Generalized Geometry, an introduction
Assignment 1

Weizmann Institute
Second Semester 2017-2018

There is no formal submission of the assignments but you are expected to work on them.

Problem 1. Consider the vector space $\mathbb{R}^4$ with the symplectic form

$$\omega((x_1, y_1, x_2, y_2), (x'_1, y'_1, x'_2, y'_2)) = \sum_{i=1}^{2}(x_i y'_i - y_i x'_i).$$

- Find a subspace U such that $U = U^\omega$.
- Find a plane U, that is, a subspace isomorphic to $\mathbb{R}^2$, such that $U + U^\omega = \mathbb{R}^4$.

Problem 2. Prove that for any subspace $U \subset V$ of a symplectic subspace (V, ω),

- we have $U \oplus U^\omega = V$ if and only if $\omega_{|U}$ is a symplectic form. In this case U is called a symplectic subspace.
- we have $(U^\omega)^\omega = U$. In particular, U is symplectic if and only if U^ω is symplectic.

Problem 3. Show the existence of a basis for a symplectic vector space (V, ω) such that ω is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that a vector space V admits a linear symplectic structure if and only if $\dim V$ is even.

Problem 4. A subspace U of a symplectic vector space (V, ω) such that $U^\omega \subseteq U$ is called a coisotropic subspace. Prove that the quotient U/U^ω naturally inherits a symplectic structure. This is called the coisotropic reduction.

Problem 5. The invertible linear transformations of $\mathbb{R}^n$ are called the general linear group and denoted by $\mathrm{GL}(n, \mathbb{R})$. If they moreover preserve the euclidean metric, we have the orthogonal group $\mathrm{O}(n, \mathbb{R})$. For a vector space V, we write $\mathrm{GL}(V)$ and $\mathrm{O}(V, g)$ when V comes with a linear riemannian metric g.

- Show that a basis $\{v_i\}$ determines a linear riemannian metric by $g(v_i, v_j) = \delta_{ij}$, and hence any vector space admits a linear riemannian metric.

- * Can two different bases determine the same riemannian metric? If so, describe the space of bases determining the same riemannian metric.

- * Conversely, given a linear riemannian metric, how do you describe the space of metrics? You can try to do this in two different ways.

- * Describe the space of all linear riemannian metrics on a given vector space.