Generalized Geometry, an introduction
Assignment 3

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. The exterior algebra $\wedge^\bullet V^*$ is formally defined as the quotient of $\otimes^\bullet V^*$ by an ideal I. For $\alpha_1, \ldots, \alpha_k \in V^*$ we denote by $\alpha_1 \wedge \ldots \wedge \alpha_k$ the element $[\alpha_1 \otimes \ldots \otimes \alpha_k] \in \wedge^\bullet V^*/I$. Recall the identification given by

$$\alpha_1 \wedge \ldots \wedge \alpha_k \mapsto Alt_k(\alpha_1 \otimes \ldots \otimes \alpha_k) := \sum_{\sigma \in \Sigma_k} sgn(\sigma) \alpha_{\sigma 1} \otimes \ldots \otimes \alpha_{\sigma k} \in \otimes^\bullet V^*,$$

which allows to see $\wedge^\bullet V^*$ as a vector subspace of $\otimes^\bullet V^*$. As we will mainly use this representation, we will also denoted it by $\wedge^\bullet V^*$.

Prove that the product induced on $\wedge^\bullet V^* \subset \otimes^\bullet V^*$ corresponds to the **wedge product** defined as follows: for decomposable

$$\alpha = \alpha_1 \wedge \ldots \wedge \alpha_p \in \wedge^p V^*, \qquad \beta = \beta_1 \wedge \ldots \wedge \beta_q \in \wedge^q V^*,$$

where $\alpha_j, \beta_j \in V^*$, the product is given by

$$(\alpha_1 \wedge \ldots \wedge \alpha_p) \wedge (\beta_1 \wedge \ldots \wedge \beta_q) = \alpha_1 \wedge \ldots \wedge \alpha_p \wedge \beta_1 \wedge \ldots \wedge \beta_q,$$

and then it is extended linearly.

Problem 2. Let $e^1, e^2 \in V^*$ be linearly independent. From $e^1 \wedge e^2 = e^1 \otimes e^2 - e^2 \otimes e^1$, we see that $e^1 \wedge e^2 = -e^2 \wedge e^1$. Is it also true that for $\alpha \in \wedge^p V^*$, $\beta \in \wedge^q V^*$,

$$\alpha \wedge \beta = -\beta \wedge \alpha?$$

Problem 3. Let V be 4-dimensional with basis (e_1, e_2, e_3, e_4) and dual basis (e^1, e^2, e^3, e^4). Let

$$\omega = e^1 \wedge e^2, \qquad \omega' = e^1 \wedge e^2 + e^2 \wedge e^3, \qquad \omega'' = e^1 \wedge e^2 + e^3 \wedge e^4$$

be elements of $\wedge^2 V^*$, that is linear presymplectic structures. Regard them as maps $V \to V^*$ and tell if any of them is a linear symplectic structure.

Problem 4. Let V be an n-dimensional vector space.

- Compute the dimension of the vector spaces $\otimes^p V$, $\mathrm{Sym}^p V$, $\wedge^p V$.
- Use the notation $\omega^m := \underbrace{\omega \wedge \ldots \wedge \omega}_{m \text{ times}}$. Let $n = 2m$. Prove that the 2-form $\omega \in \wedge^2 V^*$ is non-degenerate if and only if $\omega^m \neq 0$.

Problem 5. The contraction by X is the linear map $i_X : \otimes^k V^* \to \otimes^{k-1} V^*$ linearly extending the correspondence

$$\alpha_1 \otimes \ldots \otimes \alpha_k \mapsto \alpha_1(X)\alpha_2 \otimes \ldots \otimes \alpha_k.$$

- Prove that the contraction maps $\wedge^k V^*$ onto $\wedge^{k-1} V^*$ and find a formula for $i_X(\alpha_1 \wedge \ldots \wedge \alpha_k)$.
- Prove that $i_X i_X \alpha = 0$ for $\alpha \in \wedge^k V^*$. What about $i_X i_X \varphi$ for $\varphi \in \otimes^k V^*$?

Problem 6. Consider $V + V^*$ with the canonical pairing

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)).$$

Recall the notion of signature of a pairing and show that this pairing has signature (n, n). Find bases of $V + V^*$ such that the pairing is given by the matrix

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that the dimension of an isotropic subspace is at most $\dim V$.

Problem 7. Let $\omega \in \otimes^2 V^*$ and $\pi \in \omega^2 V$, regarded as maps $V \to V^*$ and $V^* \to V$. Denote by gr the graph of map, that is,

$$gr(\omega) = \{X + \omega(X) \mid X \in V\}, \qquad gr(\alpha) = \{\pi(\alpha) + \alpha \mid \alpha \in V^*\}.$$

Prove that

- $\omega \in \wedge^2 V^*$ if and only if $gr(\omega)$ is maximally isotropic in $V + V^*$.
- $\pi \in \wedge^2 V$ if and only if $gr(\pi)$ is maximally isotropic in $V + V^*$.

Let L be a maximally isotropic subspace of $V + V^*$. Prove that

- $L \cap V^* = \{0\}$ if and only if $L = gr(\omega)$ for a unique $\omega \in \wedge^2 V^*$.
- $L \cap V = \{0\}$ if and only if $L = gr(\pi)$ for a unique $\pi \in \wedge^2 V$.