*\sigma = \phi^* D^2\sigma = \phi^*(F \wedge \sigma) = \phi^* F \wedge \phi^*\sigma,$$

so $\bar{F} = \phi^*F$. $\square$

As a last comment, there is an alternative viewpoint to connections as horizontal distributions over the total space $E$ of the bundle. Although we will not use it in this work, it is a fundamental viewpoint, and we have decided to include an introduction to it in Appendix B.

## 1.2. Parallel transport and holonomy

One might wonder (rightly) why a connections is called so. The reason is that it allows to compare (connect) different fibers of the bundle $E$. This is done through parallel transport.

Let $E \to M$ be a vector bundle with a connection $\nabla$. Let $\gamma : [0,1] \to M$ be a smooth curve. A section of $E$ along $\gamma$ is just a section of $\gamma^*E$. Explicitly, a section along $\gamma$ is a smooth map $\sigma : [0,1] \to E$ such that $\sigma(t) \in E_{\gamma(t)}$. It is said to be **parallel along** $\gamma$ if $(\gamma^*\nabla)\sigma = 0$. For $(\gamma^*\nabla)\sigma$ we will also use the following notations interchangeably

$$(\gamma^*\nabla)\sigma = \nabla_{\dot\gamma}\sigma = \frac{\nabla}{dt}\sigma = \dot\sigma.$$

If $(x^i)_i$ are local coordinates on $M$ and $\{\sigma_i\}_i$ is a local frame for $E$, then in these coordinates and this frame we can write $\dot\gamma(t) = \dot\gamma^i(t)\frac{\partial}{\partial x^i}$ and $\sigma(t) = \sigma^i(t)\sigma_i(\gamma(t))$, for some smooth functions $\dot\gamma^i, \sigma^i : [0,1] \to \mathbb{R}$. Let $\Gamma^k_{ij}$ be smooth local functions such that $\nabla_{\frac{\partial}{\partial x^i}}\sigma_j = \Gamma^k_{ij}\sigma_k$. Then

$$\begin{aligned}\dot\sigma(t) &= (\gamma^*\nabla)_{\frac{d}{dt}}(\sigma^i\gamma^*\sigma_i) = \dot\sigma^i(t)\sigma_i(\gamma(t)) + \sigma^i(t)\nabla_{\dot\gamma(t)}\sigma_i(\gamma(t)) \\ &= (\dot\sigma^i(t) + \Gamma^i_{jk}(\gamma(t))\sigma^k(t)\dot\gamma^j(t))\sigma_i(\gamma(t)),\end{aligned}$$

so that the equation for $\sigma$ to be parallel is locallly a first order linear ODE. These always have a unique solution defined on the whole interval of definition of the equation. Hence, we have proved the following.

**Lemma 1.12.** *Let $\gamma : [0,1] \to M$ be a smooth curve. Then for every $v \in E_{\gamma(0)}$, there is a unique parallel section $\sigma_v$ along $\gamma$ such that $\sigma_v(0) = v$.* □

This gives the sought way to connect different fibers.

**Definition 1.13.** Let $\gamma : [0,1] \to M$ a smooth curve. Then **parallel transport** along $\gamma$ is the map

$$\begin{array}{rccc} \tau_\gamma : & E_{\gamma(0)} & \longrightarrow & E_{\gamma(1)} \\ & v & \longmapsto & \sigma_v(1), \end{array}$$

where $\sigma_v$ is the unique parallel section along $\gamma$ starting at $v$ given by Lemma 1.12. ♦

Observe that parallel transport can be defined as well over piecewise smooth curves, by doing sequentially parallel transport along the smooth parts of the curve.

Parallel transport has very nice properties.

**Proposition 1.14.** *Let $E$ be a vector bundle with connection. Then*

1. *parallel transport along a smooth curve $\gamma$ in $M$ is a linear isomorphism, with inverse $\tau_{\gamma^{-1}}$, where $\gamma^{-1}(t) := \gamma(1-t)$; ,*

2. *parallel transport is invariant under reparameterization, i.e., if $f : [0,1] \to [0,1]$ is a diffeomorphism with $f(0) = 0$ and $f(1) = 1$, then $\tau_{\gamma\circ f} = \tau_\gamma$,*

3. *if $\gamma$ and $\alpha$ are two composable curves in $M$, then $\tau_{\gamma\cdot\alpha} = \tau_\alpha\tau_\gamma$;*

4. *if the connection is metric, then parallel transport is isometric, i.e., $\langle \tau_\gamma v, \tau_\gamma w\rangle = \langle v, w\rangle$ for all $v, w \in E_{\gamma(0)}$.*

*Proof.* 1 and 2 follow from the fact that if $\sigma$ is parallel along $\gamma$ then $t \mapsto \sigma(1-t)$ is parallel along $\gamma^{-1}$ starting at $\sigma(1)$ and ending at $\sigma(0)$, and $\sigma \circ f$ is parallel along $\gamma \circ f$. On the other hand, if $\nu$ is a parallel section along $\alpha$ starting at $\sigma(1)$, then

$$t \mapsto \begin{cases} \sigma(2t), & 0 \le t \le \frac{1}{2} \\ \nu(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

is a parallel section along $\gamma \cdot \alpha$ starting at $\sigma(0)$ and ending at $\nu(1) = \tau_\alpha(\sigma(1)) = \tau_\alpha\tau_\gamma(\sigma(0))$. This gives 3.

For 4, assume that the connection is metric. Then the pullback connection to $\gamma^*E$ is also metric, so that, if $\sigma$ and $\nu$ are parallel sections along $\gamma$,

$$\frac{d}{dt}\langle \sigma, \nu\rangle = \langle \dot\sigma, \nu\rangle + \langle \sigma, \dot\nu\rangle = 0.$$

Hence, $\langle \sigma(0), \nu(0)\rangle = \langle \sigma(1), \nu(1)\rangle$, and this ends the proof. □

Observe that this proof also gives that if $\{e_i\}_i$ is a (orthonormal) basis for $E_{\gamma(0)}$ then there is a unique (orthonormal) parallel frame $\{\sigma_i\}_i$ along $\gamma$ such that $\sigma_i(0) = e_i$. This procedure can be used to prove a useful formula to compute the action of a connection.

**Proposition 1.15.** *1. Let $x \in M$, $v \in T_xM$ and $\sigma \in \Gamma(E)$. Let $\gamma : [0,1] \to M$ be a smooth curve with $\gamma(0) = x$ and $\dot\gamma(0) = v$, and let $\tau_t$ be parallel transport along $\gamma$ from $x$ to $\gamma(t)$. Then*

$$\nabla_v\sigma = \left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}(\sigma(\gamma(t))).$$

2. *Let $\gamma : [0,1] \to M$ be a piecewise smooth curve and let $\tau_t$ be parallel transport along $\gamma$ from $\gamma(0)$ to $\gamma(t)$. Then for any $\sigma \in \Gamma(\gamma^*E)$ we have that*

$$\dot\sigma(t) = \tau_t \frac{d}{dt}(\tau_t^{-1}\sigma(t)).$$

*Proof.* For 1, let $\{\sigma_i\}_i$ be a parallel frame along $\gamma$. Then parallel transport is given by $\tau_t(\lambda^i\sigma_i(0)) = \lambda^i\sigma_i(t)$. Write $\sigma(\gamma(t)) = \sigma^i(t)\sigma_i(t)$ for some smooth functions $\sigma^i : [0,1] \to \mathbb{R}$. Then,

$$\begin{aligned}\nabla_v\sigma = \frac{\nabla}{dt}\bigg|_{t=0}\sigma(\gamma(t)) = \frac{\nabla}{dt}\bigg|_{t=0}\sigma^i(t)\sigma_i(t) = \dot{\sigma}^i(0)\sigma_i(0)\\ = \frac{d}{dt}\bigg|_{t=0}\sigma^i(t)\sigma_i(0) = \frac{d}{dt}\bigg|_{t=0}\tau_t^{-1}(\sigma(\gamma(t))).\end{aligned}$$

For 2, let again $\{\sigma_i\}_i$ be a parallel frame along $\gamma$ and write $\sigma(t) = \sigma^i(t)\sigma_i(t)$. Then

$$\dot{\sigma}(t) = \dot{\sigma}^i(t)\sigma_i(t) = \tau_t(\dot{\sigma}^i(t)\sigma_i(0)) = \tau_t\frac{d}{dt}(\sigma^i(t)\sigma_i(0)) = \tau_t\frac{d}{dt}(\tau_t^{-1}\sigma(t)).$$ □

Parallel transport along loops at a point plays a very important role in geometry, and it gives rise to one of our main objects of study. Let $\Pi_{x,y}$ denote the set of piecewise smooth curves in $M$ from $x$ to $y$.

**Definition 1.16.** Let $E \to M$ be a vector bundle with a connection $\nabla$. The **holonomy group** of $\nabla$ at $x \in M$ is defined as

$$\operatorname{Hol}_x(\nabla) := \{\tau_\gamma : \gamma \in \Pi_{x,x}\}.$$

The **restricted holonomy group** of $\nabla$ at $x$ is defined as

$$\operatorname{Hol}_x^0(\nabla) := \{\tau_\gamma : \gamma \in \Pi_{x,x} \text{ is null-homotopic}\}.$$ ♦

**Proposition 1.17.** *Let $E \to M$ be a vector bundle with a connection $\nabla$ and $x \in M$. Then $\operatorname{Hol}_x(\nabla)$ is a Lie subgroup of $\mathrm{GL}(E_x)$ whose connected identity component is $\operatorname{Hol}_x^0(\nabla)$. In particular, $\operatorname{Hol}_x^0(\nabla)$ is normal in $\operatorname{Hol}_x(\nabla)$.*

*Proof.* That both $\operatorname{Hol}_x(\nabla)$ and $\operatorname{Hol}_x^0(\nabla)$ are subgroups of $\mathrm{GL}(E_x)$ is a direct consequence of Proposition 1.14. We now show that $\operatorname{Hol}_x^0(\nabla)$ is an arcwise connected subgroup of $\mathrm{GL}(E_x)$, which implies that it is a Lie subgroup [Yam50]. Let $\gamma : [0,1]^2 \to M$ be a smooth homotopy with fixed endpoints starting at the constant path on $x$ (every null-homotopic path is smoothly null-homotopic [Lee12, Thm. 6.29]). By a similar argument as in Lemma 1.12 and using the smooth dependence on initial conditions of ODE theory, for each $v \in E_x$ there is $\sigma \in \Gamma(\gamma^*E)$ such that $\frac{\nabla}{\partial t}\sigma = 0$ and $\sigma(s,0) = v$ for all $s$. Then, if $\tau_s$ is parallel transport along $\gamma_s := \gamma(s,\cdot)$, we have that $\tau_s v = \sigma(s,1)$, which is smooth on $s$. Since $\gamma_0$ is the constant path, then $\sigma(0,t) \in E_x$ does not depend on $t$, and therefore $\tau_0 v = \sigma(0,1) = \sigma(0,0) = v$. We conclude that $\tau_s$ is a smooth path in $\operatorname{Hol}_x^0(\nabla)$ from $\tau_1$ to the identity, as wanted.

Since $\operatorname{Hol}_x^0(\nabla)$ is a subgroup of $\operatorname{Hol}_x(\nabla)$, this also endows $\operatorname{Hol}_x(\nabla)$ with the structure of a Lie group by translating the smooth structure of $\operatorname{Hol}_x^0(\nabla)$ by left or right multiplication.

Consider now the map $\pi_1(M) \to \operatorname{Hol}_x(\nabla)/\operatorname{Hol}_x^0(\nabla)$ given by $[\gamma] \mapsto \tau_\gamma^{-1}\operatorname{Hol}_x^0(\nabla)$. It is easily seen to be a surjective group homomorphism. Since $\pi_1(M)$ is countable [Lee11, Thm. 7.21], then $\operatorname{Hol}_x(\nabla)/\operatorname{Hol}_x^0(\nabla)$ is also countable. Hence, the image of the identity component of $\operatorname{Hol}_x(\nabla)$ by the projection $\operatorname{Hol}_x(\nabla) \to \operatorname{Hol}_x(\nabla)/\operatorname{Hol}_x^0(\nabla)$ is connected and contains $\mathrm{id}\operatorname{Hol}_x^0(\nabla)$, from which we conclude that indeed $\operatorname{Hol}_x^0(\nabla)$ is the identity component of $\operatorname{Hol}_x(\nabla)$. The fact that it is normal follows from the fact that the identity component of a Lie group is always normal. □

Therefore, the following definition makes sense.

**Definition 1.18.** Let $E \to M$ be a vector bundle with a connection $\nabla$. The **holonomy algebra** $\mathfrak{hol}_x(\nabla)$ of $\nabla$ at $x \in M$ is defined as the Lie algebra of $\operatorname{Hol}_x(\nabla)$. ♦

The holonomy group is independent of the base point in the following sense.

**Proposition 1.19.** *Let $E \to M$ be a vector bundle with a connection $\nabla$ and let $x, y \in M$ be connected by a piecewise smooth curve $\gamma$ in $M$. Then*

$$\operatorname{Hol}_x(\nabla) = \tau_\gamma^{-1}\operatorname{Hol}_y(\nabla)\tau_\gamma.$$

*Proof.* If $\alpha$ is a loop at $y$, then $\gamma \cdot \alpha \cdot \gamma^{-1}$ is a loop at $x$, and all loops at $x$ can be obtained in this way. $\square$

The holonomy group is very intricately related to the geometry of $E$. It is closely related, on the one hand, to parallel sections of the bundle, and, on the other, to the curvature of $E$, as we will see in the next section. A parallel section clearly gives an $\mathrm{Hol}_x(\nabla)$-invariant vector in $E_x$. The correspondence goes the other way around as well: every $\mathrm{Hol}_x(\nabla)$-invariant vector in $E_x$ defines a parallel section, and all parallel sections of $E$ arise in this manner.

**Theorem 1.20** (Holonomy principle)**.** *Let $M$ be connected, $E \to M$ a vector bundle with a connection $\nabla$ and $x \in M$. Then the following vector spaces are isomorphic:*

1. *the space of parallel sections of $E$,*
2. *the space of $\mathrm{Hol}_x(\nabla)$-invariant vectors in $E_x$,*
3. *the space of sections invariant under parallel transport, i.e., sections $\sigma \in \Gamma(E)$ such that $\tau_\gamma(\sigma(\gamma(0))) = \sigma(\gamma(1))$ for all piecewise smooth curves $\gamma$ in $M$.*

*Proof.* We first show the equivalence of 2 and 3. Map a section $\sigma \in \Gamma(E)$ invariant under parallel transport to $\sigma(x)$. This vector is invariant under $\mathrm{Hol}_x(\nabla)$. Indeed, if $\gamma$ is a loop at $x$, then

$$\tau_\gamma(\sigma(x)) = \sigma(\gamma(1)) = \sigma(x).$$

The map is injective: if $y \in M$ and $\gamma$ is a smooth curve from $x$ to $y$, then if $\sigma(x) = 0$ we have that $\sigma(y) = \sigma(\gamma(1)) = \tau_\gamma(\sigma(x)) = 0$, so $\sigma = 0$. The map is also surjective: let $v \in E_x$ be $\mathrm{Hol}_x(\nabla)$-invariant and define $\sigma(y) := \tau_\gamma v$, where $\gamma$ is any smooth curve from $x$ to $y$. Since $v$ is $\mathrm{Hol}_x(\nabla)$-invariant, this section is well defined. It is also smooth, because around $y$ one can take concatenations of $\gamma$ with radial curves from $y$ in some chart, and parallel transport depends smoothly on these curves. Lastly, this $\sigma$ is invariant under parallel transport: if $\gamma$ is any piecewise smooth curve in $M$ and $\alpha$ is a smooth curve from $x$ to $\gamma(0)$, then

$$\sigma(\gamma(1)) = \tau_{\alpha\cdot\gamma} v = \tau_\gamma \tau_\alpha v = \tau_\gamma(\sigma(\gamma(0))).$$

We now show the equivalence of 1 and 3. If $\sigma \in \Gamma(E)$ is invariant under parallel transport, then Proposition 1.15 gives that for any $v \in T_yM$, if $\gamma$ is a smooth curve with $\gamma(0) = y$ and $\dot\gamma(0) = v$, and $\gamma_t : [0,1] \to M$ is defined by $\gamma_t(s) = \gamma(st)$, then

$$\nabla_v \sigma = \left.\frac{d}{dt}\right|_{t=0} \tau_{\gamma_t}^{-1}(\sigma(\gamma(t))) = \left.\frac{d}{dt}\right|_{t=0} \tau_{\gamma_t^{-1}} \sigma(\gamma_t^{-1}(0)) = \left.\frac{d}{dt}\right|_{t=0} \sigma(\gamma_t^{-1}(1)) = \left.\frac{d}{dt}\right|_{t=0} \sigma(x) = 0.$$

Conversely, if $\sigma \in \Gamma(E)$ is parallel and $\gamma$ is a piecewise smooth path in $M$, then, since $\gamma^*\sigma$ is parallel along $\gamma$,

$$\tau_\gamma(\sigma(\gamma(0))) = \sigma(\gamma(1)),$$

so $\sigma$ is invariant under parallel transport. $\square$

**Corollary 1.21.** *Let $M$ be connected, $E \to M$ a vector bundle with a metric connection $\nabla$ and $x \in M$. Then $\mathrm{Hol}_x(\nabla) \subseteq \mathrm{O}(E_x)$. Moreover, if $E$ is orientable, then $\mathrm{Hol}_x(\nabla) \subseteq \mathrm{SO}(E_x)$.*

*Proof.* Write $g$ or $\langle \cdot, \cdot \rangle$ for the metric on $E$. Then for every $\sigma, \nu \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$,

$$\nabla_X g(\sigma, \nu) = X\langle \sigma, \nu \rangle - \langle \nabla_X \sigma, \nu \rangle - \langle \sigma, \nabla_X \nu \rangle = 0.$$

Then, by the holonomy principle, $g_x$ is $\mathrm{Hol}_x(\nabla)$-invariant, meaning that for all $\gamma \in \Pi_{x,x}$ and $u, v \in E_x$,

$$\tau_\gamma^{-1} g_x(u,v) = \langle \tau_\gamma u, \tau_\gamma v \rangle = g_x(u,v) = \langle u, v \rangle,$$

i.e., $\mathrm{Hol}_x(\nabla) \subseteq \mathrm{O}(E_x)$.

If $E$ is orientable, let $\omega$ be the global frame for $\det E^*$ defined on $y \in M$ by $\omega(e_1, \ldots, e_n) = 1$ for any oriented orthonormal basis $\{e_i\}_i$ of $E_y$. Let $\{\sigma_i\}_i$ be any orthonormal local frame for $E$ and $X \in \mathfrak{X}(M)$. Then

$$\nabla_X \omega(\sigma_1, \ldots, \sigma_n) = X(\omega(\sigma_1, \ldots, \sigma_n)) - \sum_i \omega(\sigma_1, \ldots, \nabla_X \sigma_i, \ldots, \sigma_n).$$

The first term vanishes because $\omega(\sigma_1, \ldots, \sigma_n) = 1$ identically. On the other hand, the only term that survives in the second term is the one corresponding to the component of $\nabla_X \sigma_i$ in the direction of $\sigma_i$, which is

$$\langle \nabla_X \sigma_i, \sigma_i \rangle = \frac{1}{2} X \langle \sigma_i, \sigma_i \rangle = 0,$$

since $\langle \sigma_i, \sigma_i \rangle = 1$ identically. Hence, by the holonomy principle, $\omega_x$ is $\mathrm{Hol}_x(\nabla)$-invariant, which gives that $\mathrm{Hol}_x(\nabla) \subseteq \mathrm{SO}(E_x)$. □

## 1.3. Ambrose–Singer theorem

We will now explore the relation between holonomy and curvature. Simply put: curvature determines the holonomy. This is the celebrated Ambrose–Singer theorem, to which we now turn. In this exposition we follow [Bal02], which is elementary, avoiding the use of any integrability theorems. Let $E \to M$ be a vector bundle with connection $\nabla$ and let $\gamma : [0,1]^2 \to M$ is a smooth map. If we consider coordinates $(s,t)$ on $[0,1]^2$, then for $\sigma \in \Gamma(\gamma^* E)$ we use the notations

$$\frac{\nabla}{\partial s} \sigma := (\gamma^* \nabla)_{\frac{\partial}{\partial s}} \sigma \quad \text{and} \quad \frac{\nabla}{\partial t} \sigma := (\gamma^* \nabla)_{\frac{\partial}{\partial t}} \sigma.$$

**Lemma 1.22.** *Let $\gamma : [0,1]^2 \to M$ be a piecewise smooth homotopy. Let $\tau_{s,t}$ be parallel transport along $\gamma_s := \gamma(s, \cdot)$ from $\gamma_s(t)$ to $\gamma_s(1)$ and let*

$$F_{s,t} := \tau_{s,t} F\left(\frac{\partial}{\partial t}\gamma(s,t), \frac{\partial}{\partial s}\gamma(s,t)\right) \tau_{s,t}^{-1} \in \mathfrak{gl}(E_{\gamma_s(1)}).$$

*Then for any $\sigma \in \Gamma(\gamma^* E)$ with $\frac{\nabla}{\partial t}\sigma = 0$ and $\frac{\nabla}{\partial s}\sigma(\cdot, 0) = 0$ we have that*

$$\frac{\nabla}{\partial s}\sigma(s,1) = \left(\int_0^1 F_{s,t} dt\right) \sigma(s,1).$$

*Proof.* To make things clearer, Figure 1.1 shows a sketch of the situation. Using Proposition 1.15, Lemma 1.11 and the fact that $\frac{\nabla}{\partial t}\sigma = 0$, we compute:

$$\frac{d}{dt}\left(\tau_{s,t} \frac{\nabla}{\partial s}\sigma(s,t)\right) = \tau_{s,t} \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\sigma(s,t) = \tau_{s,t} F\left(\frac{\partial}{\partial t}\gamma(s,t), \frac{\partial}{\partial s}\gamma(s,t)\right)\sigma(s,t) = F_{s,t}\sigma(s,1).$$

Then, since $\tau_{s,1} = \mathrm{id}$ and $\frac{\nabla}{\partial s}\sigma(\cdot, 0) = 0$,

$$\frac{\nabla}{\partial s}\sigma(s,1) = \tau_{s,1}\frac{\nabla}{\partial s}\sigma(s,1) - \tau_{s,0}\frac{\nabla}{\partial s}\sigma(s,0) = \int_0^1 \frac{d}{dt}\left(\tau_{s,t}\frac{\nabla}{\partial s}\sigma(s,t)\right) dt = \left(\int_0^1 F_{s,t} dt\right)\sigma(s,1). \quad \square$$

**Corollary 1.23.** *Let $\gamma : [0,1]^2 \to M$ be a piecewise smooth homotopy with fixed endpoints and let $\tau_s$ be parallel transport along $\gamma_s$. Then*

$$\frac{d}{ds}\tau_s = \left(\int_0^1 F_{s,t} dt\right) \tau_s.$$

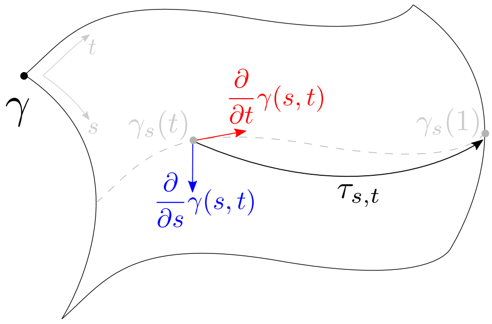


Figure 1.1: Homotopy $\gamma$.

*Proof.* Let $\sigma \in \Gamma(\gamma^* E)$ with $\frac{\nabla}{\partial t}\sigma = 0$ and $\frac{\nabla}{\partial s}\sigma(\cdot, 0) = 0$. Notice that since $\gamma$ has fixed endpoints, the covariant derivative with respect to $s$ at the endpoints is just derivation with respect to $s$. Hence, $\sigma(\cdot, 0)$ is constant and, by Lemma 1.22,

$$\begin{aligned}\frac{\nabla}{\partial s}\sigma(s,1) &= \frac{d}{ds}(\sigma(s,1)) = \frac{d}{ds}(\tau_s\sigma(s,0)) = \left(\frac{d}{ds}\tau_s\right)\sigma(s,0)\\ &= \left(\int_0^1 F_{s,t}dt\right)\sigma(s,1) = \left(\int_0^1 F_{s,t}dt\right)\tau_s\sigma(s,0),\end{aligned}$$

and this gives the result. ☐

We will now see that curvature gives information on the rate of change of parallel transport along "homotopies of square loops". To make this precise, let $z \in M$ and $u, v \in T_zM$. By a homotopy of square loops we mean the following: given a smooth map $f : U \to M$ from an open neighborhood $U$ of 0 in $\mathbb{R}^2$ containing $[0,1]^2$ such that $f(0) = z$, $\frac{\partial f}{\partial x}(0) = u$ and $\frac{\partial f}{\partial y}(0) = v$, we consider the piecewise smooth homotopy with fixed ends $\gamma : [0,1]^2 \to M$ given by

$$\gamma_s(t) = \begin{cases} f(4st, 0), & 0 \le t \le \frac{1}{4},\\ f(s, s(4t-1)), & \frac{1}{4} \le t \le \frac{1}{2},\\ f(s(3-4t), s), & \frac{1}{2} \le t \le \frac{3}{4},\\ f(0, 4s(1-t)), & \frac{3}{4} \le t \le 1. \end{cases} \tag{1.2}$$

A sketch of a homotopy of square loops can be seen in Figure 1.2.

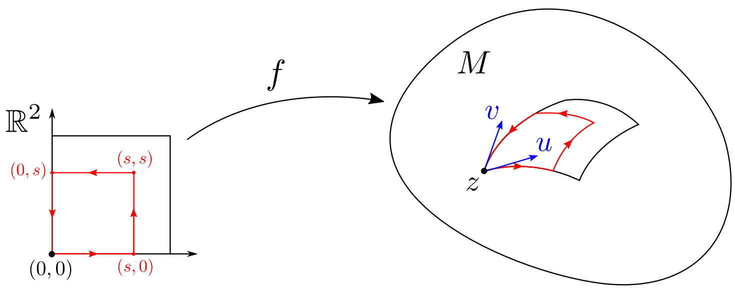


Figure 1.2: Homotopy of square loops based at $z$ in the direction of $u, v \in T_zM$.

**Proposition 1.24.** *Let $\gamma$ be a homotopy of square loops as in (1.2) and let $\tau_s$ be parallel transport along $\gamma_s$. Then*

$$\left.\frac{d}{ds}\right|_{s=0}\tau_s = 0 \quad \textit{and} \quad \left.\frac{d^2}{ds^2}\right|_{s=0}\tau_s = 2F(v,u).$$

*Proof.* Direct computation, using the skew-symmetry of $F$, gives

$$F\left(\frac{\partial}{\partial t}\gamma(s,t), \frac{\partial}{\partial s}\gamma(s,t)\right) = \begin{cases} 0, & t \leq \frac{1}{4} \text{ or } t \geq \frac{3}{4}, \\ 4sF(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})(s, s(4t-1)), & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ 4sF(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})(s(3-4t), s), & \frac{1}{2} \leq t \leq \frac{3}{4}. \end{cases}$$

Hence, by Corollary 1.23,

$$\left.\frac{d}{ds}\right|_{s=0} \tau_s = \left(\int_{1/4}^{3/4} F_{0,t} dt\right) \tau_0 = 0.$$

Also, $\frac{1}{s}F_{s,t} \to 4F(v,u)$ uniformly in $t$ as $s \to 0$ since $\tau_{0,t} = \mathrm{id}$ because $\gamma_0$ is the constant path. Then,

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \tau_s = \left(\int_{1/4}^{3/4} \left.\frac{d}{ds}\right|_{s=0} F_{s,t} dt\right) \tau_0 = 2F(v,u). \qquad \square$$

**Theorem 1.25** (Ambrose–Singer)**.** *Let $x \in M$ and denote by $\Pi_x$ the set of piecewise smooth curves $[0,1] \to M$ starting at $x$. Then*

$$\mathfrak{hol}_x(\nabla) = \mathrm{span}\{\tau_\gamma^{-1} F(u,v)\tau_\gamma : \ \gamma \in \Pi_x \text{ and } u, v \in T_{\gamma(1)}M\}.$$

*Proof.* Write $\mathfrak{g}$ for the right-hand side of the equality. Let $\gamma \in \Pi_x$ and $u, v \in T_{\gamma(1)}M$, and let $\alpha$ be a homotopy of square loops based at $\gamma(1)$ in the direction of $u$ and $v$. Write $\gamma_s := \gamma \cdot \alpha_s \cdot \gamma^{-1}$, which is a contractible loop at $x$ for each $s$. Let $g(s) := \tau_\gamma^{-1}\tau_{\sqrt{s}}\tau_\gamma$, where $\tau_s$ is parallel transport along $\alpha_s$. By Proposition 1.24 we have that $\tau_s = \mathrm{id} + F(v,u)s^2 + o(s^2)$, which implies that $g(s) = \mathrm{id} + \tau_\gamma^{-1}F(v,u)\tau_\gamma s + o(s)$, i.e.,

$$\left.\frac{d}{ds}\right|_{s=0} g(s) = \tau_\gamma^{-1}F(v,u)\tau_\gamma.$$

Hence, $g$ is smooth outside of $s = 0$ and continuously differentiable at $s = 0$, so it is a $C^1$ curve inside $\mathrm{Hol}_x^0(\nabla)$. Therefore, $\mathfrak{g} \subseteq \mathfrak{hol}_x(\nabla)$.

Now we show that $\mathfrak{g}$ is actually an ideal of $\mathfrak{hol}_x(\nabla)$. Indeed, if $t \mapsto \tau_t$ is a smooth curve in $\mathrm{Hol}_x^0(\nabla)$ starting at id with velocity $X \in \mathfrak{hol}_x(\nabla)$, then

$$\left[X, \tau_\gamma^{-1}F(u,v)\tau_\gamma\right] = \left.\frac{d}{dt}\right|_{t=0} \tau_t\tau_\gamma^{-1}F(u,v)\tau_\gamma\tau_t^{-1} \in \mathfrak{g}.$$

In particular, $\mathfrak{g}$ is a Lie subalgebra. Let $G$ be the unique connected Lie subgroup of $\mathrm{Hol}_x^0(\nabla)$ integrating $\mathfrak{g}$ [DK00, Thm. 1.10.3]. Let $\gamma$ be a piecewise smooth homotopy with fixed endpoints starting at the constant path and let $\tau_s$ be parallel transport along $\gamma_s$. Then Corollary 1.23 gives that

$$\frac{d}{ds}\tau_s = \left(\int_0^1 F_{s,t} dt\right) \tau_s.$$

Since the integrand lies in $\mathfrak{g}$ for all $s$ and $t$, the integral lies in $\mathfrak{g}$ for all $s$, and so $\tau_s \in G$ for all $s$. Indeed, if we write $X(s) \in \mathfrak{g}$ for the integral, then $\tau_s$ is a solution to a initial value problem for the time dependent vector field on $G$ given by $(g, s) \mapsto X(s)g$, and the flow of such a vector field always lies in $G$. Hence, $\mathrm{Hol}_x^0(\nabla) \subseteq G$, so $\mathfrak{hol}_x(\nabla) \subseteq \mathfrak{g}$, and this ends the proof. $\square$

With this powerful theorem at hand, we can easily prove that a bundle is flat if and only if it admits a local parallel frame around every point (there are more fundamental proofs of this result, which do not require the Ambrose–Singer theorem, see Corollary B.4).

**Corollary 1.26.** *A vector bundle $E \to M$ with a connection $\nabla$ is flat if and only if there is a parallel local frame around every point in $M$, meaning a frame $\{\sigma_i\}_i$ with $\nabla\sigma_i = 0$.*

*Proof.* If $\{\sigma_i\}_i$ is a local parallel frame, then $F(X,Y)\sigma_i = 0$ for all $X, Y \in \mathfrak{X}(M)$, where $F$ is the curvature of $\nabla$. Hence $E$ is flat. Conversely, if $E$ is flat and $x \in M$, then by the Ambrose–Singer theorem we have that $\mathrm{Hol}^0_x(\nabla) = 1$, which by the holonomy principle means that on a simply connected neighborhood of $x$ there is a parallel frame. □

## 1.4. Connections on the tangent bundle

Let $M$ be a manifold and consider its tangent bundle $TM$. Connections on $TM$ have the special feature that its space of sections is precisely $\mathfrak{X}(M)$. This allows for the following definition.

**Definition 1.27.** Let $\nabla$ be a connection on $TM$. Then its **torsion** is defined as the $\mathbb{R}$-linear operator $T : \mathfrak{X}^2(M) \to \mathfrak{X}(M)$ given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad \text{for } X, Y \in \mathfrak{X}(M). \qquad \blacklozenge$$

*Remark* 1.28. It is straightforward to see that actually $T$ is $C^\infty(M)$-linear, so that $T \in \Omega^2(M, TM)$.

A connection on $TM$ induces a connection on all spaces of tensor fields

$$\mathfrak{T}^{(k,l)}(M) := \Gamma(TM^{\otimes k} \otimes T^*M^{\otimes l})$$

by eq. (1.1).

Let $R$ be the curvature of $\nabla$. Then $R$ can be regarded as a $(1,3)$-tensor field, since $\Omega^2(M, \operatorname{End} TM) \subseteq \mathfrak{T}^{(1,3)}(M)$. It has some very interesting properties.

**Proposition 1.29.** *The following hold, if $X, Y, Z, W \in \mathfrak{X}(M)$ and $\mathfrak{S}$ stands for cyclic permutations in the arguments:*

1. $R(X,Y) = -R(Y,X)$,
2. (First Bianchi identity) $R(X,Y)Z + \mathfrak{S}(X,Y,Z) = T(T(X,Y),Z) + \nabla_X T(Y,Z) + \mathfrak{S}(X,Y,Z)$,
3. (Second Bianchi identity) $\nabla_X R(Y,Z) + R(T(X,Y),Z) + \mathfrak{S}(X,Y,Z) = 0$,
4. *if $\nabla$ is metric, then* $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,
5. *if $\nabla$ is metric and torsion-free (meaning $T = 0$), then* $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

*Proof.* 1 is just the fact that $R$ is a 2-form, whereas 4 is Proposition 1.9. For the first Bianchi identity, we explicitly write

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\ R(Y,Z)X &= \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X, \\ R(Z,X)Y &= \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y, \end{aligned}$$

and sum:

$$\begin{aligned} R(X,Y)Z + \mathfrak{S}(X,Y,Z) &= \nabla_X(T(Y,Z) + [Y,Z]) - \nabla_{[X,Y]}Z + \mathfrak{S}(X,Y,Z) \\ &= T(X,[Y,Z]) + \nabla_X(T(Y,Z)) + \mathfrak{S}(X,Y,Z) \\ &= T(X,[Y,Z]) + \nabla_X T(Y,Z) + T(\nabla_X Y, Z) + T(Y, \nabla_X Z) + \mathfrak{S}(X,Y,Z) \\ &= T(T(X,Y),Z) + \nabla_X T(Y,Z) + \mathfrak{S}(X,Y,Z). \end{aligned}$$

For the second Bianchi identity, recall that the general second Bianchi identity (Proposition 1.7) states that $DR = 0$, where $D$ is the covariant differential. Then the Koszul formula for $D$ gives

$$\begin{aligned} 0 = DR(X,Y,Z)W &= \nabla_X(R(Y,Z))W - R([X,Y],Z)W + \mathfrak{S}(X,Y,Z) \\ &= \nabla_X(R(Y,Z)W) - R(Y,Z)\nabla_X W - R([X,Y],Z) + \mathfrak{S}(X,Y,Z) \\ &= \nabla_X R(Y,Z)W + R(\nabla_X Y, Z)W + R(Y, \nabla_X Z)W - R([X,Y],Z) + \mathfrak{S}(X,Y,Z) \\ &= \nabla_X R(Y,Z)W + R(T(X,Y),Z)W + \mathfrak{S}(X,Y,Z). \end{aligned}$$

Lastly, if $\nabla$ is metric and torsion-free, then, using the first Bianchi identity and 1 and 4 repeatedly,

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= -\langle R(Y,Z)X,W\rangle - \langle R(Z,X)Y,W\rangle = \langle R(Y,Z)W,X\rangle + \langle R(Z,X)W,Y\rangle \\ &= -\langle R(Z,W)Y,X\rangle - \langle R(W,Y)Z,X\rangle - \langle R(X,W)Z,Y\rangle - \langle R(W,Z)X,Y\rangle \\ &= 2\langle R(Z,W)X,Y\rangle + \langle R(W,Y)X,Z\rangle + \langle R(X,W)Y,Z\rangle \\ &= 2\langle R(Z,W)X,Y\rangle - \langle R(Y,X)W,Z\rangle = 2\langle R(Z,W)X,Y\rangle - \langle R(X,Y)Z,W\rangle,\end{aligned}$$

and this gives the result. □

Another special feature of connections on $TM$ is that if $\gamma$ is a smooth curve in $M$, then $\dot\gamma$ is a vector field along $\gamma$, so that it makes sense to consider its acceleration $\ddot\gamma := \nabla_{\dot\gamma}\dot\gamma$.

**Definition 1.30.** A **geodesic** is a curve $\gamma$ in $M$ such that $\ddot\gamma = 0$. ♦

Written in coordinates, the equation of a curve to be geodesic is a second order ODE. Hence, there is always a unique local solution, i.e., for every $x \in M$ and $v \in T_xM$, there is a unique geodesic $\gamma_v : (-\epsilon,\epsilon) \to M$, for some $\epsilon > 0$, such that $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. Even more, let

$$U := \{v \in TM : \gamma_v \text{ is defined up to time } 1\} \subseteq TM.$$

Then $U$ is an open set [Pet16, Lem. 5.2.6] containing the zero section of $TM$, and we define the exponential map $\exp : U \to M$ by $\exp v := \gamma_v(1)$. It is a smooth map, by the smooth dependence of solutions to ODEs on initial parameters [Pet16, Thm. 5.2.3]. Notice that we can write, then, $\gamma_v(t) = \exp(tv)$ for small enough $t$. The restriction of exp to $U \cap T_xM$ is denoted by $\exp_x$.

A connection on $TM$ is said to be **(geodesically) complete** if every geodesic can be defined on the whole real line, i.e., if exp is defined on all of $TM$.

**Definition 1.31.** A **pseudo-Riemannian metric** on a manifold $M$ is a metric on $TM$. A **Riemmanian metric** on $M$ is a positive metric on $TM$. A **(pseudo-)Riemannian manifold** is a pair $(M,g)$, where $M$ is a manifold and $g$ a (pseudo-)Riemannian metric on $M$. A **(local) isometry** of $(M,g)$ is a (local) diffeomorphism $\varphi$ of $M$ such that $\varphi^*g = g$. ♦

An important class of vector fields over Riemannian manifolds, that we will use later on, are those that preserve the metric infinitesimally.

**Definition 1.32.** A **Killing vector field** on a pseudo-Riemannian manifold $(M,g)$ is a vector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X g = 0$, i.e., such that

$$X\langle Y,Z\rangle = \langle [X,Y],Z\rangle + \langle Y,[X,Z]\rangle, \qquad \text{for all } Y,Z \in \mathfrak{X}(M). \quad ♦$$

**Lemma 1.33.** *Let $(M,g)$ be a pseudo-Riemannian manifold and $X \in \mathfrak{X}(M)$. Then $X$ is Killing if and only if its flow $\{\phi_t\}_t$ acts by local isometries, by which we mean that for all $x \in M$ there is a neighborhood $U$ of $x$ in $M$ and $\epsilon > 0$ such that $\phi_t$ is an isometry on $U$ for $t \in (-\epsilon,\epsilon)$.*

*Proof.* Write $M_t := \{x \in M : \text{ the maximal integral curve of } X \text{ through } x \text{ is defined up to time } t\}$. Then $M_t$ is open and $\phi_t : M_t \to M_{-t}$ is a diffeomorphism with inverse $\phi_{-t}$ [Lee12, Thm. 9.12]. Let $x \in M$ and let $U$ be a neighborhood of $x$ in $M$ such that for all $y \in U$ the integral curve of $X$ through $y$ is defined in $(-\epsilon,\epsilon)$ (one can take $U$ to be any relatively compact neighborhood of $x$). Observe, then, that for all $t \in (-\epsilon,\epsilon)$ we have that $U \subseteq M_t$ and we can consider $\phi_t$ as a diffeomorphism from $U$ onto its image.

If the flow acts by local isometries, then

$$(\mathcal{L}_X g)_x = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* g)_x = \left.\frac{d}{dt}\right|_{t=0} g_x = 0.$$

Since this argument can be repeated for any $x \in M$, then $X$ is Killing.

Conversely, if $X$ is Killing, then for all $t \in (-\epsilon, \epsilon)$ and $y \in U$ we have that

$$\frac{d}{dt}(\phi_t^* g)_y = \frac{d}{ds}\bigg|_{s=0} (\phi_t^* \phi_s^* g)_y = (\phi_t^* \mathcal{L}_X g)_y = 0,$$

so $(\phi_t^* g)_y = (\phi_0^* g)_y = g_y$. □

Pseudo-Riemannian manifolds have the special property that there is only one metric and torsion-free connection.

**Proposition 1.34.** *Let $(M, g)$ be a pseudo-Riemannian manifold. There is a unique metric and torsion-free connection on $TM$, called the **Levi-Civita connection**. It is given by the Koszul formula*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle). \tag{1.3}$$

*Proof.* It is easy to see that if a connection on $TM$ is metric and torsion-free, then it must satisfy the Koszul formula, so it is unique. For existence, one can (patiently) check that the Koszul formula can be used to define a metric and torsion-free connection on $TM$. □

The curvature of the Levi-Civita connection is usually called the **Riemann curvature** of $M$, and its holonomy the **Riemannian holonomy** of $M$, denoted by $\mathrm{Hol}_x(M)$. Because it is metric, the Levi-Civita connection interacts nicely with isometries of $M$.

**Lemma 1.35.** *Let $(M, g)$ be a pseudo-Riemannian manifold with Levi-Civita connection $\nabla$ and $\varphi$ an isometry of $M$. Then*

$$\nabla_{\varphi_* X}(\varphi_* Y) = \varphi_*(\nabla_X Y), \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

*In particular,*

$$R(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_*(R(X, Y)Z) \quad \text{and} \quad \nabla_{\varphi_* X} R(\varphi_* Y, \varphi_* Z)\varphi_* W = \varphi_*(\nabla_X R(Y, Z)W),$$

*for all $Z, W \in \mathfrak{X}(M)$.*

*Proof.* We have that $[\varphi_* X, \varphi_* Y] = \varphi_*[X, Y]$, because for any $f \in C^\infty(M)$,

$$\begin{aligned}
[\varphi_* X, \varphi_* Y]f &= \varphi_* X(Y(f \circ \varphi) \circ \varphi^{-1}) - \varphi_* Y(X(f \circ \varphi) \circ \varphi^{-1}) \\
&= XY(f \circ \varphi) \circ \varphi^{-1} - YX(f \circ \varphi) \circ \varphi^{-1} \\
&= \varphi_*[X, Y]f.
\end{aligned}$$

We also have that

$$(\varphi_* X)\langle \varphi_* Y, \varphi_* Z\rangle = (\varphi_* X)(\langle Y, Z\rangle \circ \varphi^{-1}) = X\langle Y, Z\rangle \circ \varphi^{-1}$$

Then the Koszul formula for $\nabla$ gives

$$\langle \nabla_{\varphi_* X} \varphi_* Y, \varphi_* Z\rangle = \langle \nabla_X Y, Z\rangle \circ \varphi^{-1} = \langle \varphi_*(\nabla_X Y), \varphi_* Z\rangle.$$ □

A pseudo-Riemannian manifold is called (geodesically) complete if its Levi-Civita connection is so. The renowned Hopf–Rinow theorem [Pet16, Thm. 5.7.1] gives altervative characterizations of such a fact in the case of a Riemannian manifold. For a piecewise smooth curve $\gamma : [0, 1] \to M$ (here $M$ is Riemannian), we define its length as

$$L(\gamma) := \int_0^1 \|\dot{\gamma}(t)\| dt.$$

Then the distance from $x$ to $y$ is defined as

$$d(x, y) := \inf\{L(\gamma) : \gamma \in \Pi_{x,y}\}.$$

This actually makes $M$ into a metric space whose topology coincides with the manifold topology of $M$ [Pet16, Thm. 5.3.8]. Then the Hopf–Rinow theorem states that geodesic and metric completeness agree.

**Theorem 1.36** (Hopf–Rinow). *Let $(M, g)$ be a Riemannian manifold endowed with the Levi-Civita connection. Then the following are equivalent:*

1. *$M$ is geodesically complete,*
2. *$\exp_x$ is defined on all of $T_xM$ for some $x \in M$,*
3. *$M$ satisfies the Heine-Borel property, i.e., every closed bounded set is compact,*
4. *$M$ is metrically complete.*

*Moreover, if $M$ is complete, then any two points in $M$ can be joined by a length-minimizing geodesic.*

Infinitesimal variations of geodesics by geodesics satisfy a special equation, Jacobi's equation, which will be useful for us in Chapter 3.

**Definition 1.37.** A **Jacobi field** along a geodesic $\gamma$ in a Riemannian manifold $(M, g)$ is a vector field $J \in \Gamma(\gamma^*TM)$ such that $\ddot{J} + R(J, \dot{\gamma})\dot{\gamma} = 0$, where the covariant derivative is taken with respect to the Levi-Civita connection. ♦

**Proposition 1.38.** *Let $(M, g)$ be a Riemannian manifold and $\gamma$ a geodesic in $M$.*

1. *For every $v, w \in T_{\gamma(0)}M$, there is a unique Jacobi field $J$ along $\gamma$ (at least when $t$ is close enough to $0$) with $J(0) = v$ and $\dot{J}(0) = w$.*
2. *Let $\{\gamma_s\}_{s\in(-\epsilon,\epsilon)}$ be a smooth family of geodesics with $\gamma_0 = \gamma$. Then the vector field $t \mapsto \frac{d}{ds}\big|_{s=0} \gamma_s(t)$ is a Jacobi field along $\gamma$. Moreover, every Jacobi field along $\gamma$ arises in this way.*

*Proof.* Let $\{e_i\}_i$ be a parallel frame for $TM$ along $\gamma$, and write $J(t) = J^i(t)e_i(\gamma(t))$ and $\dot{\gamma}(t) = a^i(t)e_i(\gamma(t))$ for some smooth $J^i, a^j : [0, 1] \to \mathbb{R}$, and $R(e_i, e_j)e_k = R^l_{ijk}e_l$ for some smooth $R^l_{ijk} : [0, 1] \to \mathbb{R}$. Then

$$\ddot{J} + R(J, \dot{\gamma})\dot{\gamma} = (\ddot{J}^i + J^l a^j a^k R^i_{ljk})e_i.$$

Hence, the equation for $J$ to be Jacobi is a second order ODE, so it has a unique local solution whenever $J(0)$ and $\dot{J}(0)$ are fixed.

Let now $\{\gamma_s\}_{s\in(-\epsilon,\epsilon)}$ be a smooth family of geodesics with $\gamma_0 = \gamma$ and let $J(t) := \frac{d}{ds}\big|_{s=0} \gamma_s(t)$. Then, since $\nabla$ is torsion-free,

$$\dot{J}(t) = \frac{\nabla}{dt}\left(\frac{d}{ds}\bigg|_{s=0} \gamma_s(t)\right) = \frac{\nabla}{\partial t}\bigg|_{s=0} \frac{\partial}{\partial s}\gamma_s(t) = \frac{\nabla}{\partial s}\bigg|_{s=0} \dot{\gamma}_s(t),$$

and therefore, since $\gamma_s$ is a geodesic,

$$\ddot{J}(t) = \frac{\nabla}{dt}\left(\frac{\nabla}{\partial s}\bigg|_{s=0} \dot{\gamma}_s(t)\right) = R(\dot{\gamma}(t), J(t))\dot{\gamma}(t).$$

Conversely, let $J$ be a Jacobi field along $\gamma$. If $\gamma$ starts at $x \in M$, let $\alpha$ be the geodesic with $\alpha(0) = x$ and $\dot{\alpha}(0) = J(0)$, and let $X, Y$ be parallel vector fields along $\alpha$ with $X(0) = \dot{\gamma}(0)$ and $Y(0) = \dot{J}(0)$. Define now $\gamma_s(t) := \exp_{\alpha(s)}(t(X(s) + sY(s)))$ for small enough $s$. Then $\gamma_0 = \gamma$ and $\gamma_s$ is a geodesic for all $s$, so that $I(t) := \frac{d}{ds}\big|_{s=0} \gamma_s(t)$ defines a Jacobi field, by the above proven. Moreover, $I(0) = \dot{\alpha}(0) = J(0)$ and

$$\dot{I}(0) = \frac{\nabla}{\partial s}\bigg|_{s=0} (X(s) + sY(s)) = Y(0) = \dot{J}(0),$$

so that actually $I = J$. □

*Remark* 1.39. If $\gamma$ is a geodesic of a pseudo-Riemannian manifold $M$ and $X$ is a Killing vector field, then $\gamma^* X$ is a Jacobi field along $\gamma$, since we can write, if $\{\phi_t\}_t$ is the flow of $X$,

$$X(\gamma(t)) = \left.\frac{d}{ds}\right|_{s=0} \phi_s(\gamma(t)),$$

and $\{\phi_s \circ \gamma\}_s$ is a smooth family of geodesics around $\gamma$ for small enough $s$, by Lemma 1.33.

Back to curvature, one can define simpler curvatures on a pseudo-Riemannian manifold that carry partial information about the Riemann curvature.

**Definition 1.40.** Let $(M, g)$ be a pseudo-Riemannian manifold and $R$ its Riemann curvature. The **sectional curvature** of $M$ is the map $\kappa \in C^\infty(\mathrm{Gr}_2(TM))$ given by

$$\kappa(v, w) := \frac{2\langle R(w, v)v, w\rangle}{\|v \wedge w\|^2} = \frac{\langle R(w, v)v, w\rangle}{\|v\|^2\|w\|^2 - \langle v, w\rangle^2},$$

for $v, w \in T_xM$ linearly independent. Here $\mathrm{Gr}_2(TM)$ refers to the Grassmannian bundle of planes on $TM$.

The **Ricci curvature** of $M$ is the symmetric tensor field $\mathrm{Ric} \in \mathfrak{T}^{(0,2)}(M)$ given by

$$\mathrm{Ric}(v, w) := \mathrm{tr}(u \mapsto R(u, v)w).$$

The **scalar curvature** of $M$ is the smooth map $\mathrm{scal} \in C^\infty(M)$ given by

$$\mathrm{scal}(x) := \mathrm{tr}\,\mathrm{Ric}_x,$$

viewing $\mathrm{Ric} \in \Gamma(\mathrm{End}\,TM)$ using the metric. ♦

If $\{e_i\}_i$ is an orthonormal basis for $T_xM$, then

$$\mathrm{Ric}(v, w) = \sum_i \langle R(e_i, v)w, e_i\rangle, \quad \text{for } v, w \in T_xM,$$

and $\mathrm{scal}(x) = \sum_{i,j} \langle R(e_i, e_j)e_j, e_i\rangle = \sum_{i,j} \kappa(e_i, e_j)$.

While the Ricci and the scalar curvatures carry less information than $R$, the sectional curvature carries exactly the same amount of information.

**Proposition 1.41.** *The sectional curvature determines the Riemann curvature.*

*Proof.* Let $u, v, w, z \in T_xM$. Then an easy computation, using the first Bianchi identity, gives

$$\begin{aligned} &6\langle R(v, z)w, u\rangle = \\ &= \left.\frac{\partial^2}{\partial t \partial s}\right|_{t=s=0} (\langle R(u + tv, w + sz)(w + sz), u + tv\rangle - \langle R(u + tz, w + sv)(w + sv), u + tz\rangle), \end{aligned}$$

and the derivative on the righthand side can be computed using only the sectional curvature. □

# 2

# Riemmanian Geometry of Submanifolds

Olmos's proof of Simons's theorem on holonomy, which will be explained in Section 3.4, relies on the theory of Riemannian submanifolds. Given a Riemannian manifold $(\overline{M}, \overline{g})$, any submanifold $M$ of $\overline{M}$ inherits a metric from $\overline{g}$ and the Levi-Civita connection of $M$ is related to that of $\overline{M}$. One can also consider the normal bundle of $M$, whose fiber at $x \in M$ is composed by all the vectors in $T_x\overline{M}$ which are orthogonal to all of $T_xM$. This bundle also inherits a connection from the Levi-Civita connection on $\overline{M}$. In particular, we will show that the fundamental objects to study submanifolds in Riemannian geometry are the tangential and normal parts of $\overline{\nabla}_X Y$, where $X, Y \in \mathfrak{X}(M)$, which are, respectively, the Levi-Civita connection on $M$ and the second fundamental form of $M$, and the tangential and normal parts of $\overline{\nabla}_X \xi$, where $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}(\overline{M})$ is normal to $M$, which are, respectively, the Weingarten operator and the normal connection on $M$. All these objects are related by the fundamental equations of local submanifold theory, Theorem 2.4. They are fundamental in the sense that every four objects satisfying these relations (or, rather, their simplified version when $\overline{M}$ has constant sectional curvature) give a local isometric immersion into a space of constant sectional curvature, i.e., they completely characterize the submanifold locally [BCO16, Thm. 1.1.2].

We also study the basic properties of submanifolds of constant principal curvatures, in Section 2.2. This material will be needed in Chapter 3, in the proof of Proposition 3.47. If such proof is not to be looked at in detail, Section 2.2 may be safely skipped.

## 2.1. Fundamental equations

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and let $(M, g) \hookrightarrow (\overline{M}, \overline{g})$ be a submanifold with the induced Riemannian metric (by which we mean that $g$ is the pullback of $\overline{g}$ by the inclusion). Denote by $\overline{\nabla}$ the

Levi-Civita connection on $\overline{M}$. We will denote by $TM^\perp$ the **normal bundle** to $TM$, with fibers

$$T_xM^\perp := \{v \in T_x\overline{M} : \langle v, T_xM\rangle = 0\}.$$

Hence we have the orthogonal decomposition $T\overline{M}|_M = TM \oplus TM^\perp$. For $u \in T\overline{M}|_M$ we will denote by $u^\top$ its projection to $TM$ and by $u^\perp$ its projection to $TM^\perp$.

**Lemma 2.1.** *Let $M \hookrightarrow \overline{M}$ be a submanifold. Then the Levi-Civita connection on $M$ is given by*

$$\nabla_X Y = (\overline{\nabla}_X Y)^\top, \quad \textit{for } X, Y \in \mathfrak{X}(M),$$

*where on the right hand side $X$ and $Y$ are taken to be any extensions of $X, Y \in \mathfrak{X}(M)$ to vector fields on $\overline{M}$.*

*Proof.* Let $\overline{X}$ and $\overline{Y}$ be extensions of $X$ and $Y$, respectively. If $f \in C^\infty(M)$ and $\overline{f} \in C^\infty(\overline{M})$ is an extension of $f$, then $df_x = d\overline{f}_x|_{T_xM}$ for $x \in M$, since for any $v \in T_xM$ we can pick a smooth curve $\gamma$ in $M$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$, and

$$df_x(v) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} \overline{f}(\gamma(t)) = d\overline{f}_x(v).$$

Therefore, for any $x \in M$ we have that $Xf(x) = df_x(X(x)) = d\overline{f}_x(\overline{X}(x)) = \overline{X}\overline{f}(x)$, i.e., $\overline{X}\overline{f}$ is an extension of $Xf$, and the value of $Xf$ at $x$ can be computed using any extensions $\overline{X}$ and $\overline{f}$. Then, since $[X, Y] \in \mathfrak{X}(M)$, we have that

$$[X, Y]f(x) = X(Yf)(x) - Y(Xf)(x) = \overline{X}(\overline{Y}\overline{f})(x) - \overline{Y}(\overline{X}\overline{f})(x) = [\overline{X}, \overline{Y}]\overline{f}(x),$$

so that $[\overline{X}, \overline{Y}]$ is an extension of $[X, Y]$, independently of the chosen extensions $\overline{X}$ and $\overline{Y}$. Finally, the Koszul formula (1.3) for $\nabla$ gives, if $Z \in \mathfrak{X}(M)$ and $\overline{Z}$ is an extension,

$$\begin{aligned}
\langle \nabla_X Y, Z\rangle &= \frac{1}{2}(X\langle Y, Z\rangle - Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle) \\
&= \frac{1}{2}(\overline{X}\langle \overline{Y}, \overline{Z}\rangle - \overline{Y}\langle \overline{Z}, \overline{X}\rangle - \overline{Z}\langle \overline{X}, \overline{Y}\rangle - \langle \overline{X}, [\overline{Y}, \overline{Z}]\rangle + \langle \overline{Y}, [\overline{Z}, \overline{X}]\rangle + \langle \overline{Z}, [\overline{X}, \overline{Y}]\rangle) \\
&= \langle \overline{\nabla}_{\overline{X}} \overline{Y}, \overline{Z}\rangle = \langle (\overline{\nabla}_{\overline{X}} \overline{Y})^\top, Z\rangle,
\end{aligned}$$

so the nondegeneracy of the metric on $M$ gives the result. Since all the terms in the Koszul formula are independent of the chosen extensions, the formula for the Levi-Civita connection of $M$ does not depend on them either. □

The fundamental elements for studying the geometry of Riemannian submanifolds are the second fundamental form, the normal connection and the Weingarten or shape operator, which we now introduce. We let $\mathfrak{X}^\perp(M) := \Gamma(TM^\perp)$ stand for the space of sections of the normal bundle, called normal vector fields on $M$.

**Definition 2.2.** Let $M \hookrightarrow \overline{M}$ be a submanifold. The **second fundamental form** of $M$ is defined as the map $\mathrm{II} : \mathfrak{X}(M)^2 \to \mathfrak{X}^\perp(M)$ given by

$$\mathrm{II}(X, Y) := (\overline{\nabla}_X Y)^\perp, \quad \text{for } X, Y \in \mathfrak{X}(M).$$

We define the **normal connection** $\nabla^\perp : \mathfrak{X}(M) \times \mathfrak{X}^\perp(M) \to \mathfrak{X}^\perp(M)$ on $TM^\perp$ as

$$\nabla^\perp_X \xi := (\overline{\nabla}_X \xi)^\perp, \quad \text{for } X \in \mathfrak{X}(M) \text{ and } \xi \in \mathfrak{X}^\perp(M).$$

Finally, we define the **Weingarten operator** on the direction of $\xi \in \mathfrak{X}^\perp(M)$ as the map $W_\xi : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$W_\xi X := -(\overline{\nabla}_X \xi)^\top, \quad \text{for } X \in \mathfrak{X}(M).$$ ♦

Hence we can write, for $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^{\perp}(M)$,

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X,Y) \quad \text{and} \quad \overline{\nabla}_X \xi = \nabla^{\perp}_X \xi - W_\xi X.$$

The first equation is called **Gauss's formula** and the second **Weingarten's equation**.

**Lemma 2.3.** *1. The second fundamental form, the normal connection and the Weingarten operator are all independent of the chosen extensions of the vector fields involved.*

2. *The normal connection defines a connection on the bundle $TM^{\perp}$.*

3. II *and* $W$ *are* $C^\infty(M)$*-linear, and moreover* II *is symmetric. Hence,* II *defines a section of* $S^2T^*M \otimes TM^{\perp}$ *and* $W$ *a section of* $T^*M^{\perp} \otimes T^*M \otimes TM$.

4. *For all* $X, Y \in \mathfrak{X}(M)$ *and* $\xi \in \mathfrak{X}^{\perp}(M)$ *we have that*

$$\langle \mathrm{II}(X,Y), \xi \rangle = \langle W_\xi X, Y \rangle.$$

5. *For all* $\xi \in \mathfrak{X}^{\perp}(M)$*, the operator* $W_\xi$ *is self-adjoint.*

*Proof.* 1 follows as in the proof of Lemma 2.1. Since they are independent of extensions, we will denote the extensions by the same name as the objects on $M$. For 2, let $X \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^{\perp}(M)$. Clearly $\nabla^{\perp}_X \xi$ is $C^\infty(M)$-linear on $X$, because $\overline{\nabla}_X \xi$ is so. On the other hand, if $f \in C^\infty(M)$, then

$$\nabla^{\perp}_X(f\xi) = (\overline{\nabla}_X(f\xi))^{\perp} = (Xf)\xi^{\perp} + f(\overline{\nabla}_X\xi)^{\perp} = (Xf)\xi + f\nabla^{\perp}_X\xi.$$

To see that II is symmetric, let $X, Y \in \mathfrak{X}(M)$. Then, since $\overline{\nabla}$ is torsion-free and $[X,Y] \in \mathfrak{X}(M)$,

$$\mathrm{II}(X,Y) = (\overline{\nabla}_X Y)^{\perp} = (\overline{\nabla}_Y X + [X,Y])^{\perp} = (\overline{\nabla}_Y X)^{\perp} = \mathrm{II}(Y,X).$$

Also, $\mathrm{II}(X,Y)$ is clearly $C^\infty(M)$-linear on $X$, since $\overline{\nabla}_X Y$ is so, and since II is symmetric, it is also $C^\infty(M)$-linear on $Y$. On the other hand, if $\xi \in \mathfrak{X}^{\perp}(M)$, the expression $W_\xi X$ is again $C^\infty(M)$-linear on $X$, while

$$W_{f\xi}X = -(\overline{\nabla}_X(f\xi))^{\top} = -((Xf)\xi + f\overline{\nabla}_X\xi)^{\top} = fW_\xi X.$$

This establishes 3. A simple computation gives 4, since $\overline{\nabla}$ is metric and $\langle Y, \xi \rangle = 0$:

$$\langle \mathrm{II}(X,Y), \xi \rangle = \langle \overline{\nabla}_X Y, \xi \rangle = -\langle Y, \overline{\nabla}_X \xi \rangle = \langle W_\xi X, Y \rangle.$$

And now 5 immediately follows from the symmetry of II:

$$\langle W_\xi X, Y \rangle = \langle \mathrm{II}(X,Y), \xi \rangle = \langle \mathrm{II}(Y,X), \xi \rangle = \langle X, W_\xi Y \rangle. \qquad \square$$

Since $(TM^{\perp}, \nabla^{\perp})$ is a bundle with connection, we call its holonomy the **normal holonomy** of $M$, and denote it by $\mathrm{Hol}^{\perp}_x(M)$ (and by $\mathrm{Hol}^{\perp 0}_x(M)$ its restricted version).

Let $\overline{R}$, $R$ and $R^{\perp}$ be the curvatures of $T\overline{M}$, $TM$ and $TM^{\perp}$, respectively. The fundamental equations to study the geometry of submanifolds are the following:

**Theorem 2.4** (Gauss–Codazzi–Ricci equations)**.** *Let* $M \hookrightarrow \overline{M}$ *be a submanifold. Then for* $X, Y, Z, V \in \mathfrak{X}(M)$ *and* $\xi, \eta \in \mathfrak{X}^{\perp}(M)$*, we have that*

1. **(Gauss's equation)** $(\overline{R}(X,Y)Z)^{\top} = R(X,Y)Z - W_{\mathrm{II}(Y,Z)}X + W_{\mathrm{II}(X,Z)}Y$*, or, equivalently,*

$$\langle \overline{R}(X,Y)Z, V \rangle = \langle R(X,Y)Z, V \rangle + \langle \mathrm{II}(Y,V), \mathrm{II}(X,Z) \rangle - \langle \mathrm{II}(X,V), \mathrm{II}(Y,Z) \rangle;$$

2. **(Codazzi's equation)** $(\overline{R}(X,Y)Z)^\perp = \nabla^\perp_X \mathrm{II}(Y,Z) - \nabla^\perp_Y \mathrm{II}(X,Z)$;
3. **(Ricci's equation)** $(\overline{R}(X,Y)\xi)^\perp = R^\perp(X,Y)\xi + \mathrm{II}(W_\xi X, Y) - \mathrm{II}(X, W_\xi Y)$, *or, equivalently,*
$$\langle \overline{R}(X,Y)\xi, \eta\rangle = \langle R^\perp(X,Y)\xi,\eta\rangle - \langle [W_\xi, W_\eta]X, Y\rangle;$$
4. $(\overline{R}(X,Y)\xi)^\top = (\nabla_Y W)_\xi X - (\nabla_X W)_\xi Y$.

*Proof.* Direct computation, using that $\nabla$ is torsion-free:
$$\begin{aligned}\overline{R}(X,Y)Z &= \overline{\nabla}_X(\nabla_Y Z + \mathrm{II}(Y,Z)) - \overline{\nabla}_Y(\nabla_X Z + \mathrm{II}(X,Z)) - \overline{\nabla}_{[X,Y]}Z\\ &= R(X,Y)Z + \mathrm{II}(X,\nabla_Y Z) - W_{\mathrm{II}(Y,Z)}X + \nabla^\perp_X(\mathrm{II}(Y,Z))\\ &\qquad - \mathrm{II}(Y,\nabla_X Z) + W_{\mathrm{II}(X,Z)}Y - \nabla^\perp_Y(\mathrm{II}(X,Z)) - \mathrm{II}([X,Y],Z)\\ &= R(X,Y)Z - W_{\mathrm{II}(Y,Z)}X + W_{\mathrm{II}(X,Z)}Y + \nabla^\perp_X \mathrm{II}(Y,Z) - \nabla^\perp_Y \mathrm{II}(X,Z).\end{aligned}$$
Gauss's and Codazzi's equation immediately follow. The second form of Gauss's equation follows from Lemma 2.3(4). For Ricci's equation, we compute as well:
$$\begin{aligned}\overline{R}(X,Y)\xi &= \overline{\nabla}_X(\nabla^\perp_Y\xi - W_\xi Y) - \overline{\nabla}_Y(\nabla^\perp_X\xi - W_\xi X) - \overline{\nabla}_{[X,Y]}\xi\\ &= R^\perp(X,Y)\xi - W_{\nabla^\perp_Y\xi}X - \nabla_X(W_\xi Y) - \mathrm{II}(X, W_\xi Y)\\ &\qquad + W_{\nabla^\perp_X\xi}Y + \nabla_Y(W_\xi X) + \mathrm{II}(Y, W_\xi X) + W_\xi[X,Y]\\ &= R^\perp(X,Y)\xi + \mathrm{II}(W_\xi X, Y) - \mathrm{II}(X, W_\xi Y) + (\nabla_Y W)_\xi X - (\nabla_X W)_\xi Y.\end{aligned}$$
Ricci's and the last equation follow immediately. The second form of it is again an application of Lemma 2.3(4). □

When the ambient space $\overline{M}$ has constant sectional curvature, i.e., $\kappa = k$ for some $k \in \mathbb{R}$, it is called a **space form**. Since sectional curvature determines the Riemann curvature, it follows [Pet16, Prop. 3.1.3] that $\overline{M}$ has constant sectional curvature $k \in \mathbb{R}$ if and only if the Riemann curvature is given by
$$\overline{R}(X,Y)Z = k(\langle Y,Z\rangle X - \langle X,Z\rangle Y), \quad \text{for } X,Y,Z \in \mathfrak{X}(\overline{M}).$$
For a submanifold $M \hookrightarrow \overline{M}$ of a space form, the Codazzi and Ricci equations take particularly nice forms:
$$\nabla^\perp_X \mathrm{II}(Y,Z) = \nabla^\perp_Y \mathrm{II}(X,Z) \quad \text{and} \quad \langle R^\perp(X,Y)\xi,\eta\rangle = \langle [W_\xi, W_\eta]X, Y\rangle,$$
for $X,Y,Z \in \mathfrak{X}(M)$ and $\xi,\eta \in \mathfrak{X}^\perp(M)$.

## 2.2. Principal curvatures and curvature normals

There are some types of submanifolds with special properties that we shall consider. The first type we are interested in are those which are "composed of geodesics".

**Definition 2.5.** A submanifold $M \hookrightarrow \overline{M}$ is called **totally geodesic** if every geodesic of $M$ is also a geodesic of $\overline{M}$. ♦

**Lemma 2.6.** *A submanifold is totally geodesic if and only if its second fundamental form vanishes.*

*Proof.* Let $M$ be the submanifold with second fundamental form II. For any curve $\gamma$ on $M$, Gauss's formula gives $\overline{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \mathrm{II}(\dot\gamma,\dot\gamma)$. If II vanishes, then $\nabla_{\dot\gamma}\dot\gamma = 0$ implies that $\overline{\nabla}_{\dot\gamma}\dot\gamma = 0$, and so $M$ is totally geodesic. Conversely, if $M$ is totally geodesic, then $\mathrm{II}(v,v) = 0$ for all $v \in TM$. Since
$$2\,\mathrm{II}(u,v) = \mathrm{II}(u+v,u+v) - \mathrm{II}(u,u) - \mathrm{II}(v,v)$$
for all $u, v \in TM$, then II vanishes identically. □

For instance, the complete totally geodesic submanifolds of Euclidean space $\mathbb{R}^n$ are the affine subspaces, since geodesics are straight lines.

The second type of submanifolds we are interested in are those whose eigenvalues of the Weingarten operators are locally constant.

**Definition 2.7.** The **principal curvatures** of a submanifold $M \hookrightarrow \overline{M}$ at $x \in M$ in the direction of $\xi \in T_xM^\perp$ are the eigenvalues of $W_\xi$. We say that $M$ has **constant principal curvatures** if the eigenvalues of $W_{\xi(t)}$ are constant for any parallel normal vector field $\xi$ along any piecewise smooth curve in $M$. ♦

In particular, on a manifold with constant principal curvatures the eigenvalues of $W_\xi$ are constant for any local parallel normal vector field $\xi$. The reason why we are interested in submanifolds with constant principal curvatures is because the following construction works especially well in this case.

Let $M \hookrightarrow \overline{M}$ be a submanifold. Define

$$T_xM_0^\perp := \{\xi \in T_xM^\perp : g\xi = \xi, \text{ for all } g \in \mathrm{Hol}_x^{\perp 0}(M)\}.$$

In general, if $E \to N$ is a vector bundle with a connection $\nabla$ and $E' \to N$ is a subbundle of $E$, then $E'$ is said to be **parallel** if it is invariant under parallel transport, i.e., if $\tau_\gamma(E'_{\gamma(0)}) = E'_{\gamma(1)}$ for all piecewise smooth curves $\gamma$ in $N$. Equivalently, if for all $\sigma \in \Gamma(E')$ and $X \in \mathfrak{X}(N)$ we have that $\nabla_X\sigma \in \Gamma(E')$. It is the necessary and sufficient condition for the possibility of restricting $\nabla$ to a connection on $E'$.

**Lemma 2.8.** *The bundle $TM_0^\perp$ whose fiber at $x \in M$ is $T_xM_0^\perp$ is a smooth parallel flat subbundle of $TM^\perp$.*

*Proof.* For $x \in M$, let $\{\xi_i\}_i$ be a basis for $T_xM_0^\perp$ and $U$ be a simply connected neighborhood of $x$ in $M$. Since $\xi_i$ is $\mathrm{Hol}_x^{\perp 0}(M)$-invariant, then by the holonomy principle it can be extended to a $\nabla^\perp$-parallel smooth normal vector field on $U$ which we also call $\xi_i$. If $y \in U$ and $\eta \in T_yM_0^\perp$, let $\gamma$ be a curve from $y$ to $x$ in $U$. Then $\tau_\gamma^\perp\eta \in T_xM^\perp$ is easily seen to lie in $T_xM_0^\perp$, so $\tau_\gamma^\perp\eta$ is a linear combination of $\{\xi_i(x)\}_i$, and hence $\eta$ is a linear combination of $\{\xi_i(y)\}_i$. Therefore, $\{\xi_i(y)\}_i$ is a basis for $T_yM_0^\perp$, which gives that $TM_0^\perp$ is smooth and parallel. Moreover, since $\{\xi_i\}_i$ is a $\nabla^\perp$-parallel frame, then Corollary 1.26 gives that $TM_0^\perp$ is flat. □

From now on, let $\overline{M}$ be a space form. Let $\xi \in T_xM_0^\perp$. Then by Ricci's equation we have that $[W_\xi, W_\eta] = 0$ for all $\eta \in T_xM^\perp$. This means that $\{W_\xi : \xi \in T_xM_0^\perp\}$ is a commuting family of self-adjoint operators, which means that there is a decomposition into common eigenspaces

$$T_xM = E_1(x) \oplus \cdots \oplus E_{g(x)}(x), \tag{2.1}$$

that is, there are unique linear functionals $\lambda_i(x) \in T_x^*M_0^\perp$ and vectors $\eta_i(x) \in T_xM_0^\perp$ such that

$$W_\xi v_i = \lambda_i(x)(\xi)v_i = \langle \eta_i(x), \xi\rangle v_i, \qquad \text{for } \xi \in T_xM_0^\perp \text{ and } v_i \in E_i(x).$$

The vectors $\eta_i(x)$ are called **curvature normals** at $x$.

**Proposition 2.9.** *Let $M \hookrightarrow \overline{M}$ be a connected submanifold with constant principal curvatures, where $\overline{M}$ is a space form. Then*

1. *the function $g$ on the decomposition (2.1) is constant and the corresponding curvature normals are well-defined smooth $\nabla^\perp$-parallel vector fields;*
2. *if moreover $\overline{M} = \mathbb{R}^n$ and $M$ is not contained in a proper totally geodesic submanifold, then the curvature normals at $x$ span $T_xM_0^\perp$ for all $x \in M$.*

*Proof.* The integer $g(x)$ is the number of different common eigenspaces on $x \in M$. This means that there is some $\xi \in T_xM_0^\perp$ for which $W_\xi$ has $g(x)$ different eigenvalues. We can extend $\xi$ to a

$\nabla^\perp$-parallel normal vector field on a simply connected neighborhood $U$ of $x$, which we also call $\xi$. Since $M$ has constant principal curvatures, then $W_\xi$ has constant eigenvalues on $U$, which means that for any $y \in U$ we have that $g(y) \geq g(x)$, since $T_yM$ has to split at least into $g(x)$ eigenspaces. Let $\eta \in T_yM_0^\perp$ be such that $W_\eta$ has $g(y)$ different eigenvalues. By the same argument, we can extend it to a $\nabla^\perp$-parallel normal vector field on $U$, and since $W_\eta$ has constant eigenvalues on $U$ then $g(x) \geq g(y)$, which gives $g(y) = g(x)$. Since $M$ is connected, this finally gives that $g$ is constant.

Let $x \in M$ and let $\eta_i(x) \in T_xM_0^\perp$ be a curvature normal at $x$. On a simply connected neighborhood $U$ of $x$, $\eta_i(x)$ can be extended to a smooth $\nabla^\perp$-parallel vector field $\eta_i$. Let $\xi$ be a $\nabla^\perp$-parallel vector field on $U$, so that $W_\xi$ has constant eigenvalues. Then at $y \in U$, the number $\langle \eta_i(x), \xi(x)\rangle$ is an eigenvalue of $W_{\xi(y)}$. Let $v \in T_xM$ be an eigenvector of such an eigenvalue. The function $\langle \eta_i, \xi\rangle$ is constant on $U$, since for any $X \in \mathfrak{X}(U)$ we have that

$$X\langle \eta_i, \xi\rangle = \langle \nabla_X^\perp \eta_i, \xi\rangle + \langle \eta_i, \nabla_X^\perp \xi\rangle = 0.$$

Hence,

$$W_{\xi(y)}v = \langle \eta_i(x), \xi(x)\rangle v = \langle \eta_i(y), \xi(y)\rangle v,$$

so that $\eta_i(y)$ is also a curvature normal at $y \in U$. Since $M$ can be covered by simply connected charts, then $\eta_i$ can be globally defined.

Suppose now that $\overline{M} = \mathbb{R}^n$. For $x \in M$, define $V_x := \{\xi \in T_xM^\perp : W_\xi = 0\}$. To see that it defines a smooth subbundle of $TM^\perp$, let $\xi_1, \dots, \xi_l \in V_x$ be a basis and extend them to local smooth vector fields by parallel transporting along radial geodesics, i.e., define $\xi_i(\exp_x v) := \tau_1^\perp(\xi_i(x))$, where $\tau_t^\perp$ is parallel transport from $x$ to $\exp_x(tv)$ along $s \mapsto \exp_x(sv)$, where $v \in T_xM$. Then the vector fields $\{\xi_i\}_i$ are pointwise linearly independent. To see that they generate $V$ pointwise, let $\eta \in V_y$, where $y = \exp_x v$, and let $\eta(t) := \tau_t^\perp(\tau_1^\perp)^{-1}\eta$. Then $\eta(t)$ is parallel along the radial geodesic from $x$ to $y$ and $\eta(1) = \eta$. Since $M$ has constant principal curvatures, then $W_{\eta(t)}$ has constant eigenvalues, and since $W_{\eta(1)} = W_\eta = 0$, then $W_{\eta(0)} = 0$. Write $\eta(0) = \eta^i\xi_i(x)$, for some constants $\eta^i \in \mathbb{R}$. Then $\eta = \eta(1) = \tau_1^\perp(\eta(0)) = \eta^i\tau_1^\perp(\xi_i(x)) = \eta^i\xi_i(y)$. Therefore $\{\xi_i\}_i$ is a smooth frame for $V$. Moreover, $V$ is parallel and flat. Indeed, if $\xi$ is a parallel section of $TM^\perp$ along a piecewise smooth curve $\gamma$ with $\xi(0) \in V$, then $W_{\xi(t)}$ has constant eigenvalues, so $W_{\xi(1)} = W_{\tau_\gamma^\perp(\xi(0))} = 0$ because $W_{\xi(0)} = 0$. Moreover, if $\xi \in V_x$, then the Ricci equation implies that $R^\perp(v,w)\xi = 0$ for all $v, w \in T_xM$, so $V$ is flat.

Let $\xi \in \Gamma(V)$ be $\nabla^\perp$-parallel. Then $\overline{\nabla}_X\xi = d\xi(X) = \nabla_X^\perp\xi - W_\xi X = 0$ for every $X \in \mathfrak{X}(M)$, so actually $\xi$ is constant as a map $\xi : M \to \mathbb{R}^n$. Since $V$ is flat, there is a local parallel frame around every $x \in M$, by Corollary 1.26, which means that there is an affine subspace $W \subseteq \mathbb{R}^n$ such that $V_x = W$ for all $x \in M$. Hence, $T_xM \subseteq V_x^\perp = W^\perp$, where $^\perp$ is taken in $\mathbb{R}^n$, which in turn means that $M \subseteq W^\perp$. Since $M$ is not contained in a proper totally geodesic submanifold, then necessarily $W^\perp = \mathbb{R}^n$, i.e., $V_x = 0$ for all $x \in M$.

Now, $\xi \in T_xM_0^\perp$ satisfies $W_\xi = 0$ if and only if all its eigenvalues are zero, i.e., if and only if $\langle \eta_i(x), \xi\rangle = 0$ for all $i$. That is,

$$V_x \cap T_xM_0^\perp = \operatorname{span}\{\eta_i(x)\}_i^\perp,$$

where $^\perp$ is taken inside of $T_xM_0^\perp$. Since $V_x = 0$, we conclude that $\operatorname{span}\{\eta_i(x)\}_i = T_xM_0^\perp$, as wanted. □

**Example 2.10** (Sphere)**.** Consider the sphere $\mathbb{S}^n$ with the metric induced by the standard Euclidean metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{n+1}$. Then $T_x\mathbb{S}^n = (\mathbb{R}x)^\perp$, for $x \in \mathbb{S}^n$. Let $E \in \mathfrak{X}(\mathbb{R}^{n+1})$ denote the Euler vector field, given by $E(x) = x$. Then $E(x) \in T_x(\mathbb{S}^n)^\perp$ for all $x \in \mathbb{S}^n$. If $X \in \mathfrak{X}(\mathbb{S}^n)$, then $W_EX = -(\overline{\nabla}_XE)^\top = -X^\top = -X$, i.e., $W_E = -\mathrm{id}$. Hence, $\mathbb{S}^n$ has constant principal curvatures. Moreover, $W_EX = -X = \langle -E, E\rangle X$, so the curvature normal is $-E$. ▼

*Remark* 2.11. A manifold which is not contained in a proper totally geodesic submanifold is called full. The space $V_x^\perp$, with $^\perp$ taken inside of $T_xM^\perp$, is known as the first normal space of $M$ at $x$, and it coincides with the span of the image of II,

$$V_x^\perp = \operatorname{span}\{\mathrm{II}(v,w) : v,w \in T_xM\} \subseteq T_xM^\perp.$$

# 3

# Berger's Holonomy Theorem

We now turn to Berger's classification theorem of holonomy groups. The theorem states that if $M$ is an irreducible, not locally symmetric, oriented and connected Riemannian manifold of dimension $n \geq 2$, then one of the following must hold:

1. $\mathrm{Hol}^0(M) = \mathrm{SO}(n)$,
2. $n = 2m$ for $m > 0$ and $\mathrm{Hol}^0(M) = \mathrm{U}(n)$,
3. $n = 2m$ for $m > 0$ and $\mathrm{Hol}^0(M) = \mathrm{SU}(n)$,
4. $n = 4m$ for $m > 0$ and $\mathrm{Hol}^0(M) = \mathrm{Sp}(n)$,
5. $n = 4m$ for $m > 0$ and $\mathrm{Hol}^0(M) = \mathrm{Sp}(n)\,\mathrm{Sp}(1)$,
6. $n = 7$ and $\mathrm{Hol}^0(M) = G_2$,
7. $n = 8$ and $\mathrm{Hol}^0(M) = \mathrm{Spin}(7)$.

The theorem was first proven by Berger [Ber55]. The proof was very algebraic and relied heavily on the classification of Lie groups and its representations, which made it quite complex. Some years later Simons [Sim62] offered a new (still algebraic) proof, and it was not until forty years later that Olmos [Olm05] found a geometric proof of Simons's theorem. It is Olmos's proof that we will present here. We will not follow [Olm05] exactly, but rather a polished version of the proof one can find in [BCO16].

In this chapter we will make sense of what it means for a Riemannian manifold to be reducible and (locally) symmetric, and we will see what this tells us about its holonomy. Then we will turn to the more technical side of the proof of Simons's theorem, using the machinery developed in Chapters 1 and 2. Finally, we will give precise invariant definitions of the groups in Berger's list and describe what types of special geometries each groups gives rise to. These include Kähler, Calabi–Yau, hyperkähler and quaternionic Kähler geometries. We will give some examples of each of them.

## 3.1. Reducible spaces

The main aim of this section is to prove that having a reducible holonomy representation implies that the manifold is locally a product. If the manifold is complete, it is even globally a product. This is de Rham's decomposition theorem.

**Definition 3.1.** A connected Riemannian manifold $M$ is **reducible** if its holonomy representation is reducible. It is **irreducible** if its restricted holonomy representation is irreducible. ♦

**Proposition 3.2.** *Let $M$ be reducible, $x \in M$ and $D_x$ a nontrivial $\mathrm{Hol}_x(M)$-invariant subspace of $T_xM$. Define a distribution $D$ by setting $D_y := \tau_\gamma D_x$, where $\gamma$ is any piecewise smooth curve from $x$ to $y$. Then*

1. *$D$ is a well-defined involutive smooth distribution,*
2. *the maximal integral submanifold $N$ of $D$ through $x$ is totally geodesic.*

*Proof.* If $\alpha$ were another piecewise smooth curve from $x$ to $y$, then $\alpha \cdot \gamma^{-1}$ is a loop at $x$, so $\tau_\gamma^{-1}\tau_\alpha D_x = D_x$, i.e., $\tau_\gamma D_x = \tau_\alpha D_x$. This shows well-definedness. To see that it is smooth, let $y \in M$ and let $U$ be a normal neighborhood about $y$. Let $\{v_i\}_i$ be a basis for $D_y$ and define vector fields $\{X_i\}_i$ in $U$ by $X_i(\exp_y u) = \tau_u v_i$, where $u \in T_yM$ is small enough such that $\exp_y u \in U$ and $\tau_u$ is parallel transport from $y$ to $\exp_y u$ along the geodesic $t \mapsto \exp_y(tu)$. Since $\tau_u$ depends smoothly on $u$, the vector fields $\{X_i\}_i$ are smooth on $U$. It is clear that they form a frame for $D$.

To see involutivity, let $X, Y \in \Gamma(D)$. Since $[X,Y] = \nabla_X Y - \nabla_Y X$, it is enough to see that $\nabla_X Y \in \Gamma(D)$. Let $y \in M$ and $\gamma$ the integral curve of $X$ through $y$. Then, if $\tau_t$ is parallel transport from $y$ to $\gamma(t)$ along $\gamma$, by Proposition 1.15 we have that

$$\nabla_X Y(y) = \left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}(Y(t)).$$

Since $Y(t) \in D_{\gamma(t)}$ and $\tau_t^{-1} D_{\gamma(t)} = D_y$, we have that, indeed, $\nabla_X Y(y) \in D_y$. This gives 1.

By Frobenius's integrability theorem ([Lee12, Thm. 19.12] for instance), $D$ is integrable. Let $N$ be the maximal integral submanifold of $D$ through $x$. Let $y \in N$ and $v \in T_yN$, and let $\gamma$ be the maximal geodesic starting at $y$ with velocity $v$. Then $\dot\gamma$ is parallel along $\gamma$, so $\dot\gamma(t) = \tau_t v \in D_{\gamma(t)}$. Since $D$ is a regular distribution, this easily implies that $\gamma$ does not leave $N$. For a complete proof of this fact (also in the possibly singular setting) see Proposition 4.16. Hence, $N$ is totally geodesic. □

**Proposition 3.3.** *Let $M$ be reducible, $x \in M$ and $D_x$ a nontrivial $\mathrm{Hol}_x(M)$-invariant subspace of $T_xM$. Let $D$ be the distribution corresponding to $D_x$ and $D'$ the distribution corresponding to $D_x^\perp$. Let $N$ (resp. $N'$) be the maximal integral submanifold of $D$ (resp. $D'$) through $x$. Then there are open neighborhoods $V$ in $M$, $U$ in $N$ and $U'$ in $N'$ of $x$ such that $V$ is isometric to $U \times U'$.*

*Proof.* First observe that, because parallel transport is isometric, $D$ and $D'$ are everywhere orthogonal. Let $(W', (x^1, \ldots, x^k, y^{k+1}, \ldots, y^n))$ be coordinates about $x$ such that $\{\frac{\partial}{\partial x^i}\}_{i=1}^k$ is a frame for $D|_{W'}$ and $(W'', (z^1, \ldots, z^k, x^{k+1}, \ldots, x^n))$ coordinates about $x$ such that $\{\frac{\partial}{\partial x^i}\}_{i=k+1}^n$ is a frame for $D'|_{W''}$. Then $(W' \cap W'', (x^1, \ldots, x^n))$ are also coordinates about $x$.

Let $V$ be the cube given by $|x^i| < \epsilon$, for $i = 1, \ldots, n$ and $\epsilon > 0$ small enough such that $V \subseteq W' \cap W''$. Let $U$ be the cube in $N$ given by $|x^i| < \epsilon$, for $i = 1, \ldots, k$, and $U'$ be the cube in $N'$ given by $|x^i| < \epsilon$, for $i = k+1, \ldots, n$. Then as smooth manifolds $V = U \times U'$. To see that they are also isometric we have to show that $\langle X, Y\rangle = 0$ if $X \in \Gamma(D)$ and $Y \in \Gamma(D')$ and that $X\langle Y, Z\rangle = 0$ if either $X \in \Gamma(D)$ and $Y, Z \in \Gamma(D')$ or $X \in \Gamma(D')$ and $Y, Z \in \Gamma(D')$.

Since $D$ and $D'$ are orthogonal, it is clear that $\langle X, Y\rangle = 0$ if $X \in \Gamma(D)$ and $Y \in \Gamma(D')$. On the other hand, if $X \in \Gamma(D)$ and $Y \in \Gamma(D')$, we can assume them to be coordinate vector fields, so that $[X, Y] = 0$, and then, since the Levi-Civita connection is torsion-free,

$$\nabla_X Y - \nabla_Y X = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

By the same reasoning as in the proof of Proposition 3.2 we have that $\nabla_X Y \in \Gamma(D')$ and $\nabla_Y X \in \Gamma(D)$, so actually both vanish. Then, if $Z \in \Gamma(D')$ is another coordinate vector field,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = 0.$$

The other case is analogous. □

**Proposition 3.4.** *In the same situation as in Proposition 3.3, there are normal subgroups $G$ and $G'$ of $\mathrm{Hol}^0_x(M)$ such that $\mathrm{Hol}^0_x(M) = G \times G'$, where $G$ acts trivially on $D_x^\perp$ and $G'$ trivially on $D_x$. Moreover, $\mathrm{Hol}^0_x(N) \subseteq G$ and $\mathrm{Hol}^0_x(N') \subseteq G'$.*

*Proof.* Let $\gamma$ be a contractible loop at $x$ and let $g$ be the extension of $\tau_\gamma|_{D_x}$ to all of $T_xM$ by acting trivially on $D_x^\perp$ and $g'$ the extension of $\tau_\gamma|_{D_x^\perp}$ to all of $T_xM$ by acting trivially on $D_x$. We will show that both $g$ and $g'$ lie in $\mathrm{Hol}^0_x(M)$.

Suppose first that $\gamma$ is a lasso, that is, of the form $\alpha \cdot \beta \cdot \alpha^{-1}$, where $\alpha$ is a piecewise smooth curve from $x$ to some $y \in M$ and $\beta$ is a contractible loop at $y$ small enough so that it is contained in a decomposable open set $V = U \times U'$ as in Proposition 3.3. Let $\tilde{\beta}$ (resp. $\tilde{\beta}'$) be the projection of $\beta$ unto $U$ (resp. $U'$) and let $\tilde{\gamma} := \alpha \cdot \tilde{\beta} \cdot \alpha^{-1}$ and $\tilde{\gamma}' := \alpha \cdot \tilde{\beta}' \cdot \alpha^{-1}$. Since $\tau_\beta = \tau_{\tilde{\beta}} \times \tau_{\tilde{\beta}'}$ and both $D$ and $D'$ are preserved by parallel transport, then $\tau_\gamma = \tau_{\tilde{\gamma}} \times \tau_{\tilde{\gamma}'}$ as well. Also, $\tau_{\tilde{\gamma}}$ (resp. $\tau_{\tilde{\gamma}'}$) acts trivially on $D_x^\perp$ (resp. $D_x$). Hence, $g = \tau_{\tilde{\gamma}}$ and $g' = \tau_{\tilde{\gamma}'}$, so both lie indeed in $\mathrm{Hol}^0_x(M)$.

Parallel transport along a contractible loop always equals a finite product of parallel transport along such lassos [KN63, Vol. 1, App. 7], so $g, g' \in \mathrm{Hol}^0_x(M)$ for any contractible loop $\gamma$. Let now $G$ be the subgroup of $\mathrm{Hol}^0_x(M)$ consisting of extensions of $\tau_\gamma|_{D_x}$ to all of $T_xM$ by acting trivially on $D_x^\perp$ for all $\tau_\gamma \in \mathrm{Hol}^0_x(M)$ and similarly for $G'$. Then we just proved that $\mathrm{Hol}^0_x(M) = G \times G'$ with $G$ acting trivially on $D_x^\perp$ and $G'$ trivially on $D_x$. It is easy to check that $G$ and $G'$ are normal in $\mathrm{Hol}^0_x(M)$.

Lastly, let $\gamma$ be a contractible loop contained in $N$. Suppose first that it is a lasso $\gamma = \alpha \cdot \beta \cdot \alpha^{-1}$, with both $\alpha$ and $\beta$ in $N$ and with $\beta$ small enough so that it is contained in a decomposable open set $V = U \times U'$. Then $\tau_\beta$ is the identity on $D'_{\alpha(1)}$, so for any $v \in D_x^\perp$ we have that $\tau_\gamma v = \tau_\alpha^{-1} \tau_\beta \tau_\alpha v = \tau_\alpha^{-1} \tau_\alpha v = v$, so that $\tau_\gamma$ is the identity on $D_x^\perp$. Since parallel transport along a contractible loop equals a finite product of parallel transport along such lassos, we finally obtain that $\mathrm{Hol}^0_x(N) \subseteq G$. The case for $\mathrm{Hol}^0_x(N')$ is analogous. □

We have seen that if $M$ is reducible then it can be locally written as a product. If $M$ is 1-connected and complete, then $M$ can actually be written globally as a product. This is de Rham's decomposition theorem, originally proved in [dR52]. Several different proofs and generalizations have been given [KN63, Wu64, Mal72], but here we follow [Pan92], which seems to be the most elementary, avoiding the use of much machinery.

**Lemma 3.5.** *In the same situation as in Proposition 3.3, with $M$ connected and complete, let $v \in D_x$ and $v' \in D_x^\perp$. Let $\tau : D'_x \to D'_y$ be parallel transport along any curve in $N$ from $x$ to $y := \exp_x v$ and $\tau' : D_x \to D_z$ parallel transport along any curve in $N'$ from $x$ to $z := \exp_x v'$. Then $\exp_y(\tau v') = \exp_z(\tau' v)$.*

*Proof.* First observe that $\tau$ and $\tau'$ are well defined by Proposition 3.4. Let $V$ be the parallel vector field along $t \mapsto \exp_x(tv')$ with $V(0) = v$ and define $\gamma : [0, 1]^2 \to M$ by $\gamma(s, t) = \exp(tV(s))$. It is enough to see that $\gamma(\cdot,1)$ is geodesic and that $\frac{\partial}{\partial s}\gamma(0,\cdot)$ is parallel along $\gamma_0$ (in Figure 3.1 you can see a sketch of the situation). Indeed, if this were the case then

$$\exp_y(\tau v') = \exp_y\left(\frac{\partial}{\partial s}\gamma(0,1)\right) = \exp_y\left(\left.\frac{d}{ds}\right|_{s=0}\gamma(s,1)\right) = \gamma(1,1) = \exp_z(V(1)) = \exp_z(\tau' v).$$

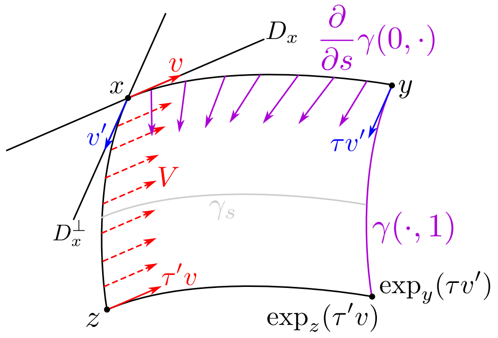


Figure 3.1: Proof of Lemma 3.5.

To this end, let $s_0 \in [0,1]$ and pick a neighborhood $W$ of $\gamma(s_0,0)$ isometric to a product $U \times U'$, in the fashion of Proposition 3.3. Projecting to $U$ and $U'$, write $\gamma(s,t) = (\beta(s,t), \beta'(s,t))$. We will see that actually $\beta$ depends only on $t$ and $\beta'$ only on $s$.

Since the integral submanifolds of $D$ and $D'$ (the distributions induced by $D_x$ and $D_x^\perp$, respectively) are totally geodesic, then for each $s$,

$$\frac{\partial}{\partial t}\gamma(s,t) = \frac{d}{dt}(\exp(tV(s))) \in D_{\gamma(s,t)}.$$

Hence, we have that $\frac{\partial}{\partial t}\beta'(s,t) = 0$, i.e., $\beta'$ depends only on $s$. Similarly,

$$\frac{\partial}{\partial s}\gamma(s,0) = \frac{d}{ds}(\exp_{\exp_x(sv')}(0)) = \frac{d}{ds}(\exp_x(sv')) \in D'_{\gamma(s,0)},$$

so $\frac{\partial}{\partial s}\beta(s,0) = 0$, i.e., $\beta(s,0)$ is constant on $s$. Finally, $\frac{\partial}{\partial t}\gamma(\cdot,0)$ is parallel along $\gamma(\cdot,0)$, because

$$\frac{\partial}{\partial t}\gamma(s,0) = \left.\frac{d}{dt}\right|_{t=0}\exp(tV(s)) = V(s),$$

which gives that $\frac{\nabla}{\partial s}\frac{\partial}{\partial t}\beta(s,0) = 0$. Since $\beta(s,0)$ is constant on $s$, then all the vectors $\frac{\partial}{\partial t}\beta(s,0)$ belong to the same tangent space of $U$, so actually $\frac{\partial}{\partial s}\frac{\partial}{\partial t}\beta(s,0) = 0$, i.e., $\frac{\partial}{\partial t}\beta(s,0)$ does not depend on $s$. Now we have that $\beta(s,\cdot)$ is a geodesic in $U$ starting at $\beta(s,0)$ with velocity $\frac{\partial}{\partial t}\beta(s,0)$; since neither of these two things depends on $s$, we conclude that $\beta(s,t)$ does not depend on $s$. Hence we can finally write (in $W$) $\gamma(s,t) = (\beta(t), \beta'(s))$.

Observe that $\beta'$ is a geodesic in $U'$, because $\gamma(s,0) = (\beta(0), \beta'(s)) = \exp_x(sv')$. Hence, for each fixed $t$ we have that $\gamma(\cdot,t) = (\beta(t), \beta'(\cdot))$ is geodesic. Along it, $\frac{\partial}{\partial t}\gamma(\cdot,t) = (\dot\beta(t), 0)$ is parallel.

We now split $[0,1]^2$ into a finite amount of sufficiently small squares such that the image by $\gamma$ of each square is contained into one such $W$ decomposable as a product. For all the squares with $t$ small enough, we just proved that $\gamma(\cdot,t)$ is geodesic and $\frac{\partial}{\partial t}\gamma(\cdot,t)$ is parallel along it, for each fixed $t$. These conditions are enough to prove the same for the following row of squares, with bigger $t$. Inductively, we can prove that $\gamma(\cdot,t)$ is geodesic and $\frac{\partial}{\partial t}\gamma(\cdot,t)$ is parallel along it for all $t \in [0,1]$. This gives that $\gamma(\cdot,1)$ is geodesic and that $\frac{\partial}{\partial s}\gamma(0,\cdot)$ is parallel along $\gamma_0$, since

$$\frac{\nabla}{\partial t}\frac{\partial}{\partial s}\gamma(0,t) = \frac{\nabla}{\partial s}\frac{\partial}{\partial t}\gamma(0,t) = 0. \qquad \square$$

**Theorem 3.6** (de Rham's decomposition)**.** *Let $M$ be a 1-connected and complete reducible Riemannian manifold. Let $x \in M$ and let $N$ and $N'$ be the maximal integral submanifolds through $x$ of the distributions induced by a nontrivial $\mathrm{Hol}_x(M)$-invariant subspace of $T_xM$ and its orthogonal. Then $M$ is isometric to $N \times N'$.*

*Proof.* Let $D$ and $D'$ be the mentioned distributions. For any $x \in M$, denote by $N(x)$ and $N'(x)$ the maximal integral submanifolds through $x$. For $y = \exp_x v$, with $v \in D_x$, let $\tau_{x,y} : D'_x \to D'_y$ be parallel transport along any curve in $N(x)$ from $x$ to $y$, and for $z = \exp_x v'$, with $v' \in D'_x$, let $\tau'_{x,z} : D_x \to D_z$ be parallel transport along any curve in $N'(x)$ from $x$ to $z$. Then Lemma 3.5 reads $\exp_y(\tau_{x,y}v') = \exp_z(\tau'_{x,z}v)$. Define now $F_x^z : N(x) \to N(z)$ by $F_x^z(y) = \exp_y(\tau_{x,y}v') = \exp_z(\tau'_{x,z}v)$. It is smooth and satisfies (as is easy to check) $F_x^a = F_z^a \circ F_x^z$ for any $a \in N'(x)$, from where we see that it is a diffeomorphism. It is moreover an isometry. Indeed, let $u \in T_yN(x) = D_y$ and $\gamma(t) = \exp_y(tu)$. Then, by Lemma 3.5, since $F_x^z(y) = \exp_y(\tau_{x,y}v')$,

$$\exp_{\gamma(t)}(\tau_{y,\gamma(t)}\tau_{x,y}v') = \exp_{F_x^z(y)}(\tau'_{y,F_x^z(y)}tu),$$

so that

$$\begin{aligned} F_{x*}^z u &= \frac{d}{dt}\bigg|_{t=0} F_x^z(\gamma(t)) = \frac{d}{dt}\bigg|_{t=0} \exp_{\gamma(t)}(\tau_{x,\gamma(t)}v') \\ &= \frac{d}{dt}\bigg|_{t=0} \exp_{\gamma(t)}(\tau_{y,\gamma(t)}\tau_{x,y}v') = \frac{d}{dt}\bigg|_{t=0} \exp_{F_x^z(y)}(\tau'_{y,F_x^z(y)}tu) \\ &= \tau'_{y,F_x^z(y)}u. \end{aligned}$$

Similarly we can define isometries $G_x^y : N'(x) \to N'(y)$ by $G_x^y(z) = \exp_z(\tau'_{x,z}v)$. Observe that $F_x^z(y) = G_x^y(z)$.

We now fix $x \in M$ and define $F : N(x) \times N'(x) \to M$ by $F(y,z) = F_x^z(y) = G_x^y(z)$. It is a local isometry, as a consequence of Proposition 3.3 and because each $F_x^z$ and $G_x^y$ are isometries. Since $M$ is complete, $N(x) \times N'(x)$ is connected and complete. Hence, $F$ is a local isometry from a complete manifold to a connected manifold, which implies that it is actually a Riemannian covering map [Pet16, Lem. 5.6.4]. Since $M$ is simply connected, necessarily $F$ is an isometry. □

## 3.2. Symmetric spaces

Symmetric spaces, as their name suggests, are Riemannian manifolds which are symmetric, in the following sense: about every point of the manifold there is an isometric involution which inverts the direction of geodesics, called a geodesic symmetry, leaving the point fixed. As we will see, these spaces are always homogeneous and their geometric properties are very intimately linked to the Lie theoretic data of their isometry group. This will allow us to compute the holonomy of irreducible symmetric spaces.

In Section 3.2.3 we prove that the orbits of the holonomy representation of irreducible symmetric spaces are submanifolds with constant principal curvatures. We will need this in the proof of Proposition 3.47.

### 3.2.1. Definitions and basic properties

**Definition 3.7.** A Riemannian manifold $(M,g)$ is a **symmetric space** if for every $x \in M$ there is an isometry $\sigma_x$ such that $\sigma_x^2 = \mathrm{id}$, $\sigma_{x*}(x) = -\mathrm{id}$ and $\sigma_x(x) = x$. It is a **locally symmetric space** if the isometry $\sigma_x$ only exists on a neighborhood of $x$. ♦

*Remark* 3.8. On a Riemannian manifold $M$, let $U_x$ be a neighborhood of $x \in M$ such that $\exp_x : B(0,\epsilon) \to U_x$ is a diffeomorphism. Then the map $\sigma_x : U_x \to U_x$ defined by $\sigma_x(\exp_x v) = \exp_x(-v)$ is called the **geodesic symmetry** at $x$. A locally symmetric space is one where such a map is an isometry. A symmetric space is one where such a map can be extended to a global isometry.

**Lemma 3.9.** *Symmetric spaces are complete.*

*Proof.* Let $M$ be a symmetric space and $x \in M$. Let $\gamma$ be a geodesic starting at $x$. Then $\sigma_x \circ \gamma$ is the unique geodesic starting at $x$ with velocity $-\dot\gamma(0)$, and so $\sigma_x(\gamma(t)) = \gamma(-t)$.

If we write $\gamma_s(t) := \gamma(t+s)$, whenever it is defined, then

$$\begin{aligned}\sigma_{\gamma(s/2)}\sigma_x(\gamma(t)) &= \sigma_{\gamma(s/2)}(\gamma(-t)) = \sigma_{\gamma(s/2)}\left(\gamma_{s/2}\left(-t-\frac{s}{2}\right)\right)\\ &= \gamma_{s/2}\left(t+\frac{s}{2}\right) = \gamma(t+s).\end{aligned}$$

Then if $\gamma$ is defined up to time $T$, it can be extended up to $2T$ by applying $\sigma_{\gamma(s/2)}\sigma_x$ for $0 \le s \le T$. Hence $M$ is complete. □

**Corollary 3.10.** *Let $M$ be a connected symmetric space and $x \in M$. Then there is only one isometry $\sigma_x$ with $\sigma_x^2 = \mathrm{id}$, $\sigma_{x*}(x) = -\mathrm{id}$ and $\sigma_x(x) = x$.*

*Proof.* Let $\sigma_x$ and $\sigma'_x$ be two such isometries. Then $g = \sigma'_x\sigma_x^{-1}$ is an isometry fixing $x$ and whose differential at $x$ is id. Let $y \in M$ and $\gamma$ a geodesic starting at $x$ with $\gamma(1) = y$, which exists because $M$ is complete and by the Hopf–Rinow theorem. Then $g \circ \gamma$ is a geodesic starting at $x$ with velocity $\dot\gamma(0)$, and so $g \circ \gamma = \gamma$. Hence, $g(y) = g(\gamma(1)) = \gamma(1) = y$. Therefore $g = \mathrm{id}$. □

Hence it makes sense to speak of $\sigma_x$ as *the* geodesic symmetry at $x$.

**Example 3.11** (Euclidean space)**.** Consider Euclidean space $\mathbb{R}^n$ with the standard Euclidean product. The map $\sigma_x(y) = 2x - y$ is a geodesic symmetry defined on all of $\mathbb{R}^n$, so it is a symmetric space. ▼

**Example 3.12** (Sphere)**.** Consider the sphere $\mathbb{S}^n$ with the metric induced by the standard Euclidean metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{n+1}$. Let $\sigma_x(y) := 2\langle y, x\rangle x - y$ (reflection about the line $\mathbb{R}x$). It is immediate to check that $\sigma_x$ is a geodesic symmetry. ▼

**Example 3.13** (Hyperbolic space)**.** Consider Minkowski space $\mathbb{R}^{1,n}$, i.e., $\mathbb{R}^{n+1}$ with the nonpositive metric

$$((t,x),(s,y)) := -ts + \langle x, y\rangle,$$

for $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ the standard Euclidean product on $\mathbb{R}^n$. Consider the hyperboloid model for hyperbolic space

$$\mathscr{H}^n := \{(t,x) \in \mathbb{R}^{1,n} : -t^2 + \|x\|^2 = -1,\, t > 0\}.$$

Then $T_{(t,x)}\mathscr{H}^n = \{(\lambda, v) \in \mathbb{R}^{n+1} : -\lambda t + \langle v, x\rangle = 0\}$. The restriction of $(\cdot,\cdot)$ to $T_{(t,x)}\mathscr{H}^n$ is positive definite. Indeed, if $\lambda = 0$, then $((\lambda,v),(\lambda,v)) = \|v\|^2 > 0$, and if $\lambda \neq 0$, then $v \neq 0$, because $-\lambda t + \langle v, x\rangle = 0$ and $t > 0$, and in this case,

$$\lambda^2 = \frac{|\langle v, x\rangle|^2}{t^2} \le \|v\|^2\frac{\|x\|^2}{t^2} = \|v\|^2\frac{\|x\|^2}{1+\|x\|^2} < \|v\|^2,$$

so that

$$((\lambda,v),(\lambda,v)) = -\lambda^2 + \|v\|^2 > 0.$$

The reflection $\sigma_x(y) = -y - 2\,(y,x)\,x$ is the geodesic symmetry in this case. ▼

**Example 3.14** (Lie groups)**.** Let $G$ be a Lie group with a bi-invariant metric, i.e., a Riemannian metric such that left and right translations, $L_g$ and $R_g$, are isometries for every $g \in G$. Let $\sigma_e(h) := h^{-1}$, for $h \in G$ and where $e \in G$ is the identity element. Then clearly $\sigma_e^2 = \mathrm{id}$, $\sigma_{e*}(e) = -\mathrm{id}$ and $\sigma_e(e) = e$. Moreover, since $\sigma_e \circ L_g = R_{g^{-1}} \circ \sigma_e$, then

$$\sigma_{e*}(g) \circ L_{g*}(e) = R_{g^{-1}*}(e) \circ \sigma_{e*}(e),$$

so that $\sigma_{e*}(g)$ is a linear isometry. Hence, $\sigma_e$ is the geodesic symmetry at $e$. For $g \in G$, we set $\sigma_g := L_g \circ \sigma_e \circ L_{g^{-1}}$, that is, $\sigma_g(h) = gh^{-1}g$. ▼

**Example 3.15** (Projective spaces)**.** Let $\mathbb{K}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ act on $\mathbb{K}^{n+1}$ by scalar multiplication. Then we define the $\mathbb{K}$ projective space as $\mathbb{KP}^n := (\mathbb{K}^{n+1} \setminus \{0\})/\mathbb{K}^\times$. By restricting to unit vectors, we have the following equalities

$$\mathbb{RP}^n = \mathbb{S}^n/\{\pm 1\}, \qquad \mathbb{CP}^n = \mathbb{S}^{2n+1}/\mathbb{S}^1, \qquad \mathbb{HP}^n = \mathbb{S}^{4n+3}/\mathbb{S}^3.$$

In general, if $G$ is a Lie group acting freely and properly by isometries on a symmetric space $(M, g)$, then it is easy to see that there is a unique metric $g'$ on the orbit space $M/G$ such that the projection $p : M \to M/G$ is a Riemmanian submersion, that is, such that the linear isomorphism $(\ker p_*(x))^\perp \cong T_{p(x)}(M/G)$ is an isometry for all $x \in M$. If in addition the action is such that $\sigma_{hx} = h\sigma_x h^{-1}$ for all $x \in M$ and $h \in G$, then $(M/G, g')$ is symmetric, with geodesic symmetry at $p(x)$ given by $\sigma_{p(x)} \circ p := p \circ \sigma_x$.

Let $\langle \cdot, \cdot \rangle$ stand for the standard Euclidean (resp. Hermitian, quaternionic) inner product on $\mathbb{R}^{n+1}$ (resp. $\mathbb{C}^{n+1}$, $\mathbb{H}^{n+1}$). Then the map $\sigma_x(y) = 2\langle y, x\rangle x - y$ is a geodesic symmetry of $\mathbb{S}^n$ (resp. $\mathbb{S}^{2n+1}$, $\mathbb{S}^{4n+3}$). The Riemannian metric on $\mathbb{S}^{2n+1}$ (resp. $\mathbb{S}^{4n+3}$) is the restriction of the Euclidean metric $\operatorname{Re}\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ (resp. of $\operatorname{Re}\langle \cdot, \cdot \rangle$ on $\mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$). Observe that this geodesic symmetry coincides with that of Example 3.12 only in the case of $\mathbb{S}^n$. It is immediate to see that they satisfy $\sigma_{hx} = h\sigma_x h^{-1}$. We conclude that $\mathbb{KP}^n$ is a symmetric space. ▼

**Lemma 3.16.** *Let $M$ be a connected symmetric space and $G = \operatorname{Isom}^0(M)$ the identity component of the isometry group of $M$. Then $G$ acts transitively on $M$, so $M = G/H$, where $H = G_x$ is the isotropy group of some $x \in M$. Moreover, $H$ is compact.*

*Proof.* Any two points $x, y \in M$ can be joined by a geodesic $\gamma : [0, 1] \to M$. Then $\sigma_{\gamma(1/2)}\sigma_x(x) = \sigma_{\gamma(1/2)}\sigma_x(\gamma(0)) = \gamma(1) = y$. By the Myers-Steenrod theorem [MS39], $\operatorname{Isom}(M)$ is a finite-dimensional Lie group. Moreover, the map $t \mapsto \sigma_{\gamma(t/2)}\sigma_x$ is a continuous map $[0, 1] \to \operatorname{Isom}(M)$ from id to $\sigma_{\gamma(1/2)}\sigma_x$, so that actually $\sigma_{\gamma(1/2)}\sigma_x \in G$. Hence $M = G/H$ with $H = G_x$. That $H$ is compact is the content of [KN63, Vol. 1, Chap. 1, Cor. 4.8]. □

From now on we fix a connected symmetric space $M$ and write $M = G/H$ for $G = \operatorname{Isom}^0(M)$ and $H = G_x$ for a fixed $x \in M$. Since $M$ is complete, then $\mathfrak{g} := \operatorname{Lie} G$ is the Lie algebra of Killing fields, i.e., vector fields $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X g = 0$. Since $H$ is the subgroup of isometries fixing $x$, then $\mathfrak{h} := \operatorname{Lie} H$ is

$$\mathfrak{h} = \{X \in \mathfrak{g} : X(x) = 0\}.$$

Let $\sigma : G \to G$ be conjugation by $\sigma_x$, that is, $\sigma(g) = \sigma_x g \sigma_x^{-1} = \sigma_x g \sigma_x$. Then $\sigma^2 = \mathrm{id}$ and $\sigma(h) = h$ for all $h \in H$. Indeed, if $h \in H$ and $y \in M$, let $\gamma : [0, 1] \to M$ be a geodesic from $x$ to $y$; then

$$\sigma(h)(y) = \sigma_x h \sigma_x(\gamma(1)) = \sigma_x(h \circ \gamma(-1)) = h(\gamma(1)) = h(y).$$

Consider as well $\sigma_* := \operatorname{Ad}(\sigma_x) : \mathfrak{g} \to \mathfrak{g}$.

**Proposition 3.17.** *We have that $\mathfrak{h} = \ker(\sigma_* - \mathrm{id})$. If $\mathfrak{p} := \ker(\sigma_* + \mathrm{id})$, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ satisfying the Cartan relations*

$$[\mathfrak{h}, \mathfrak{h}], [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h} \quad \textit{and} \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}. \tag{3.1}$$

*Moreover, $\mathfrak{p} = \{X \in \mathfrak{g} : \nabla X(x) = 0\}$ and $\mathfrak{p} \cong T_x M$.*

*Proof.* Let $X \in \mathfrak{h}$. Its flow is given by $\operatorname{Exp}(tX)$, where we use Exp to denote the Lie group exponential, not to be confused with exp, the exponential map in the Riemannian manifold $M$. Since $\operatorname{Exp}(tX) \in H$ for each $t$, we have that

$$\sigma_* X = \left.\frac{d}{dt}\right|_{t=0} \sigma(\operatorname{Exp}(tX)) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(tX) = X.$$

Conversely, if $\sigma_* X = X$, then

$$X(x) = \sigma_* X(x) = \left.\frac{d}{dt}\right|_{t=0} \sigma_x \operatorname{Exp}(tX)\sigma_x(x) = \sigma_{x*}(X(x)) = -X(x),$$

so $X(x) = 0$ and $X \in \mathfrak{h}$.

Since $\sigma_*$ is involutive, meaning $\sigma_*^2 = \mathrm{id}$, then its eigenvalues are $\pm 1$. Then the decomposition of $\mathfrak{g}$ into $\sigma_*$ eigenspaces is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. The Cartan relations follow from the fact that $\sigma_*$ is a Lie algebra homomorphism.

Finally, let $X \in \mathfrak{g}$ be such that $\nabla X(x) = 0$. Let $\gamma(t) = \exp_x(tX(x))$ be the unique geodesic starting at $x$ with velocity $X(x)$ and let $Y \in \mathfrak{X}(M)$ be given by

$$Y(y) := \left.\frac{d}{dt}\right|_{t=0} \sigma_{\gamma(t/2)}\sigma_x(y).$$

Then $Y \in \mathfrak{g}$. Let $v \in T_xM$ and let $\alpha$ be any curve starting at $x$ with velocity $v$. Then

$$\begin{aligned}\nabla_v Y(x) &= \nabla_{\frac{\partial}{\partial s}} \left.\frac{\partial}{\partial t}\right|_{s=t=0} \sigma_{\gamma(t/2)}\sigma_x(\alpha(s)) = \nabla_{\frac{\partial}{\partial t}} \left.\frac{\partial}{\partial s}\right|_{s=t=0} \sigma_{\gamma(t/2)}\sigma_x(\alpha(s)) \\ &= \left.\frac{\nabla}{dt}\right|_{t=0} (\sigma_{\gamma(t/2)}\sigma_x)_* v.\end{aligned}$$

If $V$ is a parallel vector field along $\gamma$ with $V(0) = v$, then $\sigma_{x*}V$ is parallel along $\gamma$ (traversed in the inverse direction) with $\sigma_{x*}V(0) = -v$, so that $\sigma_{x*}V(t) = -V(-t)$. Similarly, if we write $V_{t/2}(s) = V(s + \frac{t}{2})$, then $V_{t/2}$ is parallel along $\gamma_{t/2}$ with initial value $V(\frac{t}{2})$, and hence $\sigma_{\gamma(t/2)*}V_{t/2}$ is parallel along $\gamma_{t/2}$ with initial value $-V(\frac{t}{2})$. Therefore $\sigma_{\gamma(t/2)*}V_{t/2}(s) = -V_{t/2}(-s)$. This finally gives

$$\begin{aligned}(\sigma_{\gamma(t/2)}\sigma_x)_*V(s) &= \sigma_{\gamma(t/2)*}(-V(-s)) = -\sigma_{\gamma(t/2)*}\left(V_{t/2}\left(-s - \frac{t}{2}\right)\right) \\ &= V_{t/2}\left(s + \frac{t}{2}\right) = V(s+t).\end{aligned}$$

Then $(\sigma_{\gamma(t/2)}\sigma_x)_*v = (\sigma_{\gamma(t/2)}\sigma_x)_*V(0) = V(t)$, which implies $\nabla_v Y(x) = 0$. Moreover,

$$Y(x) = \left.\frac{d}{dt}\right|_{t=0} \sigma_{\gamma(t/2)}\sigma_x(x) = \dot\gamma(0) = X(x).$$

Killing fields are determined by their value and the value of their covariant derivative at a point, since so it is for Jacobi fields, and Killing fields are Jacobi fields along geodesics (they are infinitesimal variations of the geodesic by geodesics). Hence $X = Y$.

If we let $\tilde\sigma : G \times G \to G$ be given by $\tilde\sigma(g,h) = \sigma(g)h$, then

$$\tilde\sigma_*(X,X) = \sigma_*X + X = \left.\frac{d}{dt}\right|_{t=0} \tilde\sigma(\sigma_{\gamma(t/2)}\sigma_x, \sigma_{\gamma(t/2)}\sigma_x) = 0,$$

so $X \in \mathfrak{p}$.

Conversely, if $X \in \mathfrak{p}$, let $Y$ be as before. Then $X - Y \in \mathfrak{h} \cap \mathfrak{p} = 0$, so that $X = Y$. Then $\nabla X(x) = \nabla Y(x) = 0$.

Finally, consider $\mathfrak{p} \to T_xM$ the evaluation at $x$. If $X \in \mathfrak{p}$ is such that $X(x) = 0$, then $X \in \mathfrak{h} \cap \mathfrak{p} = 0$, which gives that the map is injective. For any $v \in T_xM$, let $\gamma(t) = \exp_x(tv)$ and let $Y$ be as before. Then $Y(x) = v$ and $\nabla Y(x) = 0$, so that $Y \in \mathfrak{p}$. This gives surjectivity and ends the proof. □

As said, many geometric properties of $M$ are very closely related to the Lie theoretic data of $G$ and $H$. For instance, the curvature of $M$ can be computed using the Lie bracket on $\mathfrak{p}$.

**Proposition 3.18.** *Let $R$ be the Riemann curvature of $M$. Then*

$$(R(X,Y)Z)(x) = -[[X,Y],Z](x), \qquad \textit{for } X,Y,Z \in \mathfrak{p}.$$

*Proof.* Let $X, Y \in \mathfrak{p}$ and $\gamma(t) = \exp_x(tY(x))$. As in the proof of Proposition 3.17, we have that the flow of $Y$ is given by $\sigma_{\gamma(t/2)}\sigma_x$. Then

$$Y(\gamma(t)) = \left.\frac{d}{ds}\right|_{s=0} \sigma_{\gamma(s/2)}\sigma_x(\gamma(t)) = \left.\frac{d}{ds}\right|_{s=0} \gamma(t+s) = \dot{\gamma}(t).$$

Since $X$ is Killing, and Killing fields are Jacobi fields (Remark 1.39), we have that Jacobi's equation (Definition 1.37) at time 0 gives $\nabla_Y \nabla_Y X + R(X,Y)Y = 0$ at $x$. If $Z \in \mathfrak{p}$, then we have that at $x$

$$\begin{aligned} 0 &= \nabla_{Y+Z}\nabla_{Y+Z}X + R(X, Y+Z)(Y+Z) \\ &= \nabla_Z\nabla_Y X + R(X,Z)Y + \nabla_Y\nabla_Z X + R(X,Y)Z \\ &= R(X,Z)Y + R(X,Y)Z + R(Z,Y)X + 2\nabla_Y\nabla_Z X + \nabla_{[Z,Y]}X. \end{aligned}$$

Since $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$, then $[Z,Y](x) = 0$, and hence $0 = 2(R(X,Y)Z + \nabla_Y\nabla_Z X)(x)$. Finally, at $x$,

$$\begin{aligned} R(X,Y)Z &= -R(Y,Z)X + R(X,Z)Y = \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X \\ &= \nabla_Z[X,Y] = \nabla_{[X,Y]}Z + [Z,[X,Y]] = -[[X,Y],Z], \end{aligned}$$

as wanted. □

**Corollary 3.19.** *The Ricci curvature of $M$ is given by*

$$\operatorname{Ric}(X,Y)(x) = -\operatorname{tr}((\operatorname{ad} X \operatorname{ad} Y)|_{\mathfrak{p}}), \qquad \textit{for } X, Y \in \mathfrak{p}.$$

*Proof.* Simple computation using Proposition 3.18 (here $Z \in \mathfrak{p}$):

$$\operatorname{Ric}(X,Y)(x) = \operatorname{tr}(Z(x) \mapsto (R(Z,X)Y)(x)) = \operatorname{tr}(Z \mapsto -[[Z,X]Y]) = -\operatorname{tr}((\operatorname{ad} X \operatorname{ad} Y)|_{\mathfrak{p}}).$$ □

Lastly, we show that locally symmetric spaces are determined by the parallelism properties of its curvature. Recall that if $R$ is the Riemann curvature of a Riemannian manifold $M$, then when we view $R$ as a $\operatorname{End} TM$-valued 2-form on $M$ it is $D$-closed, where $D$ is the covariant differential (Definition 1.2) of the Levi-Civita connection $\nabla$. On the other hand, $\nabla$ can be extended to act on tensor fields on $M$, and hence it makes sense to consider $\nabla R$ as a $(1,4)$-tensor field on $M$. Recall that, as we saw in Proposition 1.29, the fact that $DR = 0$ gives the second Bianchi identity for $R$:

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0, \qquad \text{for } X, Y, Z \in \mathfrak{X}(M).$$

**Proposition 3.20.** *A Riemannian space is locally symmetric if and only if its Riemann curvature is parallel.*

*Proof.* Let $M$ be locally symmetric with curvature $R$. Let $x \in M$ and $\sigma_x$ the (local) geodesic symmetry. Since $\sigma_{x*}(x) = -\mathrm{id}$ and $\sigma_x$ is a local isometry, we have that for all $u, v, w, z \in T_xM$, by Lemma 1.35,

$$-\nabla_u R(v,w)z = \sigma_{x*}(\nabla_u R(v,w)z) = \nabla_{\sigma_{x*}u} R(\sigma_{x*}v, \sigma_{x*}w)\sigma_{x*}z = \nabla_u R(v,w)z,$$

which gives $\nabla_u R = 0$.

Conversely, let $M$ be such that $\nabla R = 0$. Let $\gamma$ be a geodesic and $\{E_i\}_i$ a parallel orthonormal frame along it. Then, since $\nabla_{\dot\gamma} R = 0$, we have that $\nabla_{\dot\gamma}(R(E_i, \dot\gamma)\dot\gamma) = 0$, i.e., $R(E_i,\dot\gamma)\dot\gamma$ is parallel along $\gamma$. Hence, there are constants $\{a_i^j\}_{i,j}$ such that $R(E_i,\dot\gamma)\dot\gamma = a_i^j E_j$. The Jacobi equation for a field $J = b^i E_i$ along $\gamma$ is, then, $\ddot{b}^i + a_j^i b^j = 0$.

Let $U_x$ be a neighborhood of $x \in M$ such that $\exp_x : B(0,\epsilon) \to U_x$ is a diffeomorphism. We want to show that $\sigma_x : U_x \to U_x$ given by $\sigma_x(\exp_x v) = \exp_x(-v)$ is an isometry. Let $y = \exp_x v \in U_x$ and $u = \exp_{x*}(v)w \in T_yU_x$, for $v, w \in T_xM$. Then

$$u = \left.\frac{d}{ds}\right|_{s=0} \exp_x(v + sw).$$

Consider the Jacobi field $J(t) = \frac{d}{ds}\big|_{s=0} \exp_x(t(v+sw))$ along $t \mapsto \exp_x(tv)$. Then $u = J(1)$. Let $\{e_i\}_i$ be an orthonormal basis of $T_xM$ and extend it to a parallel orthonormal frame $\{E_i\}_i$ along the geodesic. Then the coefficients for the equation of $J$ are $\langle R(e_i, v)v, e_j\rangle$. On the other hand,

$$\sigma_{x*}u = \frac{d}{ds}\bigg|_{s=0} \sigma_x(\exp_x(v+sw)) = \frac{d}{ds}\bigg|_{s=0} \exp_x(-v-sw).$$

Consider the Jacobi field $\tilde{J}(t) = \frac{d}{ds}\big|_{s=0} \exp_x(t(-v-sw))$ along $t \mapsto \exp_x(-tv)$. Then $\sigma_{x*}u = \tilde{J}(1)$. The coefficients of the equation of $\tilde{J}$ along this geodesic with respect to the frame $\{-E_i\}_i$ are $\langle R(-e_i, -v)(-v), -e_j\rangle = \langle R(e_i, v)v, e_j\rangle$. Observe that $J(0) = \tilde{J}(0) = 0$ and that $\dot{J}(0) = w$ and $\dot{\tilde{J}}(0) = -w$. So the equations for the coefficients of $J$ with respect to $\{E_i\}_i$ and of $\tilde{J}$ with respect to $\{-E_i\}_i$ are the same with the same initial conditions. By the uniqueness of solutions to ODEs we obtain that the solution is the same. In particular, since both are coefficients with respect to an orthonormal frame, we get that their norm coincides at every time $t$. In particular, $\|\sigma_{x*}u\| = \|\tilde{J}(1)\| = \|J(1)\| = \|u\|$. Then $\sigma_x$ is an isometry, since

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \qquad \square$$

### 3.2.2. Isotropy representation and holonomy

Back to our globally symmetric space $M$, observe that, if we let $C_h$ stand for conjugation by $h$ in $G$, then, since $\sigma(h) = \sigma_x h \sigma_x = h$,

$$\sigma \circ C_h(g) = \sigma_x hgh^{-1}\sigma_x = h\sigma_x g\sigma_x h^{-1} = C_h \circ \sigma(g),$$

which gives that $\sigma_* \operatorname{Ad}(h) = (\sigma \circ C_h)_* = (C_h \circ \sigma)_* = \operatorname{Ad}(h)\sigma_*$. This implies that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is $\operatorname{Ad}(H)$-invariant.

Consider the **isotropy representation** $H \times T_xM \to T_xM$ given by $(h, v) \mapsto h_*v$.

**Lemma 3.21.** *The isotropy representation is faithful. With the identification $T_xM \cong \mathfrak{p}$, it corresponds to the adjoint representation of $H$ on $\mathfrak{p}$.*

*Proof.* Let $h \in H$ be such that $h_*(x) = \mathrm{id}$. Let $y \in M$ and $\gamma$ a starting at $x$ with $\gamma(1) = y$. Then $h \circ \gamma$ is a geodesic starting at $x$ with velocity $\dot{\gamma}(0)$, and so $h \circ \gamma = \gamma$. Hence, $h(y) = h(\gamma(1)) = \gamma(1) = y$. Therefore $h = \mathrm{id}$ and the isotropy representation is faithful.

Also, if $X \in \mathfrak{p}$, then

$$\operatorname{Ad}(h)X(x) = \frac{d}{dt}\bigg|_{t=0} h\operatorname{Exp}(tX)h^{-1}(x) = h_*(X(h^{-1}(x)) = h_*(X(x)),$$

so indeed the isotropy representation corresponds to the adjoint representation of $H$ on $\mathfrak{p}$. $\square$

**Corollary 3.22.** *Let $\mathfrak{n}$ be an ideal of $\mathfrak{g}$ with $\mathfrak{n} \subseteq \mathfrak{h}$, then $\mathfrak{n} = 0$.*

*Proof.* By the Cartan relations (3.1), we have that $[\mathfrak{n}, \mathfrak{p}] \subseteq \mathfrak{n} \cap \mathfrak{p} \subseteq \mathfrak{h} \cap \mathfrak{p} = 0$. Then $\mathfrak{n} \subseteq \ker(\operatorname{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{p}))$. By Lemma 3.21, this kernel is 0. $\square$

**Theorem 3.23.** *If $\mathfrak{g}$ is semisimple and $M$ simply connected, then the holonomy representation of $M$ is equivalent to the isotropy representation. In particular,* $\operatorname{Hol}(M) = H$.

*Proof.* Since $\nabla R = 0$ by Proposition 3.18, then the holonomy principle (Theorem 1.20) implies that $\tau_\gamma^{-1} R_y = R_x$ for any path $\gamma$ from $x$ to $y$. Then the Ambrose–Singer theorem (Theorem 1.25) gives

$$\mathfrak{hol}(M) = \operatorname{span}\{R_x(u, v) : u, v \in T_xM\} = \operatorname{im} R_x.$$

By Proposition 3.18, this is exactly $\mathrm{ad}[\mathfrak{p},\mathfrak{p}]|_{\mathfrak{p}}$, and by the Cartan relations (3.1), we conclude that $\mathfrak{hol}(M) \subseteq \mathrm{ad}\,\mathfrak{h}$, when regarding $\mathrm{ad}\,\mathfrak{h}$ inside $\mathfrak{gl}(\mathfrak{p})$.

It is easy to see that $[\mathfrak{p},\mathfrak{p}] \oplus \mathfrak{p}$ is an ideal of $\mathfrak{g}$, since by the Jacobi identity and the Cartan relations,

$$[[\mathfrak{p},\mathfrak{p}],\mathfrak{h}] \subseteq [[\mathfrak{p},\mathfrak{h}],\mathfrak{p}] \subseteq [\mathfrak{p},\mathfrak{p}].$$

Since the Killing form of $\mathfrak{g}$ is nondegenerate by Cartan's criterion ([Kna02, Thm. 1.45] for instance), then the orthogonal of $[\mathfrak{p},\mathfrak{p}] \oplus \mathfrak{p}$ with respect to the Killing form is also an ideal of $\mathfrak{g}$, and it lies inside of $\mathfrak{h}$. By Corollary 3.22, this ideal must vanish, and hence $[\mathfrak{p},\mathfrak{p}] \oplus \mathfrak{p} = \mathfrak{g}$, i.e., $[\mathfrak{p},\mathfrak{p}] = \mathfrak{h}$.

Therefore, $\mathfrak{hol}(M) = \mathrm{ad}\,\mathfrak{h}$. Since the adjoint representation corresponds to the isotropy representation by Lemma 3.21 and this one is faithful, we conclude that the holonomy representation is equivalent to the isotropy representation and $\mathfrak{hol}(M) = \mathfrak{h}$. Since $M = G/H$ is simply connected and $G$ is connected, then $H$ is also connected, and this finally gives $\mathrm{Hol}(M) = \mathrm{Hol}^0(M) = H$. □

An interesting case is when $M$ is irreducible and at least 2-dimensional, because then $\mathfrak{g}$ is semisimple. Observe that in this case $M$ has nonvanishing curvature, because if it did not we would have that $\mathfrak{hol}(M) = \mathrm{im}\, R_x = 0$, so $\mathrm{Hol}(M) = 1$, and since $M$ is irreducible it should have to be 1-dimensional.

**Proposition 3.24.** *If $M$ is irreducible of dimension at least 2 then $\mathfrak{g}$ is semisimple.*

*Proof.* The first part of the proof of Theorem 3.23 gives that $\mathfrak{hol}(M) \subseteq \mathrm{ad}\,\mathfrak{h}$. Since $\mathfrak{hol}(M)$ acts irreducibly on $T_xM \cong \mathfrak{p}$, so does $\mathrm{ad}\,\mathfrak{h}$ on $\mathfrak{p}$. Suppose the radical $\mathfrak{r}$ of $\mathfrak{g}$ is nonzero and let $k+1$ be the level where the descending series of $\mathfrak{r}$ terminates, i.e., such that $\mathfrak{r}^{(k)} \neq 0$ and $\mathfrak{r}^{(k+1)} = [\mathfrak{r}^{(k)},\mathfrak{r}^{(k)}] = 0$. Since $\mathfrak{r}^{(k)}$ is an ideal, then $[\mathfrak{h},\mathfrak{r}^{(k)} \cap \mathfrak{p}] \subseteq \mathfrak{r}^{(k)} \cap \mathfrak{p}$, so $\mathfrak{r}^{(k)} \cap \mathfrak{p}$ is $\mathrm{ad}\,\mathfrak{h}$-invariant.

If $\mathfrak{r}^{(k)} \cap \mathfrak{p} = 0$, then $\mathfrak{r}^{(k)}$ is an ideal of $\mathfrak{g}$ inside of $\mathfrak{h}$, so by Corollary 3.22 it must vanish, which is not possible. If $\mathfrak{r}^{(k)} \cap \mathfrak{p} = \mathfrak{p}$ then $\mathfrak{p} \subseteq \mathfrak{r}^{(k)}$ and $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{r}^{(k+1)} = 0$. Then by Proposition 3.18 we have that $\mathrm{im}\, R_x = \mathrm{ad}[\mathfrak{p},\mathfrak{p}]|_{\mathfrak{p}} = 0$, which cannot be because $\dim M \geq 2$. □

### 3.2.3. Orbits of $s$-representations

In submanifold theory, 1-connected semisimple symmetric spaces are of great importance, where we call a symmetric space semisimple if its algebra of Killing vector fields is semisimple.

**Definition 3.25.** A representation of a Lie group is called an ***s*-representation** if it is equivalent to the isotropy representation of a 1-connected semisimple symmetric space. ♦

In words of [BCO16], for many reasons, orbits of $s$-representations play in submanifold theory the same role as symmetric spaces in Riemannian geometry. We will now focus on showing that orbits of irreducible $s$-representations are submanifolds with constant principal curvatures.

Following the notation of the rest of the section, let $M = G/H$ be a 1-connected irreducible symmetric space of dimension at least 2, and write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ for its Cartan decomposition. Here $G = \mathrm{Isom}^0(M)$ and $H = G_x$ for some $x \in M$. Then $\mathfrak{g}$ is semisimple by Proposition 3.24. Let $B$ denote the Killing form of $\mathfrak{g}$, which is nondegenerate [Kna02, Thm. 1.45]. We recall that it is given by

$$B(X,Y) := \mathrm{tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y).$$

It is symmetric, and it is also invariant under Lie algebra automorphisms of $\mathfrak{g}$. Indeed, if $\theta$ is such an automorphism, then

$$B(\theta X,\theta Y) = \mathrm{tr}(\mathrm{ad}(\theta X)\,\mathrm{ad}(\theta Y)) = \mathrm{tr}(\theta(\mathrm{ad}\,X\,\mathrm{ad}\,Y)\theta^{-1}) = \mathrm{tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y) = B(X,Y).$$

In particular, it holds for $\sigma_* = \mathrm{Ad}(\sigma_x)$. This gives that $B(\mathfrak{p},\mathfrak{h}) = 0$, because if $X \in \mathfrak{p}$ and $Y \in \mathfrak{h}$, then

$$B(X,Y) = B(\sigma_* X,\sigma_* Y) = -B(X,Y).$$

This in turn implies that $B|_{\mathfrak{h}}$ and $B|_{\mathfrak{p}}$ are nondegenerate.

**Lemma 3.26.** *$B|_{\mathfrak{h}}$ is negative definite.*

*Proof.* Let $X \in \mathfrak{h}$. By the Cartan relations (3.1) we have that $(\operatorname{ad} X)^2|_{\mathfrak{h}}$ and $(\operatorname{ad} X)^2|_{\mathfrak{p}}$ are linear endomorphisms of $\mathfrak{h}$ and $\mathfrak{p}$, respectively. Hence,

$$B(X, X) = \operatorname{tr}(\operatorname{ad} X)^2|_{\mathfrak{h}} + \operatorname{tr}(\operatorname{ad} X)^2|_{\mathfrak{p}}.$$

The first term is $B_{\mathfrak{h}}(X, X)$, where $B_{\mathfrak{h}}$ is the Killing form of $\mathfrak{h}$. Since $H$ is compact, by Lemma 3.16, then there is an $\operatorname{Ad}(H)$-invariant inner product on $\mathfrak{h}$, with respect to which $\operatorname{ad} X|_{\mathfrak{h}}$ is skew-self-adjoint. Hence, $\operatorname{ad} X|_{\mathfrak{h}}$ diagonalizes with imaginary eigenvalues, and $B_{\mathfrak{h}}(X, X)$ is the sum of the squares of such eigenvalues, so that $B_{\mathfrak{h}}(X, X) \leq 0$. If it is 0, then $\operatorname{ad} X|_{\mathfrak{h}} = 0$ and $B(X, \cdot) = 0$. By the nondegeneracy of $B$, this can only happen if $X = 0$.

As for the second term, consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ given by

$$\langle Y, Z \rangle := \langle Y(x), Z(x) \rangle, \qquad \text{for } Y, Z \in \mathfrak{p}.$$

It is $\operatorname{Ad}(H)$-invariant, since by Lemma 3.21, if $h \in H$,

$$\langle \operatorname{Ad}(h)Y, \operatorname{Ad}(h)Z \rangle = \langle h_*(Y(x)), h_*(Z(x)) \rangle = \langle Y, Z \rangle.$$

Then $\operatorname{ad} X|_{\mathfrak{p}}$ is skew-self-adjoint with respect to this product, and the same reasoning as with $\operatorname{ad} X|_{\mathfrak{h}}$ gives that $\operatorname{tr}(\operatorname{ad} X)^2|_{\mathfrak{p}} \leq 0$ and it is 0 if and only if $X = 0$. □

For $X \in \mathfrak{p}$, consider the linear functional $B|_{\mathfrak{p}}(X, \cdot)$. Then there is $X^* \in \mathfrak{p}$ such that $B|_{\mathfrak{p}}(X, \cdot) = \langle X^*, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the $\operatorname{Ad}(H)$-invariant product given as in the proof of Lemma 3.26. Then the linear map $\mathfrak{p} \to \mathfrak{p}$ given by $X \mapsto X^*$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, by the symmetry of $B$. Therefore, there is an eigenspace decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m$. Let $\lambda_i$ be the real eigenvalue corresponding to $\mathfrak{p}_i$, and let $X_i \in \mathfrak{p}_i$. Recall that because $B$ is $\operatorname{Ad}(G)$-invariant then $\operatorname{ad} X$ is skew-self-adjoint for every $X \in \mathfrak{g}$. Then if $Y \in \mathfrak{h}$ and $Z \in \mathfrak{p}$ we have that

$$B([Y, X_i], Z) = -B(X_i, [Y, Z]) = -\lambda_i \langle X_i, [Y, Z] \rangle = \lambda_i \langle [Y, X_i], Z \rangle.$$

Hence, $\mathfrak{p}_i$ is $\operatorname{ad}\mathfrak{h}$-invariant. Since $M$ is irreducible and the holonomy representation coincides with the adjoint representation of $H$ on $\mathfrak{p}$, by Theorem 3.23 and Lemma 3.21, then $\mathfrak{p}_i = \mathfrak{p}$, i.e., there is $\lambda \in \mathbb{R}$ such that $B(X, Y) = \lambda \langle X, Y \rangle$ for all $X, Y \in \mathfrak{p}$. Since $B$ is nondegenerate, we must have that $\lambda \neq 0$.

We consider now the product $B'$ on $\mathfrak{g}$ which coincides with $-B$ if $\lambda < 0$ and which is $-B|_{\mathfrak{h}} + B|_{\mathfrak{p}}$ if $\lambda > 0$. It is $\operatorname{Ad}(G)$-invariant and positive definite. We need a final lemma.

**Lemma 3.27.** *Let $N \hookrightarrow \bar{N}$ be a submanifold and $g \in \operatorname{Isom}(\bar{N})$ with $g(N) = N$. Then if $\xi \in T_x N^\perp$ and $v \in T_x N$, we have that $W_{g_*\xi} g_* v = g_* W_\xi v$. In particular, $W_{g_*\xi}$ and $W_\xi$ have the same eigenvalues.*

*Proof.* Lemma 1.35 gives that $\overline{\nabla}_{g_* X} g_* Y = g_* \overline{\nabla}_X Y$ for $X, Y \in \mathfrak{X}(\overline{M})$. Since $g$ preserves $N$, then if $X \in \mathfrak{X}(N)$ and $\xi \in \mathfrak{X}^\perp(N)$ we finally have that

$$W_{g_*\xi} g_* X = -(\overline{\nabla}_{g_* X} g_* \xi)^\top = -(g_* \overline{\nabla}_X \xi)^\top = -g_* (\overline{\nabla}_X \xi)^\top = g_* W_\xi X. \qquad \square$$

**Proposition 3.28.** *Let $X \in \mathfrak{p}$ be nonzero. Then the $\operatorname{Ad}(H)$-orbit of $X$ in $\mathfrak{p}$ is a submanifold with constant principal curvatures.*

*Proof.* Let $B'$ be as described above and consider the $\operatorname{Ad}(H)$-orbit of $X$, call it $N$, as a submanifold of the Euclidean space $(\mathfrak{p}, B'|_{\mathfrak{p}})$. The tangent space of $N$ at $X$ is $[\mathfrak{h}, X]$, and $Z \in \mathfrak{p}$ lies in $T_X N^\perp$ if and only if for all $Y \in \mathfrak{h}$ we have that

$$B'([Y, X], Z) = B'(Y, [X, Z]) = 0.$$

Since $B'$ is nondegenerate, then $[X, Z] = 0$, i.e., $T_X N^\perp = \mathfrak{z}_\mathfrak{p}(X) := \{Z \in \mathfrak{p} : [Z, X] = 0\}$. Let $\mathfrak{n}$ be the orthogonal complement of $\mathfrak{z}_\mathfrak{h}(X) := \{Y \in \mathfrak{h} : [Y, X] = 0\}$ in $\mathfrak{h}$ with respect to $B'|_\mathfrak{h}$. Then $[\mathfrak{n}, T_X N^\perp] \subseteq T_X N$. Indeed, if $Z, W \in T_X N^\perp$, then

$$[[Z, W], X] = -[[W, X], Z] - [[X, Z], W] = 0,$$

so $[Z, W] \in \mathfrak{z}_\mathfrak{h}(X)$, and therefore if $Y \in \mathfrak{n}$ then

$$B'([Y, Z], W) = B'(Y, [Z, W]) = 0,$$

i.e., $[Y, Z] \in T_X N$.

Let now $h : [0, 1] \to H$ be a piecewise smooth curve and consider $X(t) = \mathrm{Ad}(h(t))X$. Since $\mathfrak{h} = \mathfrak{n} \oplus \mathfrak{z}_\mathfrak{h}(X)$, there are piecewise smooth curves $n, z : [0, 1] \to H$ such that $h(t) = n(t)z(t)$ and such that $L_{n(t)^{-1}*}\dot{n}(t) \in \mathfrak{n}$ and $L_{z(t)^{-1}*}\dot{z}(t) \in \mathfrak{z}_\mathfrak{h}(X)$ for all $t$, where $L_h$ is left translation by $h \in H$ in $H$. Then

$$\begin{aligned} \frac{d}{dt}(\mathrm{Ad}(z(t))X) &= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Ad}(z(t))\,\mathrm{Ad}(z(t)^{-1}z(t+s))X \\ &= \mathrm{Ad}(z(t))[L_{z(t)^{-1}*}\dot{z}(t), X] = 0, \end{aligned}$$

so actually $X(t) = \mathrm{Ad}(n(t))X$. Let $Z \in T_X N^\perp$ and consider $Z(t) = \mathrm{Ad}(n(t))Z$. Then $[X(t), Z(t)] = \mathrm{Ad}(n(t))[X, Z] = 0$, so $Z(t) \in T_{X(t)} N^\perp$. Also, by a similar computation as before,

$$\frac{d}{dt}Z(t) = \mathrm{Ad}(n(t))[L_{n(t)^{-1}*}\dot{n}(t), Z] \in \mathrm{Ad}(n(t))[\mathfrak{n}, T_X N^\perp] \subseteq \mathrm{Ad}(n(t))T_X N = T_{X(t)}N,$$

which means that $Z(t)$ is actually $\nabla^\perp$-parallel. Since $W_Z$ and $W_{\mathrm{Ad}(n(t))Z}$ have the same eigenvalues by Lemma 3.27, we conclude that $N$ has constant principal curvatures. $\square$

*Remark* 3.29. The sectional curvature of $M$ at $x$ on the plane spanned by orthonormal $X, Y \in \mathfrak{p}$ is

$$\kappa(X, Y) = -\langle [[Y, X], X], Y\rangle = -\frac{1}{\lambda}B([[Y, X], X], Y) = \frac{1}{\lambda}B([X, Y], [X, Y]).$$

Since $B|_\mathfrak{h}$ is negative definite, then $\kappa \geq 0$ if $\lambda < 0$ and $\kappa \leq 0$ if $\lambda > 0$ (and $\kappa \neq 0$ because $M$ is irreducible and $\dim M \geq 2$, as noted earlier). Symmetric spaces with $\kappa \geq 0$ (resp. $\kappa \leq 0$) are said to be of the compact (resp. noncompact) type.

Notice that, in particular, for an irreducible symmetric space $M$ the scalar curvature is nowhere vanishing, since $\mathrm{scal}(x) = \sum_{i,j} \kappa(e_i, e_j)$, for $\{e_i\}_i$ an orthonormal basis for $T_x M$.

## 3.3. Normal Holonomy Theorem

A key step towards the proof of Simons's theorem is the normal holonomy theorem, originally proved by Olmos in [Olm90]. Essentially, this theorem states that the action of the normal holonomy on the normal space splits orthogonally into a trivial action and an $s$-representation.

**Theorem 3.30** (Normal holonomy theorem)**.** *Let $M \hookrightarrow \overline{M}$ be a submanifold of a space form and $x \in M$. Then there is an orthogonal decomposition $T_x M^\perp = V_0 \oplus \cdots \oplus V_k$ and compact normal subgroups $G_i \subseteq \mathrm{Hol}_x^{\perp 0}(M)$, for $i = 1, \ldots, k$, such that*

1. $\mathrm{Hol}_x^{\perp 0}(M) = G_1 \times \cdots \times G_k$,
2. *$G_i$ acts trivially on $V_j$ if $i \neq j$,*
3. *$G_i$ acts irreducibly on $V_i$ as an $s$-representation.*

Another way to put it, which distills what we are interested in, is the following. Recall that if $M \hookrightarrow \overline{M}$ is a submanifold, we defined

$$T_xM_0^\perp := \{\xi \in T_xM^\perp : g\xi = \xi, \text{ for all } g \in \mathrm{Hol}_x^{\perp 0}(M)\}.$$

**Corollary 3.31.** *Let $M \hookrightarrow \overline{M}$ be a submanifold of a space form and $x \in M$. Let $T_xM_s^\perp$ be the orthogonal space to $T_xM_0^\perp$ inside of $T_xM^\perp$. Then $\mathrm{Hol}_x^{\perp 0}(M)$ acts on $T_xM_s^\perp$ as an s-representation.*

The proof we give here is the original proof given by Olmos as presented in [BCO16, Chap. 3]. It uses the theory of holonomy systems developed by Simons in [Sim62].

**Definition 3.32.** Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space. An **algebraic curvature tensor** on $V$ is a $(1,3)$-tensor $R : V \times V \to \operatorname{End} V$ such that for all $u, v, w, z \in V$,

1. $R(u,v) = -R(v,u)$,
2. $\langle R(u,v)w, z\rangle = -\langle R(u,v)z, w\rangle$,
3. $\langle R(u,v)w, z\rangle = \langle R(w,z)u, v\rangle$,
4. (Bianchi identity) $R(u,v)w + R(v,w)u + R(w,u)v = 0$.

We define its **scalar curvature** as the number

$$\mathrm{scal}(R) := \sum_{i,j} \langle R(e_i, e_j)e_j, e_i\rangle,$$

where $\{e_i\}_i$ is any orthonormal basis of $V$. We denote by $\mathscr{R}(V)$ the vector space of algebraic curvature tensors on $V$. ♦

Of course, algebraic curvature tensors are defined to mimic the behavior of the Riemann curvature at a point.

If $G \subseteq \mathrm{GL}(V)$ is a Lie group acting on $V$, then the corresponding action on $\mathscr{R}(V)$ is given by

$$(gR)(u,v) = g^{-1}R(gu, gv)g, \qquad \text{for } g \in G,$$

and the action of $\mathfrak{g} := \operatorname{Lie} G$ by

$$(XR)(u,v) = R(Xu, v) + R(u, Xv) + [R(u,v), X], \qquad \text{for } X \in \mathfrak{g}.$$

**Definition 3.33.** A **holonomy system** is a triple $(V, R, G)$, where $(V, \langle\cdot,\cdot\rangle)$ is a Euclidean space, $G \subseteq \mathrm{SO}(V)$ is a compact and connected Lie subgroup and $R \in \mathscr{R}(V)$ is such that $\operatorname{im} R \subseteq \mathfrak{g}$. In this case we say that $G$ is the **holonomy group** of the system and that $\mathfrak{g}$ is its **holonomy algebra**.

We say that the system is **irreducible** if $G$ acts irreducibly on $V$. We say that it is **symmetric** if $gR = R$ for all $g \in G$, or, equivalently, if $XR = 0$ for all $X \in \mathfrak{g}$. ♦

It was Cartan the first to notice, although not using the formalism of holonomy systems, that irreducible symmetric holonomy systems can always be represented by irreducible 1-connected symmetric spaces. This is known as Cartan's construction. We first need a proposition, which computes explicitly the Levi-Civita connection of a bi-invariant metric on a Lie group.

**Proposition 3.34.** *Let $G$ be a Lie group with a bi-invariant metric, meaning a metric for which left and right translations are isometries. Then the Levi-Civita connection is the connection given by*

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

*on left-invariant vector fields $X, Y \in \mathfrak{X}(G)$.*

*The Riemannian exponential is given by $\exp_g(L_{g*}\xi) = g \operatorname{Exp} \xi$, for $g \in G$ and $\xi \in \mathfrak{g}$, where $\operatorname{Exp}$ is the Lie group exponential and $L_g$ is left translation by $g$ in $G$. In particular, $G$ is geodesically complete.*

*Proof.* Since the metric is left and right invariant, then its restriction to $\mathfrak{g} := \operatorname{Lie} G$, which we call $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, is $\operatorname{Ad}(G)$-invariant, i.e., if $\xi, \eta \in \mathfrak{g}$, then, if $R_g$ denote left and right translation by $g \in G$

$$\langle \operatorname{Ad}(g)\xi, \operatorname{Ad}(g)\eta \rangle_{\mathfrak{g}} = \langle L_{g*}R_{g^{-1}*}\xi, L_{g*}R_{g^{-1}*}\eta \rangle = \langle \xi, \eta \rangle_{\mathfrak{g}}.$$

This gives, then, that $\operatorname{ad}\xi$ is skew-self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ for every $\xi \in \mathfrak{g}$. Also, if $Y, Z \in \mathfrak{X}(G)$ are left-invariant, then $\langle Y, Z \rangle = \langle Y(e), Z(e) \rangle_{\mathfrak{g}}$ identically. Hence, if $X, Y, Z \in \mathfrak{X}(G)$ are left-invariant,

$$\begin{aligned} X\langle Y, Z \rangle = 0 &= \frac{1}{2}(\langle [X(e), Y(e)], Z(e) \rangle_{\mathfrak{g}} + \langle Y(e), [X(e), Z(e)] \rangle_{\mathfrak{g}}) \\ &= \frac{1}{2}(\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \end{aligned}$$

and since being metric is a tensorial property in all three arguments and $TG$ is parallelizable by left-invariant vector fields, we conclude that $\nabla$ is metric. Also, if $X, Y \in \mathfrak{X}(G)$ are left-invariant,

$$\nabla_X Y - \nabla_Y X - [X, Y] = \frac{1}{2}[X, Y] - \frac{1}{2}[Y, X] - [X, Y] = 0,$$

so $\nabla$ is torsion-free.

Let $g \in G$ and consider $\gamma(t) := g \operatorname{Exp}(t\xi)$, with $\xi \in \mathfrak{g}$. Let $\xi^{\mathrm{L}}$ be the unique left-invariant vector field whose value at the identity $e \in G$ is $\xi$, given by $\xi^{\mathrm{L}}(g) = L_{g*}\xi$. Then

$$\dot{\gamma}(t) = \left.\frac{d}{ds}\right|_{s=0} g \operatorname{Exp}(t\xi) \operatorname{Exp}(s\xi) = L_{g \operatorname{Exp}(t\xi)*}\xi = \xi^{\mathrm{L}}(\gamma(t)),$$

so that

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \nabla_{\xi^{\mathrm{L}}}\xi^{\mathrm{L}}(\gamma(t)) = \frac{1}{2}[\xi^{\mathrm{L}}, \xi^{\mathrm{L}}](\gamma(t)) = 0.$$

Therefore, $\gamma$ is geodesic. We conclude that $\exp_g(L_{g*}\xi) = g \operatorname{Exp}\xi$. □

**Theorem 3.35** (Cartan's construction)**.** *Let $(V, R, G)$ be an irreducible symmetric holonomy system with $R \neq 0$. Then there is an irreducible 1-connected Riemannian symmetric space $M$ such that $(V, R, G) = (T_xM, R_x, \operatorname{Hol}_x(M))$ for any $x \in M$, where $R_x$ is the Riemann curvature at $x$. In particular, $\mathfrak{g} = \operatorname{im} R$.*

*Proof.* Let $\mathfrak{l} := \mathfrak{g} \oplus V$ and define a bracket on $\mathfrak{l}$ by taking the Lie bracket on $\mathfrak{g}$ and extending it to $\mathfrak{l}$ by $[X, v] := Xv$ and $[v, w] := -R(v, w)$ for $X \in \mathfrak{g}$ and $v, w \in V$. This new bracket in $\mathfrak{l}$ is in fact a Lie bracket. We must only check the Jacobi identity. For $X, Y \in \mathfrak{g}$ and $v \in V$ we have that

$$[X, [Y, v]] + [Y, [v, X]] + [v, [X, Y]] = XYv - YXv - [X, Y]v = 0.$$

For $X \in \mathfrak{g}$ and $v, w \in V$ it is

$$[X, [v, w]] + [v, [w, X]] + [w, [X, v]] = -[X, R(v, w)] + R(v, Xw) + R(Xv, w) = (XR)(v, w) = 0,$$

by symmetry of the system. For $u, v, w \in V$ it is

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = R(v, w)u + R(w, u)v + R(u, v)w = 0,$$

by the Bianchi identity. Observe that the decomposition $\mathfrak{l} = \mathfrak{g} \oplus V$ satisfies the Cartan relations

$$[V, V], [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}, \qquad [\mathfrak{g}, V] \subseteq V.$$

Actually, $\mathfrak{l}$ is semisimple. Indeed, if $\mathfrak{n} \subseteq \mathfrak{l}$ is an ideal such that $\mathfrak{n} \subseteq \mathfrak{g}$, then the Cartan relations imply that $\mathfrak{n}V = [\mathfrak{n}, V] \subseteq \mathfrak{n} \cap V = 0$, so $\mathfrak{n} = 0$. This is an analogue of Corollary 3.22. Hence, an analogue proof to that of Proposition 3.24, using that $R \neq 0$, now gives that $\mathfrak{l}$ is semisimple, since the action of $\operatorname{ad}_{\mathfrak{l}} \mathfrak{g}$ on $V$ coincides with the action of $\mathfrak{g}$ on $V$, which is irreducible. Also, an analogue of the proof of Theorem 3.23 gives that $[V, V] \oplus V = \mathfrak{l}$, so $[V, V] = \mathfrak{g}$.

Since $\mathfrak{g}$ is compact, we have that $B|_{\mathfrak{g}}$ is negative definite, where $B$ is the Killing form of $\mathfrak{l}$. Then, the same argument as that prior to Lemma 3.27 gives an $\mathrm{Ad}(L)$-invariant positive definite product $\langle\cdot,\cdot\rangle_{\mathfrak{l}}$ on $\mathfrak{l}$.

Let $L$ be the 1-connected Lie group integrating $\mathfrak{l}$ and $G'$ the unique connected subgroup of $L$ integrating $\mathfrak{g}$, i.e., $G' = \langle \operatorname{Exp}\mathfrak{g}\rangle$ [DK00, Thms. 1.14.3 & 1.10.3]. Let $M = L/G'$, which is a smooth manifold. Define $\sigma_* : \mathfrak{l} \to \mathfrak{l}$ by $X + v \mapsto X - v$. It is an involution with eigenspaces $\mathfrak{g}$ and $V$. Let $\pi : L \to M$ be the projection. For any $x = \pi(h) \in M$, elements of $T_xM$ are of the form $\pi_* R_{h*}\xi$ for $\xi \in \mathfrak{l}$, where $R_h$ is right translation by $h$ in $L$. Then we can define a Riemannian metric on $M$ by

$$\langle \pi_* R_{h*}\xi, \pi_* R_{h*}\eta\rangle := \langle \xi,\eta\rangle_{\mathfrak{l}}, \qquad \text{for } \xi,\eta\in\mathfrak{l}.$$

The $\mathrm{Ad}(L)$-invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{l}}$ ensures that this is well defined.

Actually, $M$ is a symmetric space. Let $\sigma : L \to L$ be the homomorphism lifting $\sigma_*$ to $L$ [DK00, Cor. 1.10.5]. The isometry at $x = \pi(h)$ is given by $\sigma_x \circ \pi = \pi \circ R_h \circ \sigma \circ R_{h^{-1}}$. It is clear that $\sigma_x^2 = \mathrm{id}$ and $\sigma_x(x) = x$, since $\sigma^2 = \mathrm{id}$ and $\sigma$ sends the identity to the identity. Also, since actually $T_xM \cong V$, any tangent vector is of the form $\pi_* R_{h*}v$, for $v \in V$, which then gives that

$$\sigma_{x*}(\pi_* R_{h*} v) = \pi_* R_{h*}\sigma_* v = -\pi_* R_{h*} v$$

for any $v \in V$, so $\sigma_{x*}(x) = -\mathrm{id}$. Lastly, $\sigma_x$ is an isometry: for any $\xi, \eta \in \mathfrak{l}$ and $k \in L$ we have that

$$\begin{aligned}\langle \sigma_{x*}\pi_* R_{k*}\xi, \sigma_{x*}\pi_* R_{k*}\eta\rangle &= \langle \pi_* R_{h*}\sigma_* R_{h^{-1}*} R_{k*}\xi, \pi_* R_{h*}\sigma_* R_{h^{-1}*} R_{k*}\eta\rangle \\ &= \langle \pi_* R_{R_h\circ\sigma\circ R_{h^{-1}}(k)*}\xi, \pi_* R_{R_h\circ\sigma\circ R_{h^{-1}}(k)*}\eta\rangle = \langle\xi,\eta\rangle_{\mathfrak{l}}.\end{aligned}$$

Let now $R_x$ denote the Riemann curvature of $M$ at $x$. Suppose that we have proved the following equality

$$R_x(\pi_* R_{h*}u, \pi_* R_{h*}v)\pi_* R_{h*}w = \pi_* R_{h*}(R(u,v)w), \qquad \text{for } u,v,w\in V. \tag{3.2}$$

Then, as in the proof of Theorem 3.23, the Ambrose–Singer theorem and the holonomy principle imply that

$$\mathfrak{hol}_x(M) = \operatorname{im} R_x \cong \operatorname{im} R \subseteq \mathfrak{g}.$$

But $\mathfrak{g} = [V,V] = \operatorname{im} R$, so $\mathfrak{g} = \mathfrak{hol}_x(M)$, and hence $\mathrm{Hol}_x(M) = G$. This also gives that $M$ is irreducible.

It only remains to prove (3.2). Consider in $L$ the following metric: $(L_{k*}\xi, L_{k*}\eta) := \langle\xi,\eta\rangle_{\mathfrak{l}}$, for $k \in L$. It is of course left-invariant, but it is actually also right-invariant:

$$(R_{k*}\xi, R_{k*}\eta) = \big(L_{k*}\,\mathrm{Ad}(k^{-1})\xi, L_{k*}\,\mathrm{Ad}(k^{-1})\eta\big) = \langle \mathrm{Ad}(k^{-1})\xi, \mathrm{Ad}(k^{-1})\eta\rangle_{\mathfrak{l}} = \langle\xi,\eta\rangle_{\mathfrak{l}}.$$

Let $\nabla^L$ be its Levi-Civita connection, given by Proposition 3.34. Since $M = L/G'$, any vector field $X \in \mathfrak{X}(M)$ is $\pi$-related to a vector field $\tilde{X} \in \mathfrak{X}(L)$. Consider the connection $\nabla$ on $M$ such that $\nabla_X Y$ is $\pi$-related to $\nabla^L_{\tilde{X}}\tilde{Y}$ for every $X, Y \in \mathfrak{X}(M)$, i.e.,

$$\nabla_X Y(\pi(k)) = \pi_*(\nabla^L_{\tilde{X}}\tilde{Y}(k)), \qquad \text{for } k\in L.$$

It is immediate to see that if $X, Y, Z \in \mathfrak{X}(M)$, then $\langle Y, Z\rangle \circ \pi = (\tilde{Y}, \tilde{Z})$, so that $X\langle Y, Z\rangle \circ \pi = \tilde{X}(\tilde{Y}, \tilde{Z})$. This directly implies that $\nabla$ is metric. Also, since $[X, Y]$ is $\pi$-related to $[\tilde{X}, \tilde{Y}]$, one also easily sees that $\nabla$ is torsion-free. Hence, $\nabla$ is the Levi-Civita connection of $M$.

From Proposition 3.34 we also get that

$$\exp_{\pi(h)}(\pi_* R_{h*}\xi) = \pi(h\operatorname{Exp}(\mathrm{Ad}(h^{-1})\xi)).$$

Let $v \in V$ and let $\gamma(t) := \pi(h\,\mathrm{Exp}(t\,\mathrm{Ad}(h^{-1})v))$ be the geodesic starting at $\pi(h)$ with velocity $\pi_* R_{h*} v$. According to the proof of Proposition 3.17, the corresponding Killing vector field $X$ with vanishing covariant derivative at $\pi(h)$ is given at $\pi(k)$, for $k \in L$, by

$$\begin{aligned}
X(\pi(k)) &= \left.\frac{d}{dt}\right|_{t=0} \sigma_{\gamma(t/2)}\sigma_{\pi(h)}(\pi(k)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \pi \circ R_{h\,\mathrm{Exp}(t\,\mathrm{Ad}(h^{-1})v/2)} \circ \sigma \circ R_{\mathrm{Exp}(-t\,\mathrm{Ad}(h^{-1})v/2)h^{-1}} \circ R_h \circ \sigma \circ R_{h^{-1}}(k) \\
&= \left.\frac{d}{dt}\right|_{t=0} \pi(kh^{-1}\sigma(h\,\mathrm{Exp}(-t\,\mathrm{Ad}(h^{-1})v/2)h^{-1})h\,\mathrm{Exp}(t\,\mathrm{Ad}(h^{-1})v/2)) \\
&= \frac{1}{2}\pi_* L_{k*}\,\mathrm{Ad}(h^{-1})v - \frac{1}{2}\pi_* L_{k*}\,\mathrm{Ad}(h^{-1})\sigma_*\,\mathrm{Ad}(h)\,\mathrm{Ad}(h^{-1})v \\
&= \pi_* L_{k*}\,\mathrm{Ad}(h^{-1})v = \pi_*((\mathrm{Ad}(h^{-1})v)^{\mathrm{L}}(k)),
\end{aligned}$$

where $\xi^{\mathrm{L}}$ is the unique left-invariant vector field in $L$ with value $\xi \in \mathfrak{l}$ at the identity of $L$.

If $u, v, w, z \in V$, since $[(\mathrm{Ad}(h^{-1})u)^{\mathrm{L}}, (\mathrm{Ad}(h^{-1})v)^{\mathrm{L}}] = (\mathrm{Ad}(h^{-1})[v, w])^{\mathrm{L}}$ and using Proposition 3.18 one finally sees that

$$\langle R_x(\pi_* R_{h*}u, \pi_* R_{h*}v)\pi_* R_{h*}w, \pi_* R_{h*}z\rangle = -\langle [[u, v], w], z\rangle_{\mathfrak{l}} = \langle R(u, v)w, z\rangle_{\mathfrak{l}},$$

and this gives (3.2). □

*Remark* 3.36. In the previous proof, one actually has that $\mathrm{Isom}^0(M) = L$. Indeed, first of all notice that since $R \neq 0$ necessarily $V$ is at least of dimension 2, and hence so is $M$. By Proposition 3.24, then, $M$ is also a semisimple symmetric space. The proof of Theorem 3.23 now gives that, if $\mathfrak{h} \oplus \mathfrak{p}$ is the Cartan decomposition of the algebra of Killing vector fields on $M$, as in the previous section, then $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}]$. But in the previous proof we have seen that $\mathfrak{p} \cong V$ as Lie algebras, and so $\mathfrak{l} = [V, V] \oplus V = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$, so $L = \mathrm{Isom}^0(M)$.

If $M \hookrightarrow \overline{M}$ is a submanifold of a space form and $x \in M$, we can view its normal curvature at $x$ as a linear map $R^\perp : \Lambda^2 T_x M \to \Lambda^2 T_x M^\perp$ by means of the formula

$$\langle R^\perp(u \wedge v), \xi \wedge \eta\rangle = \langle R^\perp(u, v)\xi, \eta\rangle, \quad \text{for } u, v \in T_x M \text{ and } \xi, \eta \in T_x M^\perp.$$

Then its adjoint $R^{\perp *} : \Lambda^2 T_x M^\perp \to \Lambda^2 T_x M$ with respect to $\langle\cdot,\cdot\rangle$ is given, by Ricci's equation (see Theorem 2.4), by $R^{\perp *}(\xi \wedge \eta) = [W_\xi, W_\eta]$, where here we view $\Lambda^2 T_x M \subseteq T_x M^{\otimes 2} \cong \mathrm{End}\, T_x M$, using the metric. With this identification, the product on $\Lambda^2 T_x M$ is given by $\langle A, B\rangle = -\frac{1}{2}\operatorname{tr}(AB)$. The **adapted normal curvature** at $x$ is defined to be $\tilde{R}^\perp := R^\perp \circ R^{\perp *} : \Lambda^2 T_x M^\perp \to \Lambda^2 T_x M^\perp$, and it is given by

$$\langle \tilde{R}^\perp(\xi_1 \wedge \xi_2), \xi_3 \wedge \xi_4\rangle = \langle [W_{\xi_1}, W_{\xi_2}], [W_{\xi_3}, W_{\xi_4}]\rangle = -\frac{1}{2}\operatorname{tr}([W_{\xi_1}, W_{\xi_2}][W_{\xi_3}, W_{\xi_4}]),$$

for $\xi_i \in T_x M^\perp$.

**Lemma 3.37.** *$\tilde{R}^\perp$ is an algebraic curvature tensor on $T_x M^\perp$ with nonpositive scalar curvature. Moreover, its scalar curvature vanishes if and only if $\tilde{R}^\perp$ vanishes.*

*Proof.* The three conditions on (skew)symmetry on the arguments are clear, since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. The Bianchi identity can be easily checked by writing all the terms and noting that every term cancels with some other one because again $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Also, if $\{\xi_i\}_i$ is an orthonormal basis of $T_x M^\perp$, then

$$\mathrm{scal}(\tilde{R}^\perp) = -\frac{1}{2}\sum_{i,j}\operatorname{tr}([W_{\xi_i}, W_{\xi_j}][W_{\xi_j}, W_{\xi_i}]) = \frac{1}{2}\sum_{i,j}\operatorname{tr}([W_{\xi_i}, W_{\xi_j}]^2).$$

Since both $W_{\xi_i}$ and $W_{\xi_j}$ are self-adjoint, then $[W_{\xi_i}, W_{\xi_j}]$ is skew-self-adjoint, so it diagonalizes with imaginary eigenvalues. Then $\operatorname{tr}([W_{\xi_i}, W_{\xi_j}]^2)$ equals minus the sum of some squares, which is always nonpositive and only vanishes when every eigenvalue vanishes, i.e., when $[W_{\xi_i}, W_{\xi_j}] = 0$. Thus, $\operatorname{scal}(\tilde{R}^\perp)$ is nonpositive and vanishes only when $[W_{\xi_i}, W_{\xi_j}] = 0$ for all $i, j$, which means that $\tilde{R}^\perp(\xi_i \wedge \xi_j) = 0$ for every $i, j$, that is, $\tilde{R}^\perp = 0$. ☐

Observe that

$$\operatorname{im} \tilde{R}^\perp = R^\perp(\operatorname{im} R^{\perp *}) = R^\perp(\operatorname{im} R^{\perp *} \oplus \ker R^\perp) = R^\perp(\operatorname{im} R^{\perp *} \oplus (\operatorname{im} R^{\perp *})^\perp) = \operatorname{im} R^\perp.$$

Hence, by the Ambrose–Singer theorem, $\mathfrak{g} := \mathfrak{hol}_x^\perp(M)$ is spanned by

$$\{(\tau_\gamma^\perp \tilde{R}^\perp)(\xi_1, \xi_2) : \gamma \in \Pi_x(M) \text{ and } \xi_i \in T_x M^\perp\}.$$

We now prove three lemmas that we will need in the proof of the Normal Holonomy Theorem. We let $G := \mathrm{Hol}_x^{\perp 0}(M)$.

**Lemma 3.38.** *Let $S = \{\tau_\gamma^\perp \tilde{R}^\perp : \gamma \in \Pi_x(M)\}$ and let $T_x M^\perp = V_0 \oplus \cdots \oplus V_k$ be the unique, up to order, orthogonal decomposition with $G$ acting trivially on $V_0$ and irreducibly on $V_i$, for $i \neq 0$. Then for any $R \in S$ and $\xi, \eta \in T_x M^\perp$, if we denote by $\xi_i, \eta_i$ the projections of $\xi, \eta$ onto $V_i$, respectively, we have that*

1. *$R(\xi_i, \xi_j) = 0$ if $i \neq j$,*
2. *$R(\xi, \eta) = \sum_i R(\xi_i, \eta_i)$,*
3. *$R(\xi_i, \eta_i) V_j = 0$ if $i \neq j$,*
4. *$R(\xi_i, \eta_i) V_i \subseteq V_i$.*

*Proof.* For any $\xi', \eta' \in T_x M^\perp$ we have that $\langle R(\xi_i, \xi_j)\xi', \eta' \rangle = \langle R(\xi', \eta')\xi_i, \xi_j \rangle$. Since $G$ acts on $V_i$ and $R(\xi', \eta') \in \mathfrak{g}$, then $R(\xi', \eta')\xi_i \in V_i$ and so $\langle R(\xi_i, \xi_j)\xi', \eta' \rangle = 0$ if $i \neq j$. This gives 1, and 2 follows immediately. By Bianchi's identity and 1, if $i \neq j$, then $R(\xi_i, \eta_i)\xi_j = -R(\eta_i, \xi_j)\xi_i - R(\xi_j, \xi_i)\eta_i = 0$, which proves 3. Finally, 4 is just a consequence of $V_i$ being invariant under $G$ and $R(\xi_i, \eta_i) \in \mathfrak{g}$. ☐

**Lemma 3.39.** *Using the notation of Lemma 3.38, let $\mathfrak{g}_i$ be the span of $\{R(\xi_i, \eta_i) : \xi_i, \eta_i \in V_i, R \in S\}$, for $i \neq 0$. Then*

1. *$[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$,*
2. *$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and every $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$,*
3. *$\mathfrak{g}_i V_j = 0$ if $i \neq j$,*
4. *$\mathfrak{g}_i$ acts irreducibly on $V_i$.*

*Proof.* If $i \neq j$, then $[R(\xi_i, \eta_i), R(\xi_j, \eta_j)] V_l = 0$ for every $l$, by Lemma 3.38(3). This proves 1. That $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k$ follows from Lemma 3.38(2) and from the fact that

$$\mathfrak{g} = \operatorname{span}\{R(\xi, \eta) : \xi, \eta \in T_x M^\perp, R \in S\},$$

by the Ambrose–Singer theorem. To see that the sum is actually direct, observe that if $R(\xi_i, \eta_i) = R'(\xi_j, \eta_j)$, for some $R, R' \in S$ and $\xi_i, \eta_i \in V_i$ and $\xi_j, \eta_j \in V_j$, with $i \neq j$, then $R(\xi_i, \eta_i) V_i = R'(\xi_j, \eta_j) V_i = 0$, by Lemma 3.38(3). This means that $R(\xi_i, \eta_i)$ acts trivially on $T_x M^\perp$, since it already acts trivially on any $V_j$ with $i \neq j$, by Lemma 3.38(3) again. Now,

$$[\mathfrak{g}, \mathfrak{g}_i] = [\mathfrak{g}_1, \mathfrak{g}_i] + \cdots + [\mathfrak{g}_k, \mathfrak{g}_i] = [\mathfrak{g}_i, \mathfrak{g}_i].$$

The same reasoning that showed that $\mathfrak{g}_i \cap \mathfrak{g}_j = 0$ shows as well that $[\mathfrak{g}_i, \mathfrak{g}_i] \cap \mathfrak{g}_j = 0$, for $i \neq j$, from where it follows that $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$. Therefore, $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$, giving 2. Lemma 3.38 and the fact that $G$ acts irreducibly on each $V_i$ easily give 3 and 4. ☐

**Lemma 3.40.** *Let $G \subseteq \mathrm{SO}(V)$ be a connected Lie subgroup acting irreducibly and $R \in \mathscr{R}(V)$ with* $\operatorname{im} R \subseteq \mathfrak{g}$. *Then $G$ is compact, and it acts on $V$ as an $s$-representation if* $\operatorname{scal}(R) \neq 0$.

*Proof.* That $G$ is compact follows from the fact that any connected Lie subgroup of $\mathrm{SO}(n)$ acting irreducibly on $\mathbb{R}^n$ is closed in $\mathrm{SO}(n)$ (see for instance [KN63, Vol. 1, App. 5, Thm. 2]). Since $G$ is compact, there is a Haar measure $\mu$ on $G$. Let

$$R' := \int_G (gR) d\mu(g).$$

Then, since $\operatorname{scal}(gR) = \operatorname{scal}(R)$ for any $g \in G$, we have that $\operatorname{scal}(R') = \operatorname{scal}(R)\mu(G) \neq 0$ if $\operatorname{scal}(R) \neq 0$. Also, $R'$ is $G$-invariant. Hence, $(V, R', G)$ is an irreducible symmetric holonomy system. By Cartan's construction, Theorem 3.35, the $G$-action on $V$ corresponds to the holonomy representation of a 1-connected semisimple Riemannian symmetric space. Since in this case the holonomy and the isotropy representations coincide, we conclude that the $G$-action is an $s$-representation. ☐

We can finally give the sought proof.

*Proof.* (Of the Normal Holonomy Theorem, Theorem 3.30) Let $T_xM^\perp = V_0 \oplus \cdots \oplus V_k$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be as in Lemmas 3.38 and 3.39. Let $G_i$ be the connected Lie subgroup of $G$ with Lie algebra $\mathfrak{g}_i$. These groups are normal in $G$ because the $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$ by Lemma 3.39(2). Moreover, $G = G_1 \times \cdots \times G_k$. By Lemma 3.39(3) and Lemma 3.39(4) we have that $G_i$ acts trivially on $V_j$ and irreducibly on $V_i$, for $i \neq j$. Finally, for each $i$ let $R_i \in S$ be such that $R_i|_{V_i} \neq 0$, which exists because $\mathfrak{g}_i$ does not act trivially on $V_i$. Since $\operatorname{scal}(R_i) = 0$ if and only if $R_i = 0$, by Lemma 3.37, then we can apply Lemma 3.40 to $G_i \subseteq \mathrm{SO}(V_i)$ and conclude that $G_i$ is compact (and hence $G$ is so as well) and acts on $V_i$ as an $s$-representation. ☐

## 3.4. Simons's Holonomy Theorem

In this section we finally prove Simons's holonomy theorem, from which Berger's holonomy theorem can be deduced. A holonomy system $(V, R, G)$ is called **(non)transitive** if $G$ acts (does not act) transitively on the unit sphere of $V$.

**Theorem 3.41** (Simons's holonomy theorem)**.** *An irreducible nontransitive holonomy system is necessarily symmetric.*

The proof we give here is the geometric proof found by Olmos [Olm05], or rather an enhanced version of it presented in [BCO16]. It is long and technical. The two main ingredients are: if $V$ is a Euclidean vector space equipped with the Levi-Civita connection, $G \subseteq \mathrm{SO}(V)$ is a compact connected Lie subgroup acting irreducibly on $V$ and $v \in V$ is nonzero, then:

1. for every $w \in Gv$ and $g \in G$ there is a smooth curve $\gamma$ in $Gv$ from $w$ to $gw$ such that $g_*|_{T_w(Gv)^\perp} = \tau_\gamma^\perp$; this is is Proposition 3.47, towards which we already worked in Section 2.2 and Section 3.2.3;
2. there is some normal vector $\xi \in T_v(Gv)^\perp$ with $\xi \notin \mathbb{R}v$ such that
$$V = \sum_{t \in \mathbb{R}} T_{v+t\xi}(G(v+t\xi))^\perp;$$
this is Lemma 3.49.

### 3.4.1. Exponential map and equivariant vector fields

Let $M$ be a Riemannian manifold with a connection $\nabla$ on $TM$, and denote by $\pi$ the projection $TM \to M$. Let $U$ be the subset of vectors $v \in TM$ such that the unique geodesic starting at $\pi(v)$ with velocity $v$ is defined up to time 1, and define $\exp' : U \to M \times M$ by sending $v$ to $(\pi(v), \exp v)$. It is a diffeomorphism from a neighborhood of the zero section of $TM$ onto a neigborhood of the diagonal in $M \times M$ [Pet16, Thm. 5.5.1], by the inverse function theorem.

Let now $N$ be a submanifold of $M$ and $\exp^\perp : U \cap TN^\perp \to M$ the projection unto the second component of the restriction of $\exp'$ to $U \cap TN^\perp$. It is easy to see again that $\exp^\perp$ maps diffeomorphically a neighborhood of the zero section of $TN^\perp$ onto its image in $M$. Such an image is called a tubular neighborhood for $N$. If $N$ is compact, then it can be taken to be the image by $\exp^\perp$ of the $\epsilon$-ball bundle $\{\xi \in TN^\perp : \|\xi\| < \epsilon\}$ for some $\epsilon > 0$.

This gives a way of computing the isotropy of the slice representation.

**Lemma 3.42.** *Let $M$ be a Riemannian manifold and $G \subseteq \mathrm{Isom}(M)$ a compact Lie subgroup. For $x \in M$, consider the slice representation $G_x \times T_x(Gx)^\perp \to T_x(Gx)^\perp$ given by $(g, \xi) \mapsto g_*\xi$. Then if the norm of $\xi \in T_x(Gx)^\perp$ is small enough, the isotropy group of $\xi$ is $(G_x)_\xi = G_{\exp_x \xi}$.*

*Proof.* For any $g \in G$, since $g$ takes geodesics to geodesics, we have that $g \exp_x v = \exp_{gx}(g_* v)$ for every $v \in T_xM$. In particular, if $g \in (G_x)_\xi$, then $g \exp_x \xi = \exp_x \xi$, so $g \in G_{\exp_x \xi}$.

To show the converse, let $\epsilon > 0$ be such that the $\epsilon$-ball bundle gives a tubular neighborhood of the orbit $Gx$. Let $\|\xi\| < \epsilon$ and $g \in G_{\exp_x \xi}$. Then $\|g_*\xi\| < \epsilon$ as well and

$$\exp^\perp \xi = \exp_x \xi = g \exp_x \xi = \exp_{gx}(g_*\xi) = \exp^\perp(g_*\xi).$$

Since $\exp^\perp$ is a diffeomorphism on the $\epsilon$-ball bundle, this gives that $\xi = g_*\xi$, and therefore also $x = gx$. That is, $g \in (G_x)_\xi$. □

It also gives that on compact isometric orbits any normal vector can be extended to an equivariant normal vector field.

**Proposition 3.43.** *Let $V$ be a Euclidean vector space equipped with the Levi-Civita connection $\overline{\nabla}$, let $G \subseteq \mathrm{SO}(V)$ be a compact connected Lie subgroup and $v \in V$ nonzero. Then for any $\xi \in T_v(Gv)^\perp$ the formula $\tilde{\xi}(gv) := g_*\xi$ defines a normal vector field $\tilde{\xi} \in \mathfrak{X}^\perp(Gv)$. It is the unique equivariant normal vector field with value $\xi$ at $v$.*

*Proof.* Let $C := \{w \in V : \frac{1}{2}\|v\| \leq \|w\| \leq \frac{3}{2}\|v\|\}$, which is compact in $V$. Then there is $\epsilon' > 0$ such that for all $w \in C$ the $\epsilon'$-ball bundle gives a tubular neighborhood of $Gw$. Let $\epsilon := \min(\frac{1}{2}\|v\|, \epsilon')$ and $\xi \in T_v(Gv)^\perp$ with $\|\xi\| < \epsilon$. Then we can apply Lemma 3.42 to $\xi$ to obtain that $(G_v)_\xi = G_{\exp_v \xi} = G_{v+\xi}$. Hence, $G_{v+\xi} \subseteq G_v$.

Let $X \in \mathfrak{g}$. The infinitesimal generator of the $G$-action at any $w \in V$ is given by $X_V(w) = Xw$. Therefore $T_w(Gw) = \{Xw : X \in \mathfrak{g}\}$. Then, since $\xi \in T_v(Gv)^\perp$, which means that $\langle Xv, \xi\rangle = 0$, and $\mathfrak{g} \subseteq \mathfrak{so}(V)$, we have that

$$\langle X(v + \xi), \xi\rangle = \frac{1}{2}(\langle X\xi, \xi\rangle + \langle \xi, X\xi\rangle) = 0,$$

so $\xi \in T_{v+\xi}(G(v + \xi))^\perp$. Also, $v + \xi \in C$, so that $\xi$ lies in the tubular neighborhood of $G(v + \xi)$. Hence, Lemma 3.42 now gives that $(G_{v+\xi})_{-\xi} = G_v$, so that $G_v \subseteq G_{v+\xi}$. Therefore, $(G_v)_\xi = G_{v+\xi} = G_v$, which means that the slice representation of $G_v$ on $T_v(Gv)^\perp$ is trivial. This gives that $\tilde{\xi}$ is well defined: if $gv = hv$ for some $g, h \in G$, then $h^{-1}g \in G_v$, so $h_*^{-1} g_*\xi = \xi$, i.e., $g_*\xi = h_*\xi$.

Let now $\xi \in T_v(Gv)^\perp$ be of any length. Then there is some $\eta \in T_v(Gv)^\perp$ with $\|\eta\| < \epsilon$ and $\lambda \in \mathbb{R}$ such that $\xi = \lambda\eta$. Then $\tilde{\eta}$ is well defined by the above argument. But $\tilde{\xi}(gv) = g_*\xi = \lambda g_*\eta = \lambda\tilde{\eta}(gv)$, so $\tilde{\xi}$ is well defined as well.

Finally, if $\eta$ were another equivariant vector field with value $\xi$ at $v$, then $\eta(gv) = (g_*\eta)(gv) = g_*(\eta(g^{-1}gv)) = g_*\xi = \tilde{\xi}(gv)$. □

### 3.4.2. Transvections

Let $M \hookrightarrow \overline{M}$ be a submanifold. A **transvection** of $M$ is an isometry $g$ of $\overline{M}$ such that $g(M) = M$ and such that for every $x \in M$ there is a curve $\gamma$ in $M$ from $x$ to $g(x)$ such that $g_*|_{T_xM^\perp} = \tau_\gamma^\perp$. Let now $V$ be a Euclidean vector space, $G \subseteq \mathrm{SO}(V)$ a compact connected Lie subgroup and $v \in V$ nonzero. The proof of Simons's theorem given in [BCO16] is based on the following fact: every $g \in G$ is a transvection of the orbit $Gv$ if the action is irreducible, as we will now show. We need, though, some preliminaries.

The first fact we need is that if a connected Lie subgroup of $\mathrm{SO}(n)$ acts as an $s$-representation then it equals the connected component of the identity of its normalizer. This is a consequence of the following fact, for which we need some notions that will be introduced in Section 3.6, so it can be skipped in a first reading and revisited afterwards. Recall that the normalizer of a subgroup $H$ of a group $G$ is the subgroup $\{g \in G : gHg^{-1} = H\}$.

**Proposition 3.44.** *Let $M$ be a Riemannian manifold irreducible at $x \in M$. Let $\mathfrak{n}$ be the Lie algebra of the normalizer of $\mathrm{Hol}_x^0(M)$ inside $\mathrm{SO}(T_xM)$. Then $\mathfrak{hol}_x(M) \neq \mathfrak{n}$ if and only if $M$ is Kähler and Ricci-flat in a 1-connected neighborhood of $x$.*

*Proof.* As already described in Section 3.3, the metric on $\mathrm{End}\, T_xM$ is given by $\langle A, B\rangle = -\frac{1}{2}\operatorname{tr}(AB)$. Let $\mathfrak{g} := \mathfrak{hol}_x(M)$ and let $\mathfrak{l}$ be its orthogonal complement in $\mathfrak{n}$. First, $\mathfrak{g}$ is an ideal of $\mathfrak{n}$, because if $A \in \mathfrak{n}$ and $B \in \mathfrak{g}$, then, since $\mathrm{Exp}(tA)\,\mathrm{Exp}(sB)\,\mathrm{Exp}(-tA) \in \mathrm{Hol}_x^0(M)$ for all $s$ and $t$,

$$\left.\frac{d^2}{dtds}\right|_{s=t=0} \mathrm{Exp}(tA)\,\mathrm{Exp}(sB)\,\mathrm{Exp}(-tA) = \left.\frac{d^2}{dtds}\right|_{s=t=0} \mathrm{Exp}(s\,\mathrm{Ad}(\mathrm{Exp}(tA))B)$$
$$= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\mathrm{Exp}(tA))B = [A, B] \in \mathfrak{g}.$$

Also, the metric $\langle\cdot,\cdot\rangle$ is Ad-invariant, since the adjoint action in $\mathrm{GL}(T_xM)$ is given by conjugation, so that if $g \in \mathrm{GL}(T_xM)$,

$$\langle \mathrm{Ad}(g)A, \mathrm{Ad}(g)B\rangle = -\frac{1}{2}\operatorname{tr}(gAg^{-1}gBg^{-1}) = \langle A, B\rangle.$$

Hence, $\mathfrak{l}$ is also an ideal of $\mathfrak{n}$, because if $A \in \mathfrak{n}$, $B \in \mathfrak{l}$ and $C \in \mathfrak{g}$, then

$$\langle [A, B], C\rangle = -\langle B, [A, C]\rangle = 0.$$

Therefore, $[\mathfrak{l}, \mathfrak{g}] = 0$. Assume that $\mathfrak{l} \neq 0$ and let $J_x \in \mathfrak{l}$ be nonzero. Then $J_x^2$ is self-adjoint and commutes with $\mathfrak{g}$, and so also with $\mathrm{Hol}_x^0(M)$, because the endomorphism exponential is given by a series. If $E_x$ is an eigenspace of eigenvalue $\lambda \in \mathbb{R}$ (these exist because $J_x^2$ is self-adjoint), then $E_x$ is $\mathrm{Hol}_x^0(M)$-invariant. Since $M$ is irreducible, we must have that $E_x = T_xM$, so that $J_x^2 = \lambda\mathrm{id}$. Since for all $v \in T_xM$ we have that $\langle J_x^2 v, v\rangle = \lambda\|v\|^2 = -\langle J_x v, J_x v\rangle = -\|J_x v\|^2$, we must have $\lambda < 0$, and by rescaling we may take $J_x^2 = -\mathrm{id}$.

By the holonomy principle, there is a parallel complex structure $J$ on a simply connected neighborhood of $x$. Hence, $M$ on that neighborhood is Kähler, by Proposition 3.67. Observe that if $X$ and $Y$ are local vector fields around $x$ and $\{E_i\}_i$ is a local orthonormal frame, using Proposition 3.72,

$$\langle R(X,Y), J\rangle = -\frac{1}{2}\operatorname{tr}(R(X,Y)J) = -\frac{1}{2}\sum_i \langle E_i, R(X,Y)JE_i\rangle$$
$$= \frac{1}{2}\sum_i \langle JE_i, R(X,Y)E_i\rangle = -\frac{1}{2}\sum_i \langle JE_i, R(Y,E_i)X\rangle - \frac{1}{2}\sum_i \langle JE_i, R(E_i,X)Y\rangle$$
$$= -\frac{1}{2}\mathrm{Ric}(Y, JX) + \frac{1}{2}\mathrm{Ric}(X, JY) = \mathrm{Ric}(X, JY).$$

But if $\gamma$ is a curve inside this neighborhood from $x$ to $y$, since parallel transport is a linear isometry, then

$$\langle R(X,Y)_y, J_y\rangle = \langle \tau_\gamma^{-1} R(X,Y)_y \tau_\gamma, J_x\rangle = 0,$$

because $J_x \in \mathfrak{l}$ and, by the Ambrose–Singer theorem, $\tau_\gamma^{-1} R(X,Y)_y \tau_\gamma \in \mathfrak{g}$. Hence, $M$ is locally Ricci-flat at $x$.

Conversely, assume that $M$ is locally Kähler and Ricci-flat around $x$. Let $J$ be the complex structure. Then, since $J$ is parallel, it is invariant under conjugation by elements of $\mathrm{Hol}_x^0(M)$, so $[J_x, \mathfrak{g}] = 0$, i.e. $J_x \in \mathfrak{n}$. Since $M$ is Ricci-flat, then by the previous argument using the Ambrose–Singer theorem, $J_x$ is orthogonal to $\mathfrak{g}$. We conclude, then, that $\mathfrak{g} \neq \mathfrak{n}$. □

**Corollary 3.45.** *Let $G \subseteq \mathrm{SO}(n)$ be a connected Lie subgroup acting irreducibly as an s-representation. Then $G$ equals the connected component of the identity of its normalizer in $\mathrm{SO}(n)$.*

*Proof.* Let $M$ be a 1-connected irreducible symmetric space such that the $G$-action is equivalent to the isotropy (or holonomy) representation of $M$. By Remark 3.29, $M$ cannot be Ricci-flat, so the result now follows from Proposition 3.44. □

The last ingredient we need is a result by Olmos, whose proof falls outside the scope of this thesis. Here we adapt the statement to our needs.

**Proposition 3.46** ([BCO16, Cor. 5.1.8])**.** *Let $V$ be a Euclidean vector space equipped with the Levi-Civita connection, $G \subseteq \mathrm{SO}(V)$ a compact connected Lie subgroup acting irreducibly on $V$ and $v \in V$ nonzero. If $\dim T_w(Gv)_0^\perp \geq 2$, for some $w \in Gv$, then $Gv$ is the orbit of an s-representation.*

In particular, if $\dim T_w(Gv)_0^\perp \geq 2$, then $Gv$ has constant principal curvatures by Proposition 3.28.

**Proposition 3.47.** *Let $V$ be a Euclidean vector space equipped with the Levi-Civita connection, $G \subseteq \mathrm{SO}(V)$ a compact connected Lie subgroup acting irreducibly on $V$ and $v \in V$ nonzero. Then for every $w \in Gv$ and $g \in G$ there is a piecewise smooth curve $\gamma$ in $Gv$ from $w$ to $gw$ such that $g_*|_{T_w(Gv)^\perp} = \tau_\gamma^\perp$.*

*Proof.* Let $g \in G$ and let $\overline{\nabla}$ be the Levi-Civita connection on $V$ and $\nabla$ and $\nabla^\perp$ the Levi-Civita and normal connections on $Gv$. Then, as in the proof of Lemma 3.27, we have that $g_*(\nabla_X Y) = \nabla_{g_* X} g_* Y$, for $X, Y \in \mathfrak{X}(Gv)$. This immediately gives that for every $w \in Gv$,

$$\mathrm{Hol}_w^{\perp 0}(Gv) = g_*|_{T_w(Gv)^\perp}^{-1} \mathrm{Hol}_{gw}^{\perp 0}(Gv) g_*|_{T_w(Gv)^\perp}.$$

Let $w \in Gv$ and let $\tilde{g} : [0,1] \to G$ be a smooth curve from the identity to $g$. Let $\gamma(t) := \tilde{g}(t)v$. Let $T_w(Gv)^\perp = V_0 \oplus \cdots \oplus V_k$ be the decomposition given by the normal holonomy theorem (Theorem 3.30), and let $T_{gw}(Gv)^\perp = W_0 \oplus \cdots \oplus W_k$ be the corresponding decomposition. Since both representations are equivalent, with the equivalence being given by conjugation by $\tau_\gamma^\perp$, we can assume that the representations $V_i$ and $W_i$ are equivalent. Let $i \neq 0$. Since $\mathrm{Hol}_w^{\perp 0}(Gv)$ acts on $V_i$ we can regard it as being inside of $\mathrm{SO}(V_i)$. Then $(\tau_\gamma^\perp)^{-1} g_*|_{V_i}$ is in the identity component of the normalizer of $\mathrm{Hol}_w^{\perp 0}(Gv)$ in $\mathrm{SO}(V_i)$, since

$$(\tau_\gamma^\perp)^{-1} g_*|_{V_i} \mathrm{Hol}_w^{\perp 0}(Gv) g_*|_{V_i}^{-1} \tau_\gamma^\perp = (\tau_\gamma^\perp)^{-1} \mathrm{Hol}_{gw}^{\perp 0}(Gv)^{-1} \tau_\gamma^\perp = \mathrm{Hol}_w^{\perp 0}(Gv).$$

Since $\mathrm{Hol}_w^{\perp 0}(Gv)$ acts on $V_i$ as an irreducible $s$-representation, Corollary 3.45 gives that actually $(\tau_\gamma^\perp)^{-1} g_*|_{V_i}$ lies in $\mathrm{Hol}_w^{\perp 0}(Gv)$, i.e., there is a null-homotopic curve $\alpha_i$ in $Gv$ such that $g_*|_{V_i} = \tau_\gamma^\perp \tau_{\alpha_i}^\perp$. Since $\tau_{\alpha_j}^\perp$, for $j \neq i$, acts trivially on $V_i$, we also have that $g_*|_{V_i} = \tau_\gamma^\perp \tau_\alpha^\perp$, with $\alpha = \alpha_1 \cdot \ldots \alpha_k$ null-homotopic. Hence, on $T_w(Gv)_s^\perp = V_1 \oplus \cdots \oplus V_k$ we have that

$$g_*|_{T_w(Gv)_s^\perp} = \tau_\gamma^\perp \tau_\alpha^\perp.$$

It only remains to see that it also holds on $T_w(Gv)_0^\perp$. First of all, note that $Gv \subseteq \mathbb{S}(\|v\|)$, where $\mathbb{S}(r)$ is the sphere of radius $r$ in $V$. Hence, the radial vector field $\xi(w) = w$ is a normal vector field of $Gv$. Moreover, it is $\nabla^\perp$-parallel, since $\overline{\nabla}_X \xi = X$, so that $\nabla_X^\perp \xi = X^\perp = 0$ for all $X \in \mathfrak{X}(Gv)$. Hence, $\dim T_w(Gv)_0^\perp \geq 1$.

If $\dim T_w(Gv)_0^\perp = 1$, then $\xi$ is the only normal $\nabla^\perp$-parallel vector field. Since $W_\xi = -\mathrm{id}$, we have that $W_\xi$ has constant eigenvalues, and hence $Gv$ would have constant principal curvatures. If $\dim T_w(Gv)_0^\perp > 1$, then Proposition 3.46 gives that $Gv$ has constant principal curvatures as well. Observe that if $\eta$ is a curvature normal, then $g_*\eta$ is so as well. Indeed, if $E(w)$ is the corresponding eigenspace of $\eta$, so that

$$W_\xi u = \langle \eta(w), \xi \rangle u, \quad \text{for } \xi \in T_w(Gv)_0^\perp \text{ and } u \in E(w),$$

then, by Lemma 3.27,

$$W_{g_*\xi} g_* u = g_* W_\xi u = \langle g_*\eta(gw)), g_*\xi \rangle g_* u.$$

Hence $t \mapsto \tilde{g}(t)_*\eta(w)$ is a smooth family of curvature normals at $w$. Since there is only a finite amount of curvature normals, we conclude that $g_*\eta = \eta$, i.e., $\eta$ is $G$-equivariant. Since the curvature normals span $T(Gv)_0^\perp$ pointwise and they are $\nabla^\perp$-parallel, by Proposition 2.9, then for all $\xi \in T_w(Gv)_0^\perp$ we have that the corresponding $G$-equivariant vector field given by Proposition 3.43 is $\nabla^\perp$-parallel. Hence,

$$\tau_\gamma^\perp \tau_\alpha^\perp \xi = \tau_\gamma^\perp \xi = g_*\xi,$$

and this ends the proof. □

### 3.4.3. Proof of Simons's theorem

We finally prove Simons's theorem. Here we follow [BCO16, Sec. 8.2] closely.

First of all, the fact that $G$ lies in the group of transvections of the orbit $Gv$ has the following consequence, which will be key in the proof.

**Proposition 3.48.** *Let $V$ be a Euclidean vector space equipped with the Levi-Civita connection, $G \subseteq \mathrm{SO}(V)$ a compact connected Lie subgroup acting irreducibly and $v \in V$ nonzero. Let $X \in \mathfrak{g}$ and define $\bar{X} : T_v(Gv)^\perp \to T_v(Gv)^\perp$ by*

$$\bar{X}\xi := \left.\frac{\nabla^\perp}{dt}\right|_{t=0} \mathrm{Exp}(tX)_*\xi,$$

*where $\nabla^\perp$ is the normal connection of the orbit $Gv$. Then $\bar{X} \in \mathfrak{hol}_v^\perp(Gv)$.*

*Proof.* We can write $\bar{X}$ as

$$\bar{X} = \left.\frac{d}{dt}\right|_{t=0} (\tau_t^\perp)^{-1} \mathrm{Exp}(tX)_*|_{T_v(Gv)^\perp},$$

where $\tau_t^\perp$ is $\nabla^\perp$-parallel transport along the curve $t \mapsto \mathrm{Exp}(tX)v$ up to time $t$. By Proposition 3.47, for each $t$ there is a curve $\gamma_t$ in $Gv$ such that $\mathrm{Exp}(tX)_*|_{T_v(Gv)^\perp} = \tau_{\gamma_t}^\perp$, and so

$$(\tau_t^\perp)^{-1} \mathrm{Exp}(tX)_*|_{T_v(Gv)^\perp} \in \mathrm{Hol}_v^\perp(Gv)$$

for each $t$. Then $\bar{X} \in \mathfrak{hol}_v^\perp(Gv)$. □

Observe that actually, since elements of $G$ are linear isometries,

$$\bar{X}\xi = \left(\left.\frac{\overline{\nabla}}{dt}\right|_{t=0} \mathrm{Exp}(tX)_*\xi\right)^\perp = \left(\left.\frac{d}{dt}\right|_{t=0} \mathrm{Exp}(tX)\xi\right)^\perp = X_V(\xi)^\perp,$$

that is, $\bar{X}$ is the orthogonal projection unto $T_v(Gv)^\perp$ of $X_V|_{T_v(Gv)^\perp}$.

In what follows, we prove three technical lemmas.

**Lemma 3.49.** *Let $V$ be a Euclidean vector space, $G \subseteq \mathrm{SO}(V)$ a compact connected Lie subgroup that is not transitive on the unit sphere of $V$ and $v \in V$ nonzero. Then there is some normal vector $\xi \in T_v(Gv)^\perp$ with $\xi \notin \mathbb{R}v$ such that*

$$V = \sum_{t \in \mathbb{R}} T_{v+t\xi}(G(v+t\xi))^\perp.$$

*Moreover, $v \in T_{v+t\xi}(G(v+t\xi))^\perp$ for all $t \in \mathbb{R}$.*

*Proof.* First we see that $v \in T_{v+t\xi}(G(v+t\xi))^\perp$ for all $t \in \mathbb{R}$. Since $T_{v+t\xi}(G(v+t\xi)) = \{X(v+t\xi) : X \in \mathfrak{g}\}$ and because $\mathfrak{g} \subseteq \mathfrak{so}(V)$, we have that for all $X \in \mathfrak{g}$,

$$\langle v, X(v+t\xi)\rangle = \frac{1}{2}(\langle v, Xv\rangle + \langle Xv, v\rangle) - t\langle Xv, \xi\rangle = 0.$$

Let $S$ be the sphere of $V$ of radius $\|v\|$. Since $G \subseteq \mathrm{SO}(V)$ and the orbit of $\mathrm{SO}(V)$ through $v$ is $S$, we have that $Gv$ is a submanifold of $S$. If $T_v(Gv)^\perp = \mathbb{R}v = T_vS^\perp$, then $\dim Gv = \dim S$ and $Gv$ would be an open submanifold of $S$. But it is also a compact submanifold of $S$, and hence topologically closed in $S$. The connectedness of $S$ would give then that $Gv = S$, which cannot be because $G$ is not transitive. Therefore, $T_v(Gv)^\perp \neq \mathbb{R}v$ and we can pick some $\xi \in T_v(Gv)^\perp \setminus \mathbb{R}v$.

Note as well that the Weingarten operator in the direction of $v$ is $-\mathrm{id}$, since $v$ is the value of the radial vector field at $v$. Hence, by perturbing $\xi$ by some multiple of $v$ we can assume that $\det W_\xi \neq 0$.

Define now

$$U := \left(\sum_{t\in\mathbb{R}} T_{v+t\xi}(G(v+t\xi))^\perp\right)^\perp.$$

We aim at showing that $U = 0$.

Write $\gamma(t) = v + t\xi$ for simplicity. To any $X \in \mathfrak{g}$ we can associate a vector field along $\gamma$ given by

$$J_X(t) := X_V(\gamma(t)) = Xv + tX\xi.$$

Since $U \subseteq T_{\gamma(t)}(G\gamma(t))$ for all $t \in \mathbb{R}$, any element in $U$ can be written as $Xv$ for some $X \in \mathfrak{g}$. For any such $X$, since $J_X(t) \in T_{\gamma(t)}(G\gamma(t))$ for any fixed $t \in \mathbb{R}$, and $Xv \in U$, we have that

$$\langle J_X(t), \eta\rangle = \langle Xv, \eta\rangle + t\langle \dot{J}_X(0), \eta\rangle = t\langle \dot{J}_X(0), \eta\rangle = 0,$$

for any $\eta \in T_{\gamma(t)}(G\gamma(t))^\perp$, which gives that $\dot{J}_X(0) \in T_{\gamma(t)}(G\gamma(t))$ for all $t \neq 0$. To see that also $\dot{J}_X(0) \in T_v(Gv)$, note that $T_{\gamma(t)}(G\gamma(t)) \to T_v(Gv)$ as $t \to 0$. Indeed, if we write $\gamma(t)/\|\gamma(t)\| = h(t)v$ for some smooth curve $h : \mathbb{R} \to \mathrm{SO}(V)$, then

$$\begin{aligned} T_{\gamma(t)}(G\gamma(t)) &= \{X\gamma(t) : X \in \mathfrak{g}\} = \left\{X\frac{\gamma(t)}{\|\gamma(t)\|} : X \in \mathfrak{g}\right\} = \{Xh(t)v : X \in \mathfrak{g}\} \\ &\longrightarrow \{Xv : X \in \mathfrak{g}\} = T_v(Gv). \end{aligned}$$

Hence, $\dot{J}_x(0) \in U$.

Let now $\tilde{\xi}$ be the equivariant vector field normal to $Gv$ with value $\xi$ at $v$. Then for all $g \in G$ and $f \in C^\infty(Gv)$ we have that

$$\begin{aligned} X_V(gv)(\tilde{\xi}f) &= \left.\frac{d}{dt}\right|_{t=0} \tilde{\xi}f(\mathrm{Exp}(tX)gv) = \left.\frac{d}{dt}\right|_{t=0} g_*\xi(f \circ \mathrm{Exp}(tX)) \\ &= g_*\xi\left(\left.\frac{d}{dt}\right|_{t=0} f \circ \mathrm{Exp}(tX)\right) = g_*\xi(X_Vf) = \tilde{\xi}(gv)(X_Vf), \end{aligned}$$

i.e., $[\tilde{\xi}, X_V] = 0$ on $Gv$. Therefore,

$$\dot{J}_X(0) = \left.\frac{\overline{\nabla}}{dt}\right|_{t=0} J_X(t) = \overline{\nabla}_\xi X_V = \overline{\nabla}_{X_V(v)}\tilde{\xi} = \nabla^\perp_{Xv}\tilde{\xi} - W_\xi(Xv) \in U.$$

From all this we conclude that $\nabla^\perp_U\tilde{\xi} = 0$ and $W_\xi U \subseteq U$.

Since $W_\xi$ is self-adjoint, we have as well that $W_\xi(U^\perp \cap T_v(Gv)) \subseteq U^\perp \cap T_v(Gv)$. For any $Y \in \mathfrak{g}$ such that $Yv \in U^\perp \cap T_v(Gv)$ we have that $\dot{J}_Y(0) = \nabla^\perp_{Yv}\tilde{\xi} - W_\xi(Yv) \in U^\perp$, because $\nabla^\perp_{Yv}\tilde{\xi} \in T_v(Gv)^\perp \subseteq U^\perp$. Hence, $J_Y(t) = Yv + t\dot{J}_Y(0) \in U^\perp$.

Let now $X_1, \ldots, X_k \in \mathfrak{g}$ be such that $\{X_iv\}_i$ is an eigenbasis for $W_\xi|_U$, with respective eigenvalues $\{\lambda_i\}_i$. Note that $\lambda_i \neq 0$ for all $i$, since $\det W_\xi \neq 0$. Then $J_{X_i}(t) = (1 - \lambda_it)X_iv$. For any

$Z \in \mathfrak{g}$, write $Z = X + Y$, with $X$ in the span of $\{X_i\}_i$ and with $Y$ such that $Yv \in U^\perp \cap T_v(Gv)$. Then

$$J_Z(t) = \sum_i J_{X_i}(t) + J_Y(t) = \sum_i (1 - \lambda_i t) X_i v + J_Y(t).$$

For every $i$, at time $t = 1/\lambda_i$ we have $\langle J_Z(t), X_i v\rangle = (1-\lambda_i t)\|X_i v\|^2 = 0$. Hence, $X_i v \in T_{\gamma(1/\lambda_i)}(G\gamma(1/\lambda_i))^\perp$, since $T_{\gamma(t)}(G\gamma(t)) = \{Z\gamma(t) : Z \in \mathfrak{g}\} = \{J_Z(t) : Z \in \mathfrak{g}\}$. Then $X_i v \in U \cap U^\perp \cap T_v(Gv) = 0$. This finally gives $U = 0$, as wanted. □

**Lemma 3.50.** *Let $(V, R, G)$ be a holonomy system. Then*

1. *$T_v(Gv)^\perp$ is invariant under $R$ for all $v \in V$, that is,*

$$R(T_v(Gv)^\perp, T_v(Gv)^\perp)T_v(Gv)^\perp \subseteq T_v(Gv)^\perp;$$

2. *the restriction of $R$ to $T_v(Gv)^\perp$ is invariant under $\mathrm{Hol}_v^{\perp 0}(Gv)$.*

*Proof.* Let $v \in V$. Then for any $u, w \in V$ and $\xi \in T_v(Gv)^\perp$ we have that, since $R(u, w) \in \mathfrak{g}$,

$$\langle R(u, w)v, \xi\rangle = 0 = \langle R(v, \xi)u, w\rangle,$$

so $R(v, \xi) = 0$. Then the Bianchi identity gives that $R(\xi, \eta)v = -R(\eta, v)\xi - R(v, \xi)\eta = 0$ for any $\xi, \eta \in T_v(Gv)^\perp$, which means that $R(\xi, \eta) \in \mathfrak{g}_v$, where $\mathfrak{g}_v$ is the isotropy algebra of $v$, given by $\mathfrak{g}_v = \{X \in \mathfrak{g} : Xv = 0\}$. For any $X \in \mathfrak{g}_v$, $Y \in \mathfrak{g}$ and $\xi \in T_v(Gv)^\perp$ we have that

$$\langle X\xi, Yv\rangle = -\langle \xi, XYv\rangle = -\langle \xi, [X, Y]v\rangle = 0,$$

and hence $\mathfrak{g}_v T_v(Gv)^\perp \subseteq T_v(Gv)^\perp$. This gives 1.

Let $\gamma$ be a piecewise smooth curve in $Gv$ and let $\xi$ be a $\nabla^\perp$-parallel vector field normal to $Gv$ along $\gamma$. Then

$$\frac{d}{dt}\xi(t) = \frac{\overline{\nabla}}{dt}\xi = \frac{\nabla^\perp}{dt}\xi - W_{\xi(t)}(\dot\gamma(t)) \in T_{\gamma(t)}(G\gamma(t))$$

(here $\overline{\nabla}$ is the Levi-Civita connection of $V$).

Let $\xi_i$, for $i = 1, 2, 3, 4$, be vector fields of such kind. Because $R$ is a constant tensor on $V$, then $R(\gamma(t))$ is $\overline{\nabla}$-parallel, so

$$\begin{aligned}
0 &= \langle \left(\frac{\overline{\nabla}}{dt} R(\gamma(t))\right)(\xi_1(t), \xi_2(t))\xi_3(t), \xi_4(t)\rangle \\
&= \frac{d}{dt}\langle R(\xi_1(t), \xi_2(t))\xi_3(t), \xi_4(t)\rangle - \langle R\left(\frac{d}{dt}\xi_1(t), \xi_2(t)\right)\xi_3(t), \xi_4(t)\rangle \\
&\quad - \langle R\left(\xi_1(t), \frac{d}{dt}\xi_2(t)\right)\xi_3(t), \xi_4(t)\rangle - \langle R(\xi_1(t), \xi_2(t))\frac{d}{dt}\xi_3(t), \xi_4(t)\rangle \\
&\quad - \langle R(\xi_1(t), \xi_2(t))\xi_3(t), \frac{d}{dt}\xi_4(t)\rangle.
\end{aligned}$$

Since $\frac{d}{dt}\xi_i(t) \in T_{\gamma(t)}(G\gamma(t))$, then 1 gives that $\frac{d}{dt}\langle R(\xi_1(t), \xi_2(t))\xi_3(t), \xi_4(t)\rangle = 0$, and this immediately implies 2. □

**Lemma 3.51.** *Let $(V, R, G)$ be a holonomy system, $X \in \mathfrak{g}$ and $W \subseteq V$ a subspace which is invariant under both $R$ and $XR$. Let $\bar{X}$ be the orthogonal projection unto $W$ of $X_V|_W$. Then $(XR)|_W = \bar{X} \cdot R|_W$.*

*Proof.* Let $w_1, w_2, w_3 \in W$ and $\xi \in W^\perp$. Then $\langle R(w_1, \xi)w_2, w_3\rangle = \langle R(w_2, w_3)w_1, \xi\rangle = 0$ and $\langle R(w_1, w_2)\xi, w_3\rangle = -\langle R(w_1, w_2)w_3, \xi\rangle = 0$. Let $P : V \to V$ be orthogonal projection unto $W$. Explicitly, $\bar{X}$ is given by $PXw$ on $w \in W$. Then, if $w_4 \in W$,

$$\begin{aligned}
\langle (XR)(w_1, w_2)w_3, w_4\rangle &= \langle XR(w_1, w_2)w_3 - R(Xw_1, w_2)w_3 - R(w_1, Xw_2)w_3 - R(w_1, w_2)Xw_3, w_4\rangle \\
&= \langle PXR(w_1, w_2)w_3 - R(\bar{X}w_1, w_2)w_3 - R(w_1, \bar{X}w_2)w_3 - R(w_1, w_2)\bar{X}w_3, w_4\rangle \\
&= \langle (\bar{X} \cdot R|_W)(w_1, w_2)w_3, w_4\rangle,
\end{aligned}$$

and this gives the result. □

**Theorem 3.52** (Simons)**.** *Every nontransitive irreducible holonomy system is symmetric.*

*Proof.* Let $(V, R, G)$ be a nontransitive holonomy system and $X \in \mathfrak{g}$. Let $v \in V$ be nonzero, $\xi \in T_v(Gv)^\perp$ as in Lemma 3.49 and $\gamma(t) = v + t\xi$. Write $W_t := T_{\gamma(t)}(G\gamma(t))^\perp$, $R_t := R|_{W_t}$, $(XR)_t := (XR)|_{W_t}$ and $\bar{X}_t$ for the orthogonal projection unto $W_t$ of $X_V|_{W_t}$. Since $(V, XR, G)$ is also a holonomy system, then a combination of Lemmas 3.50 and 3.51 gives that $(XR)_t = \bar{X}_t R_t$. By Proposition 3.48, $\bar{X}_t \in \mathfrak{hol}^\perp_{\gamma(t)}(G\gamma(t))$, so Lemma 3.50 gives that $\bar{X}_t R_t = 0$.

By Lemma 3.49 any $w \in V$ can be written as $w = \sum_i w_i$ with $w_i \in W_{t_i}$, for some $t_i \in \mathbb{R}$, and where this sum is finite. Then, since $v \in W_{t_i}$ again by Lemma 3.49,

$$(XR)(v, w)v = \sum_i (XR)(v, w_i)v = \sum_i (XR)_{t_i}(v, w_i)v = 0.$$

Since this holds for any $v \neq 0$, $XR$ has vanishing sectional curvatures, so $XR = 0$, by Proposition 1.41. Since $X$ was arbitrary, we conclude that indeed $(V, R, G)$ is symmetric. □

## 3.5. Berger's Holonomy Theorem

We finally prove Berger's holonomy theorem for Riemannian manifolds. We then study the transitive actions on the sphere so as to give the original version of the theorem, in the form of a list. We end by considering the special geometries that arise from having the different holonomies in Berger's list.

### 3.5.1. Berger's theorem

The argument for Berger's theorem will boil down to a contradiction on the dimension of the manifold. It will rely on the following remark.

*Remark* 3.53. If $V$ is a Euclidean vector space and $G \subseteq \mathrm{SO}(V)$ is a connected compact Lie subgroup acting irreducibly on $V$ then any orbit $Gv$ with $v \neq 0$ must have dimension at least 2 unless $\dim V \leq 2$. Indeed, if $\dim Gv = 0$ then $\mathbb{R}v$ is $G$-invariant and hence $V = \mathbb{R}v$. If $\dim Gv = 1$, then any $g \in G_v$ must be the identity on $Gv$, because by the Hopf–Rinow theorem $Gv$ is geodesically complete, so that $g|_{Gv}$ is totally determined by $g_*|_{T_v(Gv)}$, which is the identity because $T_v(Gv)$ is 1-dimensional. The subspace $\mathrm{span}(Gv)$ is $G$-invariant and contains $v$, so $V = \mathrm{span}(Gv)$, which implies that $g$ is globally the identity. Hence $G_v$ is trivial and therefore $\dim G = 1$. Since $G$ is abelian and acts irreducibly, then $\dim V$ is at most 2.

An easy consequence of Cartan's construction is the following, which we will also need in the proof of Berger's theorem.

**Lemma 3.54.** *Let $(V, R, G)$ and $(V, R', G)$ be two irreducible symmetric holonomy systems of dimension at least 2 with $R \neq 0$. Then $R'$ is a scalar multiple of $R$.*

*Proof.* If $R' = 0$, it is clear. If $R' \neq 0$, then by Cartan's construction both $R$ and $R'$ have nonvanishing scalar curvatures by Remark 3.29. Hence there is some $\lambda \in \mathbb{R}$ such that $R'' := R' - \lambda R$ has vanishing scalar curvature. Since $(V, R'', G)$ is also irreducible and symmetric of dimension at least 2, if $R'' \neq 0$ Cartan's construction would give that $R''$ has nonvanishing scalar curvature. Hence $R'' = 0$. □

**Theorem 3.55** (Berger)**.** *An irreducible Riemannian manifold of dimension at least 2 whose restricted holonomy group is not transitive on the unit sphere is locally symmetric.*

*Proof.* Let $M$ be such a manifold and let $x \in M$ be such that $R_x \neq 0$, which is possible because $M$ is irreducible and of dimension at least 2. Any connected Lie subgroup of $\mathrm{SO}(n)$ acting irreducibly on $\mathbb{R}^n$ is closed in $\mathrm{SO}(n)$ [KN63, Vol. 1, App. 5, Thm. 2], from where it follows that $\mathrm{Hol}^0_x(M)$ is compact. Then $(T_xM, R_x, \mathrm{Hol}^0_x(M))$ is an irreducible nontransitive holonomy system. By Simons's theorem it is symmetric, and so by Theorem 3.35 we have that $\mathfrak{hol}_x(M) = \operatorname{im} R_x$.

Let $v \in T_xM$ and $\gamma$ any curve with $\gamma(0) = x$ and $\dot\gamma(0) = v$. Then if we denote by $\tau_t$ the parallel transport along $\gamma$ from $x$ to $\gamma(t)$,

$$\nabla_v R = \left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}(R_{\gamma(t)}),$$

where we recall that $\tau_t^{-1}(R_{\gamma(t)})(u,w) = \tau_t^{-1} R_{\gamma(t)}(\tau_t u, \tau_t w)\tau_t$ for $u, w \in T_xM$. This formula together with the Ambrose–Singer theorem and the fact that $R_{\gamma(t)}$ is also symmetric for any $t$ by Simons's theorem gives that $\nabla_v R$ is a symmetric algebraic curvature tensor with $\operatorname{im} \nabla_v R \subseteq \mathfrak{hol}_x(M)$. Hence $(T_xM, \nabla_v R, \mathrm{Hol}^0_x(M))$ is also an irreducible symmetric holonomy system.

By Lemma 3.54 there is $\lambda \in T_x^*M$ such that $\nabla_v R = \lambda(v) R_x$ for all $v \in T_xM$. Let $u \in T_xM$ be such that $\lambda(v) = \langle u, v\rangle$ for all $v \in T_xM$ and denote by $U^\perp$ its orthogonal complement in $T_xM$. Suppose $u \neq 0$. Since $\mathrm{Hol}^0_x(M)$ acts nontransitively on the sphere we must have that $\dim T_xM \geq 3$, because the only nontrivial connected compact Lie subgroup of $\mathrm{SO}(2)$ is itself and it acts transitively on the sphere.

Let $v \in T_xM$ and $w, z \in U^\perp$. Then by the second Bianchi identity (Proposition 1.29),

$$\begin{aligned} 0 &= \nabla_v R(w,z) + \nabla_w R(z,v) + \nabla_z R(v,w) \\ &= \langle u,v\rangle R(w,z) + \langle u,w\rangle R(z,v) + \langle u,z\rangle R(v,w) \\ &= \langle u,v\rangle R(w,z), \end{aligned}$$

so $R(w,z) = 0$. So for all $v_1, v_2 \in T_xM$ we have that $\langle R(v_1,v_2)w, z\rangle = \langle R(w,z)v_1, v_2\rangle = 0$, i.e., $R(v_1,v_2)w \in \mathbb{R}u$. Since $\operatorname{im} R_x = \mathfrak{hol}_x(M)$ and $\mathfrak{hol}_x(M)w$ is the tangent space to the orbit of $\mathrm{Hol}^0_x(M)$ through $w$, we get that this orbit is at most 1-dimensional. By Remark 3.53 this would mean that $\dim T_xM \leq 2$, which cannot be. Hence, $u = 0$ and $\nabla_v R = 0$ for all $v \in T_xM$.

Let $S := \overline{\{x \in M : R_x \neq 0\}} \neq \emptyset$. Then the complement of $S$ is an open set of $M$ where $R$ vanishes, so that $\nabla R = 0$ also outside of $S$. Hence $\nabla R = 0$ everywhere and $M$ is locally symmetric by Proposition 3.20. $\square$

### 3.5.2. Transitive actions on the sphere

By Berger's theorem, the restricted holonomy group of an irreducible and not locally symmetric Riemannian manifold is transitive on the sphere. The transitive actions on the sphere were classified by Montgomery and Samelson [MS43] and Borel [Bor49]. The list is given in Table 3.1.

| **Group** | $\mathrm{SO}(n)$ | $\mathrm{U}(n)$ | $\mathrm{SU}(n)$ | $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ | $\mathrm{Sp}(n)\,\mathrm{U}(1)$ | $\mathrm{Sp}(n)$ | $G_2$ | $\mathrm{Spin}(7)$ | $\mathrm{Spin}(9)$ |
|---|---|---|---|---|---|---|---|---|---|
| **Sphere it acts on** | $\mathbb{S}^{n-1}$ | $\mathbb{S}^{2n-1}$ | $\mathbb{S}^{2n-1}$ | $\mathbb{S}^{4n-1}$ | $\mathbb{S}^{4n-1}$ | $\mathbb{S}^{4n-1}$ | $\mathbb{S}^6$ | $\mathbb{S}^7$ | $\mathbb{S}^{15}$ |

Table 3.1: Transitive actions on the sphere.

From this list, two cases can be thrown away for the classification of holonomy. On the one hand, if a Riemannian manifold has holonomy inside of $\mathrm{Sp}(n)\,\mathrm{U}(1)$, then it necessarily lies inside of $\mathrm{Sp}(n)$ [Bes02, 10.66]. On the other hand, it was shown by Alekseevskii [Ale68] and Brown and Gray [BG72] that a Riemannian manifold with holonomy $\mathrm{Spin}(9)$ is necessarily symmetric. We are left, then, with seven possible cases.

We will now define the relevant groups arising in the classification, and in the following section we will explore what special properties each type of holonomy confers to the geometry of the manifold.

#### (Special) orthogonal group

Let $V$ be an oriented Euclidean space with metric $\langle\cdot,\cdot\rangle$. Recall that the special linear group is defined as the group of orientation-preserving isomorphisms of $V$:

$$\mathrm{SL}(V) := \{\tau \in \mathrm{GL}(V) : \tau\mathrm{vol} = \mathrm{vol} \text{ for some nonzero vol} \in \Lambda^n V^*\},$$

where $\tau$ acts on $V^*$ by $\tau\lambda(v) := \lambda(\tau^{-1}v)$. Observe that if $\tau \in \mathrm{GL}(V)$ preserves some nonzero vol $\in \Lambda^n V^*$, then it preserves any other element in $\Lambda^n V^*$, since this is a real line.

Then we recall that the **orthogonal group** of $V$ is defined as the metric preserving isomorphisms:

$$\mathrm{O}(V) := \{\tau \in \mathrm{GL}(V) : \ \langle \tau v, \tau w\rangle = \langle v, w\rangle \text{ for all } v, w \in V\}$$

and the **special orthogonal group** of $V$ as

$$\mathrm{SO}(V) := \mathrm{SL}(V) \cap \mathrm{O}(V).$$

We write $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ when $V = \mathbb{R}^n$ with the standard Euclidean structure.

#### (Special) unitary group

Let now $\dim V = 2n$ and $J$ be an orthogonal linear complex structure on $V$, that is, $J \in \operatorname{End} V$ such that $J^2 = -\mathrm{id}$ and $\langle Jv, Jw\rangle = \langle v, w\rangle$, for all $v, w \in V$. Define $\omega \in \Lambda^2 V^*$ by $\omega(v, w) := \langle v, Jw\rangle$ (it is indeed a 2-form, since $J$ is orthogonal). From $V$ we can produce two different complex vector spaces. On the one hand, we can define complex scalar multiplication on $V$ by $(a + ib)v := av + bJv$, for $v \in V$. We call the resulting vector space $V(\mathbb{C})$, with complex dimension $n$. The complex general linear group of $V$ is then defined as

$$\mathrm{GL}(V, J) := \mathrm{GL}(V(\mathbb{C})) = \{\tau \in \mathrm{GL}(V) : \tau J = J\tau\}.$$

On $V(\mathbb{C})$ we can define a Hermitian metric by $h(v, w) := \langle v, w\rangle + i\omega(v, w)$. By Hermitian we mean that $h(v, w) = \overline{h(w, v)}$ for all $v, w \in V(\mathbb{C})$, that it is $\mathbb{C}$-linear in the first component and that it is positive-definite. We define, then, the **unitary group** of $V$ as

$$\begin{aligned}\mathrm{U}(V, J) &:= \{\tau \in \mathrm{GL}(V(\mathbb{C})) : \ h(\tau v, \tau w) = h(v, w) \text{ for all } v, w \in V(\mathbb{C})\} \\ &= \mathrm{GL}(V, J) \cap \mathrm{O}(V).\end{aligned}$$

To be able to define the special unitary group of $V$, we need to consider the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$, with complex dimension $2n$. We can extend $J$ by $\mathbb{C}$-linearity to a complex endomorphism of $V_{\mathbb{C}}$. Since its minimal polynomial is $x^2 + 1$, it diagonalizes with eigenvalues $\pm i$. Let $V^{1,0}$ and $V^{0,1}$ be the eigenspaces of eigenvalues $i$ and $-i$, respectively. It is very easy to see that

$$V^{1,0} = \{v - iJv : v \in V\} \quad \text{and} \quad V^{0,1} = \{v + iJv : v \in V\}.$$

Since $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, we can identify $(V^{1,0})^*$ with the annihilator $(V^{0,1})^\circ$, i.e.,

$$(V^{1,0})^* = \{\alpha \in V_{\mathbb{C}}^* : \ \alpha(Jv) = i\alpha(v) \text{ for all } v \in V\},$$

and similarly,

$$(V^{0,1})^* = \{\alpha \in V_{\mathbb{C}}^* : \ \alpha(Jv) = -i\alpha(v) \text{ for all } v \in V\}.$$

Hence it makes sense to consider elements of $(V^{1,0})^*$ as "holomorphic" forms and elements of $(V^{0,1})^*$ as "antiholomorphic" forms, and to define $(p, q)$-forms ($p$ times holomorphic and $q$ times antiholomorphic) as elements of

$$\Lambda^{p,q} V^* := \operatorname{span}\left(\Lambda^p (V^{1,0})^* \wedge \Lambda^q (V^{0,1})^*\right).$$

This gives a decomposition of forms

$$\Lambda^k V_{\mathbb{C}}^* = \bigoplus_{p+q=k} \Lambda^{p,q} V^*.$$

Conjugation can be defined on $V_{\mathbb{C}}$ as the map $\lambda v \mapsto \overline{\lambda} v$, for $\lambda \in \mathbb{C}$ and $v \in V$. It is conjugate-linear as a map $V_{\mathbb{C}} \to V_{\mathbb{C}}$. Then we can define a Hermitian metric on $V_{\mathbb{C}}$ by $h(v,w) := \langle v, \overline{w} \rangle$, for $v, w \in V_{\mathbb{C}}$, where on the right-hand side $\langle \cdot, \cdot \rangle$ has been extended by $\mathbb{C}$-linearity to $V_{\mathbb{C}}$. We call this metric $h$ as well because of the following reason: the map $v \mapsto (v - iJv)/\sqrt{2}$ gives an isomorphism of complex vector spaces between $V(\mathbb{C})$ and $V^{1,0}$, and it is actually an isomorphism of Hermitian vector spaces: for all $v, w \in V(\mathbb{C})$,

$$\frac{1}{2} h(v - iJv, w - iJw) = \frac{1}{2} \langle v - iJv, w + iJw \rangle = \langle v, w \rangle + i \langle v, Jw \rangle = h(v,w).$$

Let now $\Omega \in \Lambda^{n,0} V^*$ be nonzero. Then $\Omega \wedge \overline{\Omega} = \mu \mathrm{vol}$ for some $\mu \in \mathbb{C}^\times$. If $\tau \in \mathrm{U}(V,J)$ and we also call $\tau$ its complex extension to $V_{\mathbb{C}}$, then, since $\Lambda^{n,0} V^*$ is a complex line, there is $\lambda \in \mathbb{C}$ such that $\tau \Omega = \lambda \Omega$. Since $\tau$ preserves $\langle \cdot, \cdot \rangle$, and hence $h$, we get that

$$h(\Omega, \Omega) = h(\tau\Omega, \tau\Omega) = |\lambda|^2 h(\Omega, \Omega),$$

i.e., $|\lambda|^2 = 1$. Therefore,

$$\mu \tau \mathrm{vol} = \tau(\mu \mathrm{vol}) = \tau(\Omega \wedge \overline{\Omega}) = |\lambda|^2 \Omega \wedge \overline{\Omega} = \mu \mathrm{vol},$$

which implies that $\tau \mathrm{vol} = \mathrm{vol}$. We conclude that actually $\mathrm{U}(V,J) = \mathrm{GL}(V,J) \cap \mathrm{SO}(V)$.

We define the complex special linear group as

$$\mathrm{SL}(V,J) := \{\tau \in \mathrm{GL}(V,J) : \tau\Omega = \Omega \text{ for some nonzero } \Omega \in \Lambda^{n,0} V^*\}.$$

Observe that if $\tau \in \mathrm{GL}(V,J)$ preserves some $\Omega \in \Lambda^{n,0} V^*$, then it preserves any other element in $\Lambda^{n,0} V^*$, since this is a complex line. Finally, the **special unitary group** of $V$ is defined as

$$\mathrm{SU}(V,J) := \mathrm{SL}(V,J) \cap \mathrm{U}(V,J) = \mathrm{SL}(V,J) \cap \mathrm{SO}(V).$$

We write $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ when $V = \mathbb{R}^{2n} = \mathbb{C}^n$ with the canonical complex structure.

### Symplectic group and $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$

Let now $\dim V = 4n$ and let $I$, $J$ and $K$ be orthogonal linear complex structures such that $IJ = K$. Let $\omega_I$, $\omega_J$ and $\omega_K$ be the corresponding 2-forms, respectively. From $V$ we can construct the quaternionic vector space $V(\mathbb{H})$ by defining quaternionic scalar multiplication on $V$ by $(a + ib + jc + kd)v := av + bIv + cJv + dKv$, for $v \in V$. On it we can define a quaternionic metric given by

$$\ell(v,w) := \langle v, w \rangle + i\omega_I(v,w) + j\omega_J(v,w) + k\omega_K(v,w), \qquad \text{for } v, w \in V(\mathbb{H}).$$

By quaternionic we mean that $\ell(v,w) = \overline{\ell(w,v)}$ for all $v, w \in V(\mathbb{H})$, that it is $\mathbb{H}$-linear in the first component and that it is positive-definite. We recall that conjugation on $\mathbb{H}$ is given by $\overline{a + ib + jc + kd} = a - ib - jc - kd$. Then we define the **symplectic group** of $V$ as

$$\begin{aligned} \mathrm{Sp}(V,I,J) &:= \{\tau \in \mathrm{GL}(V(\mathbb{H})) : \ \ell(\tau v, \tau w) = \ell(v,w) \text{ for all } v, w \in V(\mathbb{H})\} \\ &= \{\tau \in \mathrm{O}(V) : \tau I = I\tau, \ \tau J = J\tau\} \\ &= \mathrm{GL}(V,I) \cap \mathrm{GL}(V,J) \cap \mathrm{O}(V). \end{aligned}$$

Just as we had that actually $\mathrm{U}(V,J) \subseteq \mathrm{SO}(V)$, we also have here that $\mathrm{Sp}(V,I,J) \subseteq \mathrm{SU}(V,I)$ (and the same holds for $\mathrm{SU}(V,J)$ and $\mathrm{SU}(V,K)$). Indeed, consider the 2-form $\omega = \omega_J + i\omega_K$. If we write $\Lambda_I^{p,q} V^*$

for the $(p,q)$-forms with respect to $I$, then we have that $\omega \in \Lambda_I^{2,0}V^*$: for every $u, v \in V$ we have that

$$\begin{aligned}
\omega(u - iIu, v + iIv) &= \langle u - iIu, Jv - iKv\rangle + i\langle u - iIu, Kv + iJv\rangle \\
&= \langle u - iIu, Jv - iKv\rangle - \langle u - iIu, Jv - iKv\rangle = 0, \\
\omega(u + iIu, v + iIv) &= \langle u + iIu, Jv - iKv\rangle + i\langle u + iIu, Kv + iJv\rangle \\
&= \langle u + iIu, Jv - iKv\rangle - \langle u + iIu, Jv - iKv\rangle = 0.
\end{aligned}$$

Also, we have that

$$\begin{aligned}
\omega(u - iIu, v - iIv) &= \langle u - iIu, Jv + iKv\rangle + i\langle u - iIu, Kv - iJv\rangle \\
&= 2\langle u - iIu, Jv + iKv\rangle = 4(\langle u, Jv\rangle + i\langle u, Kv\rangle),
\end{aligned}$$

so that $\omega$ is nondegenerate. Then $\omega^n \in \Lambda_I^{2n,0}V^*$ is nonzero and, if $\tau \in \mathrm{Sp}(V, I, J)$, then $\tau\omega_J = \omega_J$ and $\tau\omega_K = \omega_K$, because $\tau \in \mathrm{U}(V, J)$ and $\tau \in \mathrm{U}(V, K)$. Therefore $\tau\omega^n = \omega^n$. Hence, $\tau \in \mathrm{SU}(V, I)$. We conclude that actually

$$\mathrm{Sp}(V, I, J) = \mathrm{SU}(V, I) \cap \mathrm{SU}(V, J) \cap \mathrm{O}(V).$$

If $V = \mathbb{R}^{4n} = \mathbb{H}^n$ with the standard quaternionic structure, then we write $\mathrm{Sp}(n) := \mathrm{Sp}(V, I, J)$. In this case, if $A^*$ denotes the quaternionic adjoint of a matrix $A$, by which we mean its conjugate transpose, then $\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) : A^*A = \mathrm{id}\}$. Since the norm on $\mathbb{H}$ is given by $|q|^2 := \overline{q}q$, and this coincides with the standard Euclidean norm, then we have that $\mathrm{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\} = \mathbb{S}^3$.

Back to $V$, consider the action of $\mathrm{Sp}(V, I, J) \times \mathrm{Sp}(1)$ on $V$ given by $(\tau, q)v := \tau(qv)$. Observe that it is not an $\mathbb{H}$-linear action. The kernel of the action is $\{(\mathrm{id}, 1), (-\mathrm{id}, -1)\}$. Then we define the **group $\mathbf{Sp}(\boldsymbol{V, I, J})\,\mathbf{Sp}(1)$** as the image of such an action, which is isomorphic to $\mathrm{Sp}(V, I, J) \times \mathrm{Sp}(1)$ modulo $\mathbb{Z}_2$. The following characterization will be useful for the treatment of $\mathrm{Sp}(V, I, J)\,\mathrm{Sp}(1)$ for holonomy purposes. It was first considered by Kraines [Kra66].

**Proposition 3.56.** *Let $Q := \omega_I^2 + \omega_J^2 + \omega_K^2$, where $\omega^2 := \omega \wedge \omega$. Then*

$$\mathrm{Sp}(V, I, J)\,\mathrm{Sp}(1) = \{g \in \mathrm{SO}(V) : gQ = Q\}.$$

*Proof.* Let $\tau \in \mathrm{Sp}(V, I, J)$ and $q \in \mathrm{Sp}(1)$. It is clear that $\tau Q = Q$. Write $q = a + ib + jc + kd$, with $a^2 + b^2 + c^2 + d^2 = 1$. Straightforward computation gives that

$$\begin{aligned}
q\omega_I &= (a^2 + b^2 - c^2 - d^2)\omega_I + 2(bc - ad)\omega_J + 2(ac + bd)\omega_K, \\
q\omega_J &= 2(ad + bc)\omega_I + (a^2 - b^2 + c^2 - d^2)\omega_J + 2(cd - ab)\omega_K, \\
q\omega_K &= 2(bd - ac)\omega_I + 2(ab + cd)\omega_J + (a^2 - b^2 - c^2 + d^2)\omega_K.
\end{aligned}$$

Then, straightforward computation again gives that $qQ = Q$. For instance, the coefficient of $\omega_I^2$ in $qQ$ is

$$(a^2 + b^2 - c^2 - d^2)^2 + 4(ad + bc)^2 + 4(bd - ac)^2 = (a^2 + b^2 + c^2 + d^2)^2 = 1,$$

while the coefficient of $\omega_I \wedge \omega_J$ is

$$2(bc - ad)(a^2 + b^2 - c^2 - d^2) + 2(ad + bc)(a^2 - b^2 + c^2 - d^2) + 4(bd - ac)(ab + cd) = 0.$$

Therefore, we have that elements of $\mathrm{Sp}(V)\,\mathrm{Sp}(1)$ preserve $Q$.

The converse is shown in [Sal89, Lem. 9.1] using representation theoretical techniques. $\square$

### $G_2$ and Spin(7)

Typically, $G_2$ and Spin(7) are called the **exceptional holonomy groups**. Both can described using the normed division algebra of octonions $\mathbb{O}$. Recall that this is the algebra obtained by applying the Cayley-Dickson construction to $\mathbb{H}$. More specifically, we introduce a new imaginary unit $\ell$ satisfying $\ell^2 = -1$ and the relations

$$p(\ell q) = \ell(\overline{p}q), \qquad (p\ell)q = (p\overline{q})\ell, \qquad (\ell p)(q\ell) = -\overline{pq},$$

for $p, q \in \mathbb{H}$, and we consider pairs of quaternions $p + \ell q$ for $p, q \in \mathbb{H}$. The product and conjugation rules are

$$(p + \ell q)(p' + \ell q') = pp' - q'\overline{q} + \ell(p'q + \overline{p}q') \quad \text{and} \quad \overline{p + \ell q} = \overline{p} - \ell q.$$

For a very nice account of octonions and its applications to geometry and topology see [Bae02].

Recall that the norm on $\mathbb{O}$ is given by $|a|^2 := \overline{a}a$ for $a \in \mathbb{O}$. It satisfies the parallelogram law: if $a, b \in \mathbb{O}$, then

$$|a + b|^2 + |a - b|^2 = (\overline{a} + \overline{b})(a + b) + (\overline{a} - \overline{b})(a - b) = 2(|a|^2 + |b|^2),$$

so it comes from an inner product on $\mathbb{O}$ given by the polarization identity, which gives $\langle a, b\rangle = \mathrm{Re}(\overline{a}b)$, where Re is the real part, given by $\mathrm{Re}\, a := \frac{1}{2}(a + \overline{a})$. It is routine to check that actually $|\cdot|^2$ equals the Euclidean norm on $\mathbb{R}^8$, and hence the inner product is the Euclidean product on $\mathbb{R}^8$.

The algebra $\mathbb{O}$ is not associative, but it is alternative, in the sense that the associator

$$[a, b, c] := (ab)c - a(bc), \qquad \text{for } a, b, c \in \mathbb{O},$$

is alternating, meaning that it vanishes whenever two arguments coincide [Har90, Lem. 6.11]. Moreover, Artin's theorem states that any subalgebra of $\mathbb{O}$ generated by two elements is associative (see [Har90, Thm. 6.39] for a proof).

By an automorphism of $\mathbb{O}$ we mean an algebra automorphism of $\mathbb{O}$, i.e., an element $g \in \mathrm{GL}(\mathbb{O})$ such that $g(ab) = g(a)g(b)$ for all $a, b \in \mathbb{O}$. Notice that, since $g(1) = g(1)^2$, we have that $g(1) = 1$. From this one can also deduce that $g$ preserves the imaginary octonions

$$\mathrm{Im}\,\mathbb{O} := \{a \in \mathbb{O} : \mathrm{Re}\, a = 0\} = \{a \in \mathbb{O} : \overline{a} = -a\}.$$

Indeed, it is easy to show that for $a \in \mathbb{O}$ we have that $a^2 \in \mathbb{R}$ if and only if $a \in \mathbb{R}$ or $a \in \mathrm{Im}\,\mathbb{O}$. Let $a \in \mathrm{Im}\,\mathbb{O}$ be nonzero, then, since $g(1) = 1$ and $\overline{a} = -a$,

$$g(a)^2 = g(a^2) = -|a|^2 g(1) = -|a|^2,$$

so $g(a) \in \mathbb{R}$ or $g(a) \in \mathrm{Im}\,\mathbb{O}$. If $g(a) = \lambda \in \mathbb{R}$, then $g(\lambda^{-1}a) = 1$, so $a = \lambda$, which cannot be because $a \in \mathrm{Im}\,\mathbb{O}$. Hence, $g$ preserves $\mathrm{Im}\,\mathbb{O}$. This implies, then, that $g(\overline{a}) = \overline{g(a)}$, which in turn implies that $g$ preserves the norm.

**Definition 3.57.** The **group $G_2$** is the automorphism group of $\mathbb{O}$. ♦

Since the elements of $G_2$ preserve the inner product and $\mathrm{Im}\,\mathbb{O}$, we can regard $G_2$ as sitting inside $\mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \cong \mathrm{SO}(7)$.

Alternatively, one can characterize $G_2$ by a geometrical property. On $\mathrm{Im}\,\mathbb{O}$ the product $a \times b := \mathrm{Im}(\overline{a}b)$ defines a cross product, by which we mean that it is $\mathbb{R}$-bilinear, skew-symmetric and such that $\langle a, a \times b\rangle = 0$ for all $a, b \in \mathrm{Im}\,\mathbb{O}$. The last equality follows from a straightforward computation:

$$\mathrm{Re}(\overline{a}\,\mathrm{Im}(\overline{a}b)) = \frac{1}{4}(\overline{a}(\overline{a}b - \overline{b}a) + (\overline{b}a - \overline{a}b)a) = \frac{1}{4}(-|a|^2 b - aba + |a|^2 b + aba) = 0.$$

**Proposition 3.58.** $G_2 = \{g \in \mathrm{SO}(\mathrm{Im}\,\mathbb{O}) : g(a \times b) = g(a) \times g(b) \text{ for all } a, b \in \mathbb{O}\}$.

*Proof.* If $g \in G_2$, then $g(a \times b) = g(\mathrm{Im}(\overline{a}b)) = \mathrm{Im}(\overline{g(a)}g(b)) = g(a) \times g(b)$. Conversely, assume that $g \in \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ preserves the cross product, and extend $g$ to all of $\mathbb{O}$ by $g(1) = 1$. Then for $a, b \in \mathrm{Im}\,\mathbb{O}$,

$$g(ab) = -g(\overline{a}b) = -\langle a, b\rangle - g(a \times b) = -\langle g(a), g(b)\rangle - g(a) \times g(b) = -\overline{g(a)}g(b) = g(a)g(b),$$

which already gives that $g \in G_2$. □

The dimension of $G_2$ is 14 [Bae02].

To introduce Spin(7), we need the broader notion of an isotopy of $\mathbb{O}$.

**Definition 3.59.** An **isotopy** of $\mathbb{O}$ is a triple $(g_1, g_2, g_3) \in \mathrm{SO}(\mathbb{O})^3$ such that $g_1(ab) = g_2(a)g_3(b)$ for all $a, b \in \mathbb{O}$. The group of isotopies of $\mathbb{O}$ we denote by $\mathrm{Iso}\,\mathbb{O}$. ♦

Indeed, it is a group: if $(g_1, g_2, g_3), (f_1, f_2, f_3) \in \mathrm{Iso}\,\mathbb{O}$, we define their product as $(g_1f_1, g_2f_2, g_3f_3)$, which is indeed an isotopy:

$$g_1f_1(ab) = g_1(f_2(a)f_3(b) = g_2f_2(a)g_3f_3(b).$$

Observe that $G_2$ embeds into $\mathrm{Iso}\,\mathbb{O}$, by sending $g \in G_2$ to $(g, g, g)$. Actually, we have the following third characterization of $G_2$.

**Proposition 3.60.** $G_2 = \{(g_1, g_2, g_3) \in \mathrm{Iso}\,\mathbb{O} : g_2(1) = g_3(1) = 1\}$.

*Proof.* The inclusion from left to right is obvious. Conversely, let $(g_1, g_2, g_3) \in \mathrm{Iso}\,\mathbb{O}$ be such that $g_2(1) = g_3(1) = 1$. For all $a \in \mathbb{O}$, then, we have that

$$g_1(a) = g_1(a1) = g_2(a)g_3(1) = g_2(a) = g_1(1a) = g_2(1)g_3(a) = g_3(a).$$

Hence $(g_1, g_2, g_3) = (g_1, g_1, g_1)$ and $g_1 \in G_2$. □

We are now ready to introduce Spin(7).

**Definition 3.61.** The **group Spin(7)** is the subgroup $\{(g_1, g_2, g_3) \in \mathrm{Iso}\,\mathbb{O} : g_2(1) = 1\}$ of $\mathrm{Iso}\,\mathbb{O}$. ♦

From Proposition 3.60 we immediately see that $G_2$ is a subgroup of Spin(7). Observe as well that if $(g_1, g_2, g_3) \in \mathrm{Spin}(7)$, then $g_1(a) = g_1(1a) = g_2(1)g_3(a) = g_3(a)$, so $g_1 = g_3$. Actually, we also have the converse: if $(g, g_2, g) \in \mathrm{Iso}\,\mathbb{O}$, then $g(1) = g_2(1)g(1)$, and by Artin's theorem, we get that $|g(1)|^2 = g_2(1)|g(1)|^2$, i.e., $g_2(1) = 1$. Hence,

$$\mathrm{Spin}(7) = \{(g_1, g_2, g_3) \in \mathrm{Iso}\,\mathbb{O} : g_1 = g_3\}.$$

**Proposition 3.62.** *1. The map* $\mathrm{Spin}(7) \to \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ *sending* $(g_1, g_2, g_3)$ *to* $g_2|_{\mathrm{Im}\,\mathbb{O}}$ *is a two-to-one and surjective group morphism. It is the universal covering map of* $\mathrm{SO}(\mathrm{Im}\,\mathbb{O})$.

*2. The map* $\mathrm{Spin}(7) \to \mathrm{SO}(\mathbb{O})$ *sending* $(g_1, g_2, g_3)$ *to* $g_1$ *is an injective group morphism.*

*Proof.* Let $(g, g_2, g) \in \mathrm{Spin}(7)$ be such that $g_2|_{\mathrm{Im}\,\mathbb{O}} = \mathrm{id}|_{\mathrm{Im}\,\mathbb{O}}$. Since $g_2(1) = 1$, we have that $g_2 = \mathrm{id}$. From this we get that $g(a) = g(a1) = g_2(a)g(1) = ag(1)$, i.e., $g$ is multiplication by $c := g(1)$ from the right. Then $g(ab) = (ab)c = g_2(a)g(b) = a(bc)$ for all $a, b \in \mathbb{O}$. This can only happen if $c \in \mathbb{R}$. Indeed, write $c = p + \ell q$ for some $p, q \in \mathbb{H}$. Then for any $p' \in \mathbb{H}$ we have that

$$\begin{aligned}(p'\ell)c &= (\ell\overline{p'})(p + \ell q) = -qp' + \ell(p\overline{p'}),\\ p'(\ell c) &= p'(\ell(p + \ell q)) = p'(-q + \ell p) = -p'q + \ell(\overline{p'}p).\end{aligned}$$

Hence, both $p$ and $q$ commute with all of $\mathbb{H}$, which immediately gives that $p, q \in \mathbb{R}$. Moreover,

$$\begin{aligned}((\ell i)(\ell j))c &= -k(p + \ell q) = -kp + \ell(kq),\\ (\ell i)((\ell j)c) &= (\ell i)((\ell j)(p + \ell q)) = (\ell i)(qj + \ell(pj)) = pji + \ell(qji),\end{aligned}$$

which gives that actually $q = 0$. Hence, $c \in \mathbb{R}$. Since $g \in \mathrm{SO}(\mathbb{O})$, then $|c| = 1$, from where we conclude that $c = \pm 1$. Hence, the kernel of the map in item 1 is $\{(\mathrm{id}, \mathrm{id}, \mathrm{id}), (-\mathrm{id}, \mathrm{id}, -\mathrm{id})\}$, as wanted.

To see that it is a surjective morphism, recall that by the Cartan-Dieudonné theorem (for instance [Mei13, Thm. 1.1]), any element of $\mathrm{O}(\mathrm{Im}\,\mathbb{O})$ can be written as a finite product of reflections. Then any element of $\mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ is a product of an even number of reflections. The reflection in $\mathrm{Im}\,\mathbb{O}$ with respect to the hyperplane orthogonal to $a \in \mathrm{Im}\,\mathbb{O}$ with $|a| = 1$ can be simply written as

$$\sigma_a(b) = b - 2\langle b, a\rangle a = b - (\bar{b}a + \bar{a}b)a = aba, \qquad \text{for } b \in \mathrm{Im}\,\mathbb{O}.$$

Let $L_a$ denote left multiplication by $a \in \mathbb{O}$. Then $-\sigma_a \in \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$, for $a \in \mathrm{Im}\,\mathbb{O}$ with $|a| = 1$, is the image of $(L_a, s_a, L_a)$, where $s_a \in \mathrm{SO}(\mathbb{O})$ is given by $-\sigma_a$ on $\mathrm{Im}\,\mathbb{O}$ and $s_a(1) = 1$. To see that this triple is indeed an isotopy, it suffices to check it on $b, c \in \mathrm{Im}\,\mathbb{O}$. We aim at proving that

$$a(bc) = -(aba)(ac).$$

This follows from the Moufang identity $(xyx)z = x(y(xz))$, for all $x, y, z \in \mathbb{O}$, since then we have that, by Artin's theorem,

$$-(aba)(ac) = -a(b(a(ac))) = |a|^2 a(bc) = a(bc).$$

To prove Moufang's identity, notice that $(xyx)z - x(y(xz))$ vanishes whenever two of the variables coincide, by Artin's theorem, so that, since $(xyx)z - x(y(xz)) = [xy, x, z] + [x, y, xz]$ and the associator is alternating,

$$\begin{aligned} 0 &= (x(y+z)x)(y+z) - x((y+z)(x(y+z))) \\ &= (xyx)z - x(y(xz)) + (xzx)y - x(z(xy)) \\ &= [xy, x, z] + [x, y, xz] + [xz, x, y] + [x, z, xy] \\ &= 2([xy, x, z] + [x, y, xz]). \end{aligned}$$

Lastly, since the map $\mathrm{Spin}(7) \to \mathrm{SO}(\mathrm{Im}\,\mathbb{O})$ is a 2-sheeted covering map, then the index of $\pi_1(\mathrm{Spin}(7))$ in $\pi_1(\mathrm{SO}(\mathrm{Im}\,\mathbb{O}))$ is 2 [Hat02, Prop. 1.32], and since $\pi_1(\mathrm{SO}(\mathrm{Im}\,\mathbb{O})) = \mathbb{Z}_2$ [Sep07, Thm. 1.24], then $\mathrm{Spin}(7)$ is simply connected.

To see 2, assume that $g = \mathrm{id}$. Then $a = g(a1) = g_2(a)g(1) = g_2(a)$, so that $g_2 = \mathrm{id}$, and this ends the proof. □

It follows, then, that $\dim \mathrm{Spin}(7) = 21$.

For holonomy purposes, there are equivalent definitions of $G_2$ and $\mathrm{Spin}(7)$ as stabilizers of a certain 3-form and a 4-form. Define

$$\phi(a, b, c) := \langle a, bc\rangle, \qquad \text{for } a, b, c \in \mathrm{Im}\,\mathbb{O}.$$

Then $\phi \in \Lambda^3(\mathrm{Im}\,\mathbb{O})^*$, and it is immediate to see that $G_2$ lies in the stabilizer of $\phi$. Actually, we have that

$$G_2 = \{g \in \mathrm{GL}(\mathrm{Im}\,\mathbb{O}) : g\phi = \phi\}$$

[Bry87, Sec. 2, Thm. 1]. If we denote by 1 the element of $\mathbb{O}^*$ such that $1(1) = 1$ and $1(\mathrm{Im}\,\mathbb{O}) = 0$, then consider

$$\psi := 1 \wedge \phi + *\phi,$$

where $*$ is the Hodge star operator with respect to $\langle \cdot, \cdot \rangle$. Then $\psi \in \Lambda^4\mathbb{O}^*$ and

$$\mathrm{Spin}(7) = \{g \in \mathrm{GL}(\mathbb{O}) : g\psi = \psi\}$$

[Bry87, Sec. 2, Thm. 4].

### 3.5.3. Berger's list

Knowledge of the transitive actions on the sphere gives the following reformulation of Berger's theorem (which is actually the original formulation by Berger).

**Theorem 3.63** (Berger's list)**.** *Let $M$ be an irreducible, not locally symmetric, orientable and connected Riemannian manifold of dimension $n \geq 2$. Then one of the following holds:*

1. $\mathrm{Hol}^0(M) = \mathrm{SO}(n)$,
2. $n = 2m$ *for* $m > 0$ *and* $\mathrm{Hol}^0(M) = \mathrm{U}(n)$,
3. $n = 2m$ *for* $m > 0$ *and* $\mathrm{Hol}^0(M) = \mathrm{SU}(n)$,
4. $n = 4m$ *for* $m > 0$ *and* $\mathrm{Hol}^0(M) = \mathrm{Sp}(n)$,
5. $n = 4m$ *for* $m > 0$ *and* $\mathrm{Hol}^0(M) = \mathrm{Sp}(n)\,\mathrm{Sp}(1)$,
6. $n = 7$ *and* $\mathrm{Hol}^0(M) = G_2$,
7. $n = 8$ *and* $\mathrm{Hol}^0(M) = \mathrm{Spin}(7)$.

## 3.6. Special geometries

Each of the infinite families appearing in Berger's list gives rise to a special type of geometry, meaning Riemannian manifolds with special geometric properties. We will now give a review of these.

### 3.6.1. Kähler manifolds

**Definition 3.64.** An **almost complex structure** on a manifold $M$ is an endomorphism $J \in \Gamma(\operatorname{End} TM)$ such that $J^2 = -\mathrm{id}$. An **almost Hermitian structure** on a Riemannian manifold $(M, g)$ is an almost complex structure $J$ on $M$ which is orthogonal with respect to $g$, that is, such that $\langle Ju, Jv\rangle = \langle u, v\rangle$ for all $u, v \in TM$. An **almost complex** (resp. **Hermitian**) **manifold** is a pair $(M, J)$ (resp. a triple $(M, g, J)$) such that $J$ is an almost complex structure on $M$ (resp. an almost Hermitian structure on $(M, g)$). ♦

The canonical examples of almost complex structures are those induced by complex manifolds, i.e., manifolds locally modeled on $\mathbb{C}^n$ such that the transition functions are holomorphic. If $(U, (z^j)_j)$ is a local chart for a complex manifold and we write $z^j = x^j + iy^j$ so that $(U, (x^j, y^j)_j)$ is a local chart for the underlying even-dimensional real manifold, then we can define an almost complex structure $J$ by $J\frac{\partial}{\partial x^j} := \frac{\partial}{\partial y^j}$, which is easily seen to be well defined globally.

**Definition 3.65.** An almost complex structure on a manifold $M$ is called **integrable** if it can be induced by the structure of a complex manifold on $M$. In that case we call $J$ a **complex structure** and the pair $(M, J)$ a **complex manifold**. A **Hermitian structure** $J$ on $(M, g)$ is an integrable almost Hermitian structure, and in that case the triple $(M, g, J)$ is called a **Hermitian manifold**. ♦

A deep result by Newlander and Nirenberg [NN57] states that integrability is equivalent to the vanishing of the **Nijenhuis tensor** $N \in \Omega^2(M, TM)$, defined by

$$N(X, Y) := J[X, Y] - [JX, Y] - [X, JY] - J[JX, JY] \quad \text{for } X, Y \in \mathfrak{X}(M).$$

Equivalently, $J$ is integrable if and only if the distribution $TM^{1,0}$, defined by $T_xM^{1,0} := (T_xM)^{1,0} \subseteq T_xM_{\mathbb{C}}$, is involutive with respect to the Lie bracket (extended by $\mathbb{C}$-linearity).

Consider an almost Hermitian manifold $(M, g, J)$ and define a $(0,2)$-tensor field $\omega$ by $\omega(u, v) := \langle u, Jv\rangle$, for $u, v \in TM$. It is actually a 2-form, since

$$\omega(u, v) = \langle u, Jv\rangle = -\langle Ju, v\rangle = -\omega(v, u),$$

because $J$ is orthogonal and squares to $-\mathrm{id}$. It is moreover nondegenerate, because if $\omega(u, v) = 0$ for all $v \in T_xM$, then $\langle u, v\rangle = 0$ for all $v \in T_xM$, so $u = 0$. This form is the **Kähler form** of the almost Hermitian manifold.

**Definition 3.66.** A **Kähler structure** on a Riemannian manifold $(M, g)$ is a Hermitian structure $J$ whose Kähler form is closed, in which case the triple $(M, g, J)$ is called a **Kähler manifold**. ♦

We now give a characterization of Kähler structures on a manifold $M$ in terms of its holonomy. It is a consequence of the following.

**Proposition 3.67.** *An almost Hermitian manifold $(M, g, J)$ is Kähler if and only if $\nabla J = 0$, where $\nabla$ is the Levi-Civita connection of $(M, g)$. Moreover, if $\omega$ is the Kähler form of $(M, g, J)$, then $\nabla J = 0$ if and only if $\nabla\omega = 0$.*

*Proof.* Since $\nabla$ is torsion-free, then $[X, Y] = \nabla_X Y - \nabla_Y X$. Also, the induced connection on $\operatorname{End} TM$ is given by $\nabla_X J(Y) = \nabla_X(JY) - J\nabla_X Y$. This allows for the Nijenhuis tensor to be written as

$$N(X, Y) = \nabla_Y J(X) - \nabla_X J(Y) + J(\nabla_{JY} J(X) - \nabla_{JX} J(Y)).$$

On the other hand, the fact that $\nabla$ is metric and the Koszul formula for $d\omega$ give that

$$d\omega(X, Y, Z) = \langle Y, \nabla_X J(Z)\rangle - \langle X, \nabla_Y J(Z)\rangle + \langle X, \nabla_Z J(Y)\rangle.$$

It is now clear that if $\nabla J = 0$ then $M$ is Kähler.

Conversely, assume that $M$ is Kähler. Then $d\omega = 0$ implies that $\langle X, \nabla_Y J(Z) - \nabla_Z J(Y)\rangle = \langle Y, \nabla_X J(Z)\rangle$. It is easy to see, using Proposition 1.15, that if $A, B \in \Gamma(\operatorname{End} TM)$, then $\nabla_X(AB) = (\nabla_X A)B + A\nabla_X B$, from where we see that $J\nabla_X J + (\nabla_X J)J = \nabla_X J^2 = -\nabla_X \mathrm{id} = 0$. Therefore, since $N(Z, Y) = 0$,

$$\begin{aligned} 0 = \langle X, N(Z, Y)\rangle &= \langle X, \nabla_Y J(Z) - \nabla_Z J(Y)\rangle + \langle X, J(\nabla_{JY} J(Z) - \nabla_{JZ} J(Y))\rangle \\ &= \langle Y, \nabla_X J(Z)\rangle - \langle X, \nabla_{JY} J(JZ) - \nabla_{JZ} J(JY)\rangle \\ &= \langle Y, \nabla_X J(Z)\rangle - \langle JY, \nabla_X J(JZ)\rangle \\ &= 2\langle Y, \nabla_X J(Z)\rangle. \end{aligned}$$

Hence, $\nabla J = 0$, as wanted.

Lastly, an easy computation gives that $\nabla_X \omega(Y, Z) = \langle Y, \nabla_X J(Z)\rangle$, so $\nabla J = 0$ if and only if $\nabla\omega = 0$. □

**Corollary 3.68.** *A connected Riemannian manifold $M$ of dimension $2n$ admits a Kähler structure if and only if $\mathrm{Hol}(M) \subseteq \mathrm{U}(n)$.*

*Proof.* Let $J$ be a Kähler structure on $M$. Then $\nabla J = 0$, by Proposition 3.67. Let $\gamma$ be a piecewise smooth curve on $M$ and $X \in \Gamma(\gamma^* TM)$ parallel. Then $\frac{\nabla}{dt}(JX) = \dot{J}X + J\dot{X} = 0$, so $\tau_\gamma J = J\tau_\gamma$. Let $x \in M$ and consider $\mathrm{U}(n)$ as $\{\tau \in \mathrm{O}(T_xM) : \tau J = J\tau\}$. If $\gamma$ is a loop at $x$, we have that $\tau_\gamma J = J\tau_\gamma$, while $\langle \tau_\gamma u, \tau_\gamma v\rangle = \langle u, v\rangle$ by Proposition 1.14, so $\tau_\gamma \in \mathrm{U}(n)$.

Conversely, assume that $\mathrm{Hol}(M) \subseteq \mathrm{U}(n)$, by which we mean that for $x \in M$ there is an orthogonal linear complex structure $J_x$ on $T_xM$ with respect to which $\mathrm{Hol}_x(M) \subseteq \mathrm{U}(n)$. If $\gamma$ is a loop at $x$, the action of $\tau_\gamma$ on $J_x$ is $\tau_\gamma J_x \tau_\gamma^{-1}$, and this equals $J_x$ because $\tau_\gamma \in \mathrm{U}(n)$. Hence, $J_x$ is $\mathrm{Hol}_x(M)$-invariant. By the holonomy principle, Theorem 1.20, there is a unique parallel section $J$ with value $J_x$ at $x$, and it is straightforward to see that it squares to $-\mathrm{id}$ and is orthogonal with respect to the metric. By Proposition 3.67, then, $J$ is a Kähler structure on $M$. □

**Example 3.69** (Riemann surfaces)**.** Let $(\Sigma, g)$ be an orientable Riemannian surface. Let $\omega$ be its Riemannian volume form, defined by taking the value 1 on any oriented orthonormal basis. Since it is a volume form, $\omega$ is nondegenerate. It is also closed, since $d\omega \in \Omega^3(\Sigma) = 0$.

Consider the maps

$$\begin{array}{cccc} g^\flat: & T\Sigma & \longrightarrow & T^*\Sigma, \\ & v & \longmapsto & i_v g \end{array} \qquad \begin{array}{cccc} \omega^\flat: & T\Sigma & \longrightarrow & T^*\Sigma. \\ & v & \longmapsto & i_v \omega \end{array}$$

Since both $g$ and $\omega$ are nondegenerate, both maps are isomorphisms. Denote by $g^\sharp$ and $\omega^\sharp$ their inverses. They satisfy the relation $g^\sharp \circ \omega^\flat = -\omega^\sharp \circ g^\flat$. To see this, let $u \in T_x\Sigma$ be nonzero and let $v := g^\sharp \circ \omega^\flat(u)$, i.e. $v$ is the unique vector in $T_x\Sigma$ such that $\omega(u, w) = \langle v, w\rangle$ for all $w \in T_x\Sigma$. Then $\langle u, v\rangle = \omega(u, u) = 0$ and $\|v\|^2 = \omega(u, v)$. Since $\omega$ is the Riemannian volume form we have that

$$\omega\left(\frac{u}{\|u\|}, \frac{v}{\|v\|}\right) = \pm 1 = \frac{\|v\|}{\|u\|}.$$

Since $u$ is nonzero, we must have that $\|u\| = \|v\|$. Any $w \in T_x\Sigma$ can be written as $w = \lambda u + \mu v$ for $\lambda, \mu \in \mathbb{R}$. Then

$$\omega(v, w) = \lambda\omega(v, u) = -\lambda\|u\|^2 = \langle -u, w\rangle,$$

which means that $\omega^\flat(v) = -g^\flat(u)$. Therefore, $\omega^\sharp \circ g^\flat(u) = -v = -g^\sharp \circ \omega^\flat(u)$.

We define now $J := g^\sharp \circ \omega^\flat \in \mathrm{End}(T\Sigma)$. It is an almost complex structure: $J^2 = -g^\sharp \circ \omega^\flat \circ \omega^\sharp \circ g^\flat = -\mathrm{id}$. Any almost complex structure on $\Sigma$ is integrable, because if $X$ is a nonvanishing local vector field, then $\{X, JX\}$ is locally a frame for $T\Sigma$, and

$$N(X, JX) = J[X, JX] - [JX, JX] + [X, X] + J[JX, X] = 0.$$

The complex structure $J$ is characterized by the formula $i_{Ju}g = i_u\omega$ for all $u \in T\Sigma$. Hence, it is actually a Hermitian structure on $(\Sigma, g)$, since

$$g(Ju, Jv) = \omega(u, Jv) = -\omega(Jv, u) = -g(J^2v, u) = g(u, v).$$

Its Kähler form is $-\omega$, because $g(u, Jv) = -\omega(u, v)$, and since $\omega$ is closed, we conclude that $(\Sigma, g, J)$ is Kähler.

The computation above allows us to give the following more geometric definition of $J$: it rotates a vector $u \in T_x\Sigma$ an angle of $\pi/2$ such that $\{u, Ju\}$ is positively oriented. ▼

**Example 3.70** (Complex projective space)**.** Consider complex projective space $\mathbb{CP}^n$ as in Example 3.15. The orbit of $(z^0, \ldots, z^n)$, which is the complex line through $(z^0, \ldots, z^n)$ and the origin, we denote by $[z^0 : \ldots : z^n]$. Complex charts on $\mathbb{CP}^n$ are given as follows: let $U_i := \{[z^0 : \ldots : z^n] : z^i \neq 0\}$, which is open in $\mathbb{CP}^n$, and define $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i([z^0 : \ldots : z^n]) := \frac{1}{z^i}(z^0, \ldots, \widehat{z^i}, \ldots, z^n).$$

The inverse, as can be readily checked, is given by

$$\varphi_i^{-1}(z^1, \ldots, z^n) = [z^1 : \ldots : z^i : 1 : z^{i+1} : \ldots : z^n].$$

The change of coordinates $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}$ for $i > j$ is given by

$$\varphi_{ij}(z^1, \ldots, z^n) = \frac{1}{z^i}(z^1, \ldots, z^j, 1, z^{j+1}, \ldots, \widehat{z^i}, \ldots, z^n),$$

which is holomorphic. Hence, $\mathbb{CP}^n$ is a complex manifold.

Consider the **Fubini-Study form** on $\mathbb{C}^n$, defined by

$$\tilde{\omega} := \frac{i}{2}\partial\overline{\partial}\log(1 + \|z\|^2),$$

where $z = (z^0, \dots, z^n)$ and $\|z\|^2$ is computed using the standard Hermitian product on $\mathbb{C}^n$, given by $\langle z, w\rangle := \sum_j z^j \overline{w}^j$. Obviously $\tilde{\omega} \in \Omega^{1,1}(\mathbb{C}^n)$ and it is closed. Explicitly, as an easy computation shows, it is given by

$$\begin{aligned}\tilde{\omega} &= \frac{i}{2(1+\|z\|^2)^2}\left((1+\|z\|^2)\sum_j dz^j \wedge d\overline{z}^j - \sum_{j,k} \overline{z}^j z^k dz^j \wedge d\overline{z}^k\right)\\ &= \frac{i}{2(1+\|z\|^2)^2}\left((1+\|z\|^2)\partial\overline{\partial}\|z\|^2 - \partial\|z\|^2 \wedge \overline{\partial}\|z\|^2\right).\end{aligned}$$

The corresponding symmetric tensor is, then

$$\tilde{g} = \tilde{\omega}(\cdot, J\cdot) = \frac{1}{(1+\|z\|^2)^2}\left((1+\|z\|^2)\sum_j dz^j d\overline{z}^j - \sum_{j,k} \overline{z}^j z^k dz^j d\overline{z}^k\right),$$

where by $dz^j d\overline{z}^k$ we mean the symmetric product $dz^j d\overline{z}^k = \frac{1}{2}(dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j)$. Then $\tilde{g}$ is a complex Riemannian metric on $\mathbb{C}^n$, by which we mean a symmetric (0,2)-tensor on the complexification of $T^*\mathbb{C}^n$ such that $\tilde{g}(\xi, \overline{\xi}) > 0$ for all nonzero $\xi \in T^*\mathbb{C}^n \otimes \mathbb{C}$. Observe that a complex Riemannian metric contains the same information as a real Riemannian metric, since we can obtain the latter from the former by restricting to the real tangent bundle and the former from the latter by extending to the complexified tangent bundle by $\mathbb{C}$-bilinearity.

It is clear that $\tilde{g}$ is symmetric and $\mathbb{C}$-bilinear, it only remains to see that it is positive-definite. Let $\xi = \sum_j (v^j \frac{\partial}{\partial z^j} + w^j \frac{\partial}{\partial \overline{z}^j})$ be nonzero and write $v = (v^1, \dots, v^n), w = (w^1, \dots, w^n) \in \mathbb{C}^n$. Then, using the Cauchy-Schwarz inequality for $\langle\cdot,\cdot\rangle$,

$$\begin{aligned}\tilde{g}(\xi, \overline{\xi}) &= \frac{1}{2(1+\|z\|^2)^2}\left((1+\|z\|^2)\sum_j (v^j\overline{v}^j + w^j\overline{w}^j) - \sum_{j,k}(\overline{z}^j z^k v^j \overline{v}^k + \overline{z}^j z^k w^k \overline{w}^j)\right)\\ &= \frac{1}{2(1+\|z\|^2)^2}\left((1+\|z\|^2)(\|v\|^2 + \|w\|^2) - |\langle v, z\rangle|^2 - |\langle w, \overline{z}\rangle|^2\right)\\ &\geq \frac{\|v\|^2 + \|w\|^2}{2(1+\|z\|^2)^2} > 0.\end{aligned}$$

Hence, $\tilde{\omega}$ is a Kähler form on $\mathbb{C}^n$. It is invariant under $\varphi_{ij}$: if $i > j$, then

$$\begin{aligned}\log(1+\|\varphi_{ij}(z)\|^2) &= \log\left(1 + \frac{1}{|z^i|^2}(|z^1|^2 + \dots + \widehat{|z^i|^2} + \dots |z^n|^2 + 1)\right)\\ &= \log\left(\frac{1}{|z^i|^2}(1+\|z\|^2)\right) = \log(1+\|z\|^2) - \log z^i - \log \overline{z}^i,\end{aligned}$$

so $\partial\overline{\partial}\log(1+\|\varphi_{ij}(z)\|^2) = \partial\overline{\partial}\log(1+\|z\|^2)$. Hence, the local forms $\{\varphi_i^*\tilde{\omega}\}_i$ on $\mathbb{CP}^n$ glue to a well-defined Kähler form $\omega$ on $\mathbb{CP}^n$, with corresponding metric $g$.

Lastly, we want to check that this structure actually coincides with the symmetric space structure on $\mathbb{CP}^n$ given in Example 3.15. First of all, recall that the standard Euclidean metric on $\mathbb{C}^{n+1}$ is given by $\mathrm{Re}\langle\cdot,\cdot\rangle$. The tangent space to the orbit of the $\mathbb{C}^\times$-action at $z \in \mathbb{C}^{n+1}$ is given by $\{\lambda z : \lambda \in \mathbb{C}\} \subseteq T_z\mathbb{C}^{n+1}$. A vector $v \in T_z\mathbb{C}^{n+1}$ is, then, orthogonal to the orbit if and only if $\langle v, z\rangle = 0$. Indeed, if $\lambda = a + ib$, then

$$\mathrm{Re}\langle v, \lambda z\rangle = a\,\mathrm{Re}\langle v, z\rangle + b\,\mathrm{Im}\langle v, z\rangle,$$

and this vanishes for all $a, b \in \mathbb{R}$ if and only if $\langle v, z\rangle = 0$.

Define now a 2-form on $\mathbb{C}^{n+1} \smallsetminus \{0\}$ by

$$\begin{aligned}\mathring{\omega} &:= \frac{i}{2\|z\|^4}\left(\|z\|^2\partial\overline{\partial}\|z\|^2 - \partial\|z\|^2 \wedge \overline{\partial}\|z\|^2\right)\\ &= \frac{i}{2\|z\|^4}\left(\|z\|^2\sum_j dz^j \wedge d\overline{z}^j - \sum_{j,k}\overline{z}^k z^j dz^k \wedge d\overline{z}^j\right).\end{aligned}$$

It is closed and $(1,1)$. If $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is the projection, an easy computation using that

$$\|\varphi_i \circ p(z)\|^2 = \frac{\|z\|^2}{|z^i|^2} - 1.$$

shows that $p^*\omega = \mathring{\omega}$.

Let $\mathring{g}$ be the $(0,2)$-tensor corresponding to $\mathring{\omega}$. It is not complex Riemannian, since a very similar computation as before gives that if $\xi = \sum_j (v^j \frac{\partial}{\partial z^j} + w^j \frac{\partial}{\partial \bar{z}^j})$ is nonzero, then

$$\mathring{g}(\xi, \bar{\xi}) = \frac{1}{2\|z\|^4} \left( \|z\|^2(\|v\|^2 + \|w\|^2) - |\langle v, z\rangle|^2 - |\langle w, \bar{z}\rangle|^2 \right) \geq 0,$$

by Cauchy-Schwarz, but it can be zero when $v = \lambda z$ and $w = \mu \bar{z}$ for any $\lambda, \mu \in \mathbb{C}$. In the case that $\xi$ is a real vector, i.e., $\bar{\xi} = \xi$, then $\mu = \bar{\lambda}$, and we have that $\xi = \lambda z$ is tangent to the orbit of $\mathbb{C}^\times$ through $z$. On the other hand, if $\xi$ is normal to the orbit, then, since $\langle v, z\rangle = 0$,

$$\mathring{g}(\xi, \bar{\xi}) = \frac{\|v\|^2}{\|z\|^2} = \|v\|^2 = \operatorname{Re} \|v\|^2.$$

Hence, restricted to the normal space to the orbit, $\mathring{g}$ coincides with the restriction of the Euclidean metric to $\mathbb{S}^{2n+1}$ (the so-called round metric). The metric on $\mathbb{CP}^n$ given in Example 3.15 was defined as the metric whose pullback along $p$ coincided with the round metric on the normal space to the orbit. Since $p^*g = \mathring{g}$, we conclude that $g$ is the sought metric. ▼

**Example 3.71** (Complex submanifolds of Kähler manifolds)**.** Let $(M, g, J)$ be a Kähler manifold and consider $i : N \hookrightarrow M$ a complex submanifold, by which we mean a submanifold such that $J(T_xN) \subseteq T_xN$. Then $i^*J$ is a complex structure on $N$. Let $\omega$ be the Kähler form of $M$. Then $i^*\omega$ is the Kähler form of $(N, i^*g, i^*J)$. It is closed, since the differential commutes with pullbacks. Hence $N$ is Kähler as well.

In particular, smooth complex projective varieties are Kähler. ▼

The curvature of a Kähler manifold interacts nicely with the complex structure.

**Proposition 3.72.** *If $(M, g, J)$ is Kähler, then for all $X, Y, Z, W \in \mathfrak{X}(M)$ the following hold:*

1. $R(X,Y)J = JR(X,Y)$,
2. $\langle R(JX, JY)Z, W\rangle = \langle R(X,Y)JZ, JW\rangle$,
3. $\operatorname{Ric}(JX, JY) = \operatorname{Ric}(X, Y)$.

*Proof.* Since $\nabla J = 0$, then $\nabla_X(JY) = J\nabla_X Y$, from where we get that for all $Z \in \mathfrak{X}(M)$,

$$R(X,Y)JZ = \nabla_X\nabla_Y(JZ) - \nabla_Y\nabla_X(JZ) - \nabla_{[X,Y]}(JZ) = JR(X,Y)Z.$$

Using the symmetry properties of $R$ and the just proven property,

$$\begin{aligned} \langle R(JX, JY)Z, W\rangle &= \langle R(Z,W)JX, JY\rangle = \langle R(Z,W)X, Y\rangle \\ &= \langle R(X,Y)Z, W\rangle = \langle R(X,Y)JZ, JW\rangle. \end{aligned}$$

Lastly, if $\{E_i\}_i$ is a local orthonormal frame for $TM$, then

$$\operatorname{Ric}(JX, JY) = \sum_i \langle R(E_i, JX)JY, E_i\rangle = \sum_i \langle R(JE_i, X)Y, JE_i\rangle = \operatorname{Ric}(X, Y),$$

since $\{JE_i\}_i$ is also an orthonormal frame for $TM$. □

### 3.6.2. Calabi–Yau manifolds

On an almost complex manifold $(M, J)$ we can define the bundles of $(p,q)$-forms $\Lambda^{p,q}T^*M$ just as we did in Section 3.5.2. Its space of sections we denote by $\Omega^{p,q}(M)$, whose elements we call differential $(p,q)$-forms. The **canonical bundle** of $M$ is the complex line bundle $\Lambda^{n,0}T^*M$ over $M$. We also consider $(p,q)$-multivector fields, that is, sections of $\Lambda^{p,q}TM$. Recall that $\alpha \in \Omega^1(M,\mathbb{C})$ belongs to $\Omega^{1,0}(M)$ if and only if it vanishes on $\mathfrak{X}^{0,1}(M)$.

**Definition 3.73.** A **Calabi–Yau structure** on a Riemannian manifold $(M, g)$ is a Kähler structure $J$ whose canonical bundle admits a nowhere vanishing parallel section $\Omega$, in which case the tuple $(M, g, J, \Omega)$ is called a **Calabi–Yau manifold**. ♦

**Proposition 3.74.** *A connected Riemannian manifold $M$ of dimension $2n$ admits a Calabi–Yau structure if and only if* $\mathrm{Hol}(M) \subseteq \mathrm{SU}(n)$. *Moreover, if $M$ is Kähler then it is Ricci-flat if and only if* $\mathrm{Hol}^0(M) \subseteq \mathrm{SU}(n)$.

*Proof.* The first part follows easily from the holonomy principle and a similar argument to that of the proof of Corollary 3.68. For the second part, assume that $M$ is Kähler and let $\{E_j, JE_j\}_j$ be a local orthonormal frame for $TM$. Let $V_j := (E_j - iJE_j)/\sqrt{2}$, with dual forms $V^j$, and define $\Omega := V^1 \wedge \cdots \wedge V^n$, which is a local frame for the canonical bundle. Since $\Lambda^{n,0}T^*M$ is a complex line bundle, then its curvature, the one induced from the Levi-Civita connection on $M$, can be considered as a 2-form $F \in \Omega^2(M,\mathbb{C})$, whose value on $X, Y \in \mathfrak{X}(M)$ can be computed locally as $F(X,Y)\Omega(V_1,\ldots,V_n)$. It is a straightforward computation (see Proposition A.1) to show that

$$F(X,Y)\Omega(V_1,\ldots,V_n) = \sum_j \Omega(V_1,\ldots,R(Y,X)V_j,\ldots,V_n).$$

Then, since $\{V_j\}_j$ is an orthonormal frame, we have that, using Bianchi's first identity and Proposition 3.72,

$$\begin{aligned}
F(X,Y)\Omega(V_1,\ldots,V_n) &= \sum_j h(R(Y,X)V_j,V_j) = \frac{1}{2}\sum_j \langle R(Y,X)(E_j - iJE_j), E_j + iJE_j\rangle \\
&= i\sum_j \langle R(Y,X)E_j, JE_j\rangle \\
&= -i\sum_j \langle R(X,E_j)Y, JE_j\rangle - i\sum_j \langle R(E_j,Y)X, JE_j\rangle \\
&= -i\sum_j \langle R(E_j,X)JY, E_j\rangle - i\sum_j \langle R(JE_j,X)JY, JE_j\rangle \\
&= -i\,\mathrm{Ric}(X,JY).
\end{aligned}$$

Hence, $F(X,Y) = -i\,\mathrm{Ric}(X,JY)$. The form $(u,v) \mapsto \mathrm{Ric}(u,Jv)$ is sometimes called the Ricci form of $M$. Hence we see that the canonical bundle is flat if and only if $M$ is Ricci-flat. By the Ambrose–Singer theorem, this means that the restricted holonomy group of the canonical bundle is trivial if and only if $M$ is Ricci-flat. If $\gamma$ is a null-homotopic loop at $x \in M$ and $\Omega \in \Lambda^{n,0}T_x^*M$, then $\tau_\gamma\Omega$ is exactly the parallel transport along $\gamma$ of $\Omega$, so that $\Omega$ is invariant under $\tau_\gamma$ viewed as parallel transport on the canonical bundle if and only if $\tau_\gamma \in \mathrm{SU}(n)$. This finally gives the result. □

Our definition of a Calabi–Yau manifold is the most natural one from the point of view of holonomy. However, there is another typical definition of Calabi–Yau manifolds in the literature: a Kähler manifold whose canonical bundle is holomorphically trivial.

To make sense of this other definition, consider the decomposition $\Omega^k(M,\mathbb{C}) = \bigoplus_{p+q=k}\Omega^{p,q}(M)$. It gives projections $\Omega^{p+q}(M,\mathbb{C}) \to \Omega^{p,q}(M)$. The composition of $d : \Omega^{p,q}(M) \to \Omega^{p+q+1}(M,\mathbb{C})$ with the projections $\Omega^{p+q+1}(M,\mathbb{C}) \to \Omega^{p+1,q}(M)$ and $\Omega^{p+q+1}(M,\mathbb{C}) \to \Omega^{p,q+1}(M)$ give the **Dolbeault operators**

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

A $(p,0)$-form is called **holomorphic** if it lies in the kernel of $\overline{\partial}$. A nonvanishing holomorphic section of the canonical bundle is called a **holomorphic volume form**. Hence, the alternative definition of Calabi–Yau is that of a Kähler manifold admitting a holomorphic volume form. We will now see that the two notions of Calabi–Yau manifolds agree somewhat on compact manifolds. By "agree somewhat" we mean the following: a compact connected manifold admitting Kähler structures admits a Calabi–Yau structure in the first sense if and only if it admits one in the second sense, but these two structures might be different!

First of all, an alternative formulation of the Newlander-Nirenberg theorem states that an almost complex manifold $(M,J)$ is complex if and only if $d = \partial + \overline{\partial}$. In such case, from $d^2 = 0$ one directly deduces that $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

Second, we need to consider formal adjoints of some operators. To do this, first remember that the Riemannian metric on $M$ induces a metric on any tensor bundle over $M$, as in the end of Section 1.1. When $M$ is compact and orientable, we can define an inner product on $\mathfrak{T}^{(k,l)}(M)$, the $L^2$**-product**, by

$$\langle T, S\rangle_2 := \int_M \langle T, S\rangle \mathrm{vol}, \qquad \text{for } T, S \in \mathfrak{T}^{(k,l)}(M),$$

where vol is the canonical volume form of $M$, characterized by taking the value 1 on any oriented orthonormal basis. The **formal adjoint** of an operator on tensors $P$ is another operator $P^*$ such that $\langle PT, S\rangle_2 = \langle T, P^*S\rangle_2$. Such an adjoint is unique, by the positive definiteness of the $L^2$-product.

**Proposition 3.75.** *The formal adjoint of the connection $\nabla : \mathfrak{T}^{(k,l)}(M) \to \mathfrak{T}^{(k,l+1)}(M)$ is the operator $\nabla^* : \mathfrak{T}^{(k,l+1)}(M) \to \mathfrak{T}^{(k,l)}(M)$ given by*

$$\nabla^* T(\theta, X_1, \dots, X_l) = -\sum_i \nabla_{E_i} T(\theta, E_i, X_1, \dots, X_l)$$

*on $T \in \mathfrak{T}^{(k,l)}(M)$, where $\theta \in \mathfrak{T}^{(0,k)}(M)$, $X_i \in \mathfrak{X}(M)$ and $\{E_i\}_i$ is any orthonormal frame.*

*Proof.* See Proposition A.3. □

For differential forms we adapt the metric, so as to take into account their skew-symmetry: we define

$$\langle \alpha^1 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \cdots \wedge \beta^k\rangle^\wedge := \det(\langle \alpha^i, \beta^j\rangle)_{i,j}.$$

It is straightforward to check that for $\alpha, \beta \in \Omega^k(M)$ we have that $\langle \alpha, \beta\rangle = k!\langle \alpha, \beta\rangle^\wedge$. We let $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ stand for the formal adjoint of the de Rham differential $d$ with respect to $\langle \cdot, \cdot\rangle^\wedge$.

**Proposition 3.76.** *For $\alpha \in \Omega^k(M)$ we have that*

$$d\alpha(X_0, \dots, X_k) = \sum_i (-1)^i \nabla_{X_i} \alpha(X_0, \dots, \hat{X}_i, \dots, X_k), \qquad \text{for } X_i \in \mathfrak{X}(M).$$

*Moreover, $d^* = \nabla^*$.*

*Proof.* See Proposition A.4. □

With the adjoints of $\nabla$ and $d$, we can form the corresponding Laplacians: the **rough Laplacian** $\nabla^*\nabla$ and the **Hodge Laplacian** $\Delta := dd^* + d^*d$. These two are related by the Weitzenböck formula. We define the **Weitzenböck operator** on $\alpha \in \mathfrak{T}^{(0,k)}(M)$ by

$$\operatorname{Ric} \alpha(X_1, \dots, X_k) := \sum_{i,j} R(E_i, X_j)\alpha(X_1, \dots, X_{j-1}, E_i, X_{j+1}, \dots, X_k),$$

for $X_j \in \mathfrak{X}(M)$ and $\{E_i\}_i$ any orthonormal frame. We call it Ric, following [Pet16], because on 1-forms it is the Ricci curvature: if $\alpha \in \Omega^1(M)$, with dual $Y \in \mathfrak{X}(M)$, and $X \in \mathfrak{X}(M)$, then

$$\operatorname{Ric}\alpha(X) = \sum_i R(E_i, X)\alpha(E_i) = \sum_i \langle Y, R(X, E_i)E_i\rangle = \operatorname{Ric}(Y, X).$$

**Proposition 3.77** (Weitzenböck)**.** *For $\alpha \in \Omega^k(M)$ we have that $\Delta\alpha = \nabla^*\nabla\alpha + \operatorname{Ric}\alpha$.*

*Proof.* See Proposition A.5. □

The next ingredient is a very special fact for compact Kähler manifolds, where the assumption of compactness is key here. Just as we did for $d$, we can consider the formal adjoints of $\partial$ and $\overline{\partial}$ and their Laplacians $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. It is a very remarkable fact that on compact Kähler manifolds $\Delta = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. For a proof see [Wel08, Chap. V, Sec. 4]. It allows us to prove the following.

**Proposition 3.78.** *On a compact Ricci-flat Kähler manifold, being parallel, closed and holomorphic are equivalent notions for $(p,0)$-forms.*

*Proof.* Let $(M, g, J)$ be compact Ricci-flat Kähler and $\alpha \in \Omega^{p,0}(M)$. If $\nabla\alpha = 0$, then $d\alpha = 0$ by Proposition 3.76 and $\overline{\partial}\alpha = 0$ because $d = \partial + \overline{\partial}$. Assume now that $\alpha$ is holomorphic, i.e., $\overline{\partial}\alpha = 0$. Then, since $\overline{\partial}^*\alpha = 0$, because $\alpha$ is a $(p,0)$-form, we have that $\Delta_{\overline{\partial}}\alpha = 0$. But since $M$ is compact Kähler, this also means that $\Delta\alpha = 0$. If we prove that $\operatorname{Ric}\alpha = 0$, then the Weitzenböck formula gives that $\nabla^*\nabla\alpha = 0$, and this in turn would give that

$$\langle \alpha, \nabla^*\nabla\alpha\rangle_2 = \|\nabla\alpha\|_2^2 = 0,$$

which implies that $\nabla\alpha = 0$. See Proposition A.7 for a proof of $\operatorname{Ric}\alpha = 0$. □

**Corollary 3.79.** *A compact Ricci-flat Kähler structure is Calabi–Yau if and only if it admits a holomorphic volume form.* □

The last ingredient is the celebrated Calabi–Yau theorem. This theorem was conjectured by Calabi in 1954 [Cal56, Cal57], and he gave a proof of the uniqueness part. In 1976 Yau proved existence [Yau77, Yau78]. To state it, recall that the **first Chern class** of an almost complex manifold $(M, J)$ is the cohomology class

$$c_1(M) := -\frac{1}{2\pi i}[F] \in H^2(M, \mathbb{C}),$$

where $F$ denotes the curvature of the canonical bundle of $M$ with respect to any connection on $TM$, for instance the Levi-Civita connection (the first Chern class is independent of the chosen connection on $TM$ [Tu17, Thm. 23.3]). In Proposition 3.74 we proved that if $M$ is Kähler, then its Ricci form lies in $2\pi c_1(M)$.

**Theorem 3.80** (Calabi–Yau)**.** *Let $M$ be a compact Kähler manifold with Kähler form $\omega$ and $\rho$ a real closed $(1,1)$-form on $M$ with $\rho \in 2\pi c_1(M)$. Then there is a unique Kähler metric on $M$ with Kähler form in $[\omega] \in H^2(M, \mathbb{R})$ whose Ricci form is $\rho$.*

For a proof see for instance [Joy07].

**Corollary 3.81.** *A compact connected complex manifold $(M, J)$ admitting Kähler structures admits a Calabi–Yau structure if and only if it admits a holomorphic volume form.*

*Proof.* If $M$ admits a Calabi–Yau structure $(g, \Omega)$, then $\overline{\partial}\Omega = 0$ because $d\Omega = 0$ by Proposition 3.76. Conversely, assume that $M$ admits a holomorphic volume form. Then the canonical bundle of $M$ is trivial as a complex vector bundle, so $c_1(M) = 0$. By the Calabi–Yau theorem, there is a metric on $M$ which is Kähler and Ricci-flat, and the result now follows from Corollary 3.79. □

Observe that we are not claiming that if $(M, g, J)$ is compact Kähler admitting a holomorphic volume form $\Omega$, then $(M, g, J, \Omega)$ is Calabi–Yau, because to be able to use the Calabi–Yau theorem we need to be able to change $g$. What we are actually claiming is the following: if $(M, g)$ is a compact Riemannian manifold with $\mathrm{Hol}(M, g) \subseteq \mathrm{U}(n)$, then we can find a Kähler structure $J$ on $M$, by Corollary 3.68, and there is a holomorphic volume form $\Omega$ on $M$ with respect to $J$ if and only if there is some metric $g'$ such that $\mathrm{Hol}(M, g') \subseteq \mathrm{SU}(n)$, i.e., such that $(M, g', J, \Omega)$ is Calabi–Yau.

**Example 3.82** (Smooth projective varieties)**.** Let $p$ be a homogeneous polynomial of degree $d$ in $\mathbb{C}^{n+1}$. Then $Z := p^{-1}(0) \setminus \{0\}$ is invariant under the action of $\mathbb{C}^\times$, since $p$ is homogeneous. Assume that $Z$ is a smooth complex hypersurface of $\mathbb{C}^{n+1} \setminus \{0\}$. Then it defines a complex hypersurface of $\mathbb{CP}^n$, call it $Y$. Let $\mathcal{O}(-1)$ be the tautological bundle on $\mathbb{CP}^n$, defined by

$$\mathcal{O}(-1) := \{(z, \ell) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in \ell\}.$$

It is a holomorphic line bundle over $\mathbb{CP}^n$. Let $\mathcal{O}(1) := \mathcal{O}(-1)^*$ and define $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$ and $\mathcal{O}(-k) := \mathcal{O}(k)^*$ for $k \geq 0$. Then $\mathcal{O}(k+l) = \mathcal{O}(k) \otimes \mathcal{O}(l)$ and $\mathcal{O}(0)$ is the trivial bundle. By [Huy05, Prop. 2.4.1], the space of homogeneous polynomials of degree $d$ is isomorphic to the space of holomorphic sections of $\mathcal{O}(d)$. Then the adjunction formula [Huy05, Prop. 2.2.17] and the fact that the holomorphic normal bundle of $Y$ is isomorphic to $\mathcal{O}(d)$ [Huy05, Prop. 2.4.7] gives that, if $K_Y$ is the canonical bundle of $Y$ and $K_{\mathbb{CP}^n}$ that of $\mathbb{CP}^n$, then $K_Y = (K_{\mathbb{CP}^n} \otimes \mathcal{O}(d))|_Y$. Since $K_{\mathbb{CP}^n} = \mathcal{O}(-n-1)$ [Huy05, Prop. 2.4.3], we conclude that $K_Y = \mathcal{O}(d-n-1)|_Y$. Hence, if $d = n+1$, we have that $K_Y$ is trivial and so, by the Calabi–Yau theorem, $Y$ admits a Kähler Ricci-flat structure. If in addition $Y$ is simply connected, then it is Calabi–Yau.

An example of this is Fermat's quintic 3-fold: let $p(z) = \sum_{i=0}^4 (z^i)^5$. Then $dp = \sum_{i=0}^4 5(z^i)^4 dz^i$, which vanishes only at $0 \in \mathbb{C}^5$, so that $p^{-1}(0) \setminus \{0\}$ is a complex hypersurface projecting to a complex hypersurface $Y$ of $\mathbb{CP}^4$. Since $5 = 4+1$, then $Y$ admits a Kähler Ricci-flat structure. ▼

### 3.6.3. Hyperkähler and quaternionic Kähler manifolds

We finally turn to the holonomy groups related to the quaternions.

**Definition 3.83.** A **hyperkähler structure** on a Riemannian manifold $(M, g)$ is a triple of Kähler structures $(I, J, K)$ such that $IJ = K$, in which case the tuple $(M, g, I, J, K)$ is called a **hyperkähler manifold**. ♦

**Proposition 3.84.** *A connected Riemannian manifold $M$ of dimension $4n$ admits a hyperkähler structure if and only if* $\mathrm{Hol}(M) \subseteq \mathrm{Sp}(n)$.

*Proof.* It follows easily from the holonomy principle and a similar argument to that of the proof of Corollary 3.68. □

An explicit example is the following.

**Example 3.85** (Gibbons-Hawking ansatz)**.** First introduced in [GH78]. The exposition here follows [GW00]. Let $U \subseteq \mathbb{R}^3$ be an open set with $H^1(U, \mathbb{R}) = 0$ and let $\pi : M \to U$ be a principal $\mathbb{S}^1$-bundle. Let $\theta$ be a connection 1-form on $M$, i.e., $\theta \in \Omega^1(M, i\mathbb{R})$ such that $\theta(\frac{\partial}{\partial t}) = i$, where $\frac{\partial}{\partial t}(x) = \frac{d}{dt}\big|_{t=0} e^{it}x$. Then its curvature $d\theta \in \Omega^2(M, i\mathbb{R})$ is basic, meaning that there is some $\alpha \in \Omega^2(U, i\mathbb{R})$ such that $d\theta = \pi^*\alpha$. Let $V \in C^\infty(U)$ be a positive function such that $*dV = \alpha/2\pi i$, where $*$ is the Hodge star operator [Wel08, Chap. V, Sec. 1]. Define now, if $\theta_0 := \theta/2\pi i$ and we write the pullback forms $\pi^* dx^i$ simply as $dx^i$,

$$\begin{aligned}
\omega_1 &:= dx^1 \wedge \theta_0 + V dx^2 \wedge dx^3, \\
\omega_2 &:= dx^2 \wedge \theta_0 + V dx^3 \wedge dx^1, \\
\omega_3 &:= dx^3 \wedge \theta_0 + V dx^1 \wedge dx^2.
\end{aligned}$$

Then $\omega_i^2 = 2V dx^1 \wedge dx^2 \wedge dx^3 \wedge \theta_0$ is nowhere zero and $\omega_i \wedge \omega_j = 0$ for all $i \neq j$. Also, an easy computation shows that $d\omega_i = 0$; for instance:

$$d\omega_1 = -dx^1 \wedge \frac{\alpha}{2\pi i} + dV \wedge dx^2 \wedge dx^3 = 0,$$

since $*dV = \alpha/2\pi i$.

Recall from Section 3.5.2 that if $(I, J, K)$ is a hyperkähler structure, then $\omega_J + i\omega_K$ is a holomorphic symplectic $(2,0)$-form with respect to $I$ (and similarly for $\omega_K + i\omega_I$ with respect to $J$ and $\omega_I + i\omega_J$ with respect to $K$). If we write

$$\begin{aligned} \omega_2 + i\omega_3 &= (dx^2 + idx^3) \wedge (\theta_0 - iV dx^1), \\ \omega_3 + i\omega_1 &= (dx^3 + idx^1) \wedge (\theta_0 - iV dx^2), \\ \omega_1 + i\omega_2 &= (dx^1 + idx^2) \wedge (\theta_0 - iV dx^3), \end{aligned}$$

from here we can read off three integrable complex structures given on the complexified cotangent bundle of $M$ by

$$\begin{aligned} J_1(dx^2) &= dx^3, & J_1(dx^1) &= V^{-1}\theta_0, \\ J_2(dx^3) &= dx^1, & J_2(dx^2) &= V^{-1}\theta_0, \\ J_3(dx^1) &= dx^2, & J_3(dx^3) &= V^{-1}\theta_0. \end{aligned}$$

These define a hyperkähler structure on $M$:

$$\begin{aligned} J_1 J_2(dx^3) &= J_1(dx^1) = V^{-1}\theta_0 = J_3(dx^3), \\ J_1 J_2(dx^2) &= J_1(V^{-1}\theta_0) = -dx^1 = J_3(dx^2). \end{aligned}$$

The metric is given by $g = \omega_1(\cdot, J_1 \cdot) = \omega_2(\cdot, J_2 \cdot) = \omega_3(\cdot, J_3 \cdot)$. Explicitly,

$$g = V((dx^1)^2 + (dx^2)^2 + (dx^3)^2) + V^{-1}\theta_0^2.$$

Observe that, since $\mathrm{SU}(2) = \mathrm{Sp}(1)$, then these spaces are also examples of Calabi–Yau manifolds. ▼

**Definition 3.86.** A **quaternionic Kähler structure** on a Riemannian manifold $(M, g)$ is a parallel form $Q \in \Omega^4(M)$ such that around every $x \in M$ there is a neighborhood on which there are almost complex structures $I$, $J$ and $K$ with $IJ = K$ and such that $Q = \omega_I^2 + \omega_J^2 + \omega_K^2$. The triple $(M, g, Q)$ is then called a **quaternionic Kähler manifold**. ♦

**Proposition 3.87.** *A connected Riemannian manifold $M$ of dimension $4n$ admits a quaternionic Kähler structure if and only if* $\mathrm{Hol}(M) \subseteq \mathrm{Sp}(n)\,\mathrm{Sp}(1)$.

*Proof.* It follows easily from the holonomy principle and a similar argument to that of the proof of Corollary 3.68, using the characterization of $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$ in Proposition 3.56. □

The fundamental example of a quaternionic Kähler manifold is $\mathbb{HP}^n$, which is also symmetric by Example 3.15. This is one of the simplest examples of a quaternionic Kähler symmetric space, also known as Wolf spaces [Wol65]. It has been conjectured by LeBrun and Salamon [LS94] that these are the only examples of complete quaternionic Kähler manifolds with positive scalar curvature. This conjecture has been proven in dimensions 4 [Hit81], 8 [PS91] and 12 [HH02]. On the other hand, there are plentiful examples of complete non-compact quaternionic Kähler manifolds with negative scalar curvature which are not symmetric [LeB91, Ale75].

4

# Lie Algebroid Connections and Holonomy

Lie algebroids were introduced by Pradines [Pra67] as the infinitesimal version of a Lie groupoid. One way to think of Lie algebroids is as a generalization of the tangent bundle. They are vector bundles $A \to M$ endowed with a structure that mimics that of $TM$: a Lie bracket on its space of sections and a way to take derivatives of smooth functions of $M$ in the directions of $A$, i.e., a bundle map $\rho : A \to TM$, called the anchor. Both objects are related via a Leibniz rule. This framework unifies several different geometries: foliations, manifolds with boundary, Poisson geometry, principal bundles. . .

In this chapter we first introduce the basics of Lie algebroids, including the induced singular foliation on the base manifold, and then we pass to Lie algebroid connections and holonomy. The latter were introduced by Fernandes first for Poisson manifolds [Fer00] and later for general Lie algebroids [Fer02]. He proved, among other things, that the Ambrose–Singer theorem does not hold for Lie algebroid connections. Rather, there are some additional terms coming from the kernel of the anchor. This makes it possible for flat Lie algebroid connections to have non-discrete holonomy. In the final section we give a proof of the Ambrose–Singer–Fernandes theorem in the spirit of Section 1.3, different from Fernandes's proof in [Fer00], and we give original examples of a flat Lie algebroid connection with non-discrete holonomy and of holonomy jumps from leaf to leaf.

## 4.1. Basic definitions and facts

**Definition 4.1.** A **Lie algebroid** on $M$ is a vector bundle $A \to M$ together with a vector bundle morphism $\rho : A \to TM$, called the **anchor**, and a Lie bracket $[\cdot, \cdot]$ on its space of sections such that the following Leibniz rule holds:

$$[a, fb] = f[a, b] + (\rho(a)f)b, \quad \text{for } a, b \in \Gamma(A) \text{ and } f \in C^\infty(M).$$

The kernel of $\rho_x : A_x \to T_xM$ is called the **isotropy at** $x$. ♦

The Leibniz rule implies that the anchor is a Lie algebra morphism at the level of sections.

**Lemma 4.2.** *Let $A \to M$ be a Lie algebroid with anchor $\rho$. Then $\rho([a,b]) = [\rho(a), \rho(b)]$ for all $a, b \in \Gamma(A)$.*

*Proof.* On the one hand, if $a, b, c \in \Gamma(A)$ and $f \in C^\infty(M)$,

$$[[a,b], fc] = f[[a,b],c] + (\rho([a,b])f)c.$$

On the other hand,

$$\begin{aligned}[[b,fc],a] &= [f[b,c] + (\rho(b)f)c, a] = -[a, f[b,c] + (\rho(b)f)c] \\ &= -f[a,[b,c]] - (\rho(a)f)[b,c] - (\rho(b)f)[a,c] - (\rho(a)\rho(b)f)c,\end{aligned}$$

and

$$\begin{aligned}[[fc,a],b] &= -[f[a,c] + (\rho(a)f)c, b] = [b, f[a,c] + (\rho(a)f)c] \\ &= f[b,[a,c]] + (\rho(b)f)[a,c] + (\rho(a)f)[b,c] + (\rho(b)\rho(a)f)c.\end{aligned}$$

The Jacobi identity for the bracket now gives that

$$0 = (\rho([a,b])f - [\rho(a), \rho(b)]f)c,$$

which establishes the claim. □

**Example 4.3** (Tangent bundle)**.** The tangent bundle $TM \to M$ is a Lie algebroid with the identity as anchor and the Lie bracket as bracket on sections. ▼

**Example 4.4** (Lie algebras)**.** A Lie algebra is a Lie algebroid over a point, with trivial anchor and bracket given by the Lie bracket of the Lie algebra. ▼

**Example 4.5** (Regular foliations)**.** Let $F$ be a rank $r$ involutive regular distribution on $M$, i.e., a subbundle $F \subseteq TM$ such that $[\Gamma(F), \Gamma(F)] \subseteq \Gamma(F)$. This is equivalent, by the Frobenius integrability theorem [Lee12, Thm. 19.12], to a regular foliation, i.e., a decomposition of $M$ into disjoint connected embedded submanifolds $\{L_i\}_{i \in I}$ such that for every point $x \in M$ there is a chart $(U, \varphi)$ such that for each $i \in I$ there is some $\lambda_i \in \mathbb{R}^{n-r}$, where $n = \dim M$, with

$$U \cap L_i = \varphi^{-1}(\mathbb{R}^r \times \{\lambda_i\}).$$

Then $F \to M$ is a Lie algebroid with anchor the inclusion $F \to TM$ and bracket the restriction of the Lie bracket to $\Gamma(F)$. ▼

**Example 4.6** (Action Lie algebroid)**.** Let $G$ be a Lie group, with Lie algebra $\mathfrak{g}$, acting smoothly on a manifold $M$. For $\xi \in \mathfrak{g}$, denote by $\xi_M$ the corresponding infinitesimal generator of the action on $M$, i.e.,

$$\xi_M(x) := \left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(t\xi) \cdot x, \qquad \text{for } x \in M,$$

where Exp is the Lie group exponential. Then the trivial vector bundle $M \times \mathfrak{g} \to M$ can be made into a Lie algebroid with the anchor given by $\rho(x, \xi) := \xi_M(x)$ and the bracket by

$$[f,g](x) := [f(x), g(x)]_{\mathfrak{g}} + g_*(\xi_M(x)) - f_*(\xi_M(x)),$$

where we have identified $\Gamma(M \times \mathfrak{g})$ with $C^\infty(M, \mathfrak{g})$ and $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie bracket of $\mathfrak{g}$. ▼

As a generalization of $TM$, we can extend to $A$ some notions typically defined on $TM$.

**Definition 4.7.** Let $A \to M$ be a Lie algebroid with anchor $\rho$. The sections of $\Lambda^k A^*$ we call **differential** $A$**-forms** and we denote them by $\Omega^k(A)$. If $E \to M$ is a vector bundle, the sections of $\Lambda^k A^* \otimes E$ we call $E$**-valued differential** $A$**-forms** and we denote them by $\Omega^k(A, E)$.

We define the $A$**-differential** on $A$-forms as the unique linear map $d_A : \Omega^k(A) \to \Omega^{k+1}(A)$ such that

1. $d_A(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^k \alpha \wedge d_A\beta$, for $\alpha \in \Omega^k(A)$ and $\beta \in \Omega^l(A)$,

2. $d_A f = df \circ \rho$, for all $f \in C^\infty(M)$ and

$$d_A\alpha(a, b) = \rho(a)(\alpha(b)) - \rho(b)(\alpha(a)) - \alpha([a, b])$$

for all $\alpha \in \Omega^1(A)$ and $a, b \in \Gamma(A)$. ♦

When we do not wish to emphasize which Lie algebroid is being considered, we will talk of LA-forms and LA-differential.

**Lemma 4.8.** *The A-differential satisfies $d_A^2 = 0$ and it is given by the Koszul formula*

$$\begin{aligned} d_A\alpha(a_0, \dots, a_k) = &\sum_i (-1)^i \rho(a_i)(\alpha(a_0, \dots, \hat{a}_i, \dots, a_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j], a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k), \end{aligned}$$

*for $\alpha \in \Omega^k(A)$ and $a_i \in \Gamma(A)$.*

*Proof.* See Lemma A.8 □

**Definition 4.9.** Let $A \to M$ be a Lie algebroid. We define the $k$th **$A$-cohomology** as

$$H^k(A) := \frac{\ker(d_A : \Omega^k(A) \to \Omega^{k+1}(A))}{\operatorname{im}(d_A : \Omega^{k-1}(A) \to \Omega^k(A))}.$$ ♦

A Lie algebroid morphism is a vector bundle morphism that descends to cohomology.

**Definition 4.10.** Let $A \to M$ and $B \to N$ be Lie algebroids. A vector bundle morphism $\Phi : A \to B$ covering $\phi : M \to N$ is a **Lie algebroid morphism** if $d_A\Phi^* = \Phi^* d_B$, where the pullback $\Phi^* : \Omega^k(B) \to \Omega^k(A)$ is given on $\alpha \in \Omega^k(B)$ by

$$(\Phi^*\alpha)_x(a_1, \dots, a_k) = \alpha_{\phi(x)}(\Phi(a_1), \dots, \Phi(a_k)), \quad \text{for } a_i \in A_x.$$ ♦

Hence, a Lie algebroid morphism $\Phi : A \to B$ induces a map on cohomology $\Phi^* : H^k(B) \to H^k(A)$. Observe that the anchor map is always a Lie algebroid morphism: if $\alpha \in \Omega^k(M)$, then

$$\begin{aligned} d_A\rho^*\alpha(a_0, \dots, a_k) &= \sum_i (-1)^i \rho(a_i)\alpha(\rho(a_0), \dots, \hat{a}_i, \dots, \rho(a_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha(\rho([a_i, a_j]), \rho(a_0), \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, \rho(a_k)) \\ &= d\alpha(\rho(a_0), \dots, \rho(a_k)) = \rho^* d\alpha(a_0, \dots, a_k). \end{aligned}$$

At first, one might have expected a simpler definition of a Lie algebroid morphism, something along the lines of "a vector bundle morphism preserving the anchors and the brackets." This definition, though, does not make sense when the morphism is not covering a diffeomorphism. Indeed, if $A \to M$ and $B \to N$ are Lie algebroids with anchors $\rho_A$ and $\rho_B$, respectively, and $\Phi : A \to B$ is a vector bundle morphism covering $\phi : M \to N$, then "preserving anchors" would just mean that $\rho_B \circ \Phi = \phi_* \circ \rho_A$, while "preserving brackets" would mean $\Phi([a, b]) = [\Phi(a), \Phi(b)]$, for $a, b \in \Gamma(A)$. But observe that if $a \in \Gamma(A)$, then $\Phi(a) \in \Gamma(\phi^* B)$, and in general $\phi^* B$ has no canonical Lie algebroid structure. If $\phi$ is a diffeomorphism, though, then $\phi^* B \cong B$ is a Lie algebroid, and these two notions of Lie algebroid morphism agree: $\Phi : A \to B$ covering a diffeomorphism $\phi : M \to N$ is a Lie algebroid morphism if and only if $\rho_B \circ \Phi = \phi_* \circ \rho_A$ and $\Phi([a, b]) \circ \phi^{-1} = [\Phi(a) \circ \phi^{-1}, \Phi(b) \circ \phi^{-1}]$ (the insertion of $\phi^{-1}$ gives the identification of $\phi^* B$ with $B$).

### 4.1.1. Singular foliation

In this section we mainly follow [AS09]. As we will see, every Lie algebroid defines a foliation on the base manifold. This foliation can be singular, i.e., the dimension of its leaves may vary. More precisely, it defines a foliation in the sense of Stefan-Sussmann [Ste74, Sus73], which we now introduce.

For a vector bundle $E \to M$, let $\Gamma_0(E)$ denote the space of compactly supported sections. We consider $\Gamma(E)$ and $\Gamma_0(E)$ as $C^\infty(M)$-modules. If $\mathscr{F} \subseteq \Gamma_0(E)$ is a submodule and $\varphi : N \to M$ a smooth map, the pullback $\varphi^*\mathscr{F}$ of $\mathscr{F}$ along $\varphi$ is the submodule of $\Gamma_0(\varphi^*E)$ generated by the elements of the form $f\varphi^*\sigma$, for $f \in C_0^\infty(N)$ and $\sigma \in \mathscr{F}$. When $\varphi$ is the inclusion of a submanifold $N$ of $M$, we call $\varphi^*\mathscr{F}$ the restriction of $\mathscr{F}$ to $N$ and write $\mathscr{F}_N$.

**Definition 4.11.** A **foliation** $\mathscr{F}$ on a manifold $M$ is a locally finitely generated submodule of $\mathfrak{X}_0(M)$ which is closed under the Lie bracket. Explicitly, it is a submodule $\mathscr{F} \subseteq \mathfrak{X}_0(M)$ such that

1. for every $x \in M$ there is a neighborhood $U$ of $x$ such that $\mathscr{F}_U$ is finitely generated: there are vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$ such that $\mathscr{F}_U = C_0^\infty(U)X_1 + \dots + C_0^\infty(U)X_k$,
2. $[\mathscr{F}, \mathscr{F}] \subseteq \mathscr{F}$.

The pair $(M, \mathscr{F})$ is called a **foliated manifold**.

The **tangent space** $F_x$ **to the leaf at** $x$ is the image of the evaluation map $\mathscr{F} \to T_xM$ given by $X \mapsto X(x)$.

For $X \in \mathscr{F}$, let $\phi_t^X$ denote its flow at time $t$. Then the **exponential** of $X$ is $\exp X := \phi_1^X \in \mathrm{Diff}(M)$. Let $\exp \mathscr{F}$ be the subgroup of $\mathrm{Diff}(M)$ generated by the elements of the form $\exp X$ for $X \in \mathscr{F}$. Then the **leaves** of $\mathscr{F}$ are the orbits of $\exp \mathscr{F}$. ♦

By definition, $x, y \in M$ lie on the same leaf if and only if there are $X_1, \dots, X_k \in \mathscr{F}$ such that

$$y = \phi_1^{X_k} \circ \dots \circ \phi_1^{X_1}(x).$$

Observe that, as it stands now, it is not clear that the leaves are actually submanifolds of $M$ and that the so-called tangent spaces to the leaves are actually the tangent spaces to any submanifold.

**Proposition 4.12.** *Let $\mathscr{F}$ be a foliation on $M$. The dimension of the tangent spaces to the leaves is lower semi-continuous. That is, for every $x \in M$ there is a neighborhood $U$ of $x$ such that $\dim F_x \leq \dim F_y$ for every $y \in U$.*

*Proof.* Since $\mathscr{F}$ is locally finitely generated, let $U$ be a neighborhood of $x$ such that there are $X_1, \dots, X_k \in \mathfrak{X}(U)$ generating $\mathscr{F}_U$. Then $\dim F_x = \dim \operatorname{span}\{X_1(x), \dots, X_k(x)\}$. This equals the rank of the matrix whose $i$th column is the components of $X_i(x)$ with respect to some coordinate chart on $U$ (possibly making $U$ smaller). This rank is computed using the minors of the matrix. By continuity of the determinant map, a minor which is nonzero at $x$ will continue to be nonzero on $U$, possibly after shrinking $U$, so this means that the rank cannot decrease on $U$. Hence, $\dim F_x \leq \dim F_y$ for any $y \in U$. □

**Definition 4.13.** Let $\mathscr{F}$ be a foliation on $M$ and $\varphi : N \to M$ a smooth map. We denote by $\varphi^{-1}(\mathscr{F})$ the submodule of $\mathfrak{X}_0(N)$ defined by

$$\varphi^{-1}(\mathscr{F}) := \{X \in \mathfrak{X}_0(N) : \varphi_* X \in \varphi^*\mathscr{F}\},$$

where $\varphi_* X \in \Gamma_0(\varphi^*TM)$ is given by $(\varphi_* X)(x) := \varphi_*(X(x))$.

We say that $\varphi$ is **transverse** to $\mathscr{F}$ if the map $\varphi^*\mathscr{F} \times \mathfrak{X}_0(N) \to \Gamma_0(\varphi^*TM)$ given by $(X, Y) \mapsto X + \varphi_* Y$ is surjective. ♦

Observe that a submersion onto $M$ is transverse to any foliation on $M$.

**Proposition 4.14** ([AS09, Prop. 1.10])**.** *Let $\mathscr{F}$ be a foliation on $M$ and $\varphi : N \to M$ a smooth map. Then $\varphi^{-1}(\mathscr{F})$ is closed under Lie brackets, and if $\varphi$ is transverse to $\mathscr{F}$ then $\varphi^{-1}(\mathscr{F})$ is locally finitely generated.*

It is obvious that for all $x \in N$ we have that $\varphi^{-1}(\mathscr{F})_x = \{v \in T_xN : \varphi_* v \in F_{\varphi(x)}\}$.

The following local normal form for foliations allows us to define the smooth structure on the leaves.

**Proposition 4.15** ([AS09, Prop. 1.12])**.** *Let $\mathscr{F}$ be a foliation on $M$ and $x \in M$. Let $q := \dim T_xM - \dim F_x$. Then there is a neighborhood $U$ of $x$, a $q$-dimensional foliated manifold $(N, \mathscr{G})$ and a submersion $\varphi : U \to N$ with connected fibers such that $\mathscr{F}_U = \varphi^{-1}(\mathscr{G})$. Moreover, we have that the tangent space of the leaf of $N$ at $\varphi(x)$ is $0$, that $\ker \varphi_*(x) = F_x$ and that each fiber of $\varphi$ is contained in a leaf of $\mathscr{F}$.*

If $U$, $(N, \mathscr{G})$ and $\varphi$ are as in Proposition 4.15 around $x \in M$, then we take $\varphi^{-1}(\varphi(x))$ as a chart for the leaf through $x$. These charts indeed constitute a smooth atlas for the leaf through $x$ [AS09, Prop. 1.14]. From Proposition 4.15 it is clear that the tangent space to the leaf through $x$ with this smooth structure if precisely $F_x$.

Proposition 4.15 also allows us to prove that curves tangent to the leaves stay on the leaves.

**Proposition 4.16.** *Let $\mathscr{F}$ be a foliation on $M$. Then $x, y \in M$ lie on the same leaf if and only if there is a piecewise smooth curve tangent to the leaves joining them, i.e., if there is $\gamma : [0,1] \to M$ piecewise smooth with $\gamma(0) = x$ and $\gamma(1) = y$ such that $\dot\gamma(t) \in F_{\gamma(t)}$ for all $t \in [0,1]$.*

*Proof.* Let $X \in \mathscr{F}$. Then for every $z \in M$ the smooth curve given by $\gamma(t) := \exp(tX)(z)$ is tangent to the leaves. Indeed, $\exp(tX)(z) = \phi_t^X(z)$, so

$$\dot\gamma(t) = \frac{d}{dt}\exp(tX)(z) = X(\phi_t^X(z)) \in F_{\phi_t^X(z)} = F_{\gamma(t)}.$$

If $x, y \in M$ lie on the same leaf, then there are $X_1, \ldots, X_k \in \mathscr{F}$ such that $y = \phi_1^{X_k} \circ \cdots \circ \phi_1^{X_1}(x)$. Define $x_i$ recursively by $x_0 := x$ and $x_i := \phi_1^{X_i}(x_{i-1})$, so that $y = x_k$, and let $\gamma_i(t) := \exp(tX_i)(x_{i-1})$. Then the concatenation $\gamma_1 \cdot \ldots \cdot \gamma_k$ is the sought curve.

Conversely, assume there is $\gamma : [0,1] \to M$ piecewise smooth with $\gamma(0) = x$ and $\gamma(1) = y$ such that $\dot\gamma(t) \in F_{\gamma(t)}$. Let $U$, $(N, \mathscr{G})$ and $\varphi$ be as in Proposition 4.15. Shrinking $U$ if necessary, assume that $\mathscr{F}_U$ is generated by $X_1, \ldots, X_k \in \mathfrak{X}(U)$. Since $\dot\gamma(t) \in F_{\gamma(t)}$, write $\dot\gamma(t) = \dot\gamma^i(t) X_i(\gamma(t))$ for $t$ close enough to 0. Let $\{X_t\}_{t\in[0,1]}$ be the time-dependent vector field on $U$ given by $X_t := \dot\gamma^i(t) X_i$. Then $\gamma$ is an integral curve of $X_t$.

Consider $\alpha := \varphi \circ \gamma$ and write $G_z$ for the tangent space to the leaf of $\mathscr{G}$ at $z \in N$. Let $Y_t := \varphi_* X_t \in \Gamma(\varphi^* N)$. Since $\mathscr{F}_U = \varphi^{-1}(\mathscr{G})$, we have that $Y_t(\varphi(x)) \in G_{\varphi(x)} = 0$ for $t$ close enough to 0. The curve $t \mapsto (t, \alpha(t))$ in $\mathbb{R} \times N$ is an integral curve of the vector field $(t, z) \mapsto (1, Y_t(z))$ starting at $(0, \varphi(x))$. Since $Y_t(\varphi(x)) = 0$, the curve $t \mapsto (t, \varphi(x))$ is also an integral curve starting at $(0, \varphi(x))$. By the uniqueness of solutions to ODEs we have that $\alpha(t) = \varphi(x)$ for $t$ close enough to 0. Hence, $\gamma(t) \in \varphi^{-1}(\varphi(x))$, which is contained in the leaf of $\mathscr{F}$ through $x$. Since $[0,1]$ is compact, we can repeat this argument a finite amount of times to conclude that $\gamma(t)$ lies in the leaf of $x$ for all $t$. In particular, $y = \gamma(1)$ does. □

Back to Lie algebroids, any Lie algebroid $A \to M$ with anchor $\rho$ defines a foliation on $M$ by setting $\mathscr{F} := \rho(\Gamma_0(A))$. Indeed, it is locally finitely generated, since $A$ is locally trivial, and it is closed under the Lie bracket by Lemma 4.2. The leaves of $A$ are the leaves of the induced foliation.

## 4.2. Lie algebroid connections

### 4.2.1. Lie algebroid connections and parallel transport

Following the "generalization of the tangent bundle" approach to Lie algebroids, the following definition is the most natural definition of a Lie algebroid connection.

**Definition 4.17.** Let $A \to M$ be a Lie algebroid and $E \to M$ a vector bundle. An **$A$-connection** on $E$ is an $\mathbb{R}$-linear operator $\nabla : \Gamma(E) \to \Omega^1(A, E)$ satisfying the Leibniz rule

$$\nabla(f\sigma) = d_A f \otimes \sigma + f\nabla\sigma.$$

We denote $\nabla\sigma(a)$ by $\nabla_a\sigma$, for $a \in A$. A section $\sigma \in \Gamma(E)$ is called **parallel** if $\nabla\sigma = 0$.

If $\langle \cdot, \cdot \rangle$ is a metric on $E$, we say that an $A$-connection is **metric** or compatible with the metric if

$$\rho(a)\langle \sigma, \nu \rangle = \langle \nabla_a\sigma, \nu \rangle + \langle \sigma, \nabla_a\nu \rangle, \qquad \text{for all } \sigma, \nu \in \Gamma(E) \text{ and } a \in \Gamma(A).$$

The **curvature** of $\nabla$ is the 2-form $F \in \Omega^2(A, \operatorname{End} E)$ given by

$$F(a, b)\sigma := \nabla_a\nabla_b\sigma - \nabla_b\nabla_a\sigma - \nabla_{[a,b]}\sigma, \qquad \text{for } a, b \in \Gamma(A) \text{ and } \sigma \in \Gamma(E). \qquad \blacklozenge$$

When we do not wish to emphasize which Lie algebroid we are considering, we will just talk of LA-connections. For a vector bundle $E \to M$ and a Lie algebroid $A \to M$, there always exists an $A$-connection on $E$. Indeed, if $\overline{\nabla}$ is a connection on $E$, then $\nabla_a\sigma := \overline{\nabla}_{\rho(a)}\sigma$, where $\rho$ is the anchor and $\sigma \in \Gamma(E)$, defines an $A$-connection on $E$. From now on, we fix a Lie algebroid $A \to M$ with anchor $\rho$ and a vector bundle $E \to M$ with an $A$-connection $\nabla$.

**Definition 4.18.** An **$A$-path** is a curve $a : [0, 1] \to A$ such that $\dot{\gamma}_a(t) = \rho(a(t))$, where $\gamma_a(t)$ is the projection of $a(t)$ to $M$. We say that $a$ goes from $\gamma_a(0)$ to $\gamma_a(1)$. $\blacklozenge$

Intuitively, an $A$-path $a$ is just a "correct velocity" for $\gamma_a$, when we regard $A$ as a generalization of $TM$. Observe that by Proposition 4.16, there is an $A$-path from $x$ to $y$ in $M$ if and only if $x$ and $y$ lie on the same leaf.

**Lemma 4.19.** *A smooth map $a : [0, 1] \to A$ is an $A$-path if and only if $\tilde{a} : T[0, 1] \to A$ given by $\tilde{a}(\frac{d}{dt}) = a(t)$ is an LA-morphism.*

*Proof.* The map $a$ is an $A$-path if and only if $\frac{d}{dt} f(\gamma_a(t)) = df(\rho(a(t)))$ for all $f \in C^\infty(M)$. The left-hand side is $d(\tilde{a}^* f)(\frac{d}{dt})$, while the right-hand side is $\tilde{a}^* d_A f(\frac{d}{dt})$. So $a$ is an $A$-path if and only if $\tilde{a}^*$ commutes with the differentials at the level of functions. Since there are no 2-forms on $[0, 1]$ because it is 1-dimensional, $\tilde{a}^*$ always commutes with the differentials at the level of $k$-forms, for $k \geq 1$. $\square$

We will often write $\tilde{a}$ as $adt$.

To be able to define parallel transport, we need to be able to take derivatives of sections along $A$-paths.

**Lemma 4.20.** *Let $B \to N$ be a Lie algebroid and $\Phi : B \to A$ a vector bundle morphism covering $\phi : N \to M$ and preserving anchors, i.e., such that $\rho_A \circ \Phi = \phi_* \circ \rho_B$. If a vector bundle $E \to M$ carries a $A$-connection, then $\phi^* E$ inherits a $B$-connection $\Phi^*\nabla$ given by $(\Phi^*\nabla)(\phi^*\sigma) := \Phi^*(\nabla\sigma)$ for $\sigma \in \Gamma(E)$.*

*Proof.* We need only check that it is well defined by checking the Leibniz rule for a section $\phi^*(f\sigma)$, for $f \in C^\infty(M)$. Since, $\Phi^*$ commutes with differentials at the level of functions, because $\rho_A \circ \Phi = \phi_* \circ \rho_B$, then

$$(\Phi^*\nabla)(\phi^*(f\sigma)) = \Phi^*(\nabla(f\sigma)) = \Phi^*(d_A f \otimes \sigma + f\nabla\sigma) = d_B\phi^* f \otimes \phi^*\sigma + \phi^* f \Phi^*(\nabla\sigma). \qquad \square$$

A section of $E$ along the $A$-path $a$ is just a section of $\gamma_a^* E$. We say that it is **parallel along** $a$ if $(a^*\nabla)\sigma = 0$. This notion makes sense precisely because of Lemmas 4.19 and 4.20. For $(a^*\nabla)\sigma$ we will also use the following notations interchangeably

$$(a^*\nabla)\sigma = \nabla_a \sigma = \frac{\nabla}{dt}\sigma = \dot{\sigma}.$$

If $\{a_i\}_i$ is a local frame for $A$ and $\{\sigma_i\}_i$ a local frame for $E$, then in these frames we can write $a(t) = a^i(t)a_i(\gamma_a(t))$ and $\sigma(t) = \sigma^i(t)\sigma_i(\gamma_a(t))$, for some smooth functions $a^i, \sigma^i : [0,1] \to \mathbb{R}$. Let $\Gamma_{ij}^k$ be smooth local functions such that $\nabla_{a_i}\sigma_j = \Gamma_{ij}^k \sigma_k$. Then

$$\begin{aligned} \dot{\sigma}(t) &= (a^*\nabla)_{\frac{d}{dt}}(\sigma^i(t)\gamma_a^*\sigma_i(t)) = \dot{\sigma}^i(t)\sigma_i(\gamma_a(t)) + \sigma^i(t)\nabla_{a(t)}\sigma_i(\gamma_a(t)) \\ &= (\dot{\sigma}^i(t) + \Gamma_{jk}^i(\gamma_a(t))\sigma^k(t)a^j(t))\sigma_i(\gamma_a(t)), \end{aligned}$$

so that the equation for $\sigma$ to be parallel is locallly a first order linear ODE. These always have a unique solution defined on the whole interval of definition of the equation. Hence, we have proved the following.

**Lemma 4.21.** *Let $a : [0,1] \to A$ be an $A$-path. Then for every $v \in E_{\gamma(0)}$, there is a unique parallel section $\sigma_v$ along $a$ such that $\sigma_v(0) = v$.* □

With this we can define **parallel transport along** $a$ (allowed to be piecewise smooth) as in Section 1.2: the map $\tau_a : E_{\gamma_a(0)} \to E_{\gamma_a(1)}$ given by $\tau_a v := \sigma_v(1)$. The basic properties of parallel transport also hold in this case (Proposition 1.14), where the $A$-path $a$ traversed in reverse order is

$$a^{-1}(t) := -a(1-t),$$

and the concatenation of two $A$-paths $a$ and $b$ with $\gamma_a(1) = \gamma_b(0)$ is defined as

$$a \cdot b(t) := \begin{cases} 2a(2t), & 0 \le t \le \frac{1}{2} \\ 2b(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

The analog of Proposition 1.15 is the following.

**Proposition 4.22.** *1. Let $x \in M$, $b \in A_x$ and $\sigma \in \Gamma(E)$. Let $a : [0,1] \to A$ be an $A$-path with $\gamma_a(0) = x$ and $a(0) = b$, and let $\tau_t$ be parallel transport along $a$ from $x$ to $\gamma_a(t)$. Then*

$$\nabla_b \sigma = \left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}(\sigma(\gamma_a(t))).$$

*2. Let $a : [0,1] \to A$ be a piecewise smooth $A$-path and let $\tau_t$ be parallel transport along $a$ from $\gamma_a(0)$ to $\gamma_a(t)$. Then for any $\sigma \in \Gamma(\gamma_a^* E)$ we have that*

$$\dot{\sigma}(t) = \tau_t \frac{d}{dt}(\tau_t^{-1}\sigma(t)).$$

### 4.2.2. Holonomy

As in the classical case, parallel transport leads to the concept of holonomy group. If $A \to M$ is a Lie algebroid, let $\Pi_{x,y}^A$ denote the set of piecewise smooth $A$-paths $a$ with $\gamma_a(0) = x$ and $\gamma_a(1) = y$.

**Definition 4.23.** Let $A \to M$ be a Lie algebroid and $E \to M$ a vector bundle with an $A$-connection $\nabla$. The ***A*-holonomy group** of $\nabla$ at $x \in M$ is defined as

$$\mathrm{Hol}_x(\nabla) := \{\tau_a : a \in \Pi_{x,x}^A\}.$$

If we let $L$ be the leaf through $x$, then the **restricted *A*-holonomy group** at $x$ is

$$\mathrm{Hol}_x^0(\nabla) := \{\tau_a : a \in \Pi_{x,x}^A \text{ with } \gamma_a \text{ null-homotopic in } L\}. \quad ♦$$

In order to prove that $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_x^0(\nabla)$ are Lie subgroups of $\mathrm{GL}(E_x)$, we need to know how to "lift" homotopies $\gamma : [0,1]^2 \to M$ to some kind of "homotopies of $A$-paths". One way to do this is the following.

Let $x \in M$ and let $L$ be the leaf through $x$. Since the rank of $\ker \rho$ does not change along $L$, we have that $\ker \rho$ is a vector bundle over $L$. Then there is a short exact sequence of vector bundles over $L$

$$0 \longrightarrow \ker \rho \longrightarrow A_L \longrightarrow TL \longrightarrow 0, \tag{4.1}$$

where $A_L$ is the restriction of $A$ to $L$.

If we consider coordinates $(s,t)$ on $[0,1]^2$, a vector bundle morphism $\Phi : T[0,1]^2 \to A$ can be written uniquely as $\Phi = a dt + b ds$, for $a, b : [0,1]^2 \to A$. Explicitly, $a(s,t) = \Phi(\frac{\partial}{\partial t})$ and $b(s,t) = \Phi(\frac{\partial}{\partial s})$.

**Lemma 4.24.** *Let $\eta : TL \to A_L$ be a splitting of the short exact sequence (4.1), i.e., a vector bundle morphism such that $\rho \circ \eta = \mathrm{id}$. Then*

1. *if $\gamma : [0,1] \to L$ is a curve, then $a = \eta(\dot\gamma)$ is an $A$-path over $\gamma$,*
2. *if $\gamma : [0,1]^2 \to L$ is a homotopy, then $\Phi = a dt + b ds$ given by $a = \eta(\frac{\partial}{\partial t}\gamma)$ and $b = \eta(\frac{\partial}{\partial s}\gamma)$ is a vector bundle morphism with $\rho \circ \Phi = \gamma_*$. In particular, $a_s := a(s,\cdot)$ is an $A$-path over $\gamma_s := \gamma(s,\cdot)$ for every $s$, and if $\dot\gamma_s(t) = 0$ then $a_s(t) = 0$.*

*Proof.* The first claim is clear. For the second, $\rho \circ \Phi(\frac{\partial}{\partial t}) = \rho \circ \eta(\frac{\partial}{\partial t}\gamma) = \gamma_*(\frac{\partial}{\partial t})$ and similarly for $\frac{\partial}{\partial s}$, so $\rho \circ \Phi = \gamma_*$. Since $\dot\gamma_s(t) = \frac{\partial}{\partial t}\gamma(s,t)$, that $a_s$ is an $A$-path over $\gamma_s$ is a consequence of the first claim. □

**Proposition 4.25.** *Let $E$ be a vector bundle with an $A$-connection $\nabla$ and $x \in M$. Then $\mathrm{Hol}_x(\nabla)$ is a Lie subgroup of $\mathrm{GL}(E_x)$ whose connected identity component is $\mathrm{Hol}_x^0(\nabla)$. In particular, $\mathrm{Hol}_x^0(\nabla)$ is normal in $\mathrm{Hol}_x(\nabla)$.*

*Proof.* That both $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_x^0(\nabla)$ are subgroups of $\mathrm{GL}(E_x)$ is a direct consequence of the properties of parallel transport (cfr. Proposition 1.14). We now show that $\mathrm{Hol}_x^0(\nabla)$ is an arcwise connected subgroup of $\mathrm{GL}(E_x)$, which implies that it is a Lie subgroup [Yam50]. Let $L$ be the leaf through $x$ and let $\gamma : [0,1]^2 \to L$ be a smooth homotopy with fixed endpoints starting at the constant path on $x$ (every null-homotopic path is smoothly null-homotopic [Lee12, Thm. 6.29]). Let $\Phi = a dt + b ds : T[0,1]^2 \to A$ be as in Lemma 4.24, which exists because short exact sequences of vector bundles are always split. By a similar argument as in Lemma 4.21 and using the smooth dependence on initial conditions of ODE theory, for each $v \in E_x$ there is $\sigma \in \Gamma(\gamma^* E)$ such that $\frac{\nabla}{\partial t}\sigma = 0$ and $\sigma(s,0) = v$ for all $s$. Then, if $\tau_s$ is parallel transport along $a_s$, we have that $\tau_s v = \sigma(s,1)$, which is smooth on $s$. Since $a_0(t) = 0 \in A_x$ for all $t$, then $\sigma(0,t) \in E_x$ does not depend on $t$, and therefore $\tau_0 v = \sigma(0,1) = \sigma(0,0) = v$. We conclude that $\tau_s$ is a smooth path in $\mathrm{Hol}_x^0(\nabla)$ from $\tau_1$ to the identity, as wanted.

Since $\mathrm{Hol}_x^0(\nabla)$ is a subgroup of $\mathrm{Hol}_x(\nabla)$, this also endows $\mathrm{Hol}_x(\nabla)$ with the structure of a Lie group by translating the smooth structure of $\mathrm{Hol}_x^0(\nabla)$ by left or right multiplication.

Consider now the map $\pi_1(L) \to \mathrm{Hol}_x(\nabla)/\mathrm{Hol}_x^0(\nabla)$ given by $[\gamma] \mapsto \tau_a^{-1}\mathrm{Hol}_x^0(\nabla)$, where $a$ is as in Lemma 4.24 (for a fixed splitting). It is easily seen to be a surjective group homomorphism. A similar argument as in the proof of Proposition 4.25, using that $\pi_1(L)$ is countable, gives that $\mathrm{Hol}_x^0(\nabla)$ is the identity component of $\mathrm{Hol}_x(\nabla)$. □

Therefore, the following definition makes sense.

**Definition 4.26.** Let $E$ be a vector bundle with an $A$-connection $\nabla$. The **holonomy algebra** $\mathfrak{hol}_x(\nabla)$ of $\nabla$ at $x \in M$ is defined as the Lie algebra of $\mathrm{Hol}_x(\nabla)$. ♦

Because parallel transport can only be done between fibers of points on the same leaf, the holonomy principle in this case is "leafwise". The proof is identical to the classical case.

**Theorem 4.27** (Holonomy principle)**.** *Let $M$ be connected and $E \to M$ a vector bundle with an $A$-connection $\nabla$. Let $x \in M$ and $L$ the leaf through $x$. Then the following vector spaces are isomorphic:*

1. *the space of parallel sections of $E_L$,*
2. *the space of $\mathrm{Hol}_x(\nabla)$-invariant vectors in $E_x$,*
3. *the space of sections on $L$ invariant under parallel transport, i.e., sections $\sigma \in \Gamma(E_L)$ such that $\tau_a(\sigma(\gamma_a(0))) = \sigma(\gamma_a(1))$ for all piecewise smooth $A$-paths $a$ on $M$.*

### 4.2.3. Ambrose–Singer–Fernandes theorem

A remarkable fact is that the Ambrose–Singer theorem for LA-holonomy picks up some extra terms, which imply that a flat $A$-connection does not necessarily have a discrete holonomy. This was already observed by Fernandes [Fer00, Fer02]. Here we give a different proof, in the spirit of Section 1.3.

Let $A \to M$ be a Lie algebroid with anchor $\rho$ and $E \to M$ be vector bundle with an $A$-connection $\nabla$. To understand what these extra terms are, observe that $a \in \ker \rho_x$ defines and endomorphism of the fiber $E_x$, as the following shows.

**Lemma 4.28.** *Let $a \in \ker \rho_x$. Then $\nabla_a$ defines an endomorphism of $E_x$, defined by $\nabla_a v := \nabla_a \sigma(x)$, where $\sigma \in \Gamma(E)$ is such that $\sigma(x) = v$.*

*Proof.* Let $\sigma \in \Gamma(E)$ be such that $\sigma(x) = v$. Any other section of $E$ with value $v$ at $x$ can be written as $\sigma + \nu$, for some $\nu \in \Gamma(E)$ with $\nu(x) = 0$. Then

$$\nabla_a(\sigma + \nu)(x) = \nabla_a \sigma(x) + \nabla_a \nu(x).$$

So it is enough to see that $\nabla_a \nu(x) = 0$. Let $\{\sigma_i\}_i$ be a frame for $E$ and write $\nu = \nu^i \sigma_i$. Then $\nu^i(x) = 0$ for all $i$, and hence, because $\rho(a) = 0$,

$$\nabla_a \nu(x) = \nu^i(x) \nabla_a \sigma_i(x) = 0. \qquad \square$$

As we will see, the proof is completely analogous to that of Section 1.3, but these new terms suddenly appear in the computations. To understand why they appear, recall that the key for our proof of the Ambrose–Singer theorem was Lemma 1.22, and this one relied on the fact that the curvature of the pullback connection is the pullback of the curvature, Lemma 1.11. For LA-connections, though, this does not hold. We will just compute the case we are interested in.

**Lemma 4.29.** *Let $E \to M$ be a vector bundle with an $A$-connection $\nabla$ and curvature $F$ and let $\Phi = a dt + b ds : [0,1]^2 \to A$ be a vector bundle morphism over $\gamma : [0,1]^2 \to M$ with $\rho \circ \Phi = \gamma_*$. Then there is a smooth map $c : [0,1]^2 \to A$ with $\rho \circ c = 0$ such that*

$$\frac{\nabla}{\partial t}\frac{\nabla}{\partial s} - \frac{\nabla}{\partial s}\frac{\nabla}{\partial t} = F(a,b) + \nabla_c.$$

*Explicitly, it is given as follows: consider $(s,t)$-dependent sections $a_{s,t}, b_{s,t} \in \Gamma(A)$ such that $a_{s,t}(\gamma(s,t)) = a(s,t)$ and $b_{s,t}(\gamma(s,t)) = b(s,t)$, then*

$$c(s,t) = \left(\frac{\partial}{\partial t} b_{s,t} - \frac{\partial}{\partial s} a_{s,t} + [a_{s,t}, b_{s,t}]\right)(\gamma(s,t)).$$

*Proof.* The proof that $\frac{\nabla}{\partial t}\frac{\nabla}{\partial s} - \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}$ is $C^\infty([0,1]^2)$-linear is straightforward, so we need only check that the claim holds for pullback sections $\gamma^*\sigma$, for $\sigma \in \Gamma(E)$. Consider a local frame $\{e_i\}_i$ for $A$ and write $a(s,t) = a^i(s,t) e_i(\gamma(s,t))$ and $b(s,t) = b^i(s,t) e_i(\gamma(s,t))$ for some $a^i, b^i : [0,1]^2 \to \mathbb{R}$. Set

$a_{s,t} := a^i(s,t)e_i$ and $b_{s,t} = b^i(s,t)e_i$. Then

$$\begin{aligned}
\frac{\nabla}{\partial t}\frac{\nabla}{\partial s}(\gamma^*\sigma) &= \frac{\nabla}{\partial t}(\nabla_{b(s,t)}\sigma) = \frac{\nabla}{\partial t}(b^i(s,t)\gamma^*(\nabla_{e_i}\sigma)) \\
&= \frac{\partial b^i}{\partial t}(s,t)\gamma^*(\nabla_{e_i}\sigma) + b^i(s,t)\nabla_{a(s,t)}\nabla_{e_i}\sigma \\
&= \left(\nabla_{\frac{\partial}{\partial t}b_{s,t}}\sigma + \nabla_{a_{s,t}}\nabla_{b_{s,t}}\sigma\right)(\gamma(s,t)).
\end{aligned}$$

Similarly,

$$\frac{\nabla}{\partial s}\frac{\nabla}{\partial t}(\gamma^*\sigma) = \left(\nabla_{\frac{\partial}{\partial s}a_{s,t}}\sigma + \nabla_{b_{s,t}}\nabla_{a_{s,t}}\sigma\right)(\gamma(s,t)).$$

Hence,

$$\left(\frac{\nabla}{\partial t}\frac{\nabla}{\partial s} - \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\right)(\gamma^*\sigma) = F(a_{s,t},b_{s,t})(\gamma(s,t)) + \nabla_{\frac{\partial}{\partial t}b_{s,t} - \frac{\partial}{\partial s}a_{s,t} + [a_{s,t},b_{s,t}]}\sigma(\gamma(s,t)).$$

It only remains to see that $\rho \circ c = 0$. Let $T_{s,t} := \rho(a_{s,t})$ and $S_{s,t} := \rho(b_{s,t})$. Then $T_{s,t}(\gamma(s,t)) = \frac{\partial}{\partial t}\gamma(s,t)$ and $S_{s,t}(\gamma(s,t)) = \frac{\partial}{\partial s}\gamma(s,t)$, since $\rho \circ \Phi = \gamma_*$. Therefore,

$$\rho \circ c(s,t) = \left(\frac{\partial}{\partial t}S_{s,t} - \frac{\partial}{\partial s}T_{s,t} + [T_{s,t},S_{s,t}]\right)(\gamma(s,t)).$$

Since $\gamma^*$ commutes with differentials, we have that for all $\alpha \in \Omega^1(M)$,

$$\begin{aligned}
0 &= (d\gamma^* - \gamma^* d)\alpha\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right) \\
&= \frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial}{\partial s}\gamma\right)\right) - \frac{\partial}{\partial s}\left(\alpha\left(\frac{\partial}{\partial t}\gamma\right)\right) - d\alpha\left(\frac{\partial}{\partial t}\gamma,\frac{\partial}{\partial s}\gamma\right) \\
&= \frac{\partial}{\partial t}(\alpha(S_{s,t})(\gamma(s,t))) - \frac{\partial}{\partial s}(\alpha(T_{s,t})(\gamma(s,t))) - d\alpha(T_{s,t},S_{s,t})(\gamma(s,t)) \\
&= \alpha\left(\frac{\partial}{\partial t}S_{s,t}\right)(\gamma(s,t)) + \left(\frac{\partial}{\partial t}\gamma\right)(\alpha(S_{s,t})) - \alpha\left(\frac{\partial}{\partial s}T_{s,t}\right)(\gamma(s,t)) - \left(\frac{\partial}{\partial s}\gamma\right)(\alpha(T_{s,t})) \\
&\qquad - T_{s,t}(\gamma(s,t))(\alpha(S_{s,t})) + S_{s,t}(\gamma(s,t))(\alpha(T_{s,t})) + \alpha([T_{s,t},S_{s,t}])(\gamma(s,t)) \\
&= \alpha(\rho \circ c(s,t)).
\end{aligned}$$

Hence, $\rho \circ c = 0$, as wanted. □

We may now start the proof of the Ambrose–Singer–Fernandes theorem.

**Lemma 4.30.** *Let $\Phi = adt + bds : T[0,1]^2 \to M$ be a vector bundle morphism over $\gamma : [0,1]^2 \to M$ with $\rho \circ \Phi = \gamma_*$, and let $\tau_{s,t}$ be parallel transport along the $A$-path $a_s$ from $\gamma_s(t)$ to $\gamma_s(1)$. Let $c : [0,1]^2 \to A$ be as in Lemma 4.29 and let*

$$F_{s,t} := \tau_{s,t}F\left(a(s,t),b(s,t)\right)\tau_{s,t}^{-1} \in \mathfrak{gl}(E_{\gamma_s(1)}) \quad \text{and} \quad A_{s,t} := \tau_{s,t}\nabla_{c(s,t)}\tau_{s,t}^{-1} \in \mathfrak{gl}(E_{\gamma_s(1)}).$$

*Then for any $\sigma \in \Gamma(\gamma^*E)$ with $\frac{\nabla}{\partial t}\sigma = 0$ and $\frac{\nabla}{\partial s}\sigma(\cdot,0) = 0$ we have that*

$$\frac{\nabla}{\partial s}\sigma(s,1) = \left(\int_0^1 (F_{s,t} + A_{s,t})dt\right)\sigma(s,1).$$

*Proof.* Using Proposition 4.22, Lemma 4.29 and the fact that $\frac{\nabla}{\partial t}\sigma = 0$, we compute:

$$\begin{aligned}
\frac{d}{dt}\left(\tau_{s,t}\frac{\nabla}{\partial s}\sigma(s,t)\right) &= \tau_{s,t}\frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\sigma(s,t) = \tau_{s,t}(F(a(s,t),b(s,t)) + \nabla_{c(s,t)})\sigma(s,t) \\
&= (F_{s,t} + A_{s,t})\sigma(s,1).
\end{aligned}$$

Explicitly, parallel transport along $a_s$ from $\gamma_s(t)$ to $\gamma_s(1)$ is given by parallel transport along the $A$-path $r \mapsto (1-t)a_s(r(1-t)+t)$, covering $r \mapsto \gamma_s(r(1-t)+t)$. Then, since $\tau_{s,1} = \mathrm{id}$, because it is parallel transport along the $A$-path $r \mapsto 0 \in A_{\gamma_s(1)}$, and $\frac{\nabla}{\partial s}\sigma(\cdot,0) = 0$,

$$\begin{aligned}\frac{\nabla}{\partial s}\sigma(s,1) &= \tau_{s,1}\frac{\nabla}{\partial s}\sigma(s,1) - \tau_{s,0}\frac{\nabla}{\partial s}\sigma(s,0) = \int_0^1 \frac{d}{dt}\left(\tau_{s,t}\frac{\nabla}{\partial s}\sigma(s,t)\right)dt \\ &= \left(\int_0^1 (F_{s,t}+A_{s,t})dt\right)\sigma(s,1).\end{aligned}$$

□

**Corollary 4.31.** *Let $\Phi = adt + bds : T[0,1]^2 \to M$ be a vector bundle morphism over a piecewise smooth homotopy $\gamma : [0,1]^2 \to M$ with fixed endpoints, with $\rho \circ \Phi = \gamma_*$ and such that if $\frac{\partial}{\partial s}\gamma(s,t) = 0$ then $b(s,t) = 0$, and let $\tau_s$ be parallel transport along $a_s$. Then*

$$\frac{d}{ds}\tau_s = \left(\int_0^1 (F_{s,t}+A_{s,t})dt\right)\tau_s.$$

*Proof.* For any $\gamma$ with fixed endpoints, such a lift $\Phi$ always exists by Lemma 4.24. Let $\sigma \in \Gamma(\gamma^*E)$ with $\frac{\nabla}{\partial t}\sigma = 0$ and $\frac{\nabla}{\partial s}\sigma(\cdot,0) = 0$. Notice that since $\gamma$ has fixed endpoints, then $b(s,0) = 0$ and $b(s,1) = 0$ for all $s$, so that the covariant derivative with respect to $s$ at the endpoints is just derivation with respect to $s$. Hence, $\sigma(\cdot,0)$ is constant and, by Lemma 4.30,

$$\begin{aligned}\frac{\nabla}{\partial s}\sigma(s,1) &= \frac{d}{ds}(\sigma(s,1)) = \frac{d}{ds}(\tau_s\sigma(s,0)) = \left(\frac{d}{ds}\tau_s\right)\sigma(s,0) \\ &= \left(\int_0^1 (F_{s,t}+A_{s,t})dt\right)\sigma(s,1) = \left(\int_0^1 (F_{s,t}+A_{s,t})dt\right)\tau_s\sigma(s,0),\end{aligned}$$

and this gives the result. □

We now want to consider the analog of the homotopies of square loops. Let $z \in M$, let $L$ be the leaf through $z$ and let $a, b \in A_z$. Let $f : U \to L$ be a smooth map from an open neighborhood $U$ of 0 in $\mathbb{R}^2$ with $f(0) = z$ and $\bar{\Phi} = \bar{a}dx + \bar{b}dy : T[0,1]^2 \to A$ a vector bundle morphism over $f$ with $\rho \circ \bar{\Phi} = f_*$ and such that $\bar{a}(0,0) = a$ and $\bar{b}(0,0) = b$, which exists by Lemma 4.24. Then the homotopies of square loops in this case can be defined as follows: let $q : [0,1]^2 \to [0,1]^2$ be the map

$$q(s,t) = \begin{cases} (4st, 0), & 0 \le t \le \frac{1}{4}, \\ (s, s(4t-1)), & \frac{1}{4} \le t \le \frac{1}{2}, \\ (s(3-4t), s), & \frac{1}{2} \le t \le \frac{3}{4}, \\ (0, 4s(1-t)), & \frac{3}{4} \le t \le 1, \end{cases}$$

then

$$\gamma := f \circ q \qquad \text{and} \qquad \Phi := \bar{\Phi} \circ q_*. \tag{4.2}$$

Then $\Phi : T[0,1]^2 \to A$ is a vector bundle morphism with $\rho \circ \Phi = \gamma_*$. Write $\Phi = adt + bds$ (not to be confused with $a, b \in A_z$, the difference is clear from context) and consider $(x,y)$- and $(s,t)$-dependent sections $\bar{a}_{x,y}, \bar{b}_{x,y}, a_{s,t}, b_{s,t} \in \Gamma(A)$ such that

$$\bar{a}_{x,y}(f(x,y)) = \bar{a}(x,y), \quad \bar{b}_{x,y}(f(x,y)) = \bar{b}(x,y), \quad a_{s,t}(\gamma(s,t)) = a(s,t) \quad \text{and} \quad b_{s,t}(\gamma(s,t)) = b(s,t).$$

It is straightforward to see that the relations among these are

$$a_{s,t} = \begin{cases} 4s\bar{a}_{4st,0}, & 0 \le t \le \frac{1}{4}, \\ 4s\bar{b}_{s,s(4t-1)}, & \frac{1}{4} \le t \le \frac{1}{2}, \\ -4s\bar{a}_{s(3-4t),s}, & \frac{1}{2} \le t \le \frac{3}{4}, \\ -4s\bar{b}_{0,4s(1-t)}, & \frac{3}{4} \le t \le 1; \end{cases} \qquad b_{s,t} = \begin{cases} 4t\bar{a}_{4st,0}, & 0 \le t \le \frac{1}{4}, \\ \bar{a}_{s,s(4t-1)} + (4t-1)\bar{b}_{s,s(4t-1)}, & \frac{1}{4} \le t \le \frac{1}{2}, \\ (3-4t)\bar{a}_{s(3-4t),s} + \bar{b}_{s(3-4t),s}, & \frac{1}{2} \le t \le \frac{3}{4}, \\ (1-t)\bar{b}_{0,4s(1-t)}, & \frac{3}{4} \le t \le 1. \end{cases}$$

**Proposition 4.32.** *Let $(\Phi, \gamma)$ be a homotopy of square loops as in (4.2) and let $\tau_s$ be parallel transport along $a_s$. Then there is $c \in \ker \rho_z$ such that*

$$\left.\frac{d}{ds}\right|_{s=0} \tau_s = 0 \quad \text{and} \quad \left.\frac{d^2}{ds^2}\right|_{s=0} \tau_s = 2F(b,a) - 2\nabla_c.$$

*Explicitly, $c$ is given as follows: let $\bar{c} : [0,1]^2 \to A$ be as in Lemma 4.29 for $\bar{\Phi}$, then $c = \bar{c}(0,0)$.*

*Proof.* Direct computation, using the skewsymmetry of $F$ and the formulas for $a_{s,t}$ and $b_{s,t}$, gives

$$F(a(s,t), b(s,t)) = \begin{cases} 0, & t \leq \frac{1}{4} \text{ or } t \geq \frac{3}{4}, \\ 4sF(\bar{b}, \bar{a})(s, s(4t-1)), & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ 4sF(\bar{b}, \bar{a})(s(3-4t), s), & \frac{1}{2} \leq t \leq \frac{3}{4}. \end{cases}$$

On the other hand, it is an easy but somewhat lengthy computation to show that the map $c : [0,1]^2 \to A$ from Lemma 4.29 for $\Phi$ is given by

$$c(s,t) = \begin{cases} 0, & t \leq \frac{1}{4} \text{ or } t \geq \frac{3}{4}, \\ -4s\bar{c}(s, s(4t-1)), & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ -4s\bar{c}(s(3-4t), s), & \frac{1}{2} \leq t \leq \frac{3}{4}. \end{cases}$$

Hence, by Corollary 1.23,

$$\left.\frac{d}{ds}\right|_{s=0} \tau_s = \left( \int_{1/4}^{3/4} (F_{0,t} + A_{0,t}) dt \right) \tau_0 = 0.$$

Also, if we let $c := \bar{c}(0,0) \in A_z$, then $\frac{1}{s}(F_{s,t} + A_{s,t}) \to 4F(b,a) - 4\nabla_c$ uniformly in $t$ as $s \to 0$ since $\tau_{0,t} = \mathrm{id}$ because $\gamma_0$ is the constant path. Then,

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \tau_s = \left( \int_{1/4}^{3/4} \left.\frac{d}{ds}\right|_{s=0} (F_{s,t} + A_{s,t}) dt \right) \tau_0 = 2F(b,a) - 2\nabla_c. \qquad \square$$

**Theorem 4.33** (Ambrose–Singer–Fernandes)**.** *Let $x \in M$ and denote by $\Pi_x^A$ the set of piecewise smooth $A$-pahts $[0,1] \to A$ starting at $x$. Then*

$$\begin{aligned} \mathfrak{hol}_x(\nabla) = \operatorname{span}\{\tau_e^{-1} F(a,b) \tau_e : \ e \in \Pi_x^A \textit{ and } a, b \in A_{\gamma_e(1)}\} \\ + \operatorname{span}\{\tau_e^{-1} \nabla_c \tau_e : \ e \in \Pi_x^A \textit{ and } c \in \ker \rho_{\gamma_e(1)}\}. \end{aligned}$$

*Proof.* Analogous to that of the Ambrose–Singer theorem, Theorem 1.25. $\square$

The new terms are genuinely a new feature of $A$-holonomy, they cannot be absorbed into the curvature terms, since there are examples of flat $A$-connections with non-discrete holonomy. The easiest example might be the following.

**Example 4.34.** If the anchor is trivial, then an $A$-connection on a vector bundle $E \to M$ is just an element in $\Omega^1(A, \operatorname{End} E)$. Let $A = \mathfrak{g}$ be a Lie algebra over a point and $E = V$ a vector space. An $A$-connection on $E$ is then a linear map $B : \mathfrak{g} \to \mathfrak{gl}(V)$. Its curvature is

$$F(\xi, \eta) = [B(\xi), B(\eta)] - B([\xi, \eta]), \qquad \text{for } \xi, \eta \in \mathfrak{g}.$$

Hence, it is flat if and only if $B$ is a Lie algebra morphism (a representation). Any smooth curve $a : [0,1] \to \mathfrak{g}$ is an $A$-path, and a smooth curve $v : [0,1] \to V$ is parallel along $a$ if and only if $\dot{v}(t) + B(a(t))v(t) = 0$. If we write $\tau_t$ for parallel transport along $a$, then it satisfies $\dot{\tau}_t = -B(a(t))\tau_t$.

If $a$ is the constant path with value $\xi \in \mathfrak{g}$, then $v(t) = e^{-tB(\xi)}v(0)$ is the parallel section along $a$. Hence, we have that $e^{B(\mathfrak{g})} \subseteq \operatorname{Hol}(B)$. Consider now $\langle e^{B(\mathfrak{g})} \rangle$, the Lie subgroup of $\operatorname{GL}(V)$ generated by $e^{B(\mathfrak{g})}$. Its Lie algebra is the Lie subalgebra of $\mathfrak{gl}(V)$ generated by $B(\mathfrak{g})$. Consider the time-dependent vector field $(g,t) \mapsto -B(a(t))g$ defined on $\langle e^{B(\mathfrak{g})} \rangle$. Its flow is contained in $\langle e^{B(\mathfrak{g})} \rangle$, and since $\tau_t$ is a flow line of such a vector field, we conclude that $\tau_t \in \langle e^{B(\mathfrak{g})} \rangle$ for all $t$, i.e., $\operatorname{Hol}(B) = \langle e^{B(\mathfrak{g})} \rangle$.

Here we can see three different cases:

1. If $B$ is flat, let $G$ be the 1-connected Lie group integrating $\mathfrak{g}$ and let $e^B : G \to \mathrm{GL}(V)$ be the Lie group morphism integrating $\mathfrak{g}$, given by $e^B(\operatorname{Exp}\xi) = e^{B(\xi)}$. Then $\langle e^{B(\mathfrak{g})}\rangle = e^B(\langle \operatorname{Exp}\mathfrak{g}\rangle) = e^B(G)$. In this case, $\mathfrak{hol}(B) = B(\mathfrak{g})$, and this corresponds to the isotropy summand in the Ambrose–Singer–Fernandes theorem.

2. If $B$ is not flat but $B(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(V)$, then we still have $\mathfrak{hol}(B) = B(\mathfrak{g})$. In this case, the curvature summand in the Ambrose–Singer–Fernandes theorem can be absorbed into the isotropy summand.

3. If $B$ is not flat and $B(\mathfrak{g})$ is not a Lie subalgebra of $\mathfrak{gl}(V)$, then we have that $\mathfrak{hol}(B)$ is the Lie subalgebra generated by $B(\mathfrak{g})$. In this case the curvature summand cannot be absorbed into the isotropy summand. The curvature terms come from conjugation: if $\xi, \eta \in \mathfrak{g}$ then
$$\left.\frac{d^2}{dsdt}\right|_{s=t=0} e^{tB(\xi)}e^{sB(\eta)}e^{-tB(\xi)} = [B(\xi), B(\eta)],$$
and these terms might not lie in $B(\mathfrak{g})$, i.e., they might not come from the isotropy.

Let us illustrate this with a particular example. Consider $\mathfrak{g} = \mathfrak{su}(2)$ and $V = \mathbb{C}^2$. Consider the basis of $\mathfrak{su}(2)$
$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
with commutation relations $[\sigma_1, \sigma_2] = -2\sigma_3$ (and cyclic permutations).

The map $B$ sending $\sigma_j$ to $\sigma_{j+1}$ (here indices are taken mod 3) is a representation, and since $B(\mathfrak{su}(2)) = \mathfrak{su}(2)$, we have that $\mathrm{Hol}(B) = \mathrm{SU}(2)$. The map $B$ sending $\sigma_1$ to itself and permuting $\sigma_2$ and $\sigma_3$ is not a representation, but we still have $B(\mathfrak{su}(2)) = \mathfrak{su}(2)$, so still $\mathrm{Hol}(B) = \mathrm{SU}(2)$. The map $B$ sending $\sigma_1$ and $\sigma_2$ to 0 and $\sigma_3$ to itself is not a representation either, and $B(\mathfrak{su}(2)) = \mathbb{R}\sigma_3$, which is a subalgebra. In this case, $\mathrm{Hol}(B) = \mathrm{U}(1)$, where $\mathrm{U}(1) \cong e^{\mathbb{R}\sigma_3} \subseteq \mathrm{SU}(2)$. Lastly, if $B$ is the map taking $\sigma_1$ to id and letting $\sigma_2$ and $\sigma_3$ unchanged, then $B$ is not a representation and $B(\mathfrak{su}(2)) = \operatorname{span}\{\mathrm{id}, \sigma_2, \sigma_3\}$ is not a subalgebra. Then $\mathfrak{hol}(B)$ is the Lie algebra generated by $B(\mathfrak{su}(2))$, which is $\mathbb{R} \oplus \mathfrak{su}(2)$. Hence,
$$\mathrm{Hol}(B) = \mathbb{R}_{>0} \times \mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) : A^*A \in \mathbb{R}_{>0}\mathrm{id}\}. \qquad \blacktriangledown$$

Also, since $A$-holonomy is a leafwise property, it can jump from leaf to leaf, as the next examples show.

**Example 4.35.** Reconsider Example 4.34 with changing basepoint. That is, consider $A = [0,1] \times \mathfrak{g}$ as the Lie algebroid over $[0,1]$ with trivial anchor and bracket given by $[a,b](t) := [a(t), b(t)]$, for $a, b : [0,1] \to \mathfrak{g}$. The connection is now given by a smooth collection of linear maps $B_t : \mathfrak{g} \to \mathfrak{gl}(V)$.

For $\mathfrak{g} = \mathfrak{su}(2)$, consider the map given by
$$\begin{aligned} B_t(\sigma_1) &:= t\mathrm{id} + (1-t)\sigma_1, \\ B_t(\sigma_2) &:= t\sigma_2 + (1-t)\sigma_3, \\ B_t(\sigma_3) &:= t\sigma_3 + (1-t)\sigma_2. \end{aligned}$$

From Example 4.34 we already now that $\mathrm{Hol}(B_0) = \mathrm{SU}(2)$ and $\mathrm{Hol}(B_1) = \mathbb{R}_{>0} \times \mathrm{SU}(2)$. If $t \neq 0, 1$, then we have that
$$[B_t(\sigma_1), B_t(\sigma_2)] = 2(1-t)(-t\sigma_3 + (1-t)\sigma_2),$$
from where we conclude that $\mathfrak{hol}(B_t) = \mathbb{R} \oplus \mathfrak{su}(2)$. Hence,
$$\mathrm{Hol}(B_t) = \begin{cases} \mathrm{SU}(2), & t = 0 \\ \mathbb{R}_{>0} \times \mathrm{SU}(2), & t > 0 \end{cases} \qquad \blacktriangledown$$

**Example 4.36** (Action Lie algebroid)**.** Let $G$ be a Lie group acting smoothly on $M$ and consider the action Lie algebroid $A = M \times \mathfrak{g}$ (Example 4.6). A smooth map $a : [0,1] \to A$, which we write as $a = (\gamma, \xi)$ for $\gamma : [0,1] \to M$ and $\xi : [0,1] \to \mathfrak{g}$, is an $A$-path if and only if

$$\xi(t)_M(\gamma(t)) = \dot{\gamma}(t).$$

Let $E = M \times V$ be the trivial vector bundle over $M$, with $V$ a vector space. Then any $A$-connection on $E$ is of the form $\nabla = \rho^* d + \beta$, for some $\beta \in \Omega^1(A, \operatorname{End} E)$, i.e., identifying $\Gamma(E) \cong C^\infty(M, V)$,

$$\nabla_\xi w(x) = w_*(\xi_M(x)) + \beta_x(\xi) w(x), \qquad \text{for } \xi \in \mathfrak{g},\ w \in C^\infty(M,V) \text{ and } x \in M.$$

Since both $A$ and $E$ are trivial, then $\beta$ is a family of linear maps $\beta_x : \mathfrak{g} \to \mathfrak{gl}(V)$ depending smoothly on $x \in M$.

From here we get that a section $\sigma = (\gamma, v)$ of $E$ along $\gamma$, where $v : [0,1] \to V$, is parallel if and only if

$$\dot{v}(t) + \beta_{\gamma(t)}(\xi(t))v(t) = 0.$$

With this we can do some explicit computations in some easy examples.

Consider the action of $\mathbb{S}^1 = \{w \in \mathbb{C} : |w| = 1\}$ on $\mathbb{C}$ by complex scalar multiplication. Since the Lie algebra of $\mathbb{S}^1$ is $\mathbb{R}$ and $\mathfrak{gl}(\mathbb{R}) = \mathbb{R}$, we get that $\beta \in C^\infty(\mathbb{C})$. In this case the anchor is $\rho(z, \lambda) = i\lambda z$, and the parallel transport equation for a section $\sigma = (z, \mu)$ along $z$ reads

$$\dot{\mu} + \beta(z)\lambda\mu = 0.$$

Since everything is 1-dimensional, we can integrate this equation to get

$$\mu(1) = \exp\left(-\int_0^1 \beta(z(t))\lambda(t)dt\right)\mu(0).$$

If $z$ is the constant loop at 0, then parallel transport along the $A$-path given by the constant $\lambda(t) = \lambda \in \mathbb{R}$ reduces to $\mu(1) = e^{-\beta(0)\lambda}\mu(0)$. Hence,

$$\operatorname{Hol}_0(\beta) = \begin{cases} \mathbb{R}_{>0}, & \beta(0) \neq 0 \\ \{1\}, & \beta(0) = 0 \end{cases}.$$

If $z$ is a loop away from 0, then the $A$-path $a = (z, \lambda)$ satisfies $\lambda = -i\dot{z}/z$. Hence,

$$\mu(1) = \exp\left(i\int_0^1 \beta(z(t))\frac{\dot{z}(t)}{z(t)}dt\right)\mu(0).$$

Different choices of $\beta$ give now different jumping behavior of the holonomy. Here we give some examples:

1. If $\beta = 0$, then $\operatorname{Hol}_z(\beta) = \{1\}$ for all $z \in \mathbb{C}$.
2. If $\beta = 1$, then if $z(t)$ is not the constant loop at 0 and it has winding number $k \in \mathbb{Z}$ around 0, then $\mu(1) = e^{-2\pi k}\mu(0)$, so
$$\operatorname{Hol}_z(\beta) = \begin{cases} \mathbb{R}_{>0}, & z = 0 \\ \{e^{2\pi k} : k \in \mathbb{Z}\}, & z \neq 0. \end{cases}$$
3. If $\beta(z) = |z|^2$, then
$$\operatorname{Hol}_z(\beta) = \begin{cases} \{1\}, & z = 0 \\ \{e^{2\pi |z|^2 k} : k \in \mathbb{Z}\}, & z \neq 0. \end{cases}$$
4. If $\beta(z) = |z|^2 - 1$, then
$$\operatorname{Hol}_z(\beta) = \begin{cases} \mathbb{R}_{>0}, & z = 0 \\ \{e^{2\pi(|z|^2-1)k} : k \in \mathbb{Z}\}, & z \neq 0 \text{ and } |z| \neq 1 \\ \{1\}, & |z| = 1. \end{cases}$$ ▼

This last example shows that the jumps of holonomy from leaf to leaf can be as wild as we like, when we consider general LA-connections. A natural question to ask is if the LA-holonomy group can have a more controlled behavior in special situations. For instance, if $A \to M$ is a Lie algebroid, we can consider $A$-connections $\nabla$ on $A$. In such situation it makes sense to define the torsion of $\nabla$ as in the classical case: $T \in \Omega^2(A, A)$ given by

$$T(a, b) := \nabla_a b - \nabla_b a - [a, b], \qquad \text{for } a, b \in \Gamma(A).$$

Then the same proof as Proposition 1.34 gives that, if $A$ is endowed with a metric, there is a unique $A$-connection on $A$ which is metric and torsion-free, which we call the Levi-Civita connection of $A$. An open question is whether the holonomy of the Levi-Civita connection exhibits a more regular behavior.

Other open questions regarding LA-holonomy, that could be used as guiding questions for future work, are the following. Is there a Berger-type list for LA-holonomy, at least for the Levi-Civita connection? If we let $\mathrm{Hol}_x(A)$ be the LA-holonomy group at $x$ of the Levi-Civita connection of a Lie algebroid $A$, then if $\mathrm{Hol}_x(A)$ acts irreducibly on $A_x$, the triple $(A_x, R_x, \mathrm{Hol}_x(A))$, where $R_x$ is the curvature of the Levi-Civita connection, is an irreducible holonomy system. Therefore, Simons's theorem (Theorem 3.41) can be applied to this triple. Does this give any useful information on the geometry of $A$? Lastly, it is still open how relevant LA-holonomy is in applications to geometry and physics.

A

# Some computations

Here we present the proofs of some results formulated or used in the main text, whose proofs require long and uninteresting computations, which would break the flow of the main reading.

## A.1. Computations of Section 3.6

**Proposition A.1.** *Let $(M, J)$ be an almost complex manifold, let $F$ be the curvature of the Levi-Civita connection on the canonical bundle $\Lambda^{n,0}T^*M$ and let $\Omega \in \Omega^{n,0}(M)$. Then for all $X, Y \in \mathfrak{X}(M)$ and $X_i \in \mathfrak{X}^{1,0}(M)$,*

$$F(X,Y)\Omega(X_1,\dots,X_n) = \sum_j \Omega(X_1,\dots,R(Y,X)X_j,\dots,X_n),$$

*where $R$ is the Riemann curvature.*

*Proof.* Straightforward computation, where the dots $\dots$ indicate the presence of the vector fields $\{X_i\}_i$:

$$\begin{aligned}
F(X,Y)\Omega(X_1,\dots,X_n) &= X(\nabla_Y\Omega(\dots)) - \sum_j \nabla_Y\Omega(\dots,\nabla_X X_j,\dots) \\
&\quad - Y(\nabla_X\Omega(\dots)) + \sum_j \nabla_X\Omega(\dots,\nabla_Y X_j,\dots) \\
&\quad - [X,Y](\Omega(\dots)) + \sum_j \Omega(\dots,\nabla_{[X,Y]}X_j,\dots) \\
&= XY(\Omega(\dots)) - \sum_j X(\Omega(\dots,\nabla_Y X_j,\dots)) \\
&\quad - \sum_j \big(Y(\Omega(\dots,\nabla_X X_j,\dots)) - \Omega(\dots,\nabla_Y\nabla_X X_j,\dots)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j,k\neq j} \Omega(\dots, \nabla_Y X_j, \dots, \nabla_X X_k, \dots) \\
&- YX(\Omega(\dots)) + \sum_j Y(\Omega(\dots, \nabla_X X_j, \dots)) \\
&+ \sum_j \big(X(\Omega(\dots, \nabla_Y X_j, \dots)) - \Omega(\dots, \nabla_X \nabla_Y X_j, \dots)\big) \\
&- \sum_{j,k\neq j} \Omega(\dots, \nabla_X X_j, \dots, \nabla_Y X_k, \dots) \\
&- [X,Y](\Omega(\dots)) + \sum_j \Omega(\dots, \nabla_{[X,Y]} X_j, \dots) \\
= &\sum_j \Omega(\dots, \nabla_Y \nabla_X X_j - \nabla_X \nabla_Y X_j + \nabla_{[X,Y]} X_j, \dots) \\
= &\sum_j \Omega(\dots, R(Y,X) X_j, \dots),
\end{aligned}
$$

as wanted. □

To ease the computation of adjoints, we will make repeated use of the following lemma.

**Lemma A.2.** *Let $E \to M$ be a vector bundle and $\nabla$ a metric connection on $E$, where the metric can be real or Hermitian. Then around every point $x \in M$ there is a local orthonormal frame $\{\sigma_i\}_i$ for $E$ which is normal at $x$, meaning that $\nabla_w \sigma_i = 0$ for all $i$ and $w \in T_x M$.*

*Proof.* Let $(U, \varphi)$ be a chart for $M$ centered at $x$ and $\{e_i\}_i$ an orthonormal basis for $E_x$. Define local sections $\sigma_i$ by $\sigma_i(\varphi^{-1}(v)) := \tau_1 e_i$, where $\tau_t$ is parallel transport from $x$ to $\varphi^{-1}(tv)$ along $s \mapsto \varphi^{-1}(sv)$. It is a smooth frame, by the smooth dependence of solutions to ODEs on parameters. Since $\tau_t$ is a linear isometry, the frame is orthonormal. Lastly, using Proposition 1.15, we get that for any $w \in T_0(\varphi(U))$,

$$
\nabla_{\varphi_*^{-1} w} \sigma_i = \frac{d}{dt}\bigg|_{t=0} \tau_t^{-1}(\sigma_i(\varphi^{-1}(tw)) = \frac{d}{dt}\bigg|_{t=0} e_i = 0,
$$

so the frame is normal at $x$. □

Also, recall the definition of the divergence of a vector field $X \in \mathfrak{X}(M)$: it is the smooth function $\operatorname{div} X$ such that $\mathcal{L}_X \mathrm{vol} = (\operatorname{div} X)\mathrm{vol}$ (here $\mathcal{L}_X$ is the Lie derivative, with the sign convention that $\mathcal{L}_X Y = [X,Y]$, for $Y \in \mathfrak{X}(M)$). Explicitly, it is given by $\operatorname{div} X = \frac{1}{2} \operatorname{tr} \mathcal{L}_X g$, as can be easily checked using an orthonormal frame $\{E_i\}_i$:

$$
\begin{aligned}
\operatorname{div} X &= (\operatorname{div} X)\mathrm{vol}(E_1, \dots, E_n) = \mathcal{L}_X \mathrm{vol}(E_1, \dots, E_n) \\
&= X(\mathrm{vol}(E_1, \dots, E_n)) - \sum_i \mathrm{vol}(E_1, \dots, \mathcal{L}_X E_i, \dots, E_n) \\
&= -\sum_i \langle [X, E_i], E_i \rangle = \frac{1}{2} \sum_i \mathcal{L}_X g(E_i, E_i).
\end{aligned}
$$

By Stokes's theorem and Cartan's formula for the Lie derivative, we have that

$$
\int_M (\operatorname{div} X)\mathrm{vol} = \int_M d i_X \mathrm{vol} = 0.
$$

**Proposition A.3** (3.75)**.** *The formal adjoint of the connection $\nabla : \mathfrak{T}^{(k,l)}(M) \to \mathfrak{T}^{(k,l+1)}(M)$ is the operator $\nabla^* : \mathfrak{T}^{(k,l+1)}(M) \to \mathfrak{T}^{(k,l)}(M)$ given by*

$$
\nabla^* T(\theta, X_1, \dots, X_l) = -\sum_i \nabla_{E_i} T(\theta, E_i, X_1, \dots, X_l)
$$

*on $T \in \mathfrak{T}^{(k,l)}(M)$, where $\theta \in \mathfrak{T}^{(0,k)}(M)$, $X_i \in \mathfrak{X}(M)$ and $\{E_i\}_i$ is any orthonormal frame.*

*Proof.* It is straightforward to see that the formula for $\nabla^*$ does not depend on the orthonormal frame. Let $\{E_i\}_i$ be an orthonormal frame which is normal at $x \in M$, as in Lemma A.2, and let $\{E^i\}_i$ denote its dual frame. If $a = (a_1, \ldots, a_m)$ is a multiindex of length $|a| := a_1 + \cdots + a_m$, we use the shorthand notation

$$E_a := E_{a_1} \otimes \cdots \otimes E_{a_m} \quad \text{and} \quad E^a := E^{a_1} \otimes \cdots \otimes E^{a_m}.$$

Let $S \in \mathfrak{T}^{(k,l+1)}(M)$ and define a 1-form $\alpha$ by $\alpha(X) := \langle T, i_X S\rangle$. Then, at the point $x$, if $a$ runs through all the multiindices of length $k$ and $b$ through all the ones with length $l$,

$$\begin{aligned}
-\nabla^*\alpha &= \sum_j \nabla_{E_j}\alpha(E_j) = \sum_j E_j(\langle T, i_{E_j}S\rangle) \\
&= \sum_j (\langle \nabla_{E_j}T, i_{E_j}S\rangle + \langle T, \nabla_{E_j} i_{E_j} S\rangle) \\
&= \sum_j \sum_{a,b} (\nabla_{E_j}T(E^a, E_b) S(E^a, E_j, E_b) + T(E^a, E_b)\nabla_{E_j} i_{E_j} S(E^a, E_b)).
\end{aligned}$$

Since

$$\begin{aligned}
\nabla_{E_j} i_{E_j} S(E^a, E_b) &= i_{E_j}\nabla_{E_j} S(E^a, E_b) + S(E^a, \nabla_{E_j}E_j, E_b) \\
&= \nabla_{E_j} S(E^a, E_j, E_b),
\end{aligned}$$

then

$$\begin{aligned}
-\nabla^*\alpha &= \sum_j \sum_{a,b} (\nabla_{E_j}T(E^a, E_b) S(E^a, E_j, E_b) + T(E^a, E_b)\nabla_{E_j} S(E^a, E_j, E_b)) \\
&= \langle \nabla T, S\rangle - \langle T, \nabla^* S\rangle.
\end{aligned}$$

Let now $Y$ be the vector field dual to $\alpha$ with respect to $g$. Then, at $x$,

$$\begin{aligned}
\operatorname{div} Y &= -\sum_j \langle [Y, E_j], E_j\rangle = -\sum_j \langle \nabla_Y E_j - \nabla_{E_j} Y, E_j\rangle \\
&= \sum_j E_j\langle Y, E_j\rangle = \sum_j E_j(\alpha(E_j)) = -\nabla^*\alpha.
\end{aligned}$$

Hence, we conclude that

$$\langle \nabla T, S\rangle_2 - \langle T, \nabla^* S\rangle_2 = \int_M (\operatorname{div} Y)\mathrm{vol} = 0. \qquad \square$$

**Proposition A.4** (3.76)**.** *For $\alpha \in \Omega^k(M)$ we have that*

$$d\alpha(X_0, \ldots, X_k) = \sum_i (-1)^i \nabla_{X_i}\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k), \qquad \text{for } X_i \in \mathfrak{X}(M).$$

*Moreover, $d^* = \nabla^*$.*

*Proof.* First, we have that

$$\nabla_{X_i}\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k) = X_i(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)) - \sum_{j \neq i} \alpha(X_0, \ldots, \nabla_{X_i}X_j, \ldots, \hat{X}_i, \ldots, X_k).$$

Since

$$
\begin{aligned}
\sum_{i,j\neq i}(-1)^i\alpha(X_0,\dots,\nabla_{X_i}X_j,\dots,\hat{X}_i,\dots,X_k) = \\
&= \sum_{i<j}(-1)^i\alpha(X_0,\dots,\hat{X}_i,\dots,\nabla_{X_i}X_j,\dots,X_k)\\
&\qquad + \sum_{j<i}(-1)^i\alpha(X_0,\dots,\nabla_{X_i}X_j,\dots,\hat{X}_i,\dots,X_k)\\
&= \sum_{i<j}(-1)^{i+j}\alpha(-\nabla_{X_i}X_j,X_0,\dots,\hat{X}_i,\dots\hat{X}_j,\dots,X_k)\\
&\qquad + \sum_{j<i}(-1)^{i+j}\alpha(\nabla_{X_i}X_j,X_0,\dots,\hat{X}_j,\dots,\hat{X}_i,\dots,X_k)\\
&= \sum_{i<j}(-1)^{i+j}\alpha(\nabla_{X_j}X_i-\nabla_{X_i}X_j,X_0,\dots,\hat{X}_i,\dots\hat{X}_j,\dots,X_k)\\
&= -\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\hat{X}_i,\dots\hat{X}_j,\dots,X_k),
\end{aligned}
$$

then the Koszul formula for $d\alpha$ finally gives the result.

As for the second claim, we have that, if $\beta\in\Omega^{k-1}(M)$,

$$
\begin{aligned}
\langle\alpha,d\beta\rangle^\wedge &= \frac{1}{k!}\langle\alpha,d\beta\rangle\\
&= \frac{1}{k!}\sum_{i_1,\dots,i_k}\alpha(E_{i_1},\dots,E_{i_k})\sum_j(-1)^{j+1}\nabla_{E_{i_j}}\beta(E_{i_1},\dots,\hat{E}_{i_j},\dots,E_{i_k})\\
&= \frac{1}{k!}\sum_j\langle\alpha,\nabla\beta\rangle = \frac{1}{(k-1)!}\langle\alpha,\nabla\beta\rangle.
\end{aligned}
$$

Hence,

$$
\langle\alpha,d\beta\rangle_2^\wedge = \frac{1}{(k-1)!}\langle\alpha,\nabla\beta\rangle_2 = \frac{1}{(k-1)!}\langle\nabla^*\alpha,\beta\rangle_2 = \langle\nabla^*\alpha,\beta\rangle_2^\wedge. \qquad \square
$$

**Proposition A.5** (Weitzenböck, 3.77)**.** *For $\alpha\in\Omega^k(M)$ we have that $\Delta\alpha=\nabla^*\nabla\alpha+\operatorname{Ric}\alpha$.*

*Proof.* Consider an orthonormal frame $\{E_i\}_i$ which is normal at $x$, as in Lemma A.2. The formula follows from a somewhat lengthy computation. Let $X_i\in\mathfrak{X}(M)$. First of all, by Proposition 3.76,

$$
\begin{aligned}
dd^*\alpha(X_1,\dots,X_k) &= \sum_j(-1)^{j+1}\nabla_{X_j}d^*\alpha(X_1,\dots,\hat{X}_j,\dots,X_k)\\
&= -\sum_{j,i}(-1)^{j+1}X_j(\nabla_{E_i}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k))\\
&\qquad + \sum_{j,i,l\neq j}(-1)^{j+1}\nabla_{E_i}\alpha(E_i,X_0,\dots,\nabla_{X_j}X_l,\dots,\hat{X}_j,\dots,X_k)\\
&= \sum_{j,i}(-1)^j\nabla_{X_j}\nabla_{E_i}\alpha(E_i,X_0,\dots,\hat{X}_j,\dots,X_k).
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
d^*d\alpha(X_1,\dots,X_k) &= -\sum_i \nabla_{E_i} d\alpha(E_i,X_1,\dots,X_k)\\
&= -\sum_i E_i(d\alpha(E_i,X_1,\dots,X_k)) + \sum_{i,j} d\alpha(E_i,X_1,\dots,\nabla_{E_i}X_j,\dots,X_k)\\
&= -\sum_i E_i(\nabla_{E_i}\alpha(X_1,\dots,X_k)) - \sum_{i,j}(-1)^j E_i(\nabla_{X_j}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k)\\
&\quad + \sum_{i,j}\nabla_{E_i}\alpha(X_1,\dots,\nabla_{E_i}X_j,\dots,X_k)\\
&\quad + \sum_{i,j}(-1)^j\nabla_{\nabla_{E_i}X_j}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots X_k)\\
&\quad + \sum_{i,l,j\neq l}(-1)^j\nabla_{X_j}\alpha(E_i,\dots,\hat{X}_j,\dots,\nabla_{E_i}X_l,\dots,X_k)\\
&= -\sum_i \nabla_{E_i}\nabla_{E_i}\alpha(X_1,\dots,X_k) - \sum_j(-1)^j\nabla_{E_i}\nabla_{X_j}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k)\\
&\quad + \sum_{i,j}(-1)^j\nabla_{\nabla_{E_i}X_j}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots X_k).
\end{aligned}$$

That computes the left-hand side of the equality. As for the right-hand side, the first term is

$$\begin{aligned}
\nabla^*\nabla\alpha(X_1,\dots,X_k) &= -\sum_i \nabla_{E_i}\nabla\alpha(E_i,X_1,\dots,X_k)\\
&= -\sum_i E_i(\nabla_{E_i}\alpha(X_1,\dots,X_k)) + \sum_{i,j}\nabla_{E_i}\alpha(X_1,\dots,\nabla_{E_i}X_j,\dots,X_k)\\
&= -\sum_i \nabla_{E_i}\nabla_{E_i}\alpha(X_1,\dots,X_k),
\end{aligned}$$

and the second term

$$\begin{aligned}
\operatorname{Ric}\alpha(X_1,\dots,X_k) &= \sum_{i,j}(-1)^{j+1}R(E_i,X_j)\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k)\\
&= -\sum_{i,j}(-1)^j\nabla_{E_i}\nabla_{X_j}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k)\\
&\quad + \sum_{i,j}(-1)^j\nabla_{X_j}\nabla_{E_i}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k)\\
&\quad + \sum_{i,j}(-1)^j\nabla_{\nabla_{E_i}X_j}\alpha(E_i,X_1,\dots,\hat{X}_j,\dots,X_k).
\end{aligned}$$

This finally establishes the result. □

**Proposition A.6.** *Let $(M,g,J)$ be a Kähler manifold and $X,Y,Z,W\in\mathfrak{X}^{1,0}(M)$. Then $h(R(X,\overline{Y})Z,W) = h(R(X,\overline{W})Z,Y)$.*

*Proof.* Let $X,Y,Z,W\in\mathfrak{X}(M)$. Then, using Proposition 3.72 we have that

$$\begin{aligned}
&h(R(X-iJX,Y+iJY)(Z-iJZ),W-iJW) =\\
&\qquad = 4\langle R(X,Y)Z-R(JX,Y)JZ,W\rangle + 4i\langle R(X,Y)Z-R(JX,Y)JZ,JW\rangle.
\end{aligned}$$

By the Bianchi identity and again Proposition 3.72,

$$\begin{aligned}
\langle R(X,Y)Z-R(JX,Y)JZ,W\rangle &= -\langle R(Y,Z)X+R(Z,X)Y,W\rangle\\
&\quad + \langle R(Y,JZ)JX+R(JZ,JX)Y,W\rangle\\
&= \langle R(X,W)Z,Y\rangle + \langle R(JX,W)JZ,Y\rangle,
\end{aligned}$$

and the same holds with $JW$ instead of $W$. This already gives the result. □

**Proposition A.7.** *Let $(M, g, J)$ be a compact Ricci-flat Kähler manifold and $\alpha \in \Omega^{p,0}(M)$. Then* $\operatorname{Ric}\alpha = 0$.

*Proof.* It is easy to see that, in general, if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M, \mathbb{C})$, then $R(X, Y)\Omega^{1,0}(M) \subseteq \Omega^{1,0}(M)$. Hence, $\operatorname{Ric}\alpha$ can only take nonzero values when evaluated on $(1,0)$-vector fields. It will now suffice to prove that $\operatorname{Ric}(\alpha^1 \wedge \cdots \wedge \alpha^k)(X_1, \ldots, X_k) = 0$ for $\alpha^j \in \Omega^{1,0}(M)$ and $X_j \in \mathfrak{X}^{1,0}(M)$.

Let $Y_j \in \mathfrak{X}^{1,0}(M)$ be the corresponding vector field to $\alpha^j$ with respect to $h$, i.e., such that $\alpha^j(X) = h(X, Y_j)$ for all $X \in \mathfrak{X}(M, \mathbb{C})$. Notice that if $\alpha^j = \beta^j + iJ\beta^j$, for some $\beta^j \in \Omega^1(M)$, then $Y_j = W_j - iJW_j$, where $W_j \in \mathfrak{X}(M)$ is dual to $\beta^j$, so that if $\{E_l\}_l$ is an orthonormal frame, we have that

$$\begin{aligned}\sum_l \alpha^j(E_l)E_l &= \sum_l \beta^j(E_l)E_l - i\sum_l \beta^j(JE_l)\\ &= \sum_l \langle W_j, E_l\rangle E_l + i\sum_l \langle JW_j, E_l\rangle E_l\\ &= W_j + iJW_j = \overline{Y_j}.\end{aligned}$$

Then we may compute

$$\begin{aligned}\operatorname{Ric}(\alpha^1 \otimes \cdots \otimes \alpha^k)(X_1, \ldots, X_k) &= \sum_{l,j} R(E_l, X_j)(\alpha^1 \otimes \cdots \otimes \alpha^k)(X_1, \ldots, E_l, \ldots, X_k)\\ &= \sum_{l,j,i} \alpha^1 \otimes \ldots R(E_l, X_j)\alpha^i \otimes \cdots \otimes \alpha^k(X_1, \ldots, E_l, \ldots, X_k)\\ &= \sum_{l,j,i\neq j} \alpha^1(X_1)\ldots R(E_l, X_j)\alpha^i(X_i)\ldots \alpha^j(E_l)\ldots \alpha^k(X_k)\\ &\quad + \sum_{l,j} \alpha^1(X_1)\ldots R(E_l, X_j)\alpha^j(E_l)\ldots \alpha^k(X_k)\\ &= \sum_{j,i\neq j} \alpha^1(X_1)\ldots h(R(X_j, \overline{Y_j})X_i, Y_i)\ldots \widehat{\alpha^j}\ldots \alpha^k(X_k)\\ &\quad + \sum_{l,j} \alpha^1(X_1)\ldots h(R(X_j, E_l)E_l, Y_j)\ldots \alpha^k(X_k).\end{aligned}$$

The second term is determined by the Ricci curvature and the Ricci form, as follows: let $X_j = Z_j - iJZ_j$ for $Z_j \in \mathfrak{X}(M)$. Then, using Proposition 3.72,

$$\begin{aligned}\sum_l h(R(X_j, E_l)E_l, Y_j) &= \sum_l \langle R(Z_j - iJZ_j, E_l)E_l, W_j + iJW_j\rangle\\ &= 2\operatorname{Ric}(Z_j, W_j) + 2i\operatorname{Ric}(Z_j, JW_j).\end{aligned}$$

Since $M$ is Ricci-flat, this term vanishes.

We can now write

$$\begin{aligned}&\operatorname{Ric}(\alpha^1 \wedge \cdots \wedge \alpha^k)(X_1, \ldots, X_k) =\\ &\quad = \sum_{j,i\neq j}\sum_{\sigma\in\mathfrak{S}_k} (\operatorname{sgn}\sigma)\alpha^{\sigma(1)}(X_1)\ldots h(R(X_j, \overline{Y_{\sigma(j)}})X_i, Y_{\sigma(i)})\ldots \widehat{\alpha^{\sigma(j)}}\ldots \alpha^{\sigma(k)}(X_k).\end{aligned}$$

For $\sigma \in \mathfrak{S}_k$ and $j$ and $i \neq j$, let $\sigma' \in \mathfrak{S}_k$ be $\sigma$ composed with the transposition $(ji)$, i.e., such that $\sigma'(j) = \sigma(i)$, $\sigma'(i) = \sigma(j)$ and $\sigma'(l) = \sigma(l)$ for any $l \neq i$ and $l \neq j$. Then the term corresponding to $\sigma$ is

$$(\operatorname{sgn}\sigma)\alpha^{\sigma(1)}(X_1)\ldots h(R(X_j, \overline{Y_{\sigma(j)}})X_i, Y_{\sigma(i)})\ldots \widehat{\alpha^{\sigma(j)}}\ldots \alpha^{\sigma(k)}(X_k),$$

and the term corresponding to $\sigma'$ is

$$-(\operatorname{sgn}\sigma)\alpha^{\sigma(1)}(X_1)\ldots h(R(X_j, \overline{Y_{\sigma(i)}})X_i, Y_{\sigma(j)})\ldots \widehat{\alpha^{\sigma(i)}}\ldots \alpha^{\sigma(k)}(X_k).$$

Since $h(R(X, \overline{Y})Z, W) = h(R(X, \overline{W})Z, Y)$ for $X, Y, Z, W \in \mathfrak{X}^{1,0}(M)$ (Proposition A.6), we finally conclude that $\operatorname{Ric}\alpha = 0$. □

## A.2. Computations of Section 4.1

**Lemma A.8** (4.8)**.** *The A-differential satisfies $d_A^2 = 0$ and it is given by the Koszul formula*

$$
\begin{aligned}
d_A\alpha(a_0, \ldots, a_k) &= \sum_i (-1)^i \rho(a_i)(\alpha(a_0, \ldots, \hat{a}_i, \ldots, a_k)) \\
&\quad + \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j], a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k),
\end{aligned}
$$

*for $\alpha \in \Omega^k(A)$ and $a_i \in \Gamma(A)$.*

*Proof.* We first prove the Koszul formula. The formulas for functions and 1-forms in the definition of $d_A$ are exactly Koszul's formula in those degrees. We prove the general formula by induction on the degree. Assume that $d_A$ is given by the Koszul formula on $\Omega^k(A)$ and let $\alpha \in \Omega^1(A)$ and $\beta \in \Omega^k(A)$. We will make use of the following explicit formulas, which are easy to deduce: if $\gamma \in \Omega^2(A)$ and $a_i \in \Gamma(A)$, then

$$
\begin{aligned}
\alpha \wedge \beta(a_0, \ldots, a_k) &= \sum_i (-1)^i \alpha(a_i)\beta(a_0, \ldots, \hat{a}_i, \ldots, a_k) \\
\gamma \wedge \beta(a_0, \ldots, a_{k+1}) &= -\sum_{i<j} (-1)^{i+j} \gamma(a_i, a_j)\beta(a_0, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{k+1}).
\end{aligned}
$$

Then, if we use the short-hand notation $\beta(\hat{\imath})$ or $\beta(\hat{\imath}, \hat{\jmath})$ to denote the absence of the arguments $a_i$ or $a_i$ and $a_j$ in $\beta$, then

$$
\begin{aligned}
d_A\alpha \wedge \beta(a_0, \ldots, a_{k+1}) &= -\sum_{i<j} (-1)^{i+j} d_A\alpha(a_i, a_j)\beta(\hat{\imath}, \hat{\jmath}) \\
&= \sum_{i<j} (-1)^{i+j} (\alpha([a_i, a_j]) + \rho(a_j)\alpha(a_i) - \rho(a_i)\alpha(a_j))\beta(\hat{\imath}, \hat{\jmath}).
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha \wedge d_A\beta(a_0, \ldots, a_{k+1}) &= \sum_i (-1)^i \alpha(a_i) d_A\beta(\hat{\imath}) \\
&= \sum_{i<j} (-1)^{i+j} (\alpha(a_j)\rho(a_i) - \alpha(a_i)\rho(a_j))\beta(\hat{\imath}, \hat{\jmath}) \\
&\quad + \sum_i \sum_{\substack{j<l \\ i \neq j,l}} (-1)^{i+j+l} \alpha(a_i)\beta([a_j, a_l], \hat{\imath}, \hat{\jmath}, \hat{l}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d_A(\alpha \wedge \beta)(a_0, \ldots, a_{k+1}) &= (d_A\alpha \wedge \beta - \alpha \wedge d_A\beta)(a_0, \ldots, a_{k+1}) \\
&= \sum_{i<j} (-1)^{i+j} \big(\rho(a_j)(\alpha(a_i)\beta(\hat{\imath}, \hat{\jmath})) - \rho(a_i)(\alpha(a_j)\beta(\hat{\imath}, \hat{\jmath}))\big) \\
&\quad + \sum_{j<l} (-1)^{j+l} \left( (\alpha([a_j, a_l])\beta(\hat{\jmath}, \hat{l}) - \sum_{\substack{i \\ i \neq j,l}} (-1)^i \alpha(a_i)\beta([a_j, a_l], \hat{\imath}, \hat{\jmath}, \hat{l}) \right) \\
&= \sum_i (-1)^i \rho(a_i)\alpha \wedge \beta(\hat{\imath}) + \sum_{j<l} (-1)^{j+l} \alpha \wedge \beta([a_j, a_l], \hat{\imath}, \hat{\jmath}, \hat{l}),
\end{aligned}
$$

and this last line is the Koszul formula. (To be fair, in the sum involving $j < l$ and $i \neq j, l$ we should have split the sum in three pieces according to whether $i < j$ or $j < i < l$ or $l < i$, and adapted the signs, but this is irrelevant for the final result).

We now prove that $d_A^2 = 0$ by induction on the degree. For a function $f \in C^\infty(M)$ we have that for all $a, b \in \Gamma(A)$,

$$
d_A^2 f(a, b) = \rho(a)d_A f(b) - \rho(b)d_A f(a) - d_A f([a, b]) = \rho(a)\rho(b)f - \rho(b)\rho(a)f - [\rho(a), \rho(b)]f = 0,
$$

by Lemma 4.2. For $\alpha \in \Omega^1(A)$, using Koszul's formula and Jacobi's identity it is straightforward to see that $d_A^2\alpha = 0$. Assume now that $d_A^2 = 0$ up to degree $k$ and let $\alpha \in \Omega^1(A)$ and $\beta \in \Omega^k(A)$. Then

$$d_A^2(\alpha \wedge \beta) = d_A(d_A\alpha \wedge \beta - \alpha \wedge d_A\beta) = d_A^2\alpha \wedge \beta + d_A\alpha \wedge d_A\beta - d_A\alpha \wedge d_A\beta + \alpha \wedge d_A^2\beta = 0. \quad \square$$

# B

# Ehresmann connections

There is an alternative viewpoint to connections which, although we do not use it in the text, is interesting to know. In this appendix we show that the notion of a linear Ehresmann connection is equivalent to our notion of connection, and that the curvature is an obstruction to the integrability of the Ehresmann connection.

In this section we will be using local formulas for both the connection and its curvature. If $U$ is a trivializing chart for $E$, then local sections of $E$ over $U$ are identified with $C^\infty(U, \mathbb{R}^r)$, for $r$ the rank of $E$. Over the trivialization, we can always consider the canonical connection on the trivial bundle, which is just the differential, and since the difference of two connections is an End $E$-valued 1-form, we conclude that there is some $\Gamma \in \Omega^1(U, \mathfrak{gl}(r, \mathbb{R}))$ such that for $x \in U$ and $v \in T_x M$,

$$\nabla_v \sigma = \sigma_* v + \Gamma(v)(\sigma(x)), \qquad \text{for } \sigma \in C^\infty(U, \mathbb{R}^r).$$

Then we can explicitly compute the curvature as follows: for $X, Y \in \mathfrak{X}(U)$ and $\sigma \in C^\infty(U, \mathbb{R}^r)$,

$$\begin{aligned} F(X,Y)\sigma &= \nabla_X(Y\sigma + \Gamma(Y)\sigma) - \nabla_Y(X\sigma + \Gamma(X)\sigma) - [X,Y]\sigma - \Gamma([X,Y])\sigma \\ &= XY\sigma + \Gamma(X)Y\sigma + X(\Gamma(Y))\sigma + \Gamma(Y)X\sigma + \Gamma(X)\Gamma(Y)\sigma \\ &\quad - YX\sigma - \Gamma(Y)X\sigma - Y(\Gamma(X))\sigma - \Gamma(X)Y\sigma - \Gamma(Y)\Gamma(X)\sigma \\ &\quad - [X,Y]\sigma - \Gamma([X,Y])\sigma \\ &= (d\Gamma + \Gamma \wedge \Gamma)(X,Y)\sigma, \end{aligned}$$

where $\Gamma \wedge \Gamma(X,Y) := \Gamma(X)\Gamma(Y) - \Gamma(Y)\Gamma(X)$. The 1-form $\Gamma$ is called the **local connection 1-form** for the trivialization over $U$.

Denote by $VE \to E$ the **vertical bundle** of $E$, with fibers $V_v E := \ker \pi_*(v)$, where $\pi : E \to M$ is the projection and $v \in E$. Since the fibers of $E$ are vector spaces, for each $v \in E_x$ there is a canonical identification $V_v E \cong E_x$, by sending $w \in E_x$ to $\frac{d}{dt}\big|_{t=0}(v + tw)$. Actually, $VE \cong \pi^* E$. In the following we use this identification implicitly. By the definition of $VE$, there is a short exact sequence of vector bundles over $E$

$$0 \longrightarrow VE \longrightarrow TE \xrightarrow{\pi_*} \pi^*TM \longrightarrow 0. \tag{B.1}$$

A splitting of this sequence is a vector bundle morphism $\mathsf{h} : \pi^*TM \to TE$ such that $\pi_* \circ \mathsf{h} = \mathrm{id}$. Moreover, we say that it is linear if for all $v \in E$ and $t \neq 0$ we have that $\mathsf{h}_{tv} = S_{t*} \circ \mathsf{h}_v$, where $S_t : E \to E$ is fiberwise scalar multiplication by $t$.

Let us deduce the local expression for such a linear splitting. Let $U$ be a trivializing chart for $E$. Then we can assume that $M = U$, $TM = U \times \mathbb{R}^n$ (where $n = \dim M$), $E = U \times \mathbb{R}^r$ (where $r$ is the rank of $E$), $TE = U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r$ and $\pi^*TM = U \times \mathbb{R}^r \times \mathbb{R}^n$. With this notation, the map $\pi : U \times \mathbb{R}^r \to U$ is given by $(x, v) \mapsto x$ and its differential $\pi_* : U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r \to U \times \mathbb{R}^r \times \mathbb{R}^n$ by $(x, v, w, u) \mapsto (x, v, w)$, and $\mathsf{h}$ is a map $U \times \mathbb{R}^r \times \mathbb{R}^n \to U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r$. The fact that $\pi_* \circ \mathsf{h} = \mathrm{id}$ implies that $\mathsf{h}(x, v, w) = (x, v, w, g(x, v, w))$, for some $g : U \times \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^r$, and the fact that it is a vector bundle morphism implies that $g$ is linear on $w$. Under these identifications, scalar multiplication by $t$ sends $(x, v)$ to $(x, tv)$, whereas $S_{t*} : U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r \to U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r$ sends $(x, v, w, u)$ to $(x, tv, w, tu)$. Hence, being a linear splitting translates into

$$\mathsf{h}(x, tv, w) = (x, tv, w, g(x, tv, w)) = S_{t*}(\mathsf{h}(x, v, w)) = (x, tv, w, tg(x, v, w)).$$

Therefore, $g$ is homogeneous of degree 1 on $v$. This implies that it is actually linear on $v$:

$$g(x, v, w) = \left.\frac{d}{dt}\right|_{t=0} tg(x, v, w) = \left.\frac{d}{dt}\right|_{t=0} g(x, tv, w) = dg_{(x,0,w)}(0, v, 0).$$

These properties allow us to see that connections on $E$ are exactly linear splittings of (B.1).

**Proposition B.1.** *Connections on a vector bundle $\pi : E \to M$ are in bijective correspondence with linear splittings of* (B.1).

*Proof.* Let $\mathsf{h} : \pi^*TM \to TE$ be a linear splitting and define a connection on $E$ by

$$\nabla_w \sigma := \sigma_* w - \mathsf{h}_{\sigma(x)}(w) \in V_{\sigma(x)}E \cong E_x, \qquad \text{for } w \in T_xM \text{ and } \sigma \in \Gamma(E).$$

Under the above local identifications, the isomorphism $\pi^*E \cong VE$ is the map $U \times \mathbb{R}^r \times \mathbb{R}^r \to U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r$ given by $(x, v, u) \mapsto (x, v, 0, u)$. The section $\sigma$ is a map $\sigma : U \to U \times \mathbb{R}^r$ given by $\sigma(x) = (x, f(x))$, for $f : U \to \mathbb{R}^r$ smooth, and its differential $\sigma_* : U \times \mathbb{R}^n \to U \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r$ is given by $(x, w) \mapsto (x, f(x), w, df_x(w))$. Locally, then, the expression for $\nabla$ is

$$\nabla_w \sigma = (x, f(x), 0, df_x(w) - g(x, f(x), w)).$$

This defines a connection on $E|_U$.

For the converse, let $\nabla$ be a connection on $E$. Define $\mathsf{h} : \pi^*TM \to TE$ as follows: for $v \in E_x$ and $w \in T_xM$, we set $\mathsf{h}_v(w) := \sigma_* w - \nabla_w \sigma$, where $\sigma \in \Gamma(E)$ is such that $\sigma(x) = v$. Here we view $\nabla_w \sigma \in E_x$ as an element of $V_vE$. Let us write it locally: let $\sigma(y) = (y, f(y))$ and, if we now let $w \in T_xM$ be $(x, w) \in U \times \mathbb{R}^n$, we can write $\nabla_w \sigma = (x, df_x(w) + \Gamma(w)v)$, where $\Gamma \in \Omega^1(U, \mathfrak{gl}(r, \mathbb{R}))$ is the local connection 1-form of $\nabla$ and $v = f(x) \in \mathbb{R}^r$. When we view $\nabla_w \sigma$ as an element of the vertical bundle, it is $(x, v, 0, df_x(w) + \Gamma(w)v)$, so finally we have the local expression

$$\mathsf{h}(x, v, w) = (x, v, w, -\Gamma(w)v),$$

which defines a linear splitting. $\square$

An **Ehresmann connection** is a choice of a **horizontal bundle** over $E$, i.e., a subbundle $H \subseteq TE$ such that $TE = VE \oplus H$. It is said to be linear if moreover we have that $S_{t*}(H_v) = H_{tv}$, for all $v \in E$ and $t \neq 0$. As a consequence of Proposition B.1 we have that connections are the same thing as linear Ehresmann connections on $E$.

**Corollary B.2.** *Connections on a vector bundle $\pi : E \to M$ are in bijective correspondence with linear Ehresmann connections on $E$.*

*Proof.* Let $\nabla$ be a connection on $E$ and let $\mathsf{h} : \pi^*TM \to TE$ be the corresponding linear splitting of (B.1) given by Proposition B.1. Let $H := \operatorname{im} \mathsf{h}$. Since $\mathsf{h}$ is a splitting, it is fiberwise injective, so $H$ is a subbundle of $TE$. Let $v \in E_x$ and $w \in T_xM$. Then $\mathsf{h}_v(w) \in V_vE$ if and only if $\pi_*(\mathsf{h}_v(w)) = w = 0$, i.e., $H_v \cap V_vE = 0$. Since $\dim(V_vE \oplus H_v) = \dim E - \dim M + \dim M = \dim E = \dim T_vE$, we conclude that $TE = VE \oplus H$, so $H$ is an Ehresmann connection. Moreover, if $t \neq 0$,

$$S_{t*}(\mathsf{h}_v(w)) = (S_t \circ \sigma)_* w - S_{t*}\nabla_w \sigma = (t\sigma)_* w - \nabla_w(t\sigma) = \mathsf{h}_{tv}(w),$$

so $H$ is linear.

For the converse, let $H$ be a linear Ehresmann connection on $E$. Then $\pi_*|_H$ is a vector bundle isomorphism between $H$ and $\pi^*TM$. Let $\mathsf{h} := (\pi_*|_H)^{-1}$. It is a splitting of (B.1): $\pi_* \circ \mathsf{h} = \pi_*|_H \circ (\pi_*|_H)^{-1} = \mathrm{id}$. Also, since $H$ is linear, we have that

$$\pi_*|_{H_v} = (\pi \circ S_t)_*|_{H_v} = \pi_*|_{H_{tv}} \circ S_{t*},$$

which gives that $\mathsf{h}$ is a linear splitting. □

From this point of view, the curvature of $\nabla$ is the obstruction for its linear Ehresmann connection to be integrable.

**Proposition B.3.** *Let $\pi : E \to M$ be a vector bundle and $\nabla$ a connection on $E$, with corresponding linear Ehresmann connection $H$. Then $(E, \nabla)$ is flat if and only if $H$ is an integrable distribution on $E$.*

*Proof.* Let $\mathsf{h}$ be the linear splitting given by $\nabla$. Then locally we have that

$$\mathsf{h}(x, v, w) = (x, v, w, -\Gamma(w)v),$$

as in the proof of Proposition B.1. If we define $\alpha \in \Omega^1(E|_U, \mathbb{R}^r)$ by the formula $\alpha_{(x,v)}(w, u) := u + \Gamma(w)v$, for $w \in T_xM$ and $u, v \in \mathbb{R}^r$, then clearly $H_{(x,v)} = \operatorname{im} \mathsf{h}_{(x,v)} = \ker \alpha_{(x,v)}$. Let us write elements of $\mathfrak{X}(E|_U)$ as pairs $(X, V), (Y, W)$, for $X, Y \in \mathfrak{X}(U)$ and $V, W \in \mathfrak{X}(\mathbb{R}^r) \cong C^\infty(\mathbb{R}^r, \mathbb{R}^r)$. If $\phi_t^X$ and $\phi_t^V$ are the flows of $X$ and $V$, respectively, then on $(x, v) \in E|_U$ we have that

$$\begin{aligned}
(X, V)(\alpha(Y, W)) &= \left.\frac{d}{dt}\right|_{t=0} \alpha(Y, W)(\phi_t^X(x), v) + \left.\frac{d}{dt}\right|_{t=0} \alpha(Y, W)(x, \phi_t^V(v)) \\
&= \left.\frac{d}{dt}\right|_{t=0} (W(v) + \Gamma(Y(\phi_t^X(x)))v) \\
&\quad + \left.\frac{d}{dt}\right|_{t=0} (W(\phi_t^V(v)) + \Gamma(Y(x))\phi_t^V(v)) \\
&= (X(\Gamma(Y)))(x)v + W_*(V(v)) + \Gamma(Y(x))V(v).
\end{aligned}$$

Observe that $(X, V)$ is a section of $\ker \alpha$ if and only if $V(v) = -\Gamma(X(x))v$ for all $(x, v) \in E|_U$. Then, if $(X, V)$ and $(Y, W)$ are both sections of $\ker \alpha$, we have that on $(x, v)$,

$$\begin{aligned}
\alpha([(X, V), (Y, W)]) &= -d\alpha((X, V), (Y, W)) \\
&= -(X(\Gamma(Y)))(x)v - W_*(V(v)) - \Gamma(Y(x))V(v) \\
&\quad + (Y(\Gamma(X)))(x)v + V_*(W(v)) + \Gamma(X(x))W(v) \\
&\quad + [V, W](v) + \Gamma([X, Y](x))v \\
&= -(d\Gamma + \Gamma \wedge \Gamma)(X, Y)(x)v = -F(X, Y)(x)v.
\end{aligned}$$

Hence, $H$ is involutive if and only if $F = 0$, which, by the Frobenius integrability theorem [Lee12, Thm. 19.12], means that $H$ is integrable if and only if $F = 0$. □

**Corollary B.4.** *A vector bundle $E \to M$ with a connection $\nabla$ is flat if and only if there is a parallel local frame around every point in $M$, meaning a frame $\{\sigma_i\}_i$ with $\nabla\sigma_i = 0$.*

*Proof.* If $\{\sigma_i\}_i$ is a local parallel frame, then $F \wedge \sigma_i = D^2\sigma_i = 0$ for all $i$, where $F$ is the curvature of $\nabla$. Hence $E$ is flat.

Conversely, assume that $E$ is flat. Let $H$ be the linear Ehresmann connection of $\nabla$. By Proposition B.3, $H$ is integrable. First of all, note that the zero section is an integral manifold of $H$. Indeed, using the local expressions used throughout this section,

$$H_{(x,0)} = \operatorname{im} \mathsf{h}_{(x,0)} = \{(x,0)\} \times \mathbb{R}^n \times \{0\},$$

and this is exactly the tangent space to the zero section at $(x, 0)$. Second, let $L_{(x,v)}$ be the (local) leaf through $(x, v)$, and consider $\pi : L_{(x,v)} \to U$. Then the differential of $\pi$ restricted the tangent space of $L_{(x,v)}$ at $(x, v)$, which is exactly $H_{(x,v)}$, is an isomorphism onto $T_xU$, with inverse $\mathsf{h}_{(x,v)}$. Hence, locally $\pi$ gives a diffeomorphism between $L_{(x,v)}$ and $U$. This means that $L_{(x,v)}$ can be written as the graph of some smooth function $f : U \to \mathbb{R}^r$, i.e., $L_{(x,v)} = \{(x, f(x)) : x \in U\}$. This function defines a parallel section of $E$ over $U$. Indeed, since tangent vectors to the graph at $(x, f(x))$ are of the form $(w, df_x(w))$ and elements of $H_{(x,f(x))}$ are of the form $(w, -\Gamma(w)f(x))$, then $df_x(w) = -\Gamma(w)f(x)$, which leads to

$$\nabla_w f = (x, df_x(w) + \Gamma(w)f(x)) = (x, 0).$$

By the same argument, the converse is also true: if $\sigma(x) = (x, f(x))$ is a (local) parallel section, then its graph is a leaf of $H$.

Fix now $x \in U$ and let $\{e_i\}_i$ be a basis for $\mathbb{R}^r$. Let $\sigma_i$ be the parallel section corresponding to the leaf $L_{(x,e_i)}$. It only remains to see that $\{\sigma_i(y)\}_i$ is a linearly independent set for all $y \in U$. Assume that there are real numbers $\{\lambda^i\}_i$ such that $\lambda^i\sigma_i(y) = 0$ for some $y \in U$. Then the leaf defined by the parallel section $\lambda^i\sigma_i$ must be the zero section, which means that $\lambda^i\sigma_i(x) = \lambda^i e_i = 0$. Hence, $\lambda^i = 0$, and this ends the proof. $\square$

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