# Generalized Geometry, an introduction
# Final Assignment

Weizmann Institute
Second Semester 2017-2018

Please, submit electronically by email before 7pm of Monday 6th August 2018.

**Problem 1.** Let $W$ be a vector space with a pairing of signature $(n, n)$.

- Prove that there exist two isotropic subspaces $W_1$, $W_2$ such that $W = W_1 \oplus W_2$.
- Prove that for any decomposition of $W$ into two isotropic subspaces, $W = W_1 \oplus W_2$, we have that both $W_1$ and $W_2$ are maximally isotropic subspaces and the pairing induces an isomorphism $W_2 \simeq W_1^*$.

Let $V$ be a real vector space with a pairing of signature $(p, q)$.

- Prove that the dimension of any maximally isotropic subspace is $\min(p, q)$.
- Find an analogue to the decomposition $W = W_1 \oplus W_2$ above (it does not need to be necessarily into two subspaces).
- Use the decomposition you find to describe maximally isotropic subspaces in $V$.

**Problem 2.** Let $M = \mathbb{R}^3$ with coordinates $(x, y, z)$. Let, for $a, b \in \mathbb{R}$,

$$\begin{aligned} u &= \sin x\, \partial_y + \cos x\, \partial_z + a\, dx, \\ v &= \cos x\, \partial_y - \sin x\, \partial_z, \\ w &= b\, \partial_x + \sin x\, dy + \cos x\, dz \end{aligned}$$

be generalized vector fields in $\Gamma(T + T^*)$. For which $a, b \in \mathbb{R}$ does the subbundle

$$L = span(u, v, w) \subset T + T^*$$

define a maximally isotropic subbundle of $T + T^*$? On the other hand, for which $a, b \in \mathbb{R}$ is $L$ involutive? Finally, for which $a, b \in \mathbb{R}$ does $L$ define a Dirac structure?

**Problem 3.** Prove that symplectic structures $\omega \in \Omega^2(M)$ on a manifold $M$ are in one-to-one correspondence with real Dirac structures $L$ such that $L \cap T = L \cap T^* = \{0\}$.

**Problem 4.** Let $J : T \to T$ be a complex structure, and $B : T \to T^*$ be a linear bundle map covering the identity map on $M$. When does

$$\mathcal{J} := \begin{pmatrix} J & 0 \\ BJ + J^* B & -J^* \end{pmatrix}$$

define a generalized almost complex structure? Assume now that $B \in \Omega^2(M)$, when does $\mathcal{J}$ define a generalized complex structure?

**Problem 5.** Consider the real manifold $\mathbb{C}^2$ with complex coordinates $(z_1, z_2)$. Does

$$\varphi = z_1 z_2 + dz_1 \wedge dz_2$$

define a generalized almost complex structure? Does it define a generalized complex structure? In any case, what is the type of $\varphi$?

**Problem 6.** Consider the real manifold $\mathbb{C}^3$ with complex coordinates $(z_1, z_2, z_3)$. Let $f(z_1, z_2, z_3)$ be a complex function $f : \mathbb{C}^3 \to \mathbb{C}$. When does

$$\psi = f(z_1, z_2, z_3) dz_1 + dz_1 \wedge dz_2 \wedge dz_3$$

define a generalized almost complex structure? When does it define a generalized complex structure? In any case, what is the type of $\psi$?