Dirac structures and generalized geometry
Problem Sheet 2

Friday 15th January 2016,
hand in by Thursday 21st

Problem 1

We talked in the previous problem sheet about Lie algebroids, and how the bracket gives a differential on $\Gamma(\wedge^\bullet A^*)$ and the Lie derivative.

We can extend the Lie bracket to sections of $\wedge^\bullet A$, so that we get

$$[\ ,\ ] : \mathcal{C}^\infty(\wedge^k A) \times \mathcal{C}^\infty(\wedge^m A) \to \mathcal{C}^\infty(\wedge^{k+m-1} A)$$

extending the Lie bracket (when $k = m = 1$), acting on functions $f \in \mathcal{C}^\infty(M)$ by $[X, f] = \pi(X)(f)$ for $X \in \mathcal{C}^\infty(A)$, and satisfying the following properties, for $Z \in \mathcal{C}^\infty(\wedge^a A)$, $Z' \in \mathcal{C}^\infty(\wedge^b A)$ and $Z'' \in \mathcal{C}^\infty(\wedge^c A)$:

(S1): $[Z, Z'] = -(-1)^{(a-1)(b-1)}[Z', Z]$,

(S2): $[Z, [Z', Z'']] = [[Z, Z'], Z''] + (-1)^{(a-1)(b-1)}[Z', [Z, Z'']]$,

(S3): $[Z, Z' \wedge Z''] = [Z, Z'] \wedge Z'' + (-1)^{(a-1)b} Z' \wedge [Z, Z'']$.

This extension is unique and is called the Schouten bracket.

a) For $X \in \Gamma(A)$, $Z \in \Gamma(\wedge^a A)$ and $\omega \in \Gamma(\wedge^a A^*)$, we know what $\mathcal{L}_X \omega$ is, and we use the notation $\mathcal{L}_X Z = [X, Z]$ (Schouten bracket). Prove that

$$\mathcal{L}_X \langle \omega, Z \rangle = \langle \mathcal{L}_X \omega, Z \rangle + \langle \omega, \mathcal{L}_X Z \rangle.$$

(*Hint: prove it first for* $a = 1$ *and apply induction on decomposable elements.*)

b) Finally, let both A and A^* be Lie algebroids. Prove the following identity for $\xi, \eta \in \Gamma(A^*)$ and $X \in \Gamma(A)$,

$$i_X \mathcal{L}_\xi d\eta = [\xi, L_X \eta] - L_{L_\xi X} \eta + d(i_{L_\xi X} \eta) - L_\xi d\langle \eta, X \rangle.$$

(*Hint: Evaluate at* $Y \in \Gamma(A)$*, apply part a) on* $(\mathcal{L}_\xi d\eta)(X, Y)$*, then Cartan's magic formula conveniently, and rearrange in order to get the identity.*)

Problem 2

We have been enriching $\mathbb{T}M = TM + T^*M$ with some extra structure. We started by spotting the natural pairing $\langle X+\alpha, X+\alpha\rangle = i_X\alpha$, we singled out the anchor map $\rho(X+\alpha) = X$, and then the Dorfman bracket $[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - i_Y d\alpha$ joined the party. These three ingredients are related by the following equation.

a) Prove that, for $u, v, w \in \Gamma(\mathbb{T}M)$,

$$\rho(u)\langle v, w\rangle = \langle [u,v], w\rangle + \langle v, [u,w]\rangle.$$

Let us go abstract and consider a vector bundle $E \to M$ together with a bundle map $\rho : E \to TM$, a non-degenerate pairing $\langle , \rangle$, and an $\mathbb{R}$-bilinear product $[,] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ satisfying $[u,[v,w]] = [[u,v],w] + [v,[u,w]]$ for $u, v, w \in \Gamma(E)$, such that they all together satisfy

$$\rho(u)\langle v, w\rangle = \langle [u,v], w\rangle + \langle v, [u,w]\rangle.$$

b) Prove that, for $f \in \mathcal{C}^\infty(M)$ and $u, v \in \Gamma(E)$, we must have

$$[u, fv] = f[u,v] + \rho(u)(f)v.$$

c) Argue that, for $u, v \in \Gamma(E)$, we must also have

$$\rho([u,v]) = [\rho(u), \rho(v)].$$

The tuple $(E, \langle , \rangle, [,], \rho)$ is only missing a property to become a so-called Courant algebroid. But for this property, we need a differential

$$d : \mathcal{C}^\infty(M) \to \Gamma(E).$$

d) Find a differential by using ρ and the pairing $\langle , \rangle$. What property are you using? (*Hint: you may want to stop at* $\Gamma(E^*)$ *on the way.*)

e) What is the property that the bracket, the pairing and the usual differential satisfy in $TM + T^*M$?

As we will see, the generalization of this property is the remaining axiom in order to endow $(E, \langle , \rangle, [,], \rho)$ with the structure of a Courant algebroid.