On exceptional symmetric Gelfand pairs

Roberto Rubio
Weizmann Institute
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The plan

- Gelfand pairs and criterion (regularity+descendants).
- Pleasantness criteria to obtain regularity.
- Application to exceptional symmetric pairs.
- Descendants of exceptional symmetric pairs.
- Main results and outlook.

We work overall over $\mathbb{C}$, although some techniques apply with more generality.

An intuitive definition of Gelfand pair

Let $H \subseteq G$ be linear algebraic complex reductive groups.
Consider the action of G on

$$\mathcal{C}^\infty(G/H) = \{f : G/H \to \mathbb{C} \mid f \text{ is smooth } \}.$$

(G, H) is a **Gelfand pair** when $\mathcal{C}^\infty(G/H)$ is multiplicity free, that is, when looking at $\mathcal{C}^\infty(G/H)$ as a representation of H, (certain) irreducible representations appear at most once.

It can also be expressed using Schwartz functions $\mathcal{S}(G/H)$, smooth rapidly decaying functions. We will mention later tempered distributions $\mathcal{S}^*$, linear functionals on Schwartz functions.

Good candidates: symmetric pairs

A **symmetric pair** is a tuple (G, H, θ) where θ is an involutive automorphism of G and $H = G^\theta$ is the set of fixed points.

Examples:

- For $G = \mathsf{SL}_n$ and $\theta(g) = (g^T)^{-1}$, we have $H = \mathsf{SO}_n$.
- For $G = \mathsf{SO}_{r+s}$ and $\theta = \mathsf{Ad}(I_{r,s})$ with $I_{r,s} = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$,

$$\theta : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

we have $H = \mathrm{S}(\mathrm{O}_r \times \mathrm{O}_s)$.

They are both Gelfand pairs.

Conjecture (van Dijk): Any complex symmetric pair (G, H) with G/H connected is a Gelfand pair.

Aizenbud-Gourevitch criterion (based on Gelfand-Kahzdan)

A symmetric pair (G, H, θ) is called **regular** when for any $g \in \{\text{“admissible elements”}\} \subseteq \{g \in G \mid \theta(g) \in gZ(G)\}$ such that $\mathcal{S}^*(R(\mathfrak{p}))^{\mathrm{Ad}\,H} \subseteq \mathcal{S}^*(R(\mathfrak{p}))^{\mathrm{Ad}\,g}$, we have[1]

$$\mathcal{S}^*(\mathfrak{p})^{\mathrm{Ad}\,H} \subseteq \mathcal{S}^*(\mathfrak{p})^{\mathrm{Ad}\,g}.$$

A **descendant** is the centralizer $(G_x, H_x, \theta_{|G_x})$ for $x \in \{g\theta(g^{-1}) \mid g \in G \text{ semisimple}\}$.

Aizenbud-Gourevitch criterion for $k = \mathbb{C}$:
If a pair (G, H, θ) is regular and all its descendants are regular, then it is a Gelfand pair*.

+ stronger conjecture: all symmetric pairs are regular.

Used to prove many classical cases, what about **exceptional** G?

[1] Notation: Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and $R\mathfrak{p}$ are the regular elements.

A closer look at regularity

A symmetric pair (G, H, θ) is called **regular** when for any $g \in \{$ "admissible elements" $\} \subseteq \{g \in G \mid \theta(g) \in gZ(G)\}$ such that $\mathcal{S}^*(R(\mathfrak{p}))^{\operatorname{Ad} H} \subseteq \mathcal{S}^*(R(\mathfrak{p}))^{\operatorname{Ad} g}$,

$$\mathcal{S}^*(\mathfrak{p})^{\operatorname{Ad} H} \subseteq \mathcal{S}^*(\mathfrak{p})^{\operatorname{Ad} g}.$$

A closer look at regularity

ᛋᛘᛞᛞᛖᛏᚱᛁᚳᛈᛁᚱ(G, H, θ)ᛁᛋᚳᚠᛚᛚᛖᛞ**ᚱᛖᚷᚢᛚᚨᚱ**ᚹᚻᛖᚾᚠᚠᚱᚠᚾᛘ
$g \in \{$ᚠᛞᛗᛁᛋᛋᛁᛒᛚᛖᛖᛚᛖᛗᛖᚾᛏᛋ$\} \subseteq \{g \in G \mid \theta(g) \in gZ(G)\}$
ᛋᚢᚳᚻᛏᚻᚨᛏ$\mathcal{S}^*(R(\mathfrak{p}))^{\mathrm{Ad}\, H} \subseteq \mathcal{S}^*(R(\mathfrak{p}))^{\mathrm{Ad}\, g}$ ‡ ᚹᛖᚻᚨᛖ

$$\mathcal{S}^*(\mathfrak{p})^{\mathrm{Ad}\, H} \subseteq \mathcal{S}^*(\mathfrak{p})^{\mathrm{Ad}\, g}.$$

An even closer look at regularity

$$\mathcal{S}^*(\mathfrak{p})^{\mathsf{Ad}\,H} \subseteq \mathcal{S}^*(\mathfrak{p})^{\mathsf{Ad}\,g}$$

An even closer look at regularity

$$\text{ᚴ}^*(\text{ᛈ})^{\mathrm{Ad}\,H} \subseteq \text{ᚴ}^*(\text{ᛈ})^{\mathrm{Ad}\,g}$$

A sufficient condition for regularity is...

$$\operatorname{Ad} g \in \operatorname{Ad} H$$

for $g \in \mathcal{A}_\theta := \{g \in G \mid \theta(g) \in gZ(G)\}$,

or more beautifully,

$$(\operatorname{Ad} G)^\theta = \operatorname{Ad}(G^\theta).$$

Definition: a pair (G, H, θ) is called **pleasant** if...

$$\mathrm{Ad}\, g \in \mathrm{Ad}\, H$$

for $g \in \mathcal{A}_\theta := \{g \in G \mid \theta(g) \in gZ(G)\}$,

or more beautifully,

$$(\mathrm{Ad}\, G)^\theta = \mathrm{Ad}(G^\theta)$$

Criteria for pleasantness of (G, H, θ)

Pleasant: $\mathrm{Ad}\, g \in \mathrm{Ad}\, H$ for $g \in \mathcal{A}_\theta := \{g \in G \mid \theta(g) \in gZ(G)\}$.

Criterion 1

If G is centerless.

Proof: we have $\mathcal{A}_\theta = H = \{g \in G \mid \theta(g) = g\}$. $\square$

Criterion 2

If $\theta(\lambda) = \lambda$ for $\lambda \in Z(G)$ ($\leftarrow$ inner involution) and $|Z(G)|$ is odd.

Proof: for $g \in \mathcal{A}_\theta \setminus H$, we have $\theta(g) = \lambda g$ for $\lambda \in Z(G) \setminus \{1\}$.
Apply θ to get $g = \lambda^2 g$, which is not possible as $|Z(G)|$ is odd.
Hence $\mathcal{A}_\theta = H$ and the pair is pleasant. $\square$

Criteria for pleasantness of (G, H, θ)

Pleasant: $\operatorname{Ad} g \in \operatorname{Ad} H$ for $g \in \mathcal{A}_\theta = \{g \in G \mid \theta(g) \in gZ(G)\}$.

Criterion 3
If $\theta(\lambda) = \lambda^{-1}$ for $\lambda \in Z(G)$ and $|Z(G)|$ is odd.

Proof: for $\theta(g) = \lambda g$ as before, consider $\sqrt{\lambda} \in Z(G)$.
We have $\operatorname{Ad} g = \operatorname{Ad}(\sqrt{\lambda} g)$ and $\theta(\sqrt{\lambda} g) = \sqrt{\lambda} g$, i.e., $\sqrt{\lambda} g \in H$. $\square$

Criterion 4
Let $G \subset \mathrm{GL}(V)$ with $Z(G) \subseteq k \cdot \mathrm{Id}$, and θ be given by $\operatorname{Ad}(M)$,
for some $M \in \mathrm{GL}(V)$ of order two.
Let V_+ and V_- be the ± 1-eigenspaces of M on V.
If $\dim V_+ \neq \dim V_-$, the pair is pleasant.

Why criterion 4 works.

Proof: For any inner involution given by $\mathrm{Ad}(M)$ with $M^2 = \mathrm{Id}$, we can see $M = I_{r,s}$ and we get

$$\theta : \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} A & -B \\ -C & D \end{smallmatrix}\right).$$

The only thing that can go wrong is if $-\mathrm{Id} \in Z(G)$, as then

$$g = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) \in \mathcal{A}_\theta \setminus H.$$

If $r \neq s$, this element is not invertible, so the pair is pleasant. $\square$

Simply-connected pairs

For any (G, H, θ) we have a simply-connected pair $(\widetilde{G}, H', \theta')$:

- $\widetilde{G}$ is simply connected and there is a covering map $\pi : \widetilde{G} \to G$,
- the involution satisfies $\pi \circ \theta' = \theta \circ \pi$,
- $H' = (\widetilde{G})^{\theta'}$, but not necessarily $H' = \pi^{-1}(H)$.

Since we have $\pi(H') \subseteq H$, it holds that

- if the pair $(\widetilde{G}, H', \theta')$ is regular, then (G, H, θ) is regular.
- if the pair $(\widetilde{G}, H', \theta')$ is a Gelfand pair, then (G, H, θ) is a Gelfand pair.

So we will look at simply-connected pairs. We will use Lie types:

$$A_n \equiv \mathrm{SL}_{n+1},\ B_n \equiv \mathrm{SO}_{2n+1},\ C_n \equiv \mathrm{Sp}_{2n},\ D_n \equiv \mathrm{SO}_{2n}.$$

Pleasant exceptional symmetric pairs

(G, H, θ) with simply-connected $G = G_2, F_4, E_6, E_7, E_8$.
There are twelve of them (we shall see them later).

We first look at their centre: G_2, F_4 and E_8 are centreless, E_6 has centre $\mathbb{Z}_3$, and E_7 has centre $\mathbb{Z}_2$.

Proposition

Any symmetric pair with $G = G_2, F_4, E_6, E_8$ is pleasant.

Proof: for E_6, any involution θ must satisfy $\theta(\lambda) = \lambda$ or $\theta(\lambda) = \lambda^{-1}$, so we can apply Criteria 2 and 3. $\square$

What about E_7? Well... what is E_7?

What is E_7?

Cayley–Dickson construction: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Let $\mathfrak{J}$ be the space of 3×3-Hermitian matrices over $\mathbb{O}_{\mathbb{C}}$:

$$\mathfrak{J} = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix},\ \xi_j \in \mathbb{C}, x_j \in \mathbb{O}_{\mathbb{C}} \right\},$$

This is a 27-dimensional. It is the exceptional Jordan algebra $(\mathfrak{J}, \circ)$.

Actually, $G_2 = \mathrm{Aut}_{\mathbb{C}}(\mathbb{O}_{\mathbb{C}})$, $F_4 = \mathrm{Aut}_{\mathbb{C}}(\mathfrak{J}, \circ)$,

$$E_6 = \{\alpha \in Iso_{\mathbb{C}}(\mathfrak{J}) \mid \det(\alpha X) = \det X \text{ for all } X \in J\}.$$

What is E_7?

Consider the complex Freudenthal vector space (56-dimensional),

$$\mathfrak{B} = \mathfrak{J} + \mathfrak{J} + \mathbb{C} + \mathbb{C}.$$

There is a Freudenthal product: for $P, Q \in \mathfrak{B}$,

$$P \times Q : \mathfrak{B} \to \mathfrak{B}.$$

We have

$$E_7 = \{\alpha \in Iso_{\mathbb{C}}(\mathfrak{B}) \mid \alpha(P \times Q)\alpha^{-1} = (\alpha P \times \alpha Q), P, Q \in \mathfrak{B}\}.$$

Let us look at the involutions at least.

E_7 and criterion 4

There are three involutions for E_7, all inner.

For $(E_7, D_6 + A_1)$, the involution comes from $\sigma : \mathfrak{J} \to \mathfrak{J}$

$$\sigma : \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}.$$

The ± 1-eigenspaces of σ on $\mathfrak{J}$ are 11 and 16-dimensional, and we get $r = 11 \cdot 2 + 2 \neq 16 \cdot 2 = s$ in Criterion 4, so the pair is pleasant.

E_7 and criterion 4

For $(E_7, E_6 + \mathbb{C})$, the involution comes from the map $\iota : \mathfrak{B} \to \mathfrak{B}$,

$$\iota(X, Y, \xi, \eta) = (-iX, iY, -i\xi, i\eta),$$

whose ± 1-eigenspaces are of the same dimension.

Actually, we can find an element g such that $\operatorname{Ad} g \notin \operatorname{Ad} H$, namely,

$$g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

E_7 and criterion 4

Finally, (E_7, A_7). Denote by τ the conjugation on $\mathbb{C}$ and on $\mathbb{O}_{\mathbb{C}}$. Denote by γ the involution of $\mathbb{O}_{\mathbb{C}}$ given, for $m, n \in \mathbb{H}_{\mathbb{C}}$, by

$$\gamma(m + ne_4) = m - ne_4.$$

Define a linear involution $\tau\gamma : \mathfrak{B} \to \mathfrak{B}$ by

$$(X, Y, \xi, \eta) \mapsto (\tau\gamma X, \tau\gamma Y, \tau\xi, \tau\eta).$$

Again, ± 1-eigenspaces are of the same dimension, and we find

$$g = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

Exceptional pleasant pairs

Proposition (R., 2018)

All exceptional simple connected $\mathbb{C}$-symmetric pairs, apart from $(E_7, E_6 + \mathbb{C})$ and (E_7, A_7), are pleasant and hence regular.

Spin pleasant pairs

Despite $(\mathsf{SO}_{r+s}, \S(\mathrm{O}_r \times \mathrm{O}_s))$ for $r \neq s$ is always a pleasant pair, we need to look at the Spin group, where things are quite different.

Proposition (R., 2018)

A symmetric pair associated with Spin_{r+s} *is pleasant if and only if it is* $(\mathsf{Spin}_{r+s}, \mathsf{Spin}_r \times \mathsf{Spin}_s)$ *with* $r + s = 4q + 2$ *and* $r \neq s$ *odd numbers.*

Other source of regular pairs

Pleasantness is a naive but very powerful way of proving regularity.

For the use of the Aizenbud-Gourevitch on classical groups, essentially, the concept of nice symmetric pairs (originally introduced by Sekiguchi) was used for the pairs that were not clearly (or pleasantly) regular.

Let us combine these two criteria for G simple, connected and simply connected.

Nice and pleasant classical pairs

		pleasant	nice
(A_{2n}, B_n)		✓	✓
(A_{2n-1}, D_n)		$\mathsf{SL}/\mathbb{Z}_{2^{max}}$	✓
(A_{2n-1}, C_n)		✓	✗
$(A_{r+s+1}, A_r + A_s + \mathbb{C})$	$r \neq s$	✓	✗
	$r = s$	$\mathsf{SL}/\mathbb{Z}_{2^{max}}$	✓
$(C_n, A_{n-1} + \mathbb{C})$		✗	✓
$(C_{r+s}, C_r + C_s)$	$r \neq s$	✓	✗
	$r = s$	✗	✗

Nice and pleasant classical pairs

		pleasant	nice
$(B_{r+s}, B_r + D_s)$	$r \neq s$	SO	✗
	$r = s$	✗	✓
D_{4q}	$r \neq s-1, s$	SO	✗
$(D_{r+s}, D_r + D_s)$	$r = s-1$	SO	✓
$(D_{r+s+1}, B_r + B_s)$	$r = s$	✗	✓
	$r \neq s-1, s$	SO	✗
D_{4q+2}	$r = s-1$	SO	✓
$(D_{r+s}, D_r + D_s)$	$r = s$	✗	✓
	$r \neq s-1, s$	✓	✗
D_{4q+2}	$r = s-1$	✓	✓
$(D_{r+s+1}, B_r + B_s)$	$r = s$	✗	✓
D_{odd}	$r \neq s-1, s$	SO	✗
$(D_{r+s}, D_r + D_s)$	$r = s-1$	SO	✓
$(D_{r+s+1}, B_r + B_s)$	$r = s$	✗	✓
$(D_r, A_{r-1} + \mathbb{C})$		✗	✗

Nice and pleasant exceptional pairs

	pleasant	nice
$(G_2, A_1 + A_1)$	✓	✓
(F_4, B_4)	✓	✗
$(F_4, C_3 + A_1)$	✓	✓
(E_6, C_4)	✓	✓
$(E_6, A_5 + A_1)$	✓	✓
(E_6, F_4)	✓	✗
$(E_6, D_5 + \mathbb{C})$	✓	✗
(E_7, A_7)	✗	✓
$(E_7, D_6 + A_1)$	✓	✗
$(E_7, E_6 + \mathbb{C})$	✗	✗
(E_8, D_8)	✓	✓
$(E_8, E_7 + A_1)$	✓	✗

Neither pleasant nor nice pairs

Proposition

The only neither pleasant nor nice pairs correspond to

$$(\mathrm{Sp}_{2r}, \mathrm{Sp}_r \times \mathrm{Sp}_r),\ (\mathrm{SO}_{2r}, \mathrm{GL}_r),\ (E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1),$$

$$(\mathrm{Spin}_{r+s}, \mathrm{Spin}_r \times \mathrm{Spin}_s) \textit{ with } |r-s| \neq 0,1,2,$$
$$\textit{except } r+s = 4q+2 \textit{ and } r \neq s \textit{ odd numbers.}$$

Why were we doing all this?

Target: exceptional Gelfand pairs

Aizenbud-Gourevitch criterion for $k = \mathbb{C}$:
If a pair (G, H, θ) is regular and all its descendants are regular, then it is a Gelfand pair.

A **descendant** is the centralizer $(G_x, H_x, \theta_{|G_x})$ for $x \in \{g\theta(g^{-1}) \mid g \in G \text{ semisimple}\} \to x$ semisimple.

How to compute descendants of an exceptional symmetric pairs?

Centralizers of semisimple elements

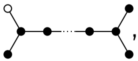
There is yet another reason to look at simply-connected groups:

- The centralizer of a semisimple element in a simply-connected group is connected and reductive (not a trivial theorem).

The only elements that can possibly spoil regularity of the descendant are inside the finite centre of the semisimple part.

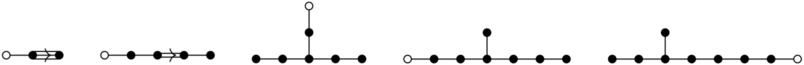
Moreover,

- the Lie type of the centralizer can be recovered by erasing nodes of the extended Dynkin diagram (Lusztig et al.).

Example: , a descendant of $\mathrm{SO}(2n)$ has at most two $\mathrm{SO}(2r)$ factors.

Centralizers of semisimple elements in exceptional groups

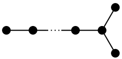
The extended Dynkin diagram of exceptional groups is:



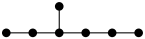
G_2 F_4 E_6 E_7 E_8

We can already see that , and hence $(C_{2r}, C_r + C_r)$ (i.e., $(\mathrm{Sp}_{4r}, \mathrm{Sp}_{2r} \times \mathrm{Sp}_{2r})$), does not appear as a descendant.

However, E_6, E_7, E_8 could have descendants of type

, $(\mathrm{Spin}_{2r}, \mathrm{GL}_r)$

or $(\mathrm{Spin}_{r+s}, \mathrm{Spin}_r \times \mathrm{Spin}_s)$

, $(E_7, E_6 + \mathbb{C})$.

We do not have to look at all semisimple elements

We are not using the information that

$$x \in \{g\theta(g^{-1}) \mid g \in G \text{ semisimple}\},$$

which implies that θ restricts to G_x.

We need to take into account the involution.
This is done visually by means of the Satake diagram.

This will give not only more information for our purposes, but a much better understanding of descendants of symmetric pairs.

Involution = Satake diagram

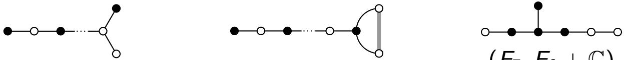
For (G, H, θ), consider the root system of G with respect to a θ-stable torus T containing a maximal θ-split torus A.

The involution θ acts on roots, permuting a choice of simple roots.

- If θ fixes a simple root, we paint it black.
- If θ permutes two simple roots, we draw a bar.

Alternatively, roots trivial on A are black, roots with the same non-trivial restriction to A are connected.

Examples:



$(D_r, A_{r-1} + \mathbb{C})$, $(\mathrm{SO}_{2r}, \mathrm{GL}_r)$ r even

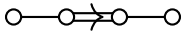
$(D_r, A_{r-1} + \mathbb{C})$, $(\mathrm{SO}_{2r}, \mathrm{GL}_r)$ r odd

$(E_7, E_6 + \mathbb{C})$ $(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$

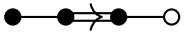
Satake diagrams classify symmetric pairs: recover the involution.

Exceptional Satake diagrams

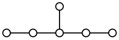
$(G_2, A_1 + A_1)$
$(G_2, \mathrm{SL}_1 \times_{\mathbb{Z}_2} \mathrm{SL}_1)$



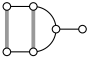
(F_4, B_4)
(F_4, Spin_9)



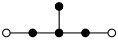
$(F_4, C_3 + A_1)$
$(F_4, \mathrm{Sp}_6 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



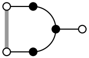
(E_6, C_4)
$(E_6, \mathrm{Sp}_8/\mathbb{Z}_2)$



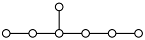
$(E_6, A_5 + A_1)$
$(E_6, \mathrm{SL}_6 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



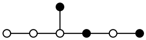
(E_6, F_4)
(E_6, F_4)



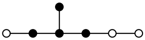
$(E_6, D_5 + \mathbb{C})$
$(E_6, \mathrm{Spin}_{10} \times_{\mathbb{Z}_4} \mathrm{GL}_1)$



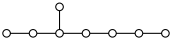
(E_7, A_7)
$(E_7, \mathrm{SL}_8/\mathbb{Z}_2)$



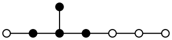
$(E_7, D_6 + A_1)$
$(E_7, \mathrm{Spin}_{12} \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



$(E_7, E_6 + \mathbb{C})$
$(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$



(E_8, D_8)
$(E_8, \mathrm{Spin}_{16}/\{\pm 1\})$



$(E_8, E_7 + A_1)$
$(E_8, E_7 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$

Visual descendants

We want to combine the Satake diagram with the extended Dynkin diagram in order to extract information about the possible descendants of an exceptional symmetric pair.

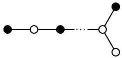
- By a result of Richardson, any semisimple element $x = g\theta(g^{-1})$ is inside a maximal θ-split torus.

Theorem (R., 2018)

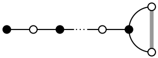
The process of erasing nodes is compatible with the colouring.

Visual descendants

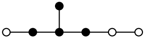
We will be looking for:



$(D_r, A_{r-1} + \mathbb{C})$,
$(\mathrm{SO}_{2r}, \mathrm{GL}_r)$
r even

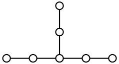


$(D_r, A_{r-1} + \mathbb{C})$,
$(\mathrm{SO}_{2r}, \mathrm{GL}_r)$
r odd

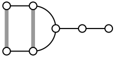


$(E_7, E_6 + \mathbb{C})$
$(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$

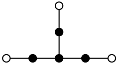
in the following extended Dynkin diagrams



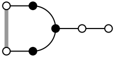
(E_6, C_4)
$(E_6, \mathrm{Sp}_8/\mathbb{Z}_2)$



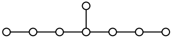
$(E_6, A_5 + A_1)$
$(E_6, \mathrm{SL}_6 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



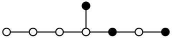
(E_6, F_4)
(E_6, F_4)



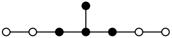
$(E_6, D_5 + \mathbb{C})$
$(E_6, \mathrm{Spin}_{10} \times_{\mathbb{Z}_4} \mathrm{GL}_1)$



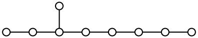
(E_7, A_7)
$(E_7, \mathrm{SL}_8/\mathbb{Z}_2)$



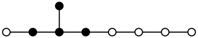
$(E_7, D_6 + A_1)$
$(E_7, \mathrm{Spin}_{12} \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



$(E_7, E_6 + \mathbb{C})$
$(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$



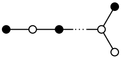
(E_8, D_8)
$(E_8, \mathrm{Spin}_{16}/\{\pm 1\})$



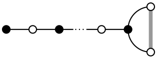
$(E_8, E_7 + A_1)$
$(E_8, E_7 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$

Visual descendants

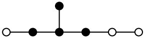
The diagrams



$(D_r, A_{r-1} + \mathbb{C})$,
$(\mathrm{SO}_{2r}, \mathrm{GL}_r)$
r even

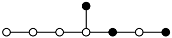


$(D_r, A_{r-1} + \mathbb{C})$,
$(\mathrm{SO}_{2r}, \mathrm{GL}_r)$
r odd

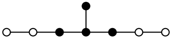


$(E_7, E_6 + \mathbb{C})$
$(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$

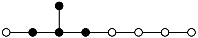
appear in



$(E_7, D_6 + A_1)$
$(E_7, \mathrm{Spin}_{12} \times_{\mathbb{Z}_2} \mathrm{SL}_2)$

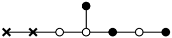


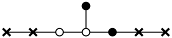
$(E_7, E_6 + \mathbb{C})$
$(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$

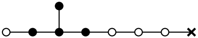


$(E_8, E_7 + A_1)$
$(E_8, E_7 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$

when doing

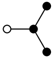




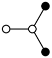


Visual descendants

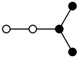
We will be looking for:



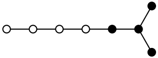
(D_4, B_3),
$(\mathrm{Spin}_8, \mathrm{Spin}_7)$,



$(D_4, D_3 + D_1)$,
$(\mathrm{Spin}_8, \mathrm{Spin}_6 \times \mathrm{Spin}_2)$,

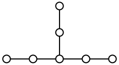


$(D_5, D_4 + \mathbb{C})$,
$(\mathrm{Spin}_{10}, \mathrm{Spin}_9) \ldots$

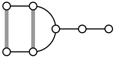


$(D_8, D_6 + D_2)$
$(\mathrm{Spin}_{16}, \mathrm{Spin}_{12} \times \mathrm{Spin}_4)$

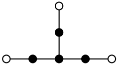
in the following extended Dynkin diagrams



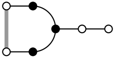
(E_6, C_4)
$(E_6, \mathrm{Sp}_8/\mathbb{Z}_2)$



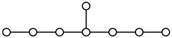
$(E_6, A_5 + A_1)$
$(E_6, \mathrm{SL}_6 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



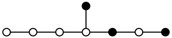
(E_6, F_4)
(E_6, F_4)



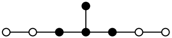
$(E_6, D_5 + \mathbb{C})$
$(E_6, \mathrm{Spin}_{10} \times_{\mathbb{Z}_4} \mathrm{GL}_1)$



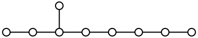
(E_7, A_7)
$(E_7, \mathrm{SL}_8/\mathbb{Z}_2)$



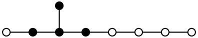
$(E_7, D_6 + A_1)$
$(E_7, \mathrm{Spin}_{12} \times_{\mathbb{Z}_2} \mathrm{SL}_2)$



$(E_7, E_6 + \mathbb{C})$
$(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$



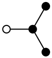
(E_8, D_8)
$(E_8, \mathrm{Spin}_{16}/\{\pm 1\})$



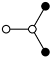
$(E_8, E_7 + A_1)$
$(E_8, E_7 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$

Visual descendants

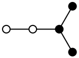
The diagrams



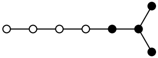
(D_4, B_3), $(\mathrm{Spin}_8, \mathrm{Spin}_7)$,



$(D_4, D_3 + D_1)$, $(\mathrm{Spin}_8, \mathrm{Spin}_6 \times \mathrm{Spin}_2)$,

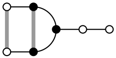


$(D_5, D_4 + \mathbb{C})$, $(\mathrm{Spin}_{10}, \mathrm{Spin}_9) \dots$

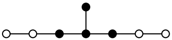


$(D_8, D_6 + D_2)$ $(\mathrm{Spin}_{16}, \mathrm{Spin}_{12} \times \mathrm{Spin}_4)$

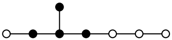
appear in



$(E_6, D_5 + \mathbb{C})$ $(E_6, \mathrm{Spin}_{10} \times_{\mathbb{Z}_4} \mathrm{GL}_1)$



$(E_7, E_6 + \mathbb{C})$ $(E_7, E_6 \times_{\mathbb{Z}_3} \mathrm{GL}_1)$



$(E_8, E_7 + A_1)$ $(E_8, E_7 \times_{\mathbb{Z}_2} \mathrm{SL}_2)$.

The main result

Theorem (R., 2018)

All exceptional complex symmetric pairs for G simple and connected, except possibly $(E_6, D_5 + \mathbb{C})$, $(E_7, E_6 + \mathbb{C})$, $(E_7, D_6 + A_1)$ and $(E_8, E_7 + A_1)$, are Gelfand pairs.

Moreover, the remaining four pairs are Gelfand pairs if the pairs (D_4, B_3), $(D_4, A_3 + \mathbb{C})$, $(D_4, D_3 + D_1)$, $(D_5, D_4 + \mathbb{C})$, $(D_6, A_5 + \mathbb{C})$, $(D_7, B_5 + B_1)$, $(E_7, E_6 + \mathbb{C})$ and $(D_8, D_6 + D_2)$ are regular.

What's next?

Joint work with Shachar Carmeli to prove that (D_4, B_3), $(D_4, A_3 + \mathbb{C})$, $(D_4, D_3 + D_1)$, $(D_5, D_4 + \mathbb{C})$, $(D_6, A_5 + \mathbb{C})$, $(D_7, B_5 + B_1)$, $(E_7, E_6 + \mathbb{C})$ and $(D_8, D_6 + D_2)$ are regular, by using distributional methods.

Some cases are easy, for instance, there is just one nilpotent orbit in the pair (D_4, B_3).

We hope to claim soon that all exceptional complex symmetric pairs are Gelfand pairs.

We will then continue with the regularity of the remaining infinite families.

Thank you!

מכון
ויצמן
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