Introduction to stochastic PDEs

Session 2

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Outline

- The stochastic heat equation in $[0, 1]$: large deviations, comparison theorem and some generalizations
- One-dimensional stochastic wave equation
- SPDEs in dimension $d > 1$: spatially correlated noise
  - The stochastic heat equation in $\mathbb{R}^d$, $d \geq 1$
  - The stochastic wave equation in $\mathbb{R}^d$, $d \in \{1, 2\}$
  - The stochastic wave equation in $\mathbb{R}^3$: *extension* of the stochastic integral
  - General result on existence and uniqueness of solution

Large deviations

Recall that we are dealing with the SPDE:

$$u(t,x) = \int_0^1 u_0(y) G_t(x,y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(ds,dy).$$

Let us consider the stochastic integral term:

$$V(t,x) := \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(ds,dy).$$

Theorem

There exist constants $C_1, C_2 > 0$ such that, for all $\lambda > 0$:

$$P\left(\sup_{(t,x)\in[0,T]\times[0,1]} |V(t,x)| > \lambda \right) \leq C_1 e^{-C_2\lambda^2}.$$

The proof is based on a generalization of the so-called Garsia-Rodemich-Rumsey lemma, thanks to Arnold-Imkeller (1996) and also Funaki-Kikuchi-Potthof (2006):

Lemma

Let $I = [0,T] \times [0,1]$ *and* $g : I \to \mathbb{R}$ *continuous. Assume that there exist two increasing functions* ψ *and* ρ *with* $\rho(0) = 0$ *and* $\lim_{z\to\infty} \psi(z) = +\infty$*, and such that:*

$$B := \int_I \int_I \psi\left(\frac{|g(t,x) - g(s,y)|}{\rho\left(|t-s|^{p_1} + |x-y|^{p_2}\right)}\right) dsdydxdt < \infty,$$

where $p_1, p_2 > 0$*. Then, for all* $(t,x), (s,y) \in I$*,*

$$|g(t,x) - g(s,y)| \le K \int_0^{|t-s|^{p_1}+|x-y|^{p_2}} \psi^{-1}\left(\frac{B}{r^{\frac{1}{p_1}+\frac{1}{p_2}}}\right) \rho(dr).$$

- This deterministic result is used to prove the Kolmogorov criterium.
- Up to some constant, $r^{\frac{1}{p_1}+\frac{1}{p_2}}$ is the area of the region determined by the points (t,x) such that $|t|^{p_1} + |x|^{p_2} \le r$.

Sketch of the proof (large deviations):

The idea is to apply the above lemma with $g = V$, $\psi(z) = e^{z^2}$, $\rho(z) = z$ and $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$. Then:

$$B = \int_I \int_I \exp\left(\frac{|V(t,x) - V(s,y)|^2}{\left(|t-s|^{\frac{1}{4}} + |x-y|^{\frac{1}{2}}\right)^2} \right) dsdydxdt.$$

- Using the martingale property of the stochastic integral and the fact that the paths of u are Hölder-continuous, we see that $E(B) < \infty$.
- Applying the lemma we end up with:

$$\sup_{(t,x)\in[0,T]\times[0,1]} |V(t,x)| \leq C|\log B|^{\frac{1}{2}}, \quad \mathbb{P}\text{-a.s.}$$

- Eventually, we apply Chebychev inequality with the function ψ:

$$P\left(\sup_{(t,x)\in[0,T]\times[0,L]} |V(t,x)| > \lambda \right) \leq C_1 e^{-C_2\lambda^2}.$$

□

Comparison theorem

Mueller (1991) and Donati-Martin and Pardoux (1993) prove the following:

Theorem

Let u_i, $i = 1, 2$, be solutions of the stochastic heat equation

$$u_i(t,x) = \int_0^1 u_{0,i}(y) G_t(x,y) dy + \int_0^t \int_0^1 G_{t-s}(x,y) \sigma(u_i(s,y)) W(ds,dy),$$

with initial condition $u_{0,i}$, $i = 1, 2$, respectively. If $u_{0,1}(x) \leq u_{0,2}(x)$ for all $x \in [0,1]$, then

$$u_1(t,x) \leq u_2(t,x),$$

for all $t \geq 0$ and $x \in [0,1]$, $\mathbb{P}$-a.s.

- *Discretize* the equations $\longrightarrow$ system of stochastic differential equations (not partial),
- apply the comparison theorem for (ordinary) SDEs.

Generalizations

We have studied the equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0,\ x\in[0,1], \tag{1}$$

$$u(0,x)=u_0(x),\ x\in[0,1], \qquad u(t,0)=u(t,1)=0,\ t>0.$$

We have given a sense to it through the mild formulation:

$$u(t,x) = \int_0^1 u_0(y)G_t(x,y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(ds,dy).$$

- We may add a non-linear term $b(u(t,x))$ (drift, b Lipschitz) to eq. (1):

$$\int_0^t\int_0^1 G_{t-s}(x,y)b(u(s,y))\ dyds \qquad \text{(pathwise integral).}$$

- The coefficients b and σ may depend on $(t,x,u(t,x))$.
- Neumann boundary conditions: $\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0$.

The stochastic heat equation in $\mathbb{R}$

Using the same techniques, we can deal with the SPDE:

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t > 0,\ x \in \mathbb{R},$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}.$$

The associated stochastic integral reads:

$$u(t,x) = \int_{\mathbb{R}} u_0(y) G_t(x-y) dy + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u(s,y)) W(ds,dy) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b(u(s,y))\, dyds.$$

where here $G_t(x)$ is the fundamental solution of the heat equation in $\mathbb{R}$ (heat kernel):

$$G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{2t}}.$$

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The stochastic wave equation

Almost all the results obtained for the stochastic heat equation are also valid for the one-dimensional stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0,\ x\in[0,1],$$

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \quad x\in[0,1],$$

with Dirichlet (or Neumann) boundary conditions:

$$u(t,0) = u(t,1) = 0, \quad t\geq 0.$$

As before, $\dot{W}$ denotes the space-time white noise. The mild solution is given by:

$$u(t,x) = \int_0^t \int_0^1 \Gamma_{t-s}(x,y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_0^1 \Gamma_{t-s}(x,y)b(u(s,y))\,dyds,$$

where $\Gamma_t(x,y)$ is the Green function of the wave equation in $[0,1]$ with Dirichlet boundary conditions (its expression is a bit nasty).

Or one may also consider the stochastic wave equation in $\mathbb{R}$:

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)b(u(s,y))\, dyds,$$

where $\Gamma_t(x)$ is the fundamental solution of the ware equation in $\mathbb{R}$:

$$\Gamma_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}, \quad t \geq 0 \quad x \in \mathbb{R}.$$

Thus, $(s,y) \mapsto \Gamma_{t-s}(x-y)$ is the indicator function of the (inverted) light cone with apex (t,x)

$\longrightarrow$ **R. Carmona and D. Nualart.** Random nonlinear wave equations: smoothness of the solutions. Probab. Theory Related Fields 79 (1988), no. 4, 469–508.

What if $x \in \mathbb{R}^2$?

Consider the following stochastic heat equation:

$$\frac{\partial u}{\partial t}(t,x) = \left[\frac{\partial^2 u}{\partial x_1^2}(t,x) + \frac{\partial^2 u}{\partial x_2^2}(t,x)\right] + \dot{W}(t,x), \quad t > 0,\ x = (x_1, x_2)^* \in \mathbb{R}^2,$$

$$u(0,x) = 0,\ x \in \mathbb{R}^2.$$

- $\dot{W}$ stands for a white noise in $\mathbb{R}_+ \times \mathbb{R}^2$.
- In general, on $\mathbb{R}^d$, the operator $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is called the Laplacian.

The mild solution would simply be

$$u(t,x) = \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y) W(ds, dy),$$

where $G_t(x)$ is the heat kernel in $\mathbb{R}^2$:

$$G_t(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{2t}}, \quad x \in \mathbb{R}^2,$$

and $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^2$.

In order that $\{u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ defines a mild solution, we only need to check that the stochastic integral

$$u(t,x) = \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x-y) W(ds,dy)$$

is well-defined.

- for all (t,x), $u(t,x)$ is the Wiener integral of $G_{t-\cdot}(x-\star)$:

$$u(t,x) = \dot{W}\left(G_{t-\cdot}(x-\star)\right).$$

- Hence, $u(t,x)$ is a mean zero Gaussian random variable.

Its variance is given by:

$$\begin{aligned} E(|u(t,x)|^2) &= \int_0^t \int_{\mathbb{R}^2} |G_{t-s}(x-y)|^2 \, dyds = \int_0^t \int_{\mathbb{R}^2} |G_s(y)|^2 \, dyds \\ &= \int_0^t \int_{\mathbb{R}^2} \frac{1}{16\pi^2 s^2} e^{-\frac{|y|^2}{2s}} \, dyds = C \int_0^t s^{-1} \, ds = +\infty!!! \end{aligned}$$

$\Longrightarrow$ there is not a mild solution of the stochastic heat equation in $\mathbb{R}^2$.

We have exactly the same problem for the stochastic wave equation in $\mathbb{R}^2$:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t>0,\ x\in\mathbb{R}^2,$$

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \quad x\in\mathbb{R}^2.$$

Mild solution:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^2} \Gamma_{t-s}(x-y) W(ds,dy) = \dot{W}\left(\Gamma_{t-\cdot}(x-\star)\right),$$

where here $\Gamma_t(x)$ is the fundamental solution of the wave equation in $\mathbb{R}^2$:

$$\Gamma_t(x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}$$

Then

$$E|u(t,x)|^2 = \frac{1}{4\pi^2}\int_0^t \int_{\mathbb{R}^2} \frac{1}{(t-s)^2-|x-y|^2}\mathbf{1}_{\{|x-y|<t-s\}}\, dyds = +\infty.$$

Spatially correlated noise (Dalang and Frangos 1998, Dalang 1999)

In order to obtain well-defined random field solutions, we consider a slightly more regular noise $\dot{W}(t,x)$:

- Mean zero Gaussian noise $\dot{W}(t,x)$ with covariance (formal):

$$E(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(t-s)f(x-y), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \qquad (d \geq 1)$$

- White correlation in time (Brownian) and colored in space.

More rigorously: let $\{\dot{W}(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ be a family of mean zero Gaussian r.v. with the following covariance functional:

$$\begin{aligned} E(\dot{W}(\varphi)\dot{W}(\psi)) &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \varphi(t,x)f(x-y)\psi(t,y) \\ &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \varphi(t,x)\delta_0(t-s)f(x-y)\psi(t,y), \end{aligned}$$

- $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous in $\mathbb{R}^d \setminus \{0\}$.
- Covariance functional $\Longrightarrow f$ symmetric.

(usually we write $W(\varphi)$)

Remarks

$$E(\dot{W}(\varphi)\dot{W}(\psi)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x) f(x-y) \psi(t,y)$$

- We say that $\dot{W}$ is a spatially homogeneous noise.
- The case f bounded has been treated by Mueller 1997.
- Applications to physics: spatial correlations of larger order than those of time $\Rightarrow$ better models than the space-time white noise.
- Indeed, one may consider even a more general correlation: non-negative tempered measure.

The spectral measure associated to the noise:

There exists a non-negative tempered measure μ such that $\mathcal{F}\mu = f$. Namely:

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^k} \mu(d\xi) < \infty, \quad \text{for some} \quad k \geq 1,$$

$$\mathcal{F}\mu = f \iff \int_{\mathbb{R}^d} f(x)\phi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(d\xi), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

By means of a change of variables and applying that $\mathcal{F}\mu = f$, we obtain:

$$
\begin{aligned}
E(\dot{W}(\varphi)\dot{W}(\psi)) &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \; \varphi(t,x) f(x-y) \psi(t,y) \\
&= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dy \; f(y) \int_{\mathbb{R}^d} dx \; \varphi(t,x)\psi(t,x-y) \\
&= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dy \; f(y) \left(\varphi(t) * \tilde{\psi}(t) \right)(y) \qquad (\tilde{\psi}(t,y) := \psi(t,-y)) \\
&= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\varphi(t)(\xi) \overline{\mathcal{F}\psi(t)(\xi)}.
\end{aligned}
$$

In particular:

$$
E(|\dot{W}(\varphi)|^2) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\varphi(t)(\xi)|^2, \quad \varphi \in \mathcal{D}(\mathbb{R}^{d+1}).
$$

Examples

1. The space-time white noise: $f(x) = \delta_0(x)$

$$E(\dot{W}(\varphi)\dot{W}(\psi)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x)\delta_0(x-y)\psi(t,y) = \langle \varphi, \psi \rangle^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}$$

The spectral measure μ is the Lebesgue measure on $\mathbb{R}^d$ and

$$E(|\dot{W}(\varphi)|^2) = \|\varphi\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d)} = \int_{\mathbb{R}_+} \|\mathcal{F}\varphi(t)\|^2_{L^2(\mathbb{R}^d)}\, dt \qquad \text{(Plancherel).}$$

2. The Riesz kernels: for $x \in \mathbb{R}^d$ and $0 < \alpha < 2 \wedge d$, let $f_\alpha(x) = |x|^{-\alpha}$. It holds that $f_\alpha = \mathcal{F} f_{d-\alpha}$, thus:

$$\begin{aligned} E(|\dot{W}(\varphi)|^2) &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x)|x-y|^{-\alpha}\varphi(t,y) \\ &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^d} d\xi\, |\xi|^{d-\alpha} |\mathcal{F}\varphi(t)(\xi)|^2. \end{aligned}$$

The isonormal process

1. Let $\mathcal{H}$ be the Hilbert space which is the closure of $\mathcal{S}(\mathbb{R}^d)$ with respect to the inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \; \phi_1(x) f(x-y) \phi_2(y), \quad \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

The space $\mathcal{H}$ is sometimes called the reproducing kernel Hilbert space. Set $\mathcal{H}_T := L^2([0,T];\mathcal{H})$.

2. Fix $T > 0$. Then the map $\varphi \mapsto \dot{W}(\varphi)$ defines an isometry:

$$\left(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_T}\right) \longrightarrow \left(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}\right)$$

where

$$\|h\|^2_{\mathcal{H}_T} = \int_0^T \|h(s)\|^2_{\mathcal{H}} \, ds.$$

3. This let us extend $\dot{W}(\varphi)$ to a mean-zero Gaussian family $\{\dot{W}(h), \; h \in \mathcal{H}_T\}$ such that:

$$E(\dot{W}(h)\dot{W}(g)) = \langle h, g \rangle_{\mathcal{H}_T} = \int_{\mathbb{R}_+} \langle g(s), h(s) \rangle_{\mathcal{H}} \, ds, \quad g, h \in \mathcal{H}_T.$$

The martingale measure

Using an approximation argument by means of *smooth* functions, we prove that $\mathcal{H}$ contains elements of the form $h = \mathbf{1}_A$ with $A \in \mathcal{B}_0(\mathbb{R}^d)$. Then

$$W_t(A) := \dot{W}\left(\mathbf{1}_{[0,t]}(\cdot)\mathbf{1}_A(\star)\right), \qquad t \geq 0,\ A \in \mathcal{B}_0(\mathbb{R}^d)$$

defines a worthy martingale measure with respect to the *natural* filtration.

The covariance measure coincides with the dominant measure and they are given by:

$$Q([0,t] \times A \times B) = \langle W(A), W(B) \rangle_t = t \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \mathbf{1}_A(x) f(x-y) \mathbf{1}_B(y).$$

Hence, we can define the stochastic integral

$$(Y \cdot W)_t(A) = \int_0^t \int_A Y(s,y) W(ds,dy), \quad t \in [0,T],\ A \in \mathcal{B}_0(\mathbb{R}^d),$$

provided that $\{Y(s,y),\ (s,y) \in [0,T] \times \mathbb{R}^d\}$ is a predictable process such that:

$$E\left(\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ |Y(s,x)| f(x-y) |Y(s,y)|\right) < +\infty.$$

Remarks

- The isonormal process $\{\dot{W}(h),\ h \in \mathcal{H}_T\}$ would let us apply the techniques of the Malliavin calculus.
- As mentioned before, in the case of the space-time white noise, we have $\mathcal{H} = L^2(\mathbb{R}^d)$. However, in the present situation, $\mathcal{H}$ is a larger space and indeed may contain distributions:

$$\begin{aligned}\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ \phi_1(x) f(x-y) \phi_2(y) \\ &= \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\phi_1(\xi) \overline{\mathcal{F}\phi_2(\xi)}, \quad \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).\end{aligned}$$

- The space $\mathcal{H}$ can be identified with the elements Φ such that

$$\|\Phi\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Phi(\xi)|^2 < +\infty.$$

Later on, we will give examples of Φ that are not functions.

The stochastic heat equation in $\mathbb{R}^d$

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t>0,\ x \in \mathbb{R}^d,$$
$$u(0,x) = 0, \quad x \in \mathbb{R}^d.$$

Here, $\dot{W}$ stands for the spatially homogeneous noise on $\mathbb{R}^d$. The mild solution, if exists, is given by the Wiener integral

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y) W(ds,dy) = \dot{W}(G_{t-\cdot}(x-\star)), \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

where $G_t(x) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right)$. This is well-defined provided that

$$\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\ G_{t-s}(x-y) f(y-z) G_{t-s}(x-z) < +\infty,$$

because, for any $t>0$, $G_t(\star) \in \mathcal{S}(\mathbb{R}^d)$. This exhibits the relation between the differential operator and the noise.

Let us express this condition in terms of the spectral measure μ (recall that $\mathcal{F}f = \mu$): again because $G_t(\star) \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, G_{t-s}(x-y) f(y-z) G_{t-s}(x-z)$$
$$= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_s(x-\cdot)(\xi)|^2 = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_s(\xi)|^2$$
$$= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-8\pi^2 s|\xi|^2} = \int_{\mathbb{R}^d} \mu(d\xi)\, \frac{1}{8\pi^2|\xi|^2} \left(1 - e^{-8\pi^2 t|\xi|^2}\right).$$

The latter expression is finite if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < +\infty \qquad \text{(for all} \quad d \geq 1)$$

Recall that $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

The condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\,\mu(d\xi) < +\infty \tag{2}$$

can be expressed in terms of f, but we do not have such a unified expression:

- If $d = 1$, (2) is always true.
- If $d = 2$, (2) is equivalent to

$$\int_{|x|\leq 1} f(x) \log \frac{1}{|x|}\, dx < +\infty.$$

- If $d \geq 3$, (2) is equivalent to

$$\int_{|x|\leq 1} f(x) \frac{1}{|x|^{d-2}}\, dx < +\infty.$$

This has been proved by Dalang 1999.

Mild solution

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0,\ x\in\mathbb{R}^d,$$
$$u(0,x) = u_0, \quad x \in \mathbb{R}^d$$

- $\dot{W}$ is the spatially homogeneous noise on $\mathbb{R}^d$.
- $b, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions.
- The initial condition $u_0 : \mathbb{R}^d \to \mathbb{R}$ is measurable and bounded.

By definition, the mild solution is an adapted process $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfying:

$$u(t,x) = (G_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)b(u(s,y))\, dyds. \qquad (3)$$

Existence and uniqueness of solution

Theorem

Assume that b, σ are Lipschitz, u_0 is measurable and bounded and that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty \qquad \left(\Leftrightarrow \int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}G_t(\xi)|^2 < \infty \right)$$

Then, there exists a unique mild solution to equation (3). Moreover, for all $p \geq 1$:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < +\infty.$$

$\longrightarrow$ Dalang 1999, Dalang and QS 2010.

The proof is based on a Picard iteration scheme and the following version of the Gronwall lemma:

Lemma (Dalang 1999)

Let $g : [0, T] \to \mathbb{R}$ *be a non-negative function such that*

$$\int_0^T g(s)ds < +\infty.$$

Let $\{f_n,\ n \geq 1\}$ *be a sequence of non-negative functions on* $[0, T]$ *and* $k_1, k_2 > 0$ *such that, for all* $t \in [0, T]$*:*

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(t-s)ds.$$

Then

$$\sup_{n\geq 0} \sup_{t\in[0,T]} f_n(t) < \infty$$

and if $k_1 = k_2 = 0$*, then* $\sum_{n\geq 0} f_n(t)$ *converges uniformly* $[0, T]$*.*

The stochastic wave equation with $d \in \{1, 2\}$

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t > 0,\ x \in \mathbb{R}^d,$$
$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \quad x \in \mathbb{R}^d \quad \text{(for simplicity)}.$$

The mild form is given by:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma^d_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_{\mathbb{R}^d} \Gamma^d_{t-s}(x-y)b(u(s,y))\, dyds,$$

where the fundamental solution $\Gamma^d_t(x)$, $d \in \{1, 2\}$ is given by:

$$\Gamma^1_t(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}, \qquad \Gamma^2_t(x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2 - |x|^2}}\mathbf{1}_{\{|x|<t\}}.$$

Dalang 1999: under the condition

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Gamma_t^d(\xi)|^2 < +\infty \qquad (d = 1, 2) \tag{4}$$

there exists a unique mild solution with uniformly finite moments of any order. Indeed, condition (4) is also equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < +\infty.$$

This is because there is a unified expression of $\mathcal{F}\Gamma_t^d$ for any dimension $d \geq 1$:

$$\mathcal{F}\Gamma_t^d(\xi) = \frac{\sin(2\pi t |\xi|)}{2\pi |\xi|},$$

so that one proves:

$$\frac{C_1}{1 + |\xi|^2} \leq \int_0^T \frac{\sin^2(2\pi t |\xi|)}{4\pi^2 |\xi|^2} dt \leq \frac{C_2}{1 + |\xi|^2}.$$

Remarks

- Stochastic heat equation with $d \geq 1$: the path properties have been studied in Sanz-Solé and Sarrà 2002.
- Stochastic wave equation with $d \in \{1, 2\}$:
  - $d = 1$ with the space-time white noise: Carmona and Nualart 1988.
  - $d = 2$ with the spatially homogeneous noise: Mueller 1997, Dalang and Frangos 1998 and Dalang 1999.
  - Path properties: Dalang and Sanz-Solé 2005.
- Extension to other types of SPDEs: the damped wave equation (Dalang 1999):

$$\frac{\partial^2 u}{\partial t^2}(t,x) + c\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x), \quad x \in \mathbb{R}^d, \; d \geq 1,$$

where $c \neq 0$. The condition $\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-1} \mu(d\xi) < \infty$ is also sufficient to have existence and uniqueness of mild solution.

Recall

- We have considered the stochastic heat and wave equations with space dimension $d \geq 1$ and $d \in \{1, 2\}$, respectively.
- The noise $\dot{W}$ is white in time and with a spatially homogeneous correlation f, whose spectral measure is μ:

$$E(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(t-s)f(x-y)$$

- We have obtained existence and uniqueness of mild solution provided that

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\,\mu(d\xi) < +\infty.$$

- The stochastic integral in the mild form is with respect to the martingale measure associated to the noise.

Problem!

Consider the stochastic wave equation in $\mathbb{R}^3$ (we take $b = 0$):

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad t > 0,\ x \in \mathbb{R}^3,$$

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, \quad x \in \mathbb{R}^3.$$

The mild solution should be given by the solution of

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma^3_{t-s}(x-y)\sigma(u(s,y))W(ds,dy)$$

where $\Gamma^3_t(x)$ is the fundamental solution of the wave equation in $\mathbb{R}^3$:

$$\Gamma^3_t = \frac{1}{4\pi t}\sigma_t,$$

where σ_t is the uniform surface measure on the 3-dimensional sphere of radius t. Thus Γ^3_t is a measure!

With respect to the martingale measure associated to $\dot{W}$, we are only able to integrate functions!

In order to be able to study mild solutions to the stochastic wave equation in $\mathbb{R}^3$, we need to extend the stochastic integral:

$$\int_0^t \int_{\mathbb{R}^3} Y(s, y) W(ds, dy),$$

- the integral takes values in $\mathbb{R}$, but
- the process Y satisfies that, for any $t > 0$, $Y(t)$ does not need to be a function

Extensions of the stochastic integral:

- Dalang 1999: extension of Walsh stochastic integral with respect to worthy martingale measures.
- Nualart and QS 2007 (also Dalang and QS 2010): stochastic integral with respect to the cylindrical Wiener process associated to $\dot{W}$, covering Walsh and Dalang's integral.

Cylindrical Wiener process associated to $\dot{W}$

- Hilbert space $\mathcal{H}$: the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \phi_1(x) f(x-y) \phi_2(y), \quad \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

  We set $\mathcal{H}_T = L^2([0,T]; \mathcal{H})$.

- We had $\{\dot{W}(h),\ h \in \mathcal{H}_T\}$ the isonormal Gaussian process associated to $\dot{W}$: mean zero Gaussian family with

$$E(\dot{W}(h_1)\dot{W}(h_2)) = \langle h_1, h_2 \rangle_{\mathcal{H}_T}, \quad h_1, h_2 \in \mathcal{H}_T.$$

Definition

We define:

$$W_t(g) = \dot{W}(\mathbf{1}_{[0,t]}(\cdot) g(\star)), \quad t \in [0,T],\ g \in \mathcal{H}.$$

Then, $\{W_t(g),\ t \in [0,T],\ g \in \mathcal{H}\}$ is called a cylindrical Wiener process on the Hilbert space $\mathcal{H}$: for any $g \in \mathcal{H}$, $\{W_t(g),\ t \in [0,T]\}$ is a Brownian motion with variance $t\|g\|^2_{\mathcal{H}}$ and

$$E(W_s(g_1) W_t(g_2)) = (s \wedge t) \langle g_1, g_2 \rangle_{\mathcal{H}}.$$

Stochastic integration

- There is a *natural and simple* procedure to define stochastic integrals with respect to $W_t(g)$.
- The space of square-integrable processes taking values in the Hilbert space $\mathcal{H}$ is denoted by $L^2(\Omega \times [0,T];\mathcal{H})$.
- Let $(e_n)_{n\geq 1}$ be a complete orthonormal system of $\mathcal{H}$.

Definition

Let $Y = \{Y(s,y),\, (s,y) \in [0,T] \times \mathbb{R}^d\}$ be a predictable process in $L^2(\Omega \times [0,T];\mathcal{H})$. The stochastic integral of Y with respect to the cylindrical Wiener process $W_t(g)$ is defined by

$$(Y \cdot W)_t := \sum_{n=1}^{\infty} \underbrace{\int_0^t \langle Y(s), e_n \rangle_{\mathcal{H}} dW_s(e_n)}_{\text{Itô integral}}$$

One proves that the series converges in $L^2(\Omega)$ and the limit does not depend on the chosen CONS.

Remarks

- We also use the notation:

$$\int_0^t \int_{\mathbb{R}^d} Y(s,y) W(ds,dy) := (Y \cdot W)_t, \quad t \in [0,T],$$

- $\{(Y \cdot W)_t, \mathcal{F}_t, \ t \in [0,T]\}$ is a square-integrable martingale.

- Isometry property: let $\mathcal{M}$ be the space square-integrable and continuous martingales $X = \{X_t, \ t \in [0,T]\}$ endowed with the norm $\|X\| := E(|X_T|^2)$. Then, the integral defines an isometry

$$L^2(\Omega \times [0,T]; \mathcal{H}) \longrightarrow \mathcal{M}$$

That is:

$$E\left(\int_0^T \|Y(s)\|^2_{\mathcal{H}} \, ds\right) = E\left(\left|\int_0^T \int_{\mathbb{R}^d} Y(s,y) W(ds,dy)\right|^2\right).$$

$$(Y \cdot W)_t := \sum_{n=1}^{\infty} \int_0^t \langle Y(s), e_n \rangle_{\mathcal{H}} dW_s(e_n)$$

- This integral could have been defined following the *usual* procedure: simple processes...
- Let $\{W_t(A), \mathcal{F}_t, \ t \in [0,T], A \in \mathcal{B}_0(\mathbb{R}^d)\}$ be the martingale measure associated to $\dot{W}$. Using Walsh integral, recall that we can integrate any function-valued process $Y(s,y)$ such that

$$E\left(\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, |Y(s,x)| f(x-y) |Y(s,y)| \right) < +\infty. \tag{5}$$

Proposition (Nualart and QS 2007, Dalang and QS 2010)

If Y is a predictable process satisfying (5), then Y belongs to $L^2(\Omega \times [0,T]; \mathcal{H})$ and the stochastic integrals coincide.

We have an analogous result for Dalang's extension of Walsh stochastic integral.

- From now on, all the stochastic integrals will be considered with respect to the cylindrical Wiener process $\{W_t(g),\ t \in [0,T], g \in \mathcal{H}\}$ associated to the noise $\dot{W}$.
- Recall that we aim to give sense to integrals of the form

$$\int_0^t \int_{\mathbb{R}^d} S(s,y) Z(s,y) W(ds,dy),$$

where $S(s)$ may not be a function and Z is some real-valued process.

- For example, in the case of the stochastic wave equation in $\mathbb{R}^3$:

$$S(s) = \frac{1}{4\pi(t-s)} \sigma_{t-s} \qquad Z(s,y) = \sigma(u(s,y))$$

and σ_t is a measure.

Examples of integrands

(H) Let $t \mapsto S(t)$ be a deterministic function such that, for all t, $S(t)$ is a non-negative distribution with rapid decrease such that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S(t)(\xi)|^2 < +\infty \qquad \sup_{t\in[0,T]} S(t, \mathbb{R}^d) < +\infty.$$

- $S(t)$ non-negative distribution with rapid decrease: non-negative measure on $\mathbb{R}^d$ such that
$$\int_{\mathbb{R}^d} (1 + |x|^2)^k \, S(t, dx) < +\infty \quad \text{for all} \quad k \in \mathbb{N}.$$
- The fundamental solutions of the heat and wave equations in $d \geq 1$ and $d \in \{1, 2, 3\}$, respectively, satisfy these assumptions and

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S(t)(\xi)|^2 < +\infty \Longleftrightarrow \int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < +\infty.$$

Proposition (Nualart and QS 2007)

- *Assume that S satisfies condition **(H)**.*
- *Let $Z = \{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ be a predictable process such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|Z(t,x)|^2) < \infty.$$

Then, the random element $G = G(t,dx) = Z(t,x)S(t,dx)$ defines a predictable process in $L^2(\Omega \times [0,T]; \mathcal{H})$. Moreover, we have that

$$E\left(\int_0^T \|G(s)\|_{\mathcal{H}}^2\, ds\right) = E\left(\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(G(s)Z(s))(\xi)|^2\right)$$
$$= E(|G\cdot W|^2) \le \int_0^T ds \sup_{y\in\mathbb{R}^d} E(|Z(s,y)|^2) \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S(s)(\xi)|^2.$$

We have used the notation $G\cdot W := (G\cdot W)_T$.

Remarks

- As already pointed out, this proposition let us define the stochastic integral of $G = G(t, dx) = Z(t, x)S(t, dx)$ with respect to the cylindrical Wiener process associated to the noise $\dot{W}$:

$$G \cdot W = \int_0^T \int_{\mathbb{R}^d} S(s, y)Z(s, y)W(ds, dy) \qquad \text{(formal notation)}$$

- If we have that, for some $p \geq 2$:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|Z(t, x)|^p) < +\infty,$$

then

$$E(|G \cdot W|^p) \leq \int_0^T ds \sup_{y\in\mathbb{R}^d} E(|Z(s, y)|^p) \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S(s)(\xi)|^2.$$

Proof of the proposition

Step 1. Prove that S belongs to $L^2([0,T];\mathcal{H})$:

- Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ be non-negative, with compact support contained in $B_1(0)$, and such that $\int_{\mathbb{R}^d} \psi(x)dx = 1$. For $n \geq 1$, we set $\psi_n(x) := n^d \psi(nx)$. Then $(\psi_n)_n$ defines an approximation of the identity.

- We smooth S:
$$S_n(t) := S(t) * \psi_n \in \mathcal{S}(\mathbb{R}^d)$$
$$|\mathcal{F}S_n(t)(\xi)| \leq |\mathcal{F}\psi_n(\xi)||\mathcal{F}S(t)(\xi)| \leq |\mathcal{F}S(t)(\xi)|.$$

- We have that $\{S_n,\ n \geq 1\} \subset L^2([0,T];\mathcal{H})$ and we check $S_n \to S$ in $L^2([0,T];\mathcal{H})$.

Step 2. Prove that $G = G(t, dx) = Z(t, x)S(t, dx)$ belongs $L^2([0, T] \times \Omega; \mathcal{H})$:

- We can assume that Z is non-negative.
- We prove that, $\mathbb{P}$-a.s., $G = Z(t, x)S(t, dx)$ defines a non-negative distribution with rapid decrease:

$$\int_{\mathbb{R}^d} (1 + |x|^2)^{\frac{k}{2}} Z(t, y)S(t, dy) < +\infty, \quad \text{for all} \quad k \in \mathbb{N}, \ \mathbb{P}\text{-a.s.}$$

- We regularize G also by means of an approximation of the identity.

□

Back to SPDEs

We are in position to state an existence and uniqueness theorem for a general class of SPDEs, including heat and wave equations (the latter with $d \in \{1, 2, 3\}$):

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x), \quad t > 0, \ x \in \mathbb{R}^d$$

where L is a partial differential operator.

Initial conditions:

- **Parabolic case:**

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d.$$

- **Hyperbolic case:**

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = v_0(x), \quad x \in \mathbb{R}^d.$$

The functions b, σ are Lipschitz and $\dot{W}$ is the spatially homogeneous noise.

Mild solution

By definition, the mild solution is an adapted process $\{u(t,x),\,(t,x)\in[0,T]\times\mathbb{R}^d\}$ satisfying

$$\begin{aligned} u(t,x) =& I_0(t,x) + \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y)b(u(s,y))\, dyds, \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

The stochastic integral is with respect to the cylindrical Wiener process obtained from $\dot{W}$, $S_t(x)$ is the fundamental solution associated to the operator L and the term $I_0(t,x)$ is the contribution of the initial conditions:

- **Parabolic case:**

$$I_0(t,x) = \int_{\mathbb{R}^d} u_0(x-y)S(t,dy),$$

- **Hyperbolic case:**

$$I_0(t,x) = \int_{\mathbb{R}^d} v_0(x-y)S(t,dy) + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^d} u_0(x-y)S(t,dy)\right).$$

Heat equation:

$$I_0(t,x)=\int_{\mathbb{R}^d} u_0(x-y)(4\pi t)^{-d/2}e^{-\frac{|y|^2}{4t}}\,dy, \quad x\in\mathbb{R}^d.$$

Wave equation:

- For $d=1$, $I_0(t,x)$ is given by the so-called d'Alembert's formula

$$I_0(t,x)=\frac{1}{2}\left[u_0(x+t)+u_0(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy, \qquad x\in\mathbb{R}.$$

- For $d=2$:

$$I_0(t,x)=\frac{1}{2\pi t}\int_{|x-y|<t}\frac{u_0(y+tv_0)+\nabla u_0(y)\cdot(x-y)}{(t^2-|x-y|^2)^{1/2}}\,dy, \qquad x\in\mathbb{R}^2.$$

- For $d=3$, $x\in\mathbb{R}^3$:

$$I_0(t,x)=\frac{1}{4\pi t^2}\int_{\mathbb{R}^3}\left(tv_0(x-y)+u_0(x-y)+\nabla u_0(x-y)\cdot y\right)\sigma_t(dy).$$

Lemma (Dalang and QS 2010)

Consider the following two sets of hypotheses:

- ***Heat equation:*** $u_0 : \mathbb{R}^d \to \mathbb{R}$ *is measurable and bounded.*
- ***Wave equation:*** *When* $d = 1$, u_0 *is bounded and continuous, and* v_0 *is bounded and measurable. When* $d = 2$, $u_0 \in C^1(\mathbb{R}^2)$ *and there is* $q \in]2, \infty]$ *such that* $u_0, \nabla u_0, v_0$ *all belong to* $L^q(\mathbb{R}^2)$*. When* $d = 3$, $u_0 \in C^1(\mathbb{R}^3)$, u_0 *and* ∇u_0 *are bounded, and* v_0 *is bounded and continuous.*

Then $(t, x) \mapsto I_0(t, x)$ *is continuous and*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |I_0(t, x)| < +\infty.$$

Main result

$$u(t,x) = I_0(t,x) + \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_{\mathbb{R}^d} S_{t-s}(x-y)b(u(s,y))\, dyds, \qquad \mathbb{P}\text{-a.s.} \qquad (6)$$

Theorem (Dalang 1999, Dalang and QS 2010)

Assume that, for all t, $S(t)$ is a non-negative distribution with rapid decrease such that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}S(t)(\xi)|^2 < +\infty \qquad \sup_{t\in[0,T]} S(t,\mathbb{R}^d) < +\infty,$$

I_0 is continuous and uniformly bounded, and b and σ are Lipschitz functions. Then there exists a unique solution $\{u(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^d\}$ of equation (6). Moreover, the process u is L^2-continuous and for all $p\geq 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < +\infty.$$

Remarks

- The proof is based on a Picard iteration scheme.
- The fundamental solution of the heat and wave equations in any space dimension and with $d \in \{1,2,3\}$, respectively, satisfy the assumptions of the theorem provided that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}S(t)(\xi)|^2 < +\infty \Leftrightarrow \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < +\infty.$$

- This result is also valid for the damped wave equation and a class of parabolic equations with time-dependent coefficients.
- We are not able to go beyond $d = 3$ in the stochastic wave equation, because the fundamental solution becomes more irregular and it is no more non-negative definite:

$$\Gamma_t^d = C \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(d-3)/2} \frac{\sigma_t}{t}, \quad d \geq 3 \quad \text{odd},$$

$$\Gamma_t^d(x) = C \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(d-3)/2} (t^2 - |x|^2)_+^{-1/2}, \quad d \geq 2 \quad \text{even}.$$

Even more extensions

- Conus and Dalang 2008 have extended the stochastic integral with respect to the martingale measure associated to the spatially homogeneous noise, in order to be able to integrate random elements of the form $S(s, y)Z(s, y)$, where
- S is a function in $[0, T]$ with values in some space of distributions, without the restriction of being non-negative definite.
- With this integral, they prove existence and uniqueness of solution for the stochastic wave equation in any space dimension.
- But they are not able to prove that the solution has moments or order $p > 2$.

$\longrightarrow$ SPDEs with spatially homogeneous noise in an infinite dimensional setting: works by Da Prato and Zabczyk, Karczewska and Zabczyk, Peszat and Zabczyk.

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