Existence and smoothness of the density for spatially homogeneous SPDEs

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Spatially homogeneous SPDEs

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x) \qquad (1)$$

$$t \geq 0,\ x \in \mathbb{R}^d,\ d \geq 1$$

- L is a second order differential operator,
- vanishing initial conditions,
- $\{u(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a real-valued process,
- $b, \sigma : \mathbb{R} \to \mathbb{R}$ Lipschitz,
- $\dot{W}$ is a Gaussian random perturbation.
- Main examples: wave equation and heat equation:

$$L = \frac{\partial^2}{\partial t^2} - \Delta \quad \text{and} \quad L = \frac{\partial}{\partial t} - \Delta,$$

Δ is the Laplacian operator on $\mathbb{R}^d$.

Description of the noise

- We want real-valued solutions, thus we need to impose some spatial correlation on the noise. Roughly speaking:

$$E(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)f(x-y).$$

- Rigorously, $\{W(\varphi), \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})\}$ is a centered Gaussian process in some $(\Omega, \mathcal{F}, P)$, with

$$E(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y)\,dxdydt,$$

  where $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d \setminus \{0\}$.
- Example: Riesz kernel $f(x) = C_{\nu,d}|x|^{\nu-d}$, $0 < \nu < d$.

$\Longrightarrow$ f is symmetric and $f = \mathcal{F}\mu$, where μ is a non-negative tempered measure on $\mathbb{R}^d$ (spectral measure):

$$\int_{\mathbb{R}^d} f(x)\phi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(d\xi), \ \forall \phi \in \mathcal{S}(\mathbb{R}^d),$$

there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^m} < \infty.$$

In particular:

$$E(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(t)(\xi)\overline{\mathcal{F}\psi(t)(\xi)}\mu(d\xi)dt.$$

$$E\left(|W(\varphi)|^2\right) = \int_0^\infty \int_{\mathbb{R}^d} |\mathcal{F}\varphi(t)(\xi)|^2\mu(d\xi)dt.$$

Mild solutions

Fix a time horizon $T > 0$. A predictable stochastic process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is a solution of (1), if

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_{\mathbb{R}^d} b(u(t-s, x-y))\Gamma(s,dy)ds.$$

- Γ is the fundamental solution associated to the operator L.
- Stochastic integral: Walsh 86, Dalang 99, Da Prato and Zabczyk 92.

Hypothesis A

- The fundamental solution Γ satisfies that, for all $t > 0$, $\Gamma(t)$ is a non-negative distribution with rapid decrease, which means:

$$\int_{\mathbb{R}^d} (1 + |x|^2)^{k/2} \Gamma(t, dx) < \infty, \quad \forall k > 0.$$

- It holds that

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt < \infty.$$

- Moreover, Γ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^d$ of the form $\Gamma(t, dy)dt$ such that $\sup_{0 \le t \le T} \Gamma(t, \mathbb{R}^d) < +\infty$.
- Wave ($d \le 3$) and heat equations:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty.$$

Stochastic integrals

We denote by $\mathcal{H}$ the completion of $\mathcal{S}(\mathbb{R}^d)$ endowed with

$$\langle \phi, \vartheta \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) f(x-y) \vartheta(y) dx dy, \quad \phi, \vartheta \in \mathcal{S}(\mathbb{R}^d).$$

$$\{W(\varphi), \varphi \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)\} \Longrightarrow \{W(g), g \in L^2([0,T]; \mathcal{H})\}$$
$$\Longrightarrow W_t(h) := W\left(\mathbf{1}_{[0,t]} h\right), t \geq 0, h \in \mathcal{H}.$$

Then $\{W_t, t \in [0,T]\}$ is a cylindrical Wiener process in $\mathcal{H}$: for any $h \in \mathcal{H}$, $\{W_t(h), t \in [0,T]\}$ is a Brownian motion with variance $t\|h\|_{\mathcal{H}}^2$:

$$E(W_t(h) W_s(g)) = (s \wedge t) \langle h, g \rangle_{\mathcal{H}}.$$

$\longrightarrow$ Métivier and Pellaumail 80.

- For any predictable process G in $L^2(\Omega \times [0, T]; \mathcal{H})$, we can define the stochastic integral

$$G \cdot W = \int_0^T \int_{\mathbb{R}^d} G \, dW.$$

- Let $(e_j)_{j \geq 1}$ be a CONS of $\mathcal{H}$:

$$G \cdot W := \sum_{j=1}^{\infty} \int_0^T \langle G_s, e_j \rangle_{\mathcal{H}} dW_s(e_j).$$

- Isometry property:

$$E\left(|G \cdot W|^2\right) = E\left(\int_0^T \|G_t\|_{\mathcal{H}}^2 dt\right).$$

- One can construct a martingale measure associated to $W \Longrightarrow$ stochastic integral in Dalang 99.

Example of integrands

Proposition (Nualart and Q-S 07)

- Γ *satisfies Hypothesis A,*
- *Let* $Z = \{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ *be a predictable process such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|Z(t,x)|^2) < \infty.$$

Then, the random element $G = G(t,dx) = Z(t,x)\Gamma(t,dx)$ *is a predictable process in* $L^2(\Omega \times [0,T]; \mathcal{H})$*. Moreover, it holds that*

$$E\left(\|G\|^2_{L^2([0,T];\mathcal{H})}\right) \leq C_Z \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt.$$

- Γ belongs to $L^2([0,T];\mathcal{H})$: regularisation.
- G defines a distribution with rapid decrease, a.s.

In the mild formulation:

$$\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))W(ds, dy).$$

- Existence and uniqueness of solution: Dalang 99.
- Stochastic wave equation d>3: Conus and Dalang 07.

Examples:

- The wave equation with $d = 1, 2, 3$.

$$\Gamma(t) = \frac{1}{4\pi t}\sigma_t,$$

  where σ_t is the surface measure on the 3d sphere of radius t.
- The heat equation with any $d \geq 1$.
- In both cases: Hypothesis A is equivalent to

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty.$$

Existence and smoothness of the density

Let $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ the solution of

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))W(ds,dy) + \int_0^t \int_{\mathbb{R}^d} b(u(t-s, x-y))\Gamma(s,dy)ds.$$

We aim to prove that, for any (t,x):

1. the law of $u(t,x)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.
2. the probability density of $u(t,x)$ is a $\mathcal{C}^\infty$ function.

Carmona and Nualart 88, Bally and Pardoux 98, Millet and Sanz-Solé 99, Márquez *et al.* 01, Q-S and Sanz-Solé 04(2), Sanz-Solé 05, Mueller and Nualart 07.

Existence of density

Theorem (Nualart and Q-S 07)

Assume that

- Γ *satisfies Hypothesis A and*

$$0 < \int_0^\delta \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt, \ \forall \delta > 0,$$

- σ *and* b *are* $\mathcal{C}^1$ *functions with bounded Lipschitz continuous derivatives,*
- *there exists* $c > 0$ *such that* $|\sigma(z)| \geq c$*, for all* $z \in \mathbb{R}$*.*

Then, the law of $u(t,x)$ *is absolutely continuous with respect to Lebesgue measure.*

The proof is based on the Bouleau-Hirsch criterion.

Examples

- The stochastic heat equation ($d \geq 1$) and the stochastic wave equation ($d \leq 3$), under the condition

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty.$$

- Thus, the same hypothesis as the existence and uniqueness theorem.
- For the 3d stochastic wave equation , Q-S and Sanz-Solé 04 imposed, for some $\eta \in (0, 1/2)$:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^{\eta}} < \infty.$$

Gaussian context

Let $\mathcal{H}_T := L^2([0,T];\mathcal{H})$ and consider the centered Gaussian family $\{W(h), h \in \mathcal{H}_T\}$:

$$E\left(W(h_1)W(h_2)\right) = \langle h_1, h_2\rangle_{\mathcal{H}_T}.$$

- The Malliavin derivative is denoted by D.
- For $N \geq 1$, the domain of D^N in $L^p(\Omega;\mathcal{H}_T^{\otimes N})$ is $\mathbb{D}^{N,p}$.
- Set $\mathbb{D}^\infty = \cap_{p\geq 1}\cap_{k\geq 1}\mathbb{D}^{k,p}$.

According to Bouleau-Hirsch's criterion, if

- $F \in \mathbb{D}^{1,2}$ and
- $\|DF\|_{\mathcal{H}_T} > 0$ a.s.,

then the law of F is absolutely continuous with respect to Lebesgue measure.

Malliavin regularity

- We make use of the results in Q-S and Sanz-Solé 04 to prove that $u(t,x) \in \mathbb{D}^{1,2}$.
- The Malliavin derivative $Du(t,x)$ defines a random variable with values in $\mathcal{H}_T = L^2([0,T];\mathcal{H})$.
- Moreover, we show that

$$
\begin{aligned}
Du(t,x) =& \sigma(u(\cdot,*))\Gamma(t-\cdot,x-*) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\sigma'(u(s,y))Du(s,y)W(ds,dy) \\
&+ \int_0^t \int_{\mathbb{R}^d} b'(u(s,x-y))Du(s,x-y)\Gamma(t-s,dy)ds,
\end{aligned}
$$

Analysis of the Malliavin matrix

In order to apply Bouleau-Hirsch criterion, it suffices to show that

$$\|Du(t,x)\|_{\mathcal{H}_T} > 0, \; a.s.$$

We show that

$$\lim_{n\to\infty} P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \frac{1}{n}\right) = 0.$$

Important point:

$$\int_{t-\delta}^t \|\Gamma(t-s,x-*)\sigma(u(s,*))\|_{\mathcal{H}}^2 ds \geq C \int_0^\delta \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds.$$

Smoothness of the density

Theorem (Nualart and Q-S 07)

Assume that

- Γ *satisfies Hypothesis A,*
- *there exists* $\gamma > 0$ *such that for all* $\tau \in (0,1]$,

$$C\tau^{\gamma} \leq \int_0^{\tau} \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds. \tag{2}$$

- σ *and* b *are* $\mathcal{C}^{\infty}$ *functions with bounded derivatives of any order greater than or equal to one,*
- $|\sigma(z)| \geq c > 0$, *for all* $z \in \mathbb{R}$,

Then, $u(t,x)$ *has a density which is a* $\mathcal{C}^{\infty}$ *function.*

Examples

- The stochastic wave equation with $d = 1, 2, 3$: condition (2) is satisfied with $\gamma = 3$.
- The stochastic heat equation with $d \geq 1$: condition (2) is satisfied for any $\gamma \geq 1$.
- For these particular examples, we improve results of Márquez *et al.* 01, Q-S and Sanz-Solé 04:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^\eta} < \infty,$$

for some $\eta \in \left(0, \frac{1}{2}\right)$.

Sketch of the proof

- $u(t,x)$ belongs to $\mathbb{D}^\infty$,
- $E\left(\|Du(t,x)\|_{\mathcal{H}_T}^{-q}\right) < \infty$, for all $q \geq 2$.

Malliavin differentiability: Q-S and Sanz-Solé 04. Moreover, we have

$$
\begin{aligned}
D^N u(t,x) &= Z^N(t,x) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-z)[\Delta^N(\sigma, u(s,z)) + D^N u(s,z)\sigma'(u(s,z))] W(ds,dz) \\
&+ \int_0^t \int_{\mathbb{R}^d} [\Delta^N(b, u(s,x-z)) + D^N u(s,x-z) b'(u(s,x-z))]\Gamma(t-s,dz)ds,
\end{aligned}
$$

$$
\Delta^N(g,X) := D^N g(X) - g'(X) D^N X,
$$

Inverse of the Malliavin matrix:

We show that (Nualart 06) for all $q \geq 2$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$,

$$P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \epsilon\right) \leq C\epsilon^q.$$

We proceed as in the proof of existence of density:

$$\begin{aligned} Du(t,x) = {} & \sigma(u(\cdot,*))\Gamma(t-\cdot, x-*) \\ & + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma'(u(s,y))Du(s,y)W(ds,dy) \\ & + \int_0^t \int_{\mathbb{R}^d} b'(u(s,x-y))Du(s,x-y)\Gamma(t-s,dy)ds. \end{aligned}$$

We apply that

$$C\delta^\gamma \leq \int_0^\delta \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi)ds.$$

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