

Faculty of Physics

# On the deformation theory of Dirac structures and the complement problem 

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#### Abstract

In this MSc thesis, we study the space of Dirac structures on a smooth manifold $X$. First, we give a description of the space of all linear Dirac structures over a given vector space in terms of linear orthogonal maps on it. We then characterize the space of maximally isotropic subbundles of $T X \oplus T^{*} X$ in terms of sections of a bundle of orthogonal groups, and use this description in order to reformulate the integrability condition of Dirac structures. In addition, we state the complement problem for Dirac structures: when does a Dirac structure admit a complement that is also a Dirac structure? Using the language of quasiLie algebroids, we analyze the different curved $L_{3}$ algebras that arise from choosing different complements to a given maximally isotropic subbundle, and the relations between them. Our main contribution is the derivation of a curved Maurer-Cartan equation that describes when such a complement exists. This is a first step towards defining new invariants for Dirac structures that may describe the obstruction for the existence of a Dirac complement. Finally, we use our representation of Dirac structures as sections of a group bundle in order to outline a possible method for constructing a Dirac complement to a given Dirac structure.


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## Chapter 1

## Introduction

Symplectic and Poisson structures arise naturally in physics. Classically, the phase space of a mechanical system is an even-dimensional manifold $X$ with dynamics governed by an energy function $H: X \rightarrow \mathbb{R}$ called the Hamiltonian. To this function we associate a vector field $V_{H}$ whose flow describes the energy-dependent trajectories via the differential equation $\dot{x}=V_{H}(x)$, where $x$ denotes the state of the system - usually the values of its position and momentum. Since $H$ and $H+c$ should give the same dynamics for any constant $c$, and the flow dictated by $H$ should preserve $H$ itself (due to the principle of conservation of energy), the assignment of a vector flow to a Hamiltonian is done via a skew bundle map

$$
\pi: T^{*} X \rightarrow T X, \quad \alpha \mapsto i_{\alpha} \pi
$$

which lets us define, for a Hamiltonian $H, V_{H}=\pi(d H)$. In many natural situations, $\pi$ is nondegenerate and its inverse $\omega:=\pi^{-1}$, the skew map

$$
\omega: T X \rightarrow T^{*} X,
$$

is considered. This map $\omega$ can be seen as a 2 -form and the fact that physical laws should be time-independent is mathematically summarized by the condition $d \omega=0$. Since $\omega$ is nondegenerate, this condition turns it into a symplectic form. A (possibly degenerate) closed 2 -form of constant rank is called a presymplectic form. Presymplectic forms describe the dynamics in the presence of constraints, which mathematically corresponds to a submanifold of the phase space. Analogously, $\pi$ above can be seen as a (skew) bivector and then $d \omega=0$ translates into $[\pi, \pi]=0$ for the Schouten bracket. When $\pi$ is possibly degenerate but still satisfies $[\pi, \pi]=0$, we talk about Poisson structures, which describe the dynamics in the presence of symmetries.

The condition $d \omega=0$ or $[\pi, \pi]=0$ is indeed very natural in physics. Noether's theorem relates the symmetries of a mechanical system to conserved quantities: for a simply connected symplectic phase space $(X, \omega)$ and a Lie group $G$ acting on $X$ via symplectomorphisms (diffeomorphisms preserving $\omega$ ), there is a Lie algebra homomorphism $\operatorname{Lie}(G) \rightarrow C^{\infty}(X)$. The existence of this homomorphism is guaranteed by the fact that $d \omega=0$. The image of $\operatorname{Lie}(G)$ consists of the subalgebra of conserved quantities in the following sense: if $G$ preserves some Hamiltonian $H$, all the functions in the image of $\operatorname{Lie}(G)$ are invariant under the flow of $H$. In many instances, these functions are related to physical parameters of a mechanical system, such as momentum, centre of mass or angular momentum.

Despite their physical origin, Poisson and especially symplectic geometry play nowadays a fundamental role on geometry and topology, as the basis for Gromov-Witten invariants [23] or Floer homology [5, 17] as well as in far-reaching applications like the infinite-dimensional analogue of symplectic reduction that endows the moduli space of flat connections with a symplectic structure [1, 22].

Dirac structures originate from the natural question of what structure describes a mechanical system with symmetries (given by a Poisson structure) when there are moreover constraints. An advantage of Dirac structures over symplectic or Poisson structures is that, under the hypothesis of so-called clean intersection, they can be both be pushed forward and pulled back, in contrast to Poisson or symplectic structures which can only be pushed forward or pulled back, respectively, via smooth maps. Indeed, when considering submanifolds of a symplectic manifold, the symplectic structure is pulled back to a presymplectic (possibly not of constant rank) via the inclusion, whereas when considering quotients of a Poisson manifold by Poisson group actions, we can push the Poisson bivector to the quotient. Concatenating these operations, we see that a submanifold of a quotient, or the quotient of a submanifold, need not be a Poisson or symplectic manifold, but it is a manifold with a Dirac structure, as shown in the following diagram:


Figure 1.1: Constraints and symmetries for presymplectic and Poisson structures

Formally, given a manifold $X$, a Dirac structure is a subbundle of $T X \oplus T^{*} X$ that is maximally isotropic under the natural pairing defined on $T X \oplus T^{*} X$ (a condition which unifies the skew-symmetry conditions of presymplectic and Poisson structures, as shown in section 2.2), and is involutive under the (possibly twisted) Dorfman bracket (section 3.1). Geometrically, Dirac structures correspond to presymplectic foliations whose leaves are not necessarily of the same dimension. The variation on this dimension is captured by the so-called 'type', a pointwise invariant of Dirac structures defined as $\operatorname{codim}\left(\pi_{T}(L)\right)$, that is, the codimension of the presymplectic leaf passing through the point

In this MSc thesis, we probe some basic properties of the space of all Dirac structures, inspired by similar approaches used for the symplectic and Poisson cases. First, we will ask what structure the space of Dirac structures over a manifold carries. In the linear case, choosing an inner product on a vector space $V$ induces a correspondence between Dirac structures $L$ over $V$ and orthogonal operators $O_{L}: V \rightarrow V$ (section 2.3). Using this description, we identify properties of Dirac structures as properties of their corresponding orthogonal maps (section 2.3). For the type, defined earlier, we find:

Proposition 2.17. The type of a Dirac structure $L$ is equal to the number of -1 eigenvalues of $O_{L}$.
As our first main result, we show the space of Dirac structures is isomorphic to the subset of sections of a certain group bundle satisfying an integrability condition:

Theorem 3.25. On a Riemmanian manifold $(X, g)$, the space of Dirac structures can be identified with the space of sections of the group bundle $O(X, g)$ satisfying the equation:

$$
g\left(Z,\left(I+O^{-1}\right)\left(\nabla_{Y} O\right) X\right)+c . p=0
$$

(where $O \in \Gamma(O(X, g))$ ) for any $X, Y, Z \in \Gamma(T X)$.
The deformation theory of Dirac structures, while having similar flavour to that of presymplectic, Poisson, or complex structures, includes in it complications which make it more involved. In the three classical cases,
a differential graded Lie algebra (DGLA) is constructed, and the integrable deformations are in bijective correspondence with the Maurer-Cartan elements of this DGLA (as we show in section 4.1). When a Dirac structure $L$ admits a complementary Dirac structure, that is, a second Dirac structure $M$ such that $M \cap L=$ $\{0\}$, its deformation theory is again governed by a DGLA. The three classical examples of presymplectic, Poisson, and complex, all share the property of admitting such a complementary Dirac structure. While every Dirac structure admits a maximally isotropic complement, some Dirac structures do not admit an integrable (Dirac) complement (proposition 5.1). In this case (as we show in section 4.2 as well as chapter $5)$, the deformation theory is not defined by a DGLA anymore, but by a curved $L_{3}$ algebra:

Theorem ([11]). Let $M$ be a maximally isotropic subbundle of $T X \oplus T^{*} X$, and let $L$ be another maximally isotropic subbundle such that $M \cap L=\{0\}$. Then $\Gamma(\bigwedge L)$ carries the structure of a curved $L_{3}$ algebra, and the integrable 'small' deformations of $L$ are in bijective correspondence with the Maurer-Cartan elements of this algebra, that is, the elements $\omega \in \Gamma\left(\bigwedge^{2} L\right)$ satisfying the equation:

$$
l_{0}+l_{1}(\omega)+\frac{1}{2} l_{2}(\omega, \omega)+\frac{1}{3!} l_{3}(\omega, \omega, \omega)=0 .
$$

By small deformations of $M$, we mean those that remain transverse to $L$. In the linear case, small deformations correspond to an open subset in the space of maximally isotropic subspaces. These can be described as graphs of skew maps $\omega: M \rightarrow L$ (proposition 4.4). Curved $L_{3}$ algebras are higher generalizations of DGLA's. They are a subcase of curved $L_{\infty}$ algebras, which consist of a graded vector space $V$ along with an infinite collection of multilinear graded skew-symmetric maps $l_{k}: \otimes^{k} V \rightarrow V$. Curved $L_{\infty}$ arise frequently in deformation theory.

In order to define the $L_{3}$ structure on $\Gamma(\bigwedge L)$, we first introduce the notion of a quasi-Lie algebroid, and show that the choice of any two such $M, L$ induces a quasi-Lie algebroid structure on both (section 5.1). This is an alternative approach to [11], where the theorem above is originally proved.

The question of when a Dirac structure admits a Dirac complement can be rephrased in terms of these curved $L_{3}$ algebras. Our main contribution is showing that the existence of an integrable complement is equivalent to the existence of a solution to the curved Maurer-Cartan equation in a curved $L_{3}$ algebra:

Theorem 5.19. Let $M$ be a Dirac structure and $L$ be a maximally isotropic complementary subbundle. Then $M$ has an integrable complement if and only if there is a solution $\omega \in \Gamma\left(\bigwedge^{2} M\right)$ for the equation

$$
N_{L}+d_{L} \omega+\frac{1}{2}[\omega, \omega]=0
$$

in the curved DGLA $\Gamma(\bigwedge M)$.
Additionally, rechoosing the complementary subbundle $L$ yields a different $L_{3}$ algebra, and the two algebras admit an invertible map between them. As we show, this map gives certain relations between the different $L_{3}$ structures (propositions 5.18 and 5.21 ), but is not a curved $L_{3}$ algebra isomorphism in one of the several ways described in the current literature [13, 15]. Thus, whenever the $L_{3}$ algebra of a Dirac structure is not isomorphic, through a map in this class, to a DGLA, the Dirac structure does not admit a Dirac complement, but the contrary does not hold. This suggests a more precise description of the obstruction for the existence of a Dirac complement is needed, perhaps by means of an invariant in some cohomology related to the Dirac structure.

Finally, through a stronger version of the complement problem, we give a sufficient condition to explicitly construct a complementary Dirac structure (section 5.4).

## Chapter 2

## Linear generalized geometry

### 2.1 The pairing on $V \oplus V^{*}$

We shall make use of the following notation:

- $\wedge V$ is the exterior algebra of a vector space $V$, including $\bigwedge^{0} V=\mathbb{F}$ where $\mathbb{F}$ is the base field of $V$.
- $p_{W}$ is the projection to a subspace $W \subseteq V$ with respect to some splitting, which has been fixed beforehand or is clear from the context.
- By using ' $+c . p$ ' in an expression evaluated on arguments, we mean the addition of the same expression evaluated on all cyclic permutations of the arguments. For example, $f(x, y, z)+c . p=f(x, y, z)+$ $f(z, x, y)+f(y, z, x)$.

As described in the introduction, Dirac structures are generalizations of both presymplectic and Poisson structures. First, we formally define presymplectic and Poisson structures on a vector space $V$.

Definition 2.1. A presymplectic structure on a vector space $V$ is any $\omega \in \Lambda^{2} V^{*}$.
Definition 2.2. A Poisson structure on a vector space $V$ is any $\pi \in \bigwedge^{2} V$.
Before seeing how these structures fit into Dirac structures in section 2.2, we need to make some basic constructions.

On $V \oplus V^{*}$ we have a natural pairing: for $X+\alpha, Y+\beta \in V \oplus V^{*}$,

$$
\langle X+\alpha, Y+\beta\rangle=\frac{1}{2}(\alpha(Y)+\beta(X)) .
$$

Lemma 2.3. The pairing $\langle$,$\rangle has signature (n, n)$, where $n=\operatorname{dim} V$.
Proof. Choose a basis $\left\{X_{i}\right\}$ for $V$ and let $\left\{\alpha_{i}\right\}$ be its dual basis. The set $\left\{X_{i}+\alpha_{i}\right\} \cup\left\{X_{i}-\alpha_{i}\right\}$ is a basis for $V \oplus V^{*}$ and satisfies $\left\langle X_{i}+\alpha_{i}, X_{j}+\alpha_{j}\right\rangle=\delta_{i j},\left\langle X_{i}-\alpha_{i}, X_{j}-\alpha_{j}\right\rangle=-\delta_{i j}$ and $\left\langle X_{i}+\alpha_{i}, X_{j}-\alpha_{j}\right\rangle=0$, which gives $\langle$,$\rangle the form \left(\begin{array}{cc}I_{n} & 0 \\ 0 & -I_{n}\end{array}\right)$.

The group of orthogonal transformations for $\langle\cdot, \cdot\rangle$ can be shown to be generated by 3 types of transformations [7]:

1. Lifts of linear transformations $G: V \rightarrow V$, given by:

$$
\left(\begin{array}{cc}
G & 0 \\
0 & \left(G^{-1}\right)^{*}
\end{array}\right)
$$

acting as $X+\alpha \mapsto G X+\left(G^{-1}\right)^{*} \alpha$.
2. $B$-transformations, given by:

$$
\left(\begin{array}{cc}
I d & 0 \\
B & I d
\end{array}\right)
$$

for some skew $B: V \rightarrow V^{*}$, acting as $X+\alpha \mapsto X+\alpha+i_{X} B$.
3. $\beta$-transformations, given by:

$$
\left(\begin{array}{cc}
I d & \beta \\
0 & I d
\end{array}\right)
$$

for some skew $\beta: V^{*} \rightarrow V$, acting as $X+\alpha \mapsto X+i_{\alpha} \beta+\alpha$.

### 2.2 Maximally isotropic subspaces

As we will show, presymplectic and Poisson structures can both be thought of as maximally isotropic subspaces of $V \oplus V^{*}$ (an isotropic subspace is a subspace on which the pairing vanishes, and a maximally isotropic subspace is an isotropic subspace which is not a subspace of any other isotropic subspace). Since the form $\langle\cdot, \cdot\rangle$ has signature $(n, n)$, the following is a known linear algebra fact:

Proposition 2.4. The dimension of any maximally isotropic subspace of $V \oplus V^{*}$ is exactly $n$.
Maximally isotropic subspaces can be represented in a unique form in terms of their projection to $V$ :
Proposition 2.5. [7] For any maximally isotropic subspace $L \leq V \oplus V^{*}$ there are unique $E \leq V, \epsilon \in \bigwedge^{2} E^{*}$ so that $L=L(E, \epsilon)=\left\{X+\alpha\left|X \in E, i_{X} \epsilon=\alpha\right|_{E}\right\}$.

Proof. Let $L \leq V \oplus V^{*}$ be maximally isotropic and let $E=p_{V}(L)$ (the projection of $L$ to $V$ ). Define the $\operatorname{map} \epsilon: E \rightarrow V^{*} / \operatorname{Ann}(E) \cong E^{*}$ as follows: for $X+\xi \in L, \epsilon(X)=[\xi]$.

Let us check this map is well defined: if $\alpha, \alpha^{\prime} \in V^{*}$ satisfy that $X+\alpha$ and $X+\alpha^{\prime}$ are in $L$, then since $L$ is a subspace, $\alpha-\alpha^{\prime} \in L$, but $L$ is maximally isotropic, so $\left\langle X+\alpha, \alpha-\alpha^{\prime}\right\rangle=0=\left(\alpha-\alpha^{\prime}\right)(X)$, so $\alpha-\alpha^{\prime} \in \operatorname{Ann}(E)$, therefore $[\alpha]$ is well defined.

Therefore, we have a well defined $\epsilon \in \bigwedge^{2} E^{*}$. Following our construction, $L$ takes the form mentioned above.

Reciprocally, for any choice of $E$ and $\epsilon \in \bigwedge^{2} E, L(E, \epsilon)$ is maximally isotropic: It is isotropic since $\langle X+\alpha, X+\alpha\rangle=\epsilon(X, X)=0$, and of maximal dimension since $L(E, \epsilon)=e^{B} L(E, 0)=e^{B}(E \oplus \operatorname{Ann}(E))$ for any $B$ with $i^{*} B=\epsilon$, where $i: E \rightarrow V$ is the inclusion.

Example 2.6. 1. $V=L(V, 0)$, and more generally $E \oplus \operatorname{Ann}(E)=L(E, 0)$
2. $V^{*}=L(\{0\}, 0)$
3. For any skew $\omega: V \rightarrow V^{*}, \operatorname{graph}(\omega)=L(V, \omega)$
4. For any skew $\pi: V^{*} \rightarrow V, \operatorname{graph}(\pi)=L(\operatorname{Im}(\pi), \tilde{\omega})$, where $\tilde{\omega}$ is given as follows: through $\pi, \operatorname{Im}(\pi) \cong$ $V^{*} / \operatorname{Ker}(\pi) \cong \operatorname{Im}(\pi)^{*}$, which gives an invertible skew map $\tilde{\omega}: \operatorname{Im}(\pi) \rightarrow \operatorname{Im}(\pi)^{*}$. This is the linear case for the known fact that Poisson structures on a manifold yield symplectic foliations.

Definition 2.7. The type of a maximally isotropic subspace is defined as type $(L)=\operatorname{codim}\left(p_{V}(L)\right)$
Later, when we move to the global case where we endow these structures with a bracket, we will see the type is an important invariant, since it is stable under the symmetry group of both the pairing and the bracket.

A useful object in linear generalized geometry is the Clifford algebra of $\left(T \oplus T^{*},\langle\cdot, \cdot\rangle\right)$. Since we will use it later, we present its construction here.

Definition 2.8. The Clifford algebra $\operatorname{Cliff}(V,\langle\cdot, \cdot\rangle)$ of a pair $(V,\langle\cdot, \cdot\rangle)$ where $V$ is a vector space and $\langle\cdot, \cdot\rangle$ is an arbitrary bilinear form on $V$, is the quotient of the tensor algebra $\otimes V=\oplus_{i} \otimes^{i} V$ by the ideal generated by elements of the form $v \otimes v-\langle v, v\rangle$ for all $v \in V$, where $\langle v, v\rangle$ is considered as an element of $\otimes^{0} V=\mathbb{F}$. The product $\cdot$ on $\operatorname{Cliff}(V,\langle\cdot, \cdot\rangle)$ is the one induced from the tensor product on $\otimes V$. For $v, u \in V$, the product satisfies

$$
v \cdot u+u \cdot v=2\langle u, v\rangle
$$

Additionally, $\operatorname{Cliff}(V,\langle\cdot, \cdot\rangle)$ has a natural grading on it. It can be shown to be isomorphic (as a vector space, not as an algebra) to $\bigwedge V$, which has a grading given by the degrees of multivectors. This grading pulls back to a grading on $\operatorname{Cliff}(V,\langle\cdot, \cdot\rangle)$. We denote the degree of a homogeneous element a by $|a|$.

Definition 2.9. Let $a, b \in \operatorname{Cliff}(V,\langle\cdot, \cdot\rangle)$ be homogeneous elements (using the grading above). The graded commutator is defined as:

$$
\{a, b\}=a \cdot b-(-1)^{|a||b|} b \cdot a
$$

More generally, the graded commutator is defined as the linear extension of the above expression.
Proposition 2.10. The graded commutator satisfies the following properties:

- Graded antisymmetry:

$$
\{a, b\}=(-1)^{|a||b|}\{b, a\}
$$

- Graded Leibniz rules:

$$
\begin{aligned}
& \{a, b \cdot c\}=\{a, b\} \cdot c+(-1)^{|a||b|} b \cdot\{a, c\} \\
& \{a \cdot b, c\}=a \cdot\{b, c\}+(-1)^{|b||c|}\{a, c\} \cdot b .
\end{aligned}
$$

Proof. The proofs of the second and third propositions are similar, so we focus on the second one. Note that:

$$
\begin{aligned}
\{a, b \cdot c\} & =a \cdot b \cdot c-(-1)^{|a|(|b|+|c|)} b \cdot c \cdot a \\
& =a \cdot b \cdot c-(-1)^{|a|(|b|+|c|)} b \cdot c \cdot a+(-1)^{|a||b|} b \cdot a \cdot c-(-1)^{|a||b|} b \cdot a \cdot c \\
& =\{a, b\} \cdot c+(-1)^{|a||b|} b \cdot\{a, c\} .
\end{aligned}
$$

For $\langle\cdot, \cdot\rangle=0$, the Clifford algebra coincides with the exterior algebra [6]. Therefore, for an isotropic subspace $L \leq V$, the inclusion $i: L \rightarrow V$ induces an inclusion of algebras $i: \bigwedge L \rightarrow \operatorname{Cliff}(V,\langle\cdot, \cdot\rangle)$.

We will now derive some identities which will be useful for us later. Our interest lies in the case $V \oplus V^{*}$ with the canonical pairing. We find that the exterior algebra of any maximally isotropic subspace can be thought of naturally as a subalgebra of the Clifford algebra of $V \oplus V^{*}$. Consider the case where we have two such maximally isotropic subspaces $M, L$ which are complementary- that is, $M \cap L=\{0\}$. Since $\langle\cdot, \cdot\rangle$ is nondegenerate, $L \cong M^{*}$ through $\langle\cdot, \cdot\rangle$. In this case, we have the following:

Proposition 2.11. Let $m \in \bigwedge M, l \in \bigwedge^{1} L$. Then $\{m, l\}=i_{l} m$.
Proof. By induction on $k=|m|$. For $k=1$, this is the Clifford algebra identity. Assume, without loss of generality, that $m$ is decomposable, and write $m=x_{1} \cdot \ldots \cdot x_{k}$. Then $\{m, l\}=\left\{x_{1}, l\right\} x_{2} \cdot \ldots \cdot x_{k}-x_{1} \cdot\left\{x_{2}\right.$. $\left.\ldots \cdot x_{k}, l\right\}=i_{l} m$ by the definition of the wedge product.

Proposition 2.12. If $l, l^{\prime} \in \Lambda L,\left\{l, l^{\prime}\right\}=0$.
Proof. For $l, l^{\prime} \in \bigwedge^{1} L$ this is the Clifford algebra defining identity. Using the Leibniz rule and bilinearity the proposition follows.

### 2.3 The space of maximally isotropic subspaces

We can now move to discussing the space of all maximally isotropic subspaces of $V \oplus V^{*}$. It turns out that this space has a simple geometric description. The original construction is by Courant [4], but here we give a basis-free description of it that uses slightly more general terms, which are also more natural in generalized geometry.

Definition 2.13. A generalized metric is a splitting $V \oplus V^{*}=P \oplus N$, where $P$ is positive definite under $\langle\cdot, \cdot\rangle, N$ is negative definite, and $P \perp N$.

Every inner product on $V$, which we refer to as a metric, defines a generalized metric in a simple way: choose an orthonormal basis $\left\{X_{i}\right\}$ for $V$ and let $\left\{\alpha_{i}\right\}$ be its dual basis. Our calculations in the proof of Theorem 2.3 show that $P=\operatorname{Span}\left(\left\{X_{i}+\alpha_{i}\right\}\right)=\{X+g(X, \cdot) \mid X \in V\}$ is positive definite, $N=\operatorname{Span}\left(\left\{X_{i}-\alpha_{i}\right\}\right)=\{X-g(X, \cdot) \mid X \in V\}$ is negative definite, and they are orthogonal to each other.

Proposition 2.14. For any generalized metric there are is a unique metric $g$ and a unique skew 2-form $B$ so that:

$$
P=\left\{X+g(X, \cdot)+i_{X} B\right\},
$$

and

$$
N=\left\{X-g(X, \cdot)+i_{X} B\right\} .
$$

Proof. Note $\operatorname{Ker}\left(\left.p_{V}\right|_{P}\right)=P \cap V^{*}$, and since $V^{*}$ is isotropic while $P$ is positive definite, this intersection is zero, so $\left.p_{V}\right|_{P}$ is an isomorphism (by dimension considerations). Similarly, $\left.p_{V}\right|_{N}$ is an isomorphism. Therefore we can define the maps:

$$
\begin{aligned}
\zeta_{1} & =\left.\pi_{V^{*}} \circ p_{V}\right|_{P} ^{-1}: V \rightarrow V^{*} \\
\zeta_{2} & =\left.\pi_{V^{*}} \circ p_{V}\right|_{N} ^{-1}: V \rightarrow V^{*},
\end{aligned}
$$

so $P=\left\{X+\zeta_{1}(X) \mid X \in V\right\}$ and $N=\left\{X+\zeta_{2}(X) \mid X \in V\right\}$. Since $P \perp N$, for any $X, Y \in V, \zeta_{1}(X, Y)+$ $\zeta_{2}(Y, X)=0$. Let $g_{i}, B_{i}$ be the symmetric and antisymmetric parts of $\zeta_{i}$. For $X=Y$, the above calculation shows $g_{1}=-g_{2}$, which leaves us with $B_{1}=B_{2}$. The fact that $P$ is positive definite assures that $g=g_{1}$ is a metric, as needed.

The techniques from this proof allow us to characterize the space of all maximally isotropic subspaces of $V \oplus V^{*}$. Note that the fact that $\left.p_{V}\right|_{P}$ is an isomorphism relied only on the fact that $V$ is isotropic. Similarly, for any maximally isotropic $L \leq V \oplus V^{*},\left.p_{P}\right|_{L}$ is an isomorphism. Therefore for any $L$ we uniquely can define the linear map:

$$
\left.p_{N} \circ p_{P}\right|_{L} ^{-1}:=\tilde{O}_{L}: P \rightarrow N
$$

which sends $p \in P$ to the unique $n \in N$ such that $p+n \in L$. Now, equip $N$ with $-\langle\cdot, \cdot\rangle$, which is positive definite. We have the following:

Proposition 2.15. The operator $\tilde{O}_{L}$ is an orthogonal map from $(P,\langle\cdot, \cdot\rangle)$ to $(N,-\langle\cdot, \cdot\rangle)$.
Proof. Write $\tilde{O}_{L}(p)=n$. So, $\left.p_{P}\right|_{L} ^{-1}(p)=p+n$. Since $L$ is isotropic and $P \perp N,\|p+n\|^{2}=\|p\|^{2}-\|n\|^{2}=0$, so $\|p\|=\left\|\tilde{O}_{L}(p)\right\|$, therefore $\tilde{O}_{L}$ is orthogonal.

For simplicity, consider the case where we have a metric on $V$ (that is, the case $B=0$ as in proposition 2.14). Considering $P=\{X+g(X, \cdot) \mid X \in V\}$ and $N=\{X-g(X, \cdot) \mid X \in V\}$, we have the two isometries (where again, we equip $N$ with minus the pairing)

$$
q_{ \pm}(X)=\frac{1}{\sqrt{2}}(X \pm g(X, \cdot))
$$

from $V$ to $P$ or $N$ respectively. The composition $O_{L}=q_{-}^{-1} \circ \tilde{O}_{L} \circ q_{+}$is therefore an orthogonal map from $V$ to itself. We find:

Corollary 2.16. Given a metric $g$ on $V$, the space of all maximally isotropic subspaces of $V \oplus V^{*}$ can be identified with $O(V, g)$.


Figure 2.1: Orthogonal transformation defining a maximally isotropic subspaces

Compared to the spaces of presymplectic and Poisson structures, which are both vector spaces, the space of maximally isotropic subspaces of $V \oplus V^{*}$ is not only compact, it also carries a (non-canonical) group structure.

Various geometric properties of a maximally isotropic subspace can be identified with geometric properties of its corresponding orthogonal transformation. Keeping the same notation as in corollary 2.16, we have for example:

Proposition 2.17. The type of $L$ is equal to the number of -1 eigenvalues of $O_{L}$, that is,

$$
\operatorname{type}(L)=n-\operatorname{rank}\left(I+O_{L}\right)
$$

Proof. Note that, under our previous choices, we can write

$$
\begin{aligned}
L & =\left\{X+g(X, \cdot)+O_{L}(X)-g\left(O_{L}(X), \cdot\right) \mid X \in V\right\} \\
& =\left\{\left(I+O_{L}\right)(X)+g\left(\left(I-O_{L}\right)(X), \cdot\right) \mid X \in V\right\},
\end{aligned}
$$

and so $\operatorname{dim}\left(p_{V}(L)\right)$ equals $\operatorname{rank}\left(I+O_{L}\right)$.
This gives us a stratification of $O(V)$, given by the type of the corresponding maximally isotropic subspace. In low dimensions, this stratification can be visualized. In figure 2.2, we represent $O(2)$ as the disjoint union of two circles, and we see that type 0 is given by a line segment in one of the circles, type 2 is given by a point in the same circle, and type 1 is given by the other circle. In figure 2.3, we represent $O(3)$ as the quotient of the disjoint union of two 3 -dimensional balls, given by identifying antipodal points on their boundaries.


Figure 2.2: Type splitting of $\mathrm{O}(2)$.

As a first example of this equivalence, consider the graph of a 2 -form $\omega$. We can write:

$$
L=\operatorname{graph}(\omega)=\left\{X+i_{X} \omega \mid X \in V\right\}=\left\{\left(I+O_{L}\right) X+g\left(\left(I-O_{L}\right) X, \cdot \mid X \in V\right\},\right.
$$



Figure 2.3: Type splitting of $O(3) \cong D^{3} /\{x \sim-x \mid\|x\|=1\}$, lifted to $D^{3}$.
and since $I+O_{L}$ is invertible according to proposition 2.17,

$$
L=\left\{X+g\left(\left(I-O_{L}\right)\left(I+O_{L}\right)^{-1} X, \cdot\right) \mid X \in V\right\} .
$$

Defining $A=g^{-1} \circ \omega: V \rightarrow V$, we find the following:
Proposition 2.18. The map $O_{\text {graph }(\omega)}$ is given by the Cayley transform of $A$, which is defined as $A \mapsto$ $(I+A)^{-1}(I-A)$.

Similarly, for the graph of a Poisson structure, $O_{L}$ is the Cayley transform of $\pi \circ g$. As special cases, we find that $O_{T}=I d$ and $O_{T^{*}}=-I d$.

Another special case is $L=E \oplus \operatorname{Ann}(E)$. It can be easily shown that $O_{L}=p_{E}-p_{E \perp}$ where $p_{E}, p_{E^{\perp}}$ are the orthogonal projections to $E, E^{\perp}$ respectively.

Proposition 2.19. Let $L$ and $L^{\prime}$ be maximally isotropic subspaces. Then $L \cap L^{\prime}=\{0\}$ if and only if $O_{L} O_{L^{\prime}}^{-1}$ has no fixed points.

Proof. The subspaces $L$ and $L^{\prime}$ are the graphs of the corresponding maps $\tilde{O}_{L}, \tilde{O}_{L^{\prime}}: P \rightarrow N$, so $L \cap L^{\prime}=$ $\left\{p \in P \mid \tilde{O}_{L}(p)=\tilde{O}_{L^{\prime}}(p)\right\}$, from which the result easily follows.

Corollary 2.20. The subspace $L^{\prime}$ corresponding to $-O_{L}$ is always complementary to the subspace $L$.
Geometrically, $L^{\prime}$ as in the corollary is given by the orthogonal complement to $L$ in $V \oplus V^{*}$ after equipping $N$ with $-\langle\cdot, \cdot\rangle$.

More generally, the space of maximally isotropic subspaces which are complementary to a given maximally isotropic subspace can be described in a simple way. Consider first the case $L=V$. If $L^{\prime}$ is a complement, $O_{L^{\prime}}$ must have no fixed points, so $O_{L^{\prime}} \in O(n) \backslash f^{-1}(0)$ where $f: O(n) \rightarrow \mathbb{R}$ is the map $f(O)=\operatorname{det}(O-I)$. Since $f$ is continuous, we find:

Proposition 2.21. The set of complements to a given maximally isotropic subspace is an open subset in the set of maximally isotropic subspaces, under the natural topology on $O(V, g)$.

Finally, we are interested in what happens when we rechoose $g$.
Proposition 2.22. For a given maximally isotropic subspace $L$, let $O_{L}$ and $0_{L}^{\prime}$ be the orthogonal operators equivalent to $L$ when choosing metrics $g$ and $g^{\prime}$ on $V$, respectively. Then:

$$
O_{L}^{\prime}=\left(I+O_{L}+g^{\prime-1} \circ g\left(I-O_{L}\right)\right)\left(I+O_{L}-g^{\prime-1} \circ g\left(I-O_{L}\right)\right)^{-1} .
$$

Proof. As we have seen, $L=\left\{\left(I+O_{L}\right) X+g\left(\left(I-O_{L}\right) X, \cdot\right) \mid X \in V\right\}$. Rechoosing $g \mapsto g^{\prime}$ and getting $O_{L} \mapsto O_{L}^{\prime}$, we have $\left\{\left(I+O_{L}^{\prime}\right) X^{\prime}+g^{\prime}\left(\left(I-O_{L}^{\prime}\right) X^{\prime}, \cdot\right) \mid X^{\prime} \in V\right\}$. Since this is the same subspace, any vector $v \in L$ can be written as:

$$
v=\left(I+O_{L}\right) X+g\left(\left(I-O_{L}\right) X, \cdot\right)=\left(I+O_{L}^{\prime}\right) X^{\prime}+g^{\prime}\left(\left(I-O_{L}^{\prime}\right) X^{\prime}, \cdot\right)
$$

Projecting to $V, V^{*}$ we have the pair of equations:

$$
\left.I+O_{L}\right) X=\left(I+O_{L}^{\prime}\right) X, \quad g\left(\left(I-O_{L}\right) X\right)=g^{\prime}\left(\left(I-O_{L}^{\prime}\right) X^{\prime}\right)
$$

where we consider $g, g^{\prime}$ as maps $V \rightarrow V^{*}$. We can rewrite the second equation as:

$$
g^{\prime-1} \circ g\left(I-O_{L}\right) X=\left(I-O_{L}^{\prime}\right) X^{\prime} .
$$

Summing the two equations we find:

$$
X^{\prime}=\frac{1}{2}\left(I+O_{L}+g^{\prime-1} \circ g\left(I-O_{L}\right)\right) X
$$

By a symmetry argument, the map $\frac{1}{2}\left(I+O_{L}+g^{\prime-1} \circ g\left(I-O_{L}\right)\right)$ is invertible. Plugging into the first equation we have:

$$
\left(I+O_{L}\right) X=\frac{1}{2}\left(I+O_{L}^{\prime}\right)\left(I+O_{L}+g^{\prime-1} \circ g\left(I-O_{L}\right)\right) X
$$

Therefore:

$$
O_{L}^{\prime}=\left(I+O_{L}+g^{\prime-1} \circ g\left(I-O_{L}\right)\right)\left(I+O_{L}-g^{\prime-1} \circ g\left(I-O_{L}\right)\right)^{-1} .
$$

Remark 2.1. This shows the topology on the set of maximally isotropic subspaces is well defined. We can choose it to be the natural topology on $O(V, g)$ for some $g$, and since the map $O(V, g) \rightarrow O\left(V, g^{\prime}\right)$ is a homeomorphism, this topology is independent of $g$.

Remark 2.2. The map $O \rightarrow O^{\prime}$ is, generally, not a group homomorphism. So, while the set of maximally isotropic subspaces carries a group structure, it is not canonical.

## Chapter 3

## Dirac structures and generalized geometry

### 3.1 Dorfman bracket and Dirac structures

Presymplectic and Poisson structures on a manifold both come with their own integrability conditions. We will see both of these conditions can be unified to one - the involutivity of a certain vector subbundle under the so-called the Dorfman bracket.

Our discussion from section 2.1 can be generalized to the tangent and cotangent bundles of a manifold $X$. We define the generalized tangent bundle as $T X \oplus T^{*} X$, usually dropping $X$ and writing simply $T \oplus T^{*}$. Again, $T \oplus T^{*}$ carries on it a natural pairing on sections, given by the same formula as in section 2.1.

Both $T$ and $T^{*}$ carry on them additional geometric structure - we can take the Lie bracket of two sections of $T$, and we can take the exterior derivative of sections of $T^{*}$. Both of these structures are actually combined to give a bracket on $T \oplus T^{*}$.

Definition 3.1. The Dorfman bracket of two sections $X+\alpha, Y+\beta$ of $T \oplus T^{*}$ is given by:

$$
[X+\alpha, Y+\beta]_{\text {Dorfman }}=[X, Y]_{L i e}+L_{X} \beta-i_{Y} d \alpha
$$

From now on, we will simply use the notations $[X+\alpha, Y+\beta]_{\text {Dorfman }}=[X+\alpha, Y+\beta]$ and $[X, Y]_{\text {Lie }}=$ $[X, Y]$. Since the Lie and Dorfman brackets coincide whenever $\alpha=\beta=0$, there is no ambiguity.

Proposition 3.2. [7] Let $x_{1}, x_{2}, x_{3} \in \Gamma\left(T \oplus T^{*}\right)$. The Dorfman bracket has the following properties:

1. $p_{T}([X+\alpha, Y+\beta])=[X, Y]$
2. $p_{T}\left(x_{1}\right)\left\langle x_{2}, x_{3}\right\rangle=\left\langle\left[x_{1}, x_{2}\right], x_{3}\right\rangle+\left\langle x_{2},\left[x_{1}, x_{3}\right]\right\rangle$
3. $\left[x_{1}, x_{2}\right]+\left[x_{2}, x_{1}\right]=2 d\left\langle x_{1}, x_{2}\right\rangle$
4. $\left[x_{1},\left[x_{2}, x_{3}\right]\right]=\left[\left[x_{1}, x_{2}\right], x_{3}\right]+\left[x_{2},\left[x_{1}, x_{3}\right]\right]$
5. $\left[x_{1}, f x_{2}\right]=f\left[x_{1}, x_{2}\right]+p_{T}\left(x_{1}\right)(f) x_{2}$
6. $\left[f x_{1}, x_{2}\right]=f\left[x_{1}, x_{2}\right]-p_{T}\left(x_{2}\right)(f) x_{1}+2\left\langle x_{1}, x_{2}\right\rangle d f$

The above properties of the Dorfman bracket are almost enough to define it completely. As we will show in definition 3.11, it is also possible to 'twist' the Dorfman bracket while still maintaining these properties.

Definition 3.3. A Dirac structure is a maximally isotropic subbundle $L$ of $T \oplus T^{*}$ which is involutive under the Dorfman bracket. That is, for any $x_{1}, x_{2} \in \Gamma(L),\left[x_{1}, x_{2}\right] \in \Gamma(L)$.

Proposition 3.4. Let $\omega \in \Gamma\left(\bigwedge^{2} T^{*}\right)$. Then $d \omega=0$ if and only if graph $(\omega)$ is a Dirac structure.
Proposition 3.5. Let $\pi \in \Gamma\left(\bigwedge^{2} T\right)$. Then $[\pi, \pi]=0$ (where $[\cdot, \cdot]$ is the Schouten bracket, defined as the unique graded bracket on the space of alternating multivector fields that makes the alternating multivector fields into a Gerstenhaber algebra) if and only if $\operatorname{graph}(\pi)$ is a Dirac structure.

The integrability of Dirac structures can equivalently be encoded by the vanishing of a certain tensor, similar to the Nijenhuis tensor of an almost complex structure. As such, they carry the same name:

Definition 3.6. Let $L$ be a maximally isotropic subbundle. The Nijenhuis tensor $N_{L}$ is defined via

$$
N_{L}\left(x_{1}, x_{2}, x_{3}\right)=\left\langle\left[x_{1}, x_{2}\right], x_{3}\right\rangle
$$

where $x_{1}, x_{2}, x_{3}$ are extended to local sections of $L$.
Proposition 3.7. The operator $N_{L}$ is tensorial, that is, independent of the local extensions of the $x_{i}$, and skew symmetric.

Proof. We clearly have:

$$
N_{L}\left(x_{1}, x_{2}, f x_{3}\right)=f N_{L}\left(x_{1}, x_{2}, x_{3}\right)
$$

Therefore, showing $N_{L}$ is skew symmetric would also show it is tensorial. $N_{L}\left(x_{1}, x_{2}, x_{3}\right)=-N_{L}\left(x_{2}, x_{1}, x_{3}\right)$ is guaranteed by property 3 of the Dorfman bracket, since $L$ is isotropic. By property 2 , and again due to $L$ being isotropic,

$$
\left\langle\left[x_{1}, x_{2}\right], x_{3}\right\rangle=p_{T}\left(x_{1}\right)\left\langle x_{2}, x_{3}\right\rangle-\left\langle x_{2},\left[x_{1}, x_{3}\right]\right\rangle=-\left\langle\left[x_{1}, x_{3}\right], x_{2}\right\rangle
$$

so $N_{L}\left(x_{1}, x_{2}, x_{3}\right)=-N_{L}\left(x_{1}, x_{3}, x_{2}\right)$, as needed.
The Nijenhuis tensor measures the failure of a maximally isotropic subbundle to be integrable. For presymplectic and Poisson structures, the quantities $d \omega$ and $[\pi, \pi]$ play the same role. The following examples show this is no coincidence (as shown in [7]):

Example 3.8. $N_{\operatorname{graph}(\omega)}\left(x_{1}+i_{x_{1}} \omega, x_{2}+i_{x_{2}} \omega, x_{3}+i_{x_{3}} \omega\right)=d \omega\left(x_{1}, x_{2}, x_{3}\right)$
Example 3.9. $N_{\text {graph }(\pi)}\left(\alpha_{1}+i_{\alpha_{1}} \pi, \alpha_{2}+i_{\alpha_{2}} \pi, \alpha_{3}+i_{\alpha_{3}} \pi\right)=\frac{1}{2}[\pi, \pi]\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$
Proposition 3.10. [7] $N_{L}=0 \leftrightarrow L$ is a Dirac structure
The Dorfman bracket is not antisymmetric. Its failure to be antisymmetric is encoded by $[X, Y]+[Y, X]=$ $2 d\langle X, Y\rangle$. Therefore, its restriction to involutive maximally isotropic subbundles is indeed antisymmetric.

The Dorfman bracket, along with the natural pairing $\langle\cdot, \cdot\rangle$ and the projection to $T$, endow $T \oplus T^{*}$ with the structure of a Courant algebroid.

Definition 3.11. A Courant algebroid over a manifold $X$ is given by a vector bundle $E$, a nondegenerate pairing $\langle\cdot, \cdot\rangle$ on $E$, a bundle map $a: E \rightarrow T X$, called the anchor, and a bracket $[\cdot, \cdot]$ on the sections of $E$, satisfying the following compatibility conditions: Let $x_{i} \in \Gamma(E), f \in C^{\infty}(X)$

1. $\left[x_{1},\left[x_{2}, x_{3}\right]\right]=\left[\left[x_{1}, x_{2}\right], x_{3}\right]+\left[x_{2},\left[x_{1}, x_{3}\right]\right]$.
2. $a\left(x_{1}\right)\left\langle x_{2}, x_{3}\right\rangle=\left\langle\left[x_{1}, x_{2}\right], x_{3}\right\rangle+\left\langle x_{2},\left[x_{1}, x_{3}\right]\right\rangle$.
3. $\left[x_{1}, f x_{2}\right]=f\left[x_{1}, x_{2}\right]+\left(a\left(x_{1}\right)(f)\right) x_{2}$.
4. $[x, x]=2 D\langle x, x\rangle$, where $D$ is defined via $\langle D f, x\rangle=a(X)(f)$ for any $x \in \Gamma(E)$.

All properties of the Dorfman bracket as in proposition 3.2 can be deduced from the above properties. One could also consider the twisted version of the Dorfman bracket - given by

$$
[X+\alpha, Y+\beta]_{H}=[X+\alpha, Y+\beta]_{\text {Dorfman }}+i_{X} i_{Y} H
$$

where $H$ is a closed 3-form. The twisted Dorfman bracket again endows $T \oplus T^{*}$ with the structure of a Courant algebroid. Furthermore, an exact Courant algebroid $E$, that is, one that fits into a short exact sequence:

$$
0 \longrightarrow T^{*} \xrightarrow{a^{*}} E \xrightarrow{a} T \longrightarrow 0
$$

is isomorphic to the Courant algebroid given by $T \oplus T^{*}$ with the twisted Dorfman bracket for some $H$, which is unique up to the addition of an exact 3 -form [21, 20]. Using the definitions above, we can now give a more general definition of a Dirac structure:

Definition 3.12. Let $E$ be a Courant algebroid with split signature pairing. A Dirac structure is a maximally isotropic involutive subbundle $L \subseteq E$.

Dirac structures themselves carry the structure of a Lie algebroid.
Definition 3.13. A Lie algebroid over a manifold $X$ is given by a vector bundle L, a bundle map a:L $\rightarrow T X$ called the anchor, and a Lie bracket $[\cdot, \cdot]$ on the sections of $L$, such that:

$$
\left[l_{1}, f l_{2}\right]=f\left[l_{1}, l_{2}\right]+\left(a\left(l_{1}\right)(f)\right) l_{2}
$$

for any $l_{1}, l_{2} \in \Gamma(L), f \in C^{\infty}(X)$.
Example 3.14. The tangent bundle $T X$ with the usual Lie bracket and $a=I d$ is a Lie algebroid.
Example 3.15. A Lie algebroid over $X=\{p t\}$ is a Lie algebra.
Example 3.16. Since the Dorfman bracket on $T \oplus T^{*}$ does not satisfy the Jacobi identity whenever $\operatorname{dim} X \geq$ $2, T \oplus T^{*}$ is, generically, not a Lie algebroid.

Remark 3.1. The fact that $[\cdot, \cdot]$ is a Lie bracket satisfying the compatibility condition with the anchor also guarantees that $a\left(\left[l_{1}, l_{2}\right]\right)=\left[a\left(l_{1}\right), a\left(l_{2}\right)\right]$, as we will show in corollary 5.9.

As a final remark, we define the generalized diffeomorphism group of a manifold $X$.
Definition 3.17. The generalized diffeomorphism group GDiff $(X)$ is the group of orthogonal bundle automorphisms of $\left(T \oplus T^{*},\langle\cdot, \cdot\rangle\right)$ preserving the Dorfman bracket.

As mentioned before, the automorphism group of $\left(V \oplus V^{*},\langle\cdot, \cdot\rangle\right)$ is generated by lifted linear transformations, $B$-transformations and $\beta$-transformations. The addition of the condition of preserving the Dorfman bracket further restricts these transformations:

Theorem 3.18. [7] GDiff( $X$ ) is generated by two kinds of transformations:

- Lifted diffeomorphisms, given by the bundle map

$$
(x, v, \alpha) \mapsto\left(f(x), f_{*} v,\left(f^{-1}\right)^{*} \alpha\right)
$$

for $f \in \operatorname{Diff}(X)$ (where $x \in X$ and $\left.(v, \alpha) \in T \oplus T^{*}\right|_{x}$ ).

- B-field transformations, given by the bundle map $(x, v, \alpha) \mapsto\left(x, v, \alpha+i_{v} B\right)$, for $B$ closed.


### 3.2 Integrability in terms of sections of $\Gamma(O(T))$

Our previous representation in section 2.3 of maximally isotropic subspaces as orthogonal operators can also be moved to the global setting. Choosing a Riemmanian metric $g$ on $X$, we have a correspondence between maximally isotropic subbundles of $T \oplus T^{*}$ and smooth sections of the group bundle $O(X)$.

The technique used in proving this correspondence gives the following result on the global structure of a maximally isotropic subbundle:

Theorem 3.19. Let $L$ be a maximally isotropic subbundle. Then, as a vector bundle, $L \cong T$.
Proof. Consider $g$ as a map $g: T \rightarrow T^{*}$, and consider the map $P_{g}: T \oplus T^{*} \rightarrow T$ given by $X+\alpha \mapsto X+g^{-1} \alpha$. We show $\left.P_{g}\right|_{L}$ is injective on each fiber: if $P_{g}(X+\alpha)=0$, we have:

$$
g\left(P_{g}(X+\alpha), P_{g}(X+\alpha)\right)=0=g(X, X)+g^{*}(\alpha, \alpha)+2 \alpha(X)=g(X, X)+g^{*}(\alpha, \alpha)
$$

since $L$ is isotropic. Therefore, $X=0, \alpha=0$ and by dimension considerations $\left.P_{g}\right|_{L}$ is an isomorphism on each fiber.

Just as in section 2.3 , the pointwise linear invariants of maximally isotropic subbundles can be encoded as invariants of their corresponding orthogonal transformations. The type of a maximally isotropic subbundle at a point is again defined as

$$
\operatorname{type}\left(\left.L\right|_{p}\right):=\operatorname{codim}\left(\left.p_{T}(L)\right|_{p}\right) .
$$

Proposition 3.20. The type is an upper-semicontinuous function.
Proof. Let $p$ be a point in $X$. Let $\left\{l_{i}\right\}$ be a local basis of sections of $L$. Then, $\operatorname{codim}\left(\operatorname{Span}\left(\left\{a\left(l_{i}(p)\right)\right\}\right)\right)=$ type $\left(\left.L\right|_{p}\right)$. Assume, without loss of generality, that $a\left(l_{1}\right), \ldots, a\left(l_{k}\right)$ are linearly independent at $p$, where $k=\operatorname{dim}\left(\operatorname{Span}\left(\left\{a\left(l_{i}(p)\right)\right\}\right)\right)$. Then, by continuity, $a\left(l_{1}\right), \ldots, a\left(l_{k}\right)$ are linearly independent at a neighbourhood $U$ of $p$, and so $\operatorname{dim}\left(\operatorname{Span}\left(\left\{a\left(l_{i}\left(p^{\prime}\right)\right)\right\}\right)\right) \geq \operatorname{dim}\left(\operatorname{Span}\left(\left\{a\left(l_{i}(p)\right)\right\}\right)\right)$ for any $p^{\prime} \in U$, which gives type $\left(\left.L\right|_{p^{\prime}}\right) \leq$ type $\left(\left.L\right|_{p}\right)$.

The topology of $O(X, g)$ gives us the following result:
Theorem 3.21. The parity of $L$ at a point $p$, given by type $\left(\left.L\right|_{p}\right) \bmod 2$ is constant (independent of $p$ ).
Proof. By proposition 2.17, the type of $L$ at $p$ is equal to the number of -1 eigenvalues of $O_{L}$ at $p$. So, $\operatorname{type}\left(\left.L\right|_{p}\right) \bmod 2$ is proportional to $\operatorname{det}\left(O_{\left.L\right|_{p}}\right)$, which is constant since $O_{L}$ is a continuous function of $p$.

In addition, the integrability of a maximally isotropic subbundle can be restated in terms of its corresponding orthogonal operator. Let $\nabla$ be the Levi-Civita connection for $g$. We need the following lemma:

Lemma 3.22. Let $X, Y, Z \in \Gamma(T), \alpha \in \Gamma\left(T^{*}\right)$ defined by $\alpha(Y)=g(Z, Y)$. Then:

$$
d \alpha(X, Y)=g\left(Y, \nabla_{X} Z\right)-g\left(X, \nabla_{Y} Z\right) .
$$

Proof. Applying Koszul's formula, we have:

$$
\begin{aligned}
d \alpha(X, Y) & =X \alpha(Y)-Y \alpha(X)-\alpha([X, Y]) \\
& =X g(Z, Y)-Y g(Z, X)-g([X, Y], Z) \\
& =g\left(\nabla_{X} Z, Y\right)+g\left(Z, \nabla_{X} Y\right)-g\left(\nabla_{Y} Z, X\right)-g\left(Z, \nabla_{Y} X\right)-g(Z,[X, Y]) .
\end{aligned}
$$

Since $\nabla$ is torsion free, $[X, Y]=\nabla_{X} Y-\nabla_{Y} X[16]$, so:

$$
d \alpha(X, Y)=g\left(\nabla_{X} Z, Y\right)-g\left(\nabla_{Y} Z, X\right)
$$

As needed.
Using Cartan's magic formula $L_{X}=d i_{X}+i_{X} d$ we find:
Corollary 3.23. Using the above notation:

$$
L_{X} \alpha=g(\nabla \cdot X, Z)+g\left(\nabla_{X} Z, \cdot\right) .
$$

Finally, this allows us to express the Nijenhuis tensor in these terms.
Proposition 3.24. The Nijenhuis tensor of $L_{O}$ can be expressed in terms of $O$ as:

$$
\left(\left.2 P_{g}\right|_{L_{O}}{ }^{-1}\right)^{*} N_{L}(X, Y, Z)=g\left(Z,\left(I+O^{-1}\right)\left(\nabla_{Y} O\right) X\right)+c . p .
$$

Proof. Our calculations give the following expression for the Dorfman bracket:

$$
\left[x_{1}+i_{x_{2}} g, x_{3}+i_{x_{4}} g\right]=\nabla_{x_{1}} x_{3}-\nabla_{x_{3}} x_{1}+g\left(\nabla x_{1}, x_{4}\right)+g\left(\nabla_{x_{1}} x_{4}, \cdot\right)-g\left(\nabla_{x_{3}} x_{2}, \cdot\right)+g\left(x_{3}, \nabla x_{2}\right)
$$

Plugging in $x_{1}+i_{x_{2}} g=x=(I+O) X+g((I-O) X, \cdot), x_{3}+i_{x_{4}} g=y=(I+O) Y+g((I-O) Y, \cdot)$, we have:

$$
\begin{aligned}
{[(I+O) X+} & g((I-O) X, \cdot),(I+O) Y+g((I-O) Y, \cdot)] \\
= & \nabla_{(I+O) X}((I+O) Y)-\nabla_{(I+O) Y}((I+O) X)+g(\nabla((I+O) X),(I-O) Y) \\
& +g\left(\nabla_{(I+O) X}((I-O) Y), \cdot\right)-g\left(\nabla_{(I+O) Y}((I-O) X, \cdot)+g((I+O) Y, \nabla((I-O) X)) .\right.
\end{aligned}
$$

Pairing this expression with $z=(I+O) Z+g((I-O) Z, \cdot)$ to get the Nijenhuis tensor of $L_{O}$, we have:

$$
\begin{aligned}
2 N_{L}(x, y, z)= & g\left((I-O) Z, \nabla_{(I+O) X}((I+O) Y)-\nabla_{(I+O) Y}((I+O) X)\right) \\
& +g\left(\nabla_{(I+O) Z}((I+O) X),(I-O) Y\right)+g\left(\nabla_{(I+O) X}((I-O) Y),(I+O) Z\right) \\
& -g\left(\nabla_{(I+O) Y}((I-O) X,(I+O) Z)+g\left((I+O) Y, \nabla_{(I+O) Z}((I-O) X)\right) .\right.
\end{aligned}
$$

Focus on all terms containing $\nabla_{X}$. Using $\nabla_{X}(A(Y))=\left(\nabla_{X} A\right)(Y)+A\left(\nabla_{X}(Y)\right)$ for any $A \in \Gamma(E n d(T))$, we have that they sum to:

$$
\begin{aligned}
g\left((I-O) Z, \nabla_{(I+O) X}((I+O) Y)\right) & +g\left(\nabla_{(I+O) X}((I-O) Y),(I+O) Z\right) \\
= & g\left((I-O) Z,(I+O) \nabla_{(I+O) X}(Y)\right)+g\left((I-O) \nabla_{(I+O) X}(Y),(I+O) Z\right) \\
& +g\left((I-O) Z,\left(\nabla_{(I+O) X} O\right) Y\right)-g\left(\left(\nabla_{(I+O) X} O\right) Y,(I+O) Z\right) .
\end{aligned}
$$

Since $O$ is orthogonal, the above expression equals:

$$
-2 g\left(Z, O^{-1}\left(\nabla_{(I+O) X} O\right) Y\right)
$$

Similarly, the rest of the terms sum to:

$$
2 g\left(Z, O^{-1}\left(\nabla_{(I+O) Y} O\right) X\right)-2 g\left(Y,\left(O^{-1}\right)\left(\nabla_{(I+O) Z} O\right) X\right)
$$

Note, that the map $P_{g}$ defined earlier gives $P_{g}((I+O) X+g((I-O) X, \cdot))=2 X$. Overall we find:

$$
\left(\left.2 P_{g}\right|_{L_{O}}{ }^{-1}\right)^{*} N_{L}(X, Y, Z)=g\left(Z,\left(I+O^{-1}\right)\left(\nabla_{Y} O\right) X\right)+c . p .
$$

Theorem 3.25. The space of Dirac structures is isomorphic to the space of sections of $O(T)$ satisfying:

$$
g\left(Z,\left(I+O^{-1}\right)\left(\nabla_{Y} O\right) X\right)+\text { c.p. }=0 \quad \forall X, Y, Z \in \Gamma(T)
$$

Remark 3.2. After having proved this theorem, we were told that a similar approach to ours of calculating the Nijenhuis tensor in terms of $O$ is taken in [24], where an equivalent result is achieved.

Corollary 3.26. Let $O$ be a section of orthogonal operators such that $\nabla O=0$. Then $L_{O}$ is integrable.
Example 3.27. Let $(X, g, \omega, J)$ be a Kahler manifold. First, note $J=g^{-1} \circ \omega$. In addition, $\nabla J=0[2]$. Following the notation in section 2.3, $O_{\operatorname{graph}(\omega)}$ is the Cayley transform of $J=A$. Since $\nabla J=0$, one easily shows $\nabla O_{\operatorname{graph}(\omega)}=0$ by the Leibniz rule for $\nabla$. The integrable complement given by $-O_{\operatorname{graph}(\omega)}$ in this case is simply $\operatorname{graph}(-\omega)$.

## Chapter 4

## Deformation theory

First, we briefly recall the deformation theories of presymplectic, Poisson, and complex structures, drawing the similarities between them. Then, we will give a unified description of the first two in the framework of Dirac structures.

### 4.1 Deformations of symplectic, Poisson, and complex structures

As we will see, deformations of symplectic, Poisson, and complex structures share similarities. They are all controlled by a Maurer-Cartan type equation, or more precisely, they are all governed by a differential graded Lie algebra (DGLA), defined formally in definition 4.2. As we will show, deformations of a given Dirac structure will have a similar flavor, with a possible complication arising from the failure of a certain technical requirement.

Definition 4.1. A deformation of a (pre)symplectic structure $\omega_{0}$ is a 1-parameter family $\omega_{t}$ of (pre)symplectic structures, that is smooth as a map $\omega_{t}: T X \times \mathbb{R} \rightarrow T^{*} X$.

Of course, since $\omega_{t}$ are all (pre)symplectic structures, the only integrability condition they must satisfy is $d \omega_{t}=0$ for all $t$. If we write $\omega_{t}=\omega_{0}+\sigma_{t}$, this is simply captured by:

$$
d \sigma_{t}=0
$$

A similar definition can be given for Poisson structures. Here, the description of integrable deformations is a bit more complicated: a Poisson structure must satisfy $[\pi, \pi]=0$. Again, considering a deformation $\pi_{t}=\pi_{0}+\eta_{t}$, we see $\eta_{t}$ must satisfy $2\left[\pi_{0}, \eta_{t}\right]+\left[\eta_{t}, \eta_{t}\right]=0$. Defining $d_{\pi} \lambda=[\pi, \lambda]$ for any $\lambda \in \Gamma(\bigwedge T)$, we find $\eta_{t}$ must satisfy the equation:

$$
d_{\pi_{0}} \eta_{t}+\frac{1}{2}\left[\eta_{t}, \eta_{t}\right]=0
$$

Note, in addition, that the Jacobi identity for the Schouten bracket gives $d_{\pi}[x, y]=\left[d_{\pi} x, y\right]-(-1)^{|x|}\left[x, d_{\pi} y\right]$ for $x \in \Gamma(\bigwedge T)$ homogeneous.

For complex structures, the constructions are more complicated, but are even more similar to those used when deforming Dirac structures, as we will see in section 4.2 . For a given complex structure $J_{0}$, let $T^{1,0}, T^{0,1} \subseteq T \otimes \mathbb{C}$ be the $+i$ and $-i$ eigenbundles, respectively. A small enough deformation $J_{t}$ can be shown to be equivalent to the graph of a bundle map $\phi_{t}: T^{0,1} \rightarrow T^{1,0}$, that is, a section of $\bigwedge^{0,1} \otimes T^{1,0}$. It can be shown $[10,12]$ that $J_{t}$ is a complex structure if and only if $\phi_{t}$ satisfies the Maurer-Cartan equation:

$$
\bar{\partial} \phi_{t}+\left[\phi_{t}, \phi_{t}\right]=0
$$

### 4.2 Deformations of Dirac structures

Note the common denominator in all three of the deformation problems presented in section 4.1:

1. First, a graded vector bundle is chosen, and small deformations are encoded by sections of this vector bundle of a given rank. For presymplectic structures the vector bundle is $\Lambda T^{*}$, for Poisson structures it is $\bigwedge T$, and for complex structures it is $\bigoplus_{p} \bigwedge^{0, p} \otimes T^{1,0}$.
2. Second, both a differential and a bracket are defined on sections of this bundle (for the presymplectic case, the bracket is zero). The differential chosen in all three cases is also a graded derivation of the bracket. This gives the space of sections of the aforementioned bundle the structure of a DGLA.
3. The integrable deformations are now those satisftying the Maurer-Cartan equation in this DGLA.

This phenomena is actually generic in geometry, and could actually be captured by the following vague statement, originating in Quillen's work [9]:

Many deformation problems over characteristic 0 can be captured by Maurer-Cartan elements in some DGLA.

We now formally present the definition of a DGLA:
Definition 4.2. A differential graded Lie algebra ( $D G L A$ ) is given by a graded vector space $V$, a linear map $d: V \rightarrow V$ satisfying $d^{2}=0$, and a graded Lie bracket on $V$, that is, a bilinear map $[\cdot, \cdot]: V \otimes V \rightarrow V$ satisfying graded-antisymmetry

$$
[x, y]+(-1)^{|x||y|}[y, x]=0
$$

and the graded Jacobi identity

$$
(-1)^{|x||z|}[x,[y, z]]+(-1)^{|y||z|}\left[y,[z, x]+(-1)^{|y||z|}[z,[x, y]]=0\right.
$$

such that $d$ and $[\cdot, \cdot]$ satisfy the graded Leibniz rule:

$$
d[x, y]=[d x, y]+(-1)^{|x|}[x, d y]
$$

When a Dirac structure admits a complementary Dirac structure, the exterior algebra of the latter can be given the structure of a DGLA. This DGLA, as we will show, is exactly the DGLA controlling the deformations of the Dirac structure.

Let $M$ be a Dirac structure over a manifold $X$. Assume there exists a Dirac structure $L$ complementary to $M$, that is, $M \oplus L=T \oplus T^{*}$. In this case, we call $L$ a 'Dirac complement' to $M$.

Definition 4.3. A small deformation of $M$ is a maximally isotropic subbundle $M^{\prime}$ that is complementary to $L$.

The term 'small' here is captured by the fact that, on each fiber, the set of subspaces complementary to $L$ is open, as shown in proposition 2.21.

We now show how to describe these small deformations as sections of a certain vector bundle. Since $\langle\cdot, \cdot\rangle$ is nondegenerate, the fibers of $L$ can be identified through $\langle\cdot, \cdot\rangle$ with the duals of the fibers of $M$. Therefore, maps $M \rightarrow L$ can be considered as elements of $\otimes^{2} M^{*}$, and vice-versa, and we have the notion of 'skew' maps $M \rightarrow L$, those identified with elements of $\bigwedge^{2} M^{*}$. We have the following:

Proposition 4.4. A maximally isotropic subbundle $M^{\prime}$ is complementary to $L$ if and only if $M^{\prime}$ is the graph of a skew map $\omega: M \rightarrow L$.

Proof. If $M^{\prime}$ is complementary to $L, p_{M}\left(M^{\prime} \oplus L\right)=p_{M}\left(M^{\prime}\right)=M$, so $\left.p_{M}\right|_{M^{\prime}}$ gives an isomorphism, and we can define $\omega=\left.p_{L} \circ p_{M}^{-1}\right|_{M^{\prime}}: M \rightarrow L$, and by definition $M^{\prime}=\{m+\omega(m) \mid m \in M\}$. Since $M^{\prime}$ is maximally isotropic, it is easy to check $\omega$ is skew. The other direction is obvious.

Using the notation from before, we write $M^{\prime}=e^{\omega} M$. We can now ask when $M^{\prime}$ is integrable in terms of $\omega$. Recall that proposition 3.10 showed integrability is equivalent to the vanishing of the Nijenhuis tensor $N(x, y, z)=\langle[x, y], z\rangle$. We can consider the pullback of this tensor to $M$ along the isomorphism $e^{\omega}$. We have the following:

$$
\begin{align*}
\tilde{N}=e^{\omega *} N_{M^{\prime}}(x, y, z)= & \langle[x, y], z\rangle \\
& +\left\langle\left[i_{x} \omega, y\right], z\right\rangle+\left\langle\left[x, i_{y} \omega\right], z\right\rangle+\left\langle[x, y], i_{z} \omega\right\rangle  \tag{4.1}\\
& +\left\langle\left[i_{x} \omega, i_{y} \omega\right], z\right\rangle+\left\langle\left[i_{x} \omega, y\right], i_{z} \omega\right\rangle+\left\langle\left[x, i_{y} \omega\right], i_{z} \omega\right\rangle \\
& +\left\langle\left[i_{x} \omega, i_{y} \omega\right], i_{z} \omega\right\rangle .
\end{align*}
$$

The equation splits into terms of order of homogeneity $0,1,2$, and 3 in $\omega$. We note that the degree 0 term is the Nijenhuis tensor of $M$, whereas the degree 3 term is the pullback of the Nijenhuis tensor of $L$ along $\omega: M \rightarrow L$. Since $M, L$ are both integrable, these terms vanish, and we are left with the order 1 and 2 terms in $\omega$. It is a known fact, also appearing in [8], that these terms can be identified with the following operations on $\Gamma(\bigwedge L)$.

Proposition 4.5. The order 1 term in $\omega$ can be written as:

$$
\left\langle\left[i_{x} \omega, y\right], z\right\rangle+\left\langle\left[x, i_{y} \omega\right], z\right\rangle+\left\langle[x, y], i_{z} \omega\right\rangle=d_{M} \omega(x, y, z)
$$

Proposition 4.6. The order 2 term in $\omega$ can be written as:

$$
\left\langle\left[i_{x} \omega, i_{y} \omega\right], z\right\rangle+\left\langle\left[i_{x} \omega, y\right], i_{z} \omega\right\rangle+\left\langle\left[x, i_{y} \omega\right], i_{z} \omega\right\rangle=\frac{1}{2}[\omega, \omega](x, y, z)
$$

where $[\cdot, \cdot]$ is the extension of the Dorfman bracket to the sections of the exterior algebra of $T \oplus T^{*}$, similar to how the Schouten bracket is defined in terms of the Lie bracket. As a consequence of $L$ being integrable, $[\omega, \omega] \in \Gamma\left(\bigwedge^{3} L\right)$.

The bracket and exterior derivative also satisfy the following important relation:
Lemma 4.7. The exterior derivative is a graded derivation of the bracket. That is, for any homogenous $\alpha, \beta \in \Gamma(\bigwedge L)$,

$$
d_{M}[\alpha, \beta]=\left[d_{M} \alpha, \beta\right]+(-1)^{|\alpha|}\left[\alpha, d_{M} \beta\right]
$$

where $|\alpha|$ is the degree of $\alpha$ as an element of $\bigwedge L$.
Proposition 4.8. The two operations $d_{M},[\cdot, \cdot]$, along with the relation between them, endow $\Gamma(\bigwedge L)$ with the structure of a differential graded Lie algebra (DGLA).

Observing equation 4.1 again, and using the above lemmas, we have:

Theorem 4.9. [7] The small deformation $M^{\prime}=e^{\omega} M$ is integrable $\leftrightarrow \omega$ is a Maurer-Cartan element in the $\operatorname{DGLA}\left(\Gamma(\bigwedge L), d_{M},[\cdot, \cdot]\right)$, that is,

$$
d_{M} \omega+\frac{1}{2}[\omega, \omega]=0 .
$$

The deformation problems for presymplectic and Poisson structures, as described in section 4.1, can be now described in terms of deformation problems of a Dirac structure. The Dirac structures for both presymplectic and Poisson (and, in fact, complex structures as well) admit a natural choice for a Dirac complement.

Example 4.10. Consider $M=\operatorname{graph}(\sigma)$ and $L=T^{*}$, for $\sigma$ a closed 2-form. Since the bracket on $M$ is given by $\left[x+i_{x} \sigma, y+i_{y} \sigma\right]=[x, y]_{L i e}+i_{[x, y]} \sigma$, the Lie algebroid exterior derivative on $\Lambda T^{*}$ coincides with the usual one, and so we have the DGLA $\left(\Gamma\left(\bigwedge T^{*}\right), d, 0\right)$. The integrable deformations $e^{\omega} \operatorname{graph}(\sigma)=\operatorname{graph}(\sigma+\omega)$ are now those with $d \omega=0$, as expected.

Example 4.11. Let $M=\operatorname{graph}(\pi)$ and $L=T$, for $\pi$ a Poisson structure. The Lie algebroid exterior derivative on sections of $L$ can be shown to be $d_{M} \alpha=[\pi, \alpha]$, so the deformation equation reads:

$$
[\pi, \alpha]+\frac{1}{2}[\alpha, \alpha]=0,
$$

which is the usual deformation equation for Poisson structures [18].

## Chapter 5

## The complement problem

In the derivation of the deformation equation in theorem 4.9, we made two crucial assumptions: integrability of $M$, and integrability of $L$. These assumptions guaranteed the terms of order 0 and 3 in $\omega$ vanished. However, the assumption of the existence of an integrable $L$ complementing $M$ is nontrivial, and in fact impossible in some cases.

Proposition 5.1. Let $H$ be a 3-form which is not exact, that is, the de Rham cohomology class $[H]$ is nonzero. Then the Dirac structure $T^{*}$, under the twisted Courant bracket $[\cdot, \cdot]_{H}$, has no Dirac complement.

Proof. Any complementary maximally isotropic subbundle of $T^{*}$ is of the form $\operatorname{graph}(\omega)$ for some 2-form $\omega$. A calculation shows the Nijenhuis tensor of $\operatorname{graph}(\omega)$ is given by $N\left(x+i_{x} \omega, y+i_{y} \omega, z+i_{z} \omega\right)=(d \omega+H)(x, y, z)$. Since $H$ is not exact, $N$ will not vanish for any $\omega$. Therefore, $T^{*}$ has no integrable complement.

A natural question therefore arises:
In an exact Courant algebroid, which Dirac structures admit Dirac complements?
This question, currently, remains unanswered in the literature. It is in our interest to examine the effect of choosing a nonintegrable complement, making the first steps towards defining obstructing the existence of an integrable complement. In order to do so, we will first need to introduce some basic concepts and constructions related to a pair of complementary maximally isotropic subbundles.

### 5.1 Quasi-Lie algebroids and deformations revisited

One could also ask what geometric structure a nonintegrable maximally isotropic subbundle carries. Choosing a maximally isotropic complement endows it with a structure similar to that of an almost Lie algebroid, which we will soon introduce. Based on the definitions in [19], we introduce the concept of a quasi-Lie algebroid.

Definition 5.2. A quasi-Lie algebroid over a manifold $X$ is given by a vector bundle $M$ over $X$, along with a bundle map $a: M \rightarrow T X$ called the anchor and an antisymmetric bracket $[\cdot, \cdot]^{\prime}$ on the sections of $M$, satisfying the following compatibility condition: for any $m_{1}, m_{2} \in \Gamma(M)$ and $f \in C^{\infty}(X)$,

$$
\left[m_{1}, f m_{2}\right]=\left(a\left(m_{1}\right) f\right) m_{2}+f\left[m_{1}, m_{2}\right]
$$

Remark 5.1. A slightly stronger notion is that of an almost Lie algebroid, differing by the addition of the assumption $a\left([X, Y]^{\prime}\right)=[a(X), a(Y)]$, that is, the anchor is a morphism of Lie algebras from $\Gamma(M)$ to $\Gamma(T)$.

Remark 5.2. Note that the bracket $[\cdot, \cdot]^{\prime}$ does not necessarily satisfy the Jacobi identity. In the case it does, the notions of quasi-Lie algebroids, almost-Lie algebroids, and Lie algebroids, coincide.

For any maximally isotropic subbundle $M$, choosing a maximally isotropic complement $L$ endows it with a natural quasi-Lie algebroid structure.

Proposition 5.3. Let $M, L$ be complementary maximally isotropic subbundles of $T \oplus T^{*}$, and let $p_{M}$ be the projection to $M$ with respect to the splitting $M \oplus L$. Then $a=p_{T}$ and $[\cdot, \cdot]^{\prime}=p_{M}([\cdot, \cdot])$ (where $[\cdot, \cdot]$ is the Dorfman bracket) give a quasi-Lie algebroid structure on $M$.

Proof. Since the Dorfman bracket $\left[x_{1}, x_{2}\right]$ is antisymmetric whenever $\left\langle x_{1}, x_{2}\right\rangle=0$, so is $[\cdot, \cdot]^{\prime}$. In addition,

$$
\begin{aligned}
p_{M}\left(\left[m_{1}, f m_{2}\right]\right) & \left.=p_{M}\left(a\left(m_{1}\right) f\right) m_{2}+f\left[m_{1}, m_{2}\right]\right) \\
& =\left(a\left(m_{1}\right) f\right) p_{M}\left(m_{2}\right)+f p_{M}\left(\left[m_{1}, m_{2}\right]\right) \\
& =\left(a\left(m_{1}\right) f\right) m_{2}+f\left[m_{1}, m_{2}\right]^{\prime} .
\end{aligned}
$$

Remark 5.3. We make the following observations:

- Of course, the splitting also endows $L$ with a quasi-Lie algebroid structure.
- In the case where at least one of $M$ or $L$ is a Dirac structure, $(M, L)$ forms a quasi-Lie bialgebroid, in the sense of [19].
- We wrote $[,,]^{\prime}$ to differentiate it from the usual Dorfman bracket $[\cdot, \cdot]$. One can easily verify the difference $[\cdot, \cdot]-[\cdot, \cdot]^{\prime}: \Gamma(M) \otimes \Gamma(M) \rightarrow \Gamma(M)$ is tensorial.
- As mentioned before, the Dorfman bracket satisfies $\left[a\left(m_{1}\right), a\left(m_{2}\right)\right]=a\left(\left[m_{1}, m_{2}\right]\right)$, which guaranteed that a maximally isotropic integrable subbundle of $T \oplus T^{*}$ carries the structure of a Lie algebroid. Since we now include a projection, this is no longer guaranteed to hold.

Example 5.4. For the graph of a 2-form $M=\operatorname{graph}(\omega)$, choosing $L=T^{*}$, it can be shown that $p_{M}([X+$ $\left.\left.i_{X} \omega, Y+i_{Y} \omega\right]\right)=[X, Y]+i_{[X, Y]} \omega$.

Example 5.5. Let $E$ be a regular distribution and choose a metric $g$. Consider $M=E \oplus A n n(E), L=$ $E^{\perp} \oplus A n n\left(E^{\perp}\right)$. We can use the metric to identify $M \cong E \oplus E^{\perp} \cong T$. Let $e_{1}+f_{1}, e_{2}+f_{2} \in \Gamma\left(E \oplus E^{\perp}\right)=\Gamma(T)$. Using corollary 3.23 we find the bracket:

$$
\begin{aligned}
p_{M}\left(\left[e_{1}+f_{1}, e_{2}+f_{2}\right]\right)= & p_{E}^{\perp}\left(\left[e_{1}, e_{2}\right]\right) \\
& +p_{E^{\perp}}^{\perp}\left(\nabla_{e_{1}} f_{2}\right)-p_{E^{\perp}}^{\perp}\left(\nabla_{e_{2}} f_{1}\right)-p_{E^{\perp}}^{\perp} \circ g^{-1}\left(g\left(e_{1}, \nabla f_{2}\right)-g\left(e_{2}, \nabla f_{1}\right)\right),
\end{aligned}
$$

where $g^{-1}: T^{*} \rightarrow T$ is the inverse of $g$.
The remarks above lead us to define two natural operators on sections of a quasi-Lie algebroid:
Definition 5.6. The Alm tensor is defined as

$$
\operatorname{Alm}\left(m_{1}, m_{2}\right)=\left[a\left(m_{1}\right), a\left(m_{2}\right)\right]-a\left(\left[m_{1}, m_{2}\right]^{\prime}\right) .
$$

Remark 5.4. One can easily verify $A l m$ is tensorial, that is, defines a bilinear operation $M \otimes M \rightarrow T$. The name $A l m$ is chosen since this tensor exactly measures the failure of $M$ to be an almost Lie algebroid.

Remark 5.5. In the notation of proposition 5.3, we have

$$
\operatorname{Alm}\left(m_{1}, m_{2}\right)=a\left(p_{L}\left(\left[m_{1}, m_{2}\right]\right)\right) .
$$

Definition 5.7. The Jacobiator of sections $m_{1}, m_{2}, m_{3} \in \Gamma(L)$ is defined as

$$
\operatorname{Jac}\left(m_{1}, m_{2}, m_{3}\right)=\left[\left[m_{1}, m_{2}\right]^{\prime}, m_{3}\right]^{\prime}+c . p .
$$

In contrast to $A l m$, the operator $J a c$ is generally non-tensorial. However, the obstruction for Jac to be tensorial is exactly Alm:

Proposition 5.8. For $f \in C^{\infty}(X)$,

$$
\operatorname{Jac}\left(m_{1}, m_{2}, f m_{3}\right)=f \operatorname{Jac}\left(m_{1}, m_{2}, m_{3}\right)-\left(\operatorname{Alm}\left(m_{1}, m_{2}\right) f\right) m_{3},
$$

that is, Alm measures the failure of Jac to be tensorial.
Proof. Note that

$$
\begin{aligned}
{\left[\left[m_{1}, f m_{3}\right]^{\prime}, m_{2}\right]^{\prime}=} & -\left(a\left(m_{1}\right) a\left(m_{2}\right) f\right) m_{3}+\left(a\left(m_{2}\right) f\right)\left[m_{3}, m_{1}\right]^{\prime} \\
& -\left(a\left(m_{1}\right) f\right)\left[m_{2}, m_{3}\right]^{\prime}+f\left[\left[m_{2}, m_{3}\right]^{\prime}, m_{1}\right]^{\prime},
\end{aligned}
$$

therefore a quick calculation shows

$$
\left[\left[m_{1}, m_{2}\right]^{\prime}, f m_{3}\right]^{\prime}+c \cdot p=\left(a\left(\left[m_{1}, m_{2}\right]^{\prime}\right) f\right) m_{3}-\left(\left[a\left(m_{1}\right), a\left(m_{2}\right)\right]^{\prime} f\right) m_{3}+f J a c\left(m_{1}, m_{2}, m_{3}\right) .
$$

Corollary 5.9. If $[\cdot, \cdot]^{\prime}$ satisfies the Jacobi identity, the anchor $a$ is a Lie algebra homomorphism.
Another basic object in the theory of quasi-Lie algebroids is the quasi-Lie algebroid exterior derivative.
Definition 5.10. Let $M$ be a quasi-Lie algebroid. The quasi-Lie algebroid exterior derivative $d_{M}$ is the linear map $\Gamma\left(\bigwedge^{k} M^{*}\right) \rightarrow \Gamma\left(\bigwedge^{k+1} M^{*}\right)$ defined via the Koszul formula: For $\alpha \in \Gamma\left(\bigwedge^{k} M^{*}\right), X_{0}, \ldots, X_{k} \in \Gamma(M)$,

$$
\begin{aligned}
d_{M} \alpha\left(X_{0}, \ldots, X_{k}\right)= & \sum_{i} a\left(X_{i}\right) \alpha\left(X_{0}, \ldots, \hat{X}_{i}, \ldots, X_{k}\right) \\
& +\sum_{i<j}(-1)^{i+j} \alpha\left(\left[X_{i}, X_{j}\right]^{\prime}, X_{0}, \ldots, \hat{X}_{i}, \ldots, \hat{X}_{j}, \ldots, X_{k}\right) .
\end{aligned}
$$

While the quasi-Lie algebroid exterior derivative is a graded derivation on $\Gamma\left(\bigwedge L^{*}\right)$, its failure to be a differential is encoded by both Alm and Jac:

Proposition 5.11. For $\alpha \in \Gamma\left(M^{*}\right)$,

$$
d_{M}^{2} \alpha\left(m_{1}, m_{2}, m_{3}\right)=\alpha\left(\operatorname{Jac}\left(m_{1}, m_{2}, m_{3}\right)\right)+\left(\operatorname{Alm}\left(m_{1}, m_{2}\right) \alpha\left(m_{3}\right)+c . p\right) .
$$

Proof. We calculate:

$$
\begin{aligned}
d_{M}^{2} \alpha\left(m_{1}, m_{2}, m_{3}\right)= & a\left(m_{1}\right)\left\{a\left(m_{2}\right) \alpha\left(m_{3}\right)-a\left(m_{3}\right) \alpha\left(m_{2}\right)-\alpha\left(\left[m_{2}, m_{3}\right]^{\prime}\right)\right\} \\
& -a\left(m_{2}\right)\left\{a\left(m_{1}\right) \alpha\left(m_{3}\right)-a\left(m_{3}\right) \alpha\left(m_{1}\right)-\alpha\left(\left[m_{1}, m_{3}\right]^{\prime}\right)\right\} \\
& +a\left(m_{3}\right)\left\{a\left(m_{1}\right) \alpha\left(m_{2}\right)-a\left(m_{2}\right) \alpha\left(m_{1}\right)-\alpha\left(\left[m_{1}, m_{2}\right]^{\prime}\right)\right\} \\
& -a\left(\left[m_{1}, m_{2}\right]^{\prime}\right) \alpha\left(m_{3}\right)-a\left(m_{3}\right) \alpha\left(\left[m_{1}, m_{2}\right]^{\prime}\right)-\alpha\left(\left[\left[m_{1}, m_{2}\right]^{\prime}, m_{3}\right]^{\prime}\right) \\
& +a\left(\left[m_{1}, m_{3}\right]^{\prime}\right) \alpha\left(m_{2}\right)-a\left(m_{2}\right) \alpha\left(\left[m_{1}, m_{3}\right]^{\prime}\right)-\alpha\left(\left[\left[m_{1}, m_{3}\right]^{\prime}, m_{2}\right]^{\prime}\right) \\
& -a\left(\left[m_{2}, m_{3}\right]^{\prime}\right) \alpha\left(m_{1}\right)-a\left(m_{1}\right) \alpha\left(\left[m_{2}, m_{3}\right]^{\prime}\right)-\alpha\left(\left[\left[m_{2}, m_{3}\right]^{\prime}, m_{1}\right]^{\prime}\right) .
\end{aligned}
$$

All terms of the form $a\left(m_{i}\right) \alpha\left(\left[m_{j}, m_{k}\right]^{\prime}\right)$ cancel out, The terms of the form $a\left(\left[\left[m_{i}, m_{j}\right]^{\prime}, m_{k}\right]^{\prime}\right)$ add up to give $\alpha \circ J a c$, and the terms left sum up to give $\left(\operatorname{Alm}\left(m_{1}, m_{2}\right) \alpha\left(m_{3}\right)+c . p\right)$ as needed.

In the special case of a pair of complementary maximally isotropic subbundles $M, L$ of $T \oplus T^{*}$, the quasi-Lie algebroid exterior derivative has the property of allowing us to calculate the bracket of a section of $M$ with a section of $L$. We have:

Proposition 5.12. Let $X \in \Gamma(M), \omega \in \Gamma(\bigwedge L)$. Then:

$$
p_{\wedge L}([X, \omega])=L_{X} \omega
$$

where $L_{X}=d_{M} i_{X}+i_{X} d_{M}$.
Proof. It is not hard to show that $L_{X}$ is again a graded derivation, so it is enough to prove this for $\alpha \in \Gamma(L)$. Indeed:

$$
p_{L}([X, \alpha])(Y)=\langle[X, \alpha], Y\rangle=a(X) \alpha(Y)-\langle\alpha,[X, Y]\rangle=a(X) \alpha(Y)-\alpha([X, Y]),
$$

and

$$
L_{X} \alpha(Y)=d_{M} \alpha(X, Y)+a(Y) \alpha(X)=a(X) \alpha(Y)-\alpha([X, Y]),
$$

just as we needed.
In section 4.2, a pair of transverse Dirac structures endowed the exterior algebra of either of them with DGLA structure. Now, dropping the assumption of integrability, a pair of maximally isotropic subbundles, along with their quasi-Lie algebroid structures, will endow the exterior algebra of either of them with a curved $L_{3}$-algebra structure.

To define curved $L_{3}$ algebras formally, we first need to recall a few concepts:
Definition 5.13. An (i,j) shuffle is a permutation $\sigma \in S_{i+j}$ satisfying $\sigma(1)<\ldots<\sigma(i)$ and $\sigma(i+1)<\ldots<$ $\sigma(i+j)$. The set of all $(i, j)$ shuffles is denoted $\operatorname{Sh}(i, j)$.

Definition 5.14. The Koszul sign $\chi\left(\sigma, v_{1}, \ldots, v_{n}\right)$ of a permutation $\sigma \in S_{n}$ and a collection $v_{1}, \ldots, v_{n}$ of homogeneous elements of a graded vector space $V$ is the product of the factors $(-1)^{\left|v_{i}\right|\left|v_{j}\right|}$ for any interchange of neighbours $i, j$ when decomposing $\sigma$ to a product of interchanges, multiplied by the sign of the permutation $\sigma$.

Example 5.15. If all of the $v_{i}$ 's are of even degree, the Koszul $\operatorname{sign} \chi\left(\sigma, v_{1}, \ldots, v_{n}\right)$ is simply $\operatorname{sgn}(\sigma)$. If all of the $v_{i}$ 's are of odd degree, the Koszul sign $\chi\left(\sigma, v_{1}, \ldots, v_{n}\right)$ is 1 .

Put in simple terms, a curved $L_{3}$ algebra is given by a graded vector space, along with four gradedantisymmetric brackets $l_{k}: \otimes V \rightarrow V$ for $k=0,1,2,3$, satisfying certain relations between them. One can think of these relations as a sequence of obstructions for the $L_{3}$ algebra to reduce to a simpler structuresuch as a differential complex, a differential graded Lie algebra, and so on.

Definition 5.16. A curved $L_{3}$ algebra is a $\mathbb{Z}$-graded vector space $V=\bigoplus_{i} V_{i}$ along with multilinear gradedantisymmetric maps $l_{k}: \otimes^{k} V \rightarrow V$ for $k=0,1,2,3$, satisfying the following relations between them for $n=0,1,2,3$ :

$$
\begin{equation*}
\sum_{k=0}^{n} \sum_{\sigma \in S h(k, n-k)} l_{n-k+1}\left(l_{k}\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right), v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}\right)=0 \tag{5.1}
\end{equation*}
$$

Remark 5.6. By 'graded antisymmetric', we mean

$$
l_{k}\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)=\chi\left(\sigma, v_{1}, \ldots, v_{k}\right) l_{k}\left(v_{1}, \ldots, v_{k}\right)
$$

Remark 5.7. A curved $L_{3}$ algebra is a special case of a curved $L_{\infty}$ algebra, where all of the higher brackets $l_{k}$ vanish for $k>3$.

Properties. - For $n=0$, we have $l_{1}\left(l_{0}\right)=0$, showing the curvature is always 'closed' under $l_{1}$.

- For $n=1$, we have $l_{1}^{2}(\cdot)=l_{2}\left(l_{0}, \cdot\right)$, showing the failure of $l_{1}$ to be a differential is exactly measured by $l_{2}$ and $l_{0}$.
- For $n=2$, we have $l_{3}\left(l_{0}, v_{1}, v_{2}\right)+l_{2}\left(l_{1}\left(v_{1}\right), v_{2}\right)+l_{2}\left(l_{1}\left(v_{2}\right), v_{1}\right)+l_{1}\left(l_{2}\left(v_{1}, v_{2}\right)\right)=0$, showing the failure of $l_{1}$ to be a graded derivation of the bracket $l_{2}$ is measured by $l_{0}, l_{3}$.
- The explicit $n=3$ is cumbersome, but it shows the failure of the graded Jacobiator of $l_{2}$ to vanish is given by an expression involving $l_{0}, l_{1}$ and $l_{3}$.
- When $l_{0}$ and $l_{3}$ both vanish, we have a DGLA.

We are now ready to construct the curved $L_{3}$ algebra for a pair of complementary maximally isotropic subbundles $M$, $L$ of $T \oplus T^{*}$. Recall first our notation: the splitting $T \oplus T^{*}=M \oplus L$ gives projections $p_{M}, p_{L}$, which can be naturally extended to projections $p_{\wedge M}, p_{\wedge L}$ from $\wedge T \oplus T^{*}$ to $\bigwedge M, \bigwedge L$ respectively. In addition, since $\langle\cdot, \cdot\rangle$ is nondegenerate, we can use it to identify $L \cong M^{*}$, so by $\alpha(X)$ for $\alpha \in L, X \in M$ we mean $2\langle\alpha, X\rangle$, and again we naturally extend this pairing to $\wedge M, \wedge L$.

Theorem 5.17. ([8, 11]) Using our previous notation, given a pair of complementary maximally isotropic subbundles $M, L$ of $T \oplus T^{*}$, the following objects form a curved $L_{3}$ algebra:

1. $V=\Gamma(\bigwedge L)[1]$ with grading given by the natural grading on $\bigwedge L$ shifted by +1 . For example, sections of $L$ are considered to be of degree 2 . Using this conventions, $l_{k}$ is a degree $3-2 k$ map, meaning the degree of $l_{k}\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ is $\left|\alpha_{1}\right|+\ldots+\left|\alpha_{k}\right|+3-2 k$.
2. $l_{0}=N_{M}$ which is naturally an element of $\bigwedge^{3} L$.
3. $l_{1}=d_{M}$ is the quasi-Lie algebroid exterior derivative.
4. $l_{2}$ given by the Schouten bracket given by extending $p_{L}[\alpha, \beta]$ from $\Gamma(L)$ to $\Gamma(\bigwedge L)$.
5. $l_{3}$ given as follows: consider $N_{L}$ as an element of the Clifford algebra $\operatorname{Cliff}\left(T \oplus T^{*},\langle\cdot, \cdot\rangle\right)$. Define $l_{3}(\alpha, \beta, \gamma)$ as the graded-antisymmetrization of $\left\{\left\{\left\{N_{L}, \alpha\right\}, \beta\right\}, \gamma\right\}$, where $\{\cdot, \cdot\}$ is the graded bracket. (Note that, when taking the graded commutator here, we use the original grading on $\Gamma(\bigwedge L)$ as in definition 2.9).

Remark 5.8. (a) The shift in degree is to guarantee the curved $L_{3}$ algebra defined above satisfies the same graded-antisymmetry conditions that appear in the literature. Compare for example, the requirement $l_{2}(x, y)=-(-1)^{|x||y|} l_{2}(y, x)$, with the fact that for the Schouten bracket, $[x, y]=-(-1)^{(|x|-1)(|y|-1)}[y, x]$.
(b) At first glance it might not be obvious why $l_{3}$ returns an element of $\Gamma(\bigwedge L)$. If any of $\alpha, \beta, \gamma$ is a degree 0 element, the expression vanishes. If $\alpha, \beta, \gamma$ are all degree 1 elements, $l_{3}(\alpha, \beta, \gamma)=3!N_{L}(\alpha, \beta, \gamma)$. By the Leibniz rule the bracket of higher degree elements remains an element of $\bigwedge L$.
(c) In the literature, sometimes instead of the Koszul sign, the brackets are assumed to be 'gradedsymmetric', dropping the factor $\operatorname{sign}(\sigma)$ from $\chi$. The two definitions are equivalent, in the sense that an $L_{\infty}$ structure which is graded-antisymmetric in the first sense uniquely defines an $L_{\infty}$ structure that is graded-antisymmetric in the second sense and vice versa.

A possible invariant, which controls whether or not a Dirac structure admits a Dirac complement, could possibly exist in one of the curved $L_{3}$ algebras associated with choosing a complementary maximally isotropic subbundle. Of course, different maximally isotropic complements $L, L^{\prime}$ give different $L_{3}$ structures. Since our definition of the $L_{3}$ algebra on $\bigwedge M^{*}$ heavily relied on the splitting itself, we get two different structures, given by two different sets of brackets.

### 5.2 Rechoosing $M$

Let $M, M^{\prime}$ be maximally isotropic complements for a given maximally isotropic subbundle $L$. Since $M^{\prime}$ is transverse to $L$, it can be written as the graph of some $\epsilon: M \rightarrow L$, that is, $M^{\prime}=\{m+\omega(m) \mid m \in M\}$, and since $M^{\prime}$ is isotropic, we have that $\omega$ is skew when considered as an element of $\bigwedge^{2} M^{*}$ after identifying $L \cong M^{*}$. We write $M^{\prime}=e^{\omega} M$. Note that by our choices, the following diagram commutes:


We have two splittings of $T \oplus T^{*}$, given by $M \oplus L$ and $M^{\prime} \oplus L$, thus two sets of projections, one which we will mark $p_{M}, p_{L}$ and one which we will mark $p_{M^{\prime}}, p_{L}^{\prime}$. We have for $m+l \in M \oplus L$ :

$$
p_{L}^{\prime}(m+l)=p_{L}^{\prime}(m+l+\omega(m)-\omega(m))=l-\omega(m)
$$

Therefore,

$$
p_{L}^{\prime}=p_{L}-\omega \circ p_{M}
$$

and

$$
p_{M^{\prime}}=p_{M}+\omega \circ p_{L}
$$

We can extend the natural isomorphism $e^{\omega}: M \rightarrow M^{\prime}$ to an isomorphism $\bigwedge M \rightarrow \bigwedge M^{\prime}$. We find the following:

Proposition 5.18. The two different $L_{3}$ structures on $\Gamma(\bigwedge L)$ differ by the following operations:

1. For the Nijenhuis tensors of $M, M^{\prime}$,

$$
N_{M^{\prime}}=N_{M}+d_{M} \omega+\frac{1}{2}[\omega, \omega]_{L}+\omega^{*} N_{L}
$$

If $L$ is integrable,

$$
N_{M^{\prime}}=N_{M}+d_{M} \omega+\frac{1}{2}[\omega, \omega] .
$$

2. For a 1-form $\alpha$ :

$$
\begin{aligned}
\left(d_{M^{\prime}}-d_{M}\right) \alpha(X, Y)= & a(\omega(X)) \alpha(Y)-a(\omega(Y)) \alpha(X) \\
& -\alpha\left(p_{M}\left(\left[i_{X} \omega, Y\right]+\left[X, i_{Y} \omega\right]+\left[i_{X} \omega, i_{Y} \omega\right]\right)\right.
\end{aligned}
$$

If $L$ is integrable,

$$
\begin{aligned}
\left(d_{M^{\prime}}-d_{M}\right) \alpha(X, Y)= & a(\omega(X)) \alpha(Y)-a(\omega(Y)) \alpha(X) \\
& +a(\alpha) \omega(X, Y)+d_{L} Y(\omega(X), \alpha)-d_{L} X(\omega(Y), \alpha)
\end{aligned}
$$

3. For two sections $\alpha, \beta \in \Gamma(L)$,

$$
p_{L}^{\prime}([\alpha, \beta])=p_{L}([\alpha, \beta])-\omega \circ p_{M}([\alpha, \beta])
$$

If $M$ is integrable, the difference vanishes.
4. For the Nijenhuis tensor of $L, N_{L}=N_{L}^{\prime}$.

Proof. 1. $N_{M}$ versus $N_{M^{\prime}}$ : Our previous derivation of the deformation equation 4.1 allows us to find:

$$
N_{M^{\prime}}=N_{M}+d_{M} \omega+\frac{1}{2}[\omega, \omega]_{L}+\omega^{*} N_{L}
$$

If $L$ is integrable, the last term drops, and $[\omega, \omega]_{L}$ is defined independently of the splitting $M \oplus L$, so we can write

$$
N_{M^{\prime}}=N_{M}+d_{M} \omega+\frac{1}{2}[\omega, \omega]
$$

2. $d_{M}$ versus $d_{M^{\prime}}$ : For a 1-form $\alpha$, we have:

$$
d_{M} \alpha(X, Y)=a(X) \alpha(Y)-a(Y) \alpha(X)-\alpha\left(p_{M}[X, Y]\right)
$$

and,

$$
\begin{aligned}
d_{M^{\prime}} \alpha\left(X+i_{X} \omega, Y+i_{Y} \omega\right)= & a(X) \alpha(Y)-a(Y) \alpha(X)+a(\omega(X)) \alpha(Y) \\
& -a(\omega(Y)) \alpha(X)-\alpha\left(p_{M}([X, Y]\right. \\
& \left.+\left[i_{X} \omega, Y\right]+\left[X, i_{Y} \omega\right]+\left[i_{X} \omega, i_{Y} \omega\right]\right),
\end{aligned}
$$

where we used the fact that $L$ is isotropic.
Assuming $L$ is integrable, we can further simplify this expression. Since $d_{M} \alpha, d_{M^{\prime}} \alpha$ are both sections of $\bigwedge L$, they are completely described by their pairing with sections of $\bigwedge^{2} M$. Since the pairing with
a section of $L$ is unaffected by $e^{\omega}$, we have:

$$
\begin{aligned}
\left(d_{M^{\prime}}-d_{M}\right) \alpha(X, Y)= & a(\omega(X)) \alpha(Y)-a(\omega(Y)) \alpha(X)-\alpha\left(p_{M}\left(\left[i_{X} \omega, Y\right]+\left[X, i_{Y} \omega\right]\right)\right) \\
= & a(\omega(X)) \alpha(Y)-a(\omega(Y)) \alpha(X) \\
& +\alpha\left(d_{L} \omega(X, Y)+i_{\omega(X)} d_{L} Y-i_{\omega(Y)} d_{L} X\right) \\
= & a(\omega(X)) \alpha(Y)-a(\omega(Y)) \alpha(X) \\
& +a(\alpha) \omega(X, Y)+d_{L} Y(\omega(X), \alpha)-d_{L} X(\omega(Y), \alpha)
\end{aligned}
$$

3. $p_{L}([\cdot, \cdot])$ versus $p_{L}^{\prime}([\cdot, \cdot])$ : For two sections $\alpha, \beta \in \Gamma(L)$,

$$
p_{L}^{\prime}([\alpha, \beta])=p_{L}([\alpha, \beta])-\omega \circ p_{M}([\alpha, \beta])
$$

If $M$ is integrable, the difference vanishes.
4. $N_{L}$ versus $N_{L}^{\prime}$ : Since $N_{L}$ is defined independently of $M, M^{\prime}$, we again have $N_{L}=N_{L^{\prime}}$

A curved $L_{3}$-algebra with $l_{3}=0$ is called a curved DGLA. As we have shown, if $L$ is integrable, any complement $M$ to $L$ gives $\Gamma(\bigwedge L)$ the structure of a curved DGLA. Therefore, we find a first equivalent formulation of the complement problem:

Theorem 5.19. Let $L$ be a Dirac structure and $M$ be a maximally isotropic complementary subbundle. Then $L$ has an integrable complement if and only if there is a solution $\omega \in \Gamma\left(\bigwedge^{2} L\right)$ for the equation

$$
N_{M}+d_{M} \omega+\frac{1}{2}[\omega, \omega]=0
$$

in the curved $D G L A \Gamma(\bigwedge L)$.
Remark 5.9. The equation above is called the Maurer-Cartan equation. It is defined generally, for a curved $L_{\infty}$ algebra, as:

$$
\sum_{k} \frac{1}{k!} l_{k}(\omega)=0
$$

Solutions to this equation are called Maurer-Cartan elements.
Remark 5.10. This contrasts with the usual Maurer-Cartan equation derived in the literature, for example in [11], where the authors deform a Dirac structure by choosing a maximally isotropic complement. In that case, the deformation is given by a section $\omega$ of the second exterior power of the complement, which was generally nonintegrable, and the Maurer-Cartan equation was of the form:

$$
d_{M} \omega+\frac{1}{2}[\omega, \omega]_{L}+\frac{1}{3!} l_{3}(\omega, \omega, \omega)
$$

On the contrary, we are deforming a Dirac structure's complement into an integrable one. Note the difference in the equations - the one derived here does not contain an order 3 term, whereas the one derived in [11] does not contain an order 0 term.

We can also define the nonlinear operator $Q: \Gamma(\bigwedge M) \rightarrow \Gamma(\bigwedge M)$, acting as

$$
Q(\alpha)=N_{L}+d_{L} \alpha+\frac{1}{2}[\alpha, \alpha]_{M}
$$

In this notation, we have the following:
Proposition 5.20. The complement problem is equivalent to the following question:

$$
\text { When is } \operatorname{ker}\left(\left.Q\right|_{\wedge^{2} M}\right) \neq 0 \text { ? }
$$

Although seemingly mysterious, the operator $Q$ is related to an equivalent, but more complicated, definition of an $L_{3}$ algebra, involving a derivation on the symmetric coalgebra of a graded vector space which squares to 0 .

### 5.3 Rechoosing $L$

It is also in our interest to consider the different curved $L_{3}$ algebra structures on $\Gamma\left(\bigwedge M^{*}\right) \sim \Gamma(\bigwedge L)$ given by re-choosing $L$. Compared to the previous case where we got a curved DGLA, in this case we get a flat $L_{3}$ algebra, which has slightly more complicated operations and relations to keep track of. Consider $L, L^{\prime}$ two maximally isotropic subbundles, both complementary to $M$. Again, we could write $L^{\prime}=\{l+\epsilon(l) \mid l \in L\}$ for some skew $\epsilon: L \rightarrow M$ which can be thought of as an element of $\bigwedge^{2} M$.

Define $\left(p_{M}, p_{L}\right)$ and $\left(p_{M}^{\prime}, p_{L^{\prime}}\right)$ as before. Again, we have the relations

$$
p_{L^{\prime}}=p_{L}+\epsilon \circ p_{L}
$$

and

$$
p_{M}^{\prime}=p_{M}-\epsilon \circ p_{L} .
$$

Proposition 5.21. The two different $L_{3}$ structures on $\Gamma\left(\bigwedge M^{*}\right) \sim \Gamma(\bigwedge L)$ differ by the following operations:

1. For the Nijenhuis tensor of $M, N_{M}=N_{M}^{\prime}$.
2. For $\alpha \in \Gamma(L)$,

$$
\left(d_{M}^{\prime} \alpha-d_{M} \alpha\right)(X, Y)=N_{M}\left(X, Y, \epsilon^{*} \alpha\right) .
$$

If $M$ is integrable, the difference vanishes.
3. For two sections $X, Y \in \Gamma(L)$ :

$$
e^{-\epsilon} p_{L^{\prime}}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right)-p_{L}([X, Y])=p_{L}\left(\left[X, i_{Y} \epsilon\right]+\left[i_{X} \epsilon, Y\right]+\left[i_{X} \epsilon, i_{Y} \epsilon\right]\right) .
$$

If $M$ is integrable,

$$
p_{L^{\prime}}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right)-e^{\epsilon} p_{L}([X, Y])=e^{\epsilon}\left(d_{M}(\epsilon(X, Y))+i_{\epsilon(X)} d_{M} Y-i_{\epsilon(Y)} d_{M} X\right) .
$$

4. For the Nijenhuis tensors of $L, L^{\prime}$,

$$
N_{L^{\prime}}=e^{-\epsilon *}\left(N_{L}+d_{L} \epsilon+\frac{1}{2}[\epsilon, \epsilon]_{M}+\epsilon^{*} N_{M}\right) .
$$

If $M$ is integrable,

$$
N_{L^{\prime}}=e^{-\epsilon *}\left(N_{L}+d_{L} \epsilon+\frac{1}{2}[\epsilon, \epsilon]_{M}\right) .
$$

Proof. 1. $N_{M}^{\prime}$ versus $N_{M}$ : The element $N_{M} \in \Lambda^{3} M^{*}$ depends only on $M$, thus does not change.
2. $d_{M}$ versus $d_{M}^{\prime}$ : For $d_{M}$, since it is a derivation, it is enough to examine how it changes for 1 -forms. By the Koszul formula,

$$
\left(d_{M}^{\prime} \alpha-d_{M} \alpha\right)(X, Y)=\alpha\left(\epsilon\left(p_{L}([X, Y])\right)\right) .
$$

Note that $p_{L}([X, Y])=[X, Y]-p_{M}([X, Y])$ is tensorial, and is also related to $N_{M}(X, Y, Z)=$ $\langle[X, Y], Z\rangle=\left\langle p_{L}([X, Y]), Z\right\rangle$ (since $M$ is isotropic). Therfore we could write

$$
\left(d_{M}^{\prime} \alpha-d_{M} \alpha\right)(X, Y)=N_{M}\left(X, Y, \epsilon^{*} \alpha\right) .
$$

Of course, if $M$ is integrable, the difference vanishes as expected. In that case, $e^{\epsilon}$ is a cochain complex morphism.
 is given by $p_{L^{\prime}}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right)=(I+\epsilon) p_{L}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right)$. The difference is therefore:

$$
e^{-\epsilon} p_{L^{\prime}}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right)-p_{L}([X, Y])=p_{L}\left(\left[X, i_{Y} \epsilon\right]+\left[i_{X} \epsilon, Y\right]+\left[i_{X} \epsilon, i_{Y} \epsilon\right]\right)
$$

In the case that $M$ is integrable, we have:

$$
\begin{aligned}
e^{-\epsilon} p_{L^{\prime}}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right)-p_{L}([X, Y]) & =p_{L}\left(\left[X, i_{Y} \epsilon\right]+\left[i_{X} \epsilon, Y\right]\right) \\
& =e^{\epsilon}\left(p_{L}\left(\left[i_{X} \epsilon, Y\right]-\left[i_{Y} \epsilon, X\right]\right)-d_{M}(\epsilon(X, Y))\right) .
\end{aligned}
$$

Using proposition 5.12, we find that this term reads:

$$
\begin{aligned}
e^{-\epsilon} p_{L^{\prime}}\left(\left[X+i_{X} \epsilon, Y+i_{Y} \epsilon\right]\right) & -p_{L}([X, Y]) \\
& =d_{M}(\epsilon(X, Y))+i_{\epsilon(X)} d_{M} Y-i_{\epsilon(Y)} d_{M} X .
\end{aligned}
$$

4. $N_{L}$ versus $e^{\epsilon *} N_{L^{\prime}}$ : For $N_{L}$, our previous derivation of the deformation equation 4.1 allows us to immediately find:

$$
e^{\epsilon *} N_{L^{\prime}}=N_{L}+d_{L} \epsilon+\frac{1}{2}[\epsilon, \epsilon]_{M}+\epsilon^{*} N_{M} .
$$

Again, if $M$ is integrable, the last term vanishes.

A second equivalent statement of the complement problem is, therefore:
Remark 5.11. Again, in the case where $M$ is Dirac, we find the same Maurer-Cartan equation as in section 5.2 dictating the condition on $L^{\prime}$ to be integrable in terms of $\epsilon$. However, this Maurer-Cartan equation is that of the curved DGLA $\Gamma(\bigwedge M)$, and not of the flat $L_{3}$ algebra $\Gamma(\bigwedge L)$. The difference occurs since in the previous section we derived it by observing the bracket $l_{0}=N_{M}$, whereas now we observe how the bracket $l_{3}=N_{L}$ changes. We find that, generically, when deforming a nonintegrable ('curved') structure, a curved DGLA controls the deformations. We can compare this to the deformation of Dirac structures with a nonintegrable complement (such as in $[8,11]$ ) where a flat $L_{3}$ algebra controls the deformations instead.

### 5.4 Choosing a complement in terms of $O$

In corollary 2.20 we saw that for a given orthogonal operator $O$, the subspace $L_{-O}$ is always complementary to $L_{O}$ as maximally isotropic subspaces. Of course, the same is true for maximally isotropic subbundles:
for a given section $O \in O(X, g)$, the maximally isotropic subbundle $L_{-O}$ is always complementary to $L_{O}$. The integrability condition presented in theorem 3.25 is not invariant under $O \mapsto-O$. That is, given that $L_{O}$ is integrable, $L_{-O}$ is not necessarily integrable. However, if the stronger condition $\nabla O=0$ is satisfied, $L_{-O}$ is guaranteed to be integrable. A third possible direction for answering the complement problem could come from answering the following question:

Which Dirac structures $L$ admit a metric $g$ such that $\nabla O_{L}=0$ ?
In general, the holonomy principle [3] states that there is a bijective correspondence between sections of orthogonal operators $O$ with $\nabla O=0$ and orthogonal operators at a point which are invariant under the holonomy group. For instance, we have the following examples:

Example 5.22. On $\mathbb{R}^{n}$ using the euclidean metric, the only sections of orthogonal operators with $\nabla O=0$ are the constant ones. These correspond to Dirac structures that admit a basis of constant sections.

Example 5.23. On a Riemannian manifold with holonomy $S O(n)$ or $O(n)$, the only sections of orthogonal operators with $\nabla O=0$ are $\pm I d$, corresponding to the Dirac structures $T$ and $T^{*}$.

We thus have a possible method for constructing a Dirac complement for a Dirac structure $L$ by finding a metric which satisfies $\nabla O_{L}=0$. Note that the condition $\nabla O=0$ is stronger than the condition enforced by the vanishing of the tensor in proposition 3.24, and so it is ambitious to expect such a metric exists for every Dirac structure. On the other hand, disproving the existence of such a metric does not disprove the existence of a Dirac complement. Although not giving a precise criterion for the existence of an integrable complement, this approach could help find more examples of complementary Dirac structures.

## Chapter 6

## Discussion and conclusions

The results in sections 5.3 and 5.2 give the first steps for constructing new invariants for Dirac structures related to the two possible $L_{3}$ algebras arising from choosing a complement. It remains unclear in which category the maps $e^{\epsilon}$ and $e^{\omega}$ fall into, when considering them as maps between $L_{3}$ algebras. It is possible that one could frame them as (iso)morphisms of $L_{3}$ (or, more generally, $L_{\infty}$ ) algebras in some (possibly new) sense. In such a case, it would be fruitful to construct invariants of $L_{3}$ algebras that are stable under such maps, which would again give invariants of the Dirac structure considered. A first step would be answering the questions:

What is the obstruction for a flat $L_{3}$ algebra to be isomorphic, in the sense described above (that is, through a map in the same category as $e^{\epsilon}$ ) to a flat DGLA?
and
What is the obstruction for a curved DGLA to be isomorphic, in the sense described above (that is, through a map in the same category as $e^{\omega}$ ) to a flat DGLA?

Of course, it is possible that no such obstruction exists, and that the problem remains in constructing a section $\omega$ or a section $\epsilon$ generating this (iso)morphism. We can compare this to a classical result in the theory of $L_{\infty}$ algebras, stating that for any $L_{\infty}$ algebra $F$ there exists a quasi-isomorphism (that is, a morphism of $L_{\infty}$ algebras which induces an isomorphism between cohomologies) $F \rightarrow G$, where $G$ is a DGLA [14]. Therefore, further understanding of these morphisms is still required. In the case of section 5.3, we had that the map $e^{\epsilon}$ was a chain map descending to an isomorphism on cohomology. Therefore, it could be possible that the above obstruction exists in the form of a cohomology class. The theory of characteristic classes of $L_{\infty}$ algebras is well established [15], and so it is possible the methods from this theory could be applied to the complement problem.

In addition, the results from section 3.2 as well as the discussion in section 5.4 could be further developed. First, the geometric interpretation of the tensor given in proposition 3.24 is yet to be understood. It is possible that, through further exploration of these structures, a method of constructing a section $O^{\prime}$ satisfying both the vanishing of the tensor in proposition 3.24 as well as the conditions of proposition 2.19 could be developed. The vanishing of the tensor in proposition 3.24 could also be interpreted as a partial differential equation on the components of $O$. Analyzing the initial conditions of this PDE could, again, yield a method of constructing a complement by defining one on some lower dimensional manifold and extending it to a solution of this PDE.

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