Dirac structures and generalized geometry

Problem Sheet 3

Thursday 21st January 2016,
hand in by Thursday 28th

Problem 1

Let $L \subset TM + T^*M$ be a Dirac structure on a manifold M. We say that a function $f \in \mathcal{C}^\infty(M)$ is admissible when there exists a vector field X_f such that $X_f + df \in \Gamma(L)$. Show that the admissible functions form a Poisson algebra with the usual product fg and the bracket defined by $\{f, g\} = X_f(g)$.

Problem 2

Find the two examples of distributions that we were missing:

a) a regular and smooth distribution that is not integrable,

b) a regular distribution that is neither smooth nor integrable.

Problem 3 Apart from the pairing $\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(i_X\beta + i_Y\alpha)$, we can define a skew-symmetric bilinear form by

$$\langle X+\alpha, Y+\beta\rangle_- = \frac{1}{2}(i_Y\alpha - i_X\beta).$$

Given a Dirac structure L with associated 2-form Ω_L, prove the identity

$$(\rho_{|L})^*\Omega_L = i_L^*(\langle\ ,\ \rangle_-),$$

where $\rho_{|L}$ is the restriction of ρ to L and i_L is the inclusion $L \to TM + T^*M$.

Problem 4

Let W be a subspace of the vector space V. We would like to be able to define a (linear) Dirac structure on $W + W^*$ from a (linear) Dirac structure $L \subset V + V^*$.

a) Can we just do $L \cap (W + W^*)$ in order to define a Dirac structure?

b) Look at L as $L(E, \varepsilon)$ for some E and ε. When will $L(E \cap W, \varepsilon_{|W})$ define a Dirac structure on $W + W^*$?

c) Consider the projection $\pi : W + V^* \to W + W^*$, and define $L_W = \pi(L \cap (W + V^*))$. Prove that $(L_W)^\perp = L_W$.

d) Prove that

$$L_W \cong \frac{L \cap (W + V^*)}{L \cap (0 + \mathrm{Ann}(W))}.$$

Problem 5

When working with linear Dirac structures, we showed that, by choosing a splitting $V + V^* = P + N$ into positive and negative-definite subspaces orthogonal to each other, the set of all Dirac structures could be seen as the Lie group $\mathrm{O}(n)$.

a) What can you say about the Dirac structure corresponding to $\mathrm{Id} \in \mathrm{O}(n)$?

b) Do you think there is a structure that is more suitable than Lie group? Looking at the possible splittings $P + N$ will be relevant.

c) Once we choose P, does the subspace $P^\perp$ necessarily define a negative-definite subspace, complementary to P?

d) Parameterize the possible choices of P. (*Hint: notice that* $P \cap V^* = 0$ *implies that* P *can be seen as the graph of a certain map, take the symmetric and skew-symmetric components of this and check what they must satisfy.*)

Problem 6

We define an action of $X + \alpha \in V + V^*$ on $\varphi \in \wedge V^*$ by

$$(X + \alpha) \cdot \varphi = i_X \varphi + \alpha \wedge \varphi.$$

a) Compute $(X + \alpha)^2 \cdot \varphi := (X + \alpha) \cdot ((X + \alpha) \cdot \varphi)$.

b) Show that

$$\mathrm{Ann}(\varphi) = \{(X + \alpha) \in V + V^* \mid (X + \alpha) \cdot \varphi = 0\}$$

is an isotropic space. Is it necessarily maximally isotropic?

Problem 7

Consider $V + V^*$ with the usual pairing. The linear transformations of $V + V^*$ preserving the pairing are the Lie group $\mathrm{O}(V + V^*)$.

a) What are the elements preserving both V and V^*? (i.e., $g \in \mathrm{O}(V + V^*)$ such that $g(V) \subset V$, $g(V^*) \subset V^*$)

b) Describe the elements of $\mathrm{O}(V + V^*)$ commuting with $\rho : V + V^* \to V$.

c) Let L be a Dirac structure and $g \in \mathrm{O}(V + V^*)$, is $g(L)$ a Dirac structure?

d) What is the image of a Dirac structure $L(E, \varepsilon)$ by an element $b \in \mathrm{O}(V + V^*)$ commuting with ρ?