Malliavin calculus applied to the 3-dimensional stochastic wave equation

Lluís Quer-Sardanyons
INRIA Lorraine
Institut Élie Cartan de Nancy

joint work with Marta Sanz-Solé (Universitat de Barcelona)

Journées de Probabilités Nancy 2005
8 Septembre 2005

Outline

- The 3-dimensional stochastic wave equation
- Aim and Gaussian context
- Existence of density
- Smoothness of the density
- Final comments and references

The stochastic wave equation

We consider the non-linear stochastic partial differential equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x), \tag{1}$$

$(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, x \in \mathbb{R}^3.$$

- Δ is the Laplacian on $\mathbb{R}^3$,
- $\{u(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\}$ is a real-valued process,
- $b, \sigma : \mathbb{R} \to \mathbb{R}$,
- $\dot{F}$ is a random perturbation.

We are interested in mild solutions.

Equation (1) is a particular case of

$$Lu(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x),\ x \in \mathbb{R}^d$$

where the fundamental solution Λ of $Lu = 0$ is a Schwartz distribution.

- We want real-valued solutions, thus we need to impose some spatial correlation on the noise.
- $\{F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ is a L^2-valued centered Gaussian process with

$$E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx) \left(\varphi(s,\cdot) * \tilde{\psi}(s,\cdot)\right)(x).$$

  $\tilde{\psi}(s,x) = \psi(s,-x)$ and Γ is a non-negative tempered measure; we denote by μ its spectral measure.
- The noise is white in time and the measure Γ corresponds to the spatial correlation.

Mild solutions

- Walsh (1986) and Dalang (1999) formulation,
- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ filtered probability space,
- $(\mathcal{F}_t)_t$ generated by M, the martingale measure extension of F,

A predictable stochastic process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^3\}$, $T > 0$, solves (1) if

$$u(t,x) = \int_0^t \int_{\mathbb{R}^3} S_3(t-s, x-y)\sigma(u(s,y))M(ds,dy) + \int_0^t ds \int_{\mathbb{R}^3} S_3(s,dy) b(u(t-s, x-y)).$$

S_3 is the fundamental solution associated to the wave equation on $\mathbb{R}^3$:

$$S_3(t) = \frac{1}{4\pi t}\sigma_t,$$

σ_t is the uniform measure on the 3d sphere of radius t.

Existence and uniqueness of solution

Hypothesis D The fundamental solution Λ of $Lu = 0$ is a deterministic function in t taking values in the space of non-negative distributions with rapid decrease such that

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 < \infty.$$

Moreover, Λ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^d$ of the form $\Lambda(t, dy)dt$ such that $\sup_{0 \le t \le T} \Lambda(t, \mathbb{R}^d) < +\infty$.

The stochastic wave equation:

- $d = 3$: $\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty \Leftrightarrow$ Hypothesis D (Dalang (1999)),
- $d = 1$ Carmona and Nualart (1988),
- $d = 2$ Dalang and Mueller (1998), Millet and Sanz-Solé (1999).

Aim

For any fixed $(t, x) \in [0, T] \times \mathbb{R}^3$, we want to show that

- the probability law of the random variable $u(t, x)$ is absolutely continuous w. r. to Lebesgue measure (Q-S and Sanz-Solé (JFA 2004)),
- and that the density is a smooth function (Q-S and Sanz-Solé (Bernoulli 2004)).

Related references: Carmona and Nualart (1988), Millet and Sanz-Solé (1999) and Márquez, Mellouk and Sarrà (2001).

- We use the techniques of the Malliavin Calculus.
- Main difficulty: the fundamental solution of the 3d stochastic wave equation is a distribution.

Gaussian context

Let $\mathcal{E}$ be the inner-product space of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ endowed with

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\psi})(x),$$

where $\tilde{\psi}(x) = \psi(-x)$. Let $\mathcal{H}$ denote the completion of $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$. Then, $\mathcal{H}_T = L^2([0,T];\mathcal{H})$ is a real separable Hilbert space.
The Gaussian family is given by

$$W(h) = \int_0^T \int_{\mathbb{R}^d} h(s,x) M(ds,dx), \; h \in \mathcal{H}_T.$$

Then, $(W(h), h \in \mathcal{H}_T)$ is an isonormal Gaussian process.
We denote by D the Malliavin derivative operator and $\mathbb{D}^{N,p}$ the domain of D^N in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, $N \geq 1$, $p \in [1, \infty)$.

Extension of the stochastic integral

A formal differentiation of the *mild* form of Equation (1) yields

$$Du(t,x) = Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz)$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b'(u(t-s,x-z))Du(t-s,x-z).$$

- $Z(t,x)$ is some Hilbert-valued stochastic process.
- We extend Dalang's stochastic integral allowing Hilbert-valued integrands (Q-S and Sanz-Solé (JFA 2004)).
- We will need a recently proved existence and uniqueness result for these equations (Sanz-Solé (2005)).

Existence of density

Theorem (Q-S and Sanz-Solé(JFA 2004))

Assume that

- *the coefficients* σ *and* b *are* $\mathcal{C}^1$ *functions with bounded Lipschitz continuous derivatives,*
- *there exists* $\sigma_0 > 0$ *such that* $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$*,*
- *for some* $\eta \in (0, \frac{1}{2})$*,*

$$\sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(dx)\mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x-y) < \infty.$$

Then, the law of the random variable $u(t,x)$*,* $(t,x) \in (0,T] \times \mathbb{R}^3$*, is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}$*.*

We apply Bouleau-Hirsch criterion:

- $u(t,x) \in \mathbb{D}^{1,p}$, for all $p \in [1,\infty)$ and
- the Malliavin matrix $\|Du(t,x)\|_{\mathcal{H}_T}$ is non-degenerate.

For the differentiability:

- σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives,
- this result is valid for a more general operator L whose associated fundamental solution Λ satisfies Hypothesis D,

Lemma (Lépingle, Nualart and Sanz-Solé (1989))

Let $(F_n)_{n\geq 1} \in \mathbb{D}^{1,p}$. Assume that $(F_n)_{n\geq 1}$ converges in $L^p(\Omega)$ to F and that $\sup_{n\geq 1} E(\|DF_n\|^p_{\mathcal{H}_T}) < +\infty$. Then $F \in \mathbb{D}^{1,p}$ and there is a subsequence of $(DF_n)_{n\geq 1}$ converging to DF in the weak topology of $L^p(\Omega;\mathcal{H}_T)$.

Theorem (Q-S and Sanz-Solé (JFA 2004))

For any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u(t,x)$ *belongs to* $\mathbb{D}^{1,p}$, *for any* $p \in [1,\infty)$, *and there exists an* $\mathcal{H}_T$*-valued stochastic process* $\{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ *such that*

$$Du(t,x) = Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma'(u(s,z))Du(s,z)M(ds,dz)$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u(t-s,x-z))Du(t-s,x-z). \qquad (2)$$

If $\Lambda(t)$ were a real-valued function, $Z(t,x)$ would be

$$\Lambda(t-\cdot, x-*)\sigma(u(\cdot,*)).$$

Thus, the approach for $d = 1,2$ does not work.

Regularisation of $\Lambda(t)$: $\Lambda_n(t) = \psi_n * \Lambda(t)$, $(\psi_n)_n$ approximation of the identity.
In the preceding lemma, we take $F_n = u_n(t,x)$, where $\{u_n(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ is the solution to

$$u_n(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\sigma(u_n(s,z))M(ds,dz) + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz) b(u_n(t-s,x-z)).$$

We first prove that

$$\lim_{n\to\infty} \left(\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u_n(t,x) - u(t,x)|^p) \right) = 0.$$

To prove the convergence in $L^p(\Omega)$, $p \in [1, \infty)$:

- uniformly boundedness in $L^p(\Omega)$, for all $p \in [1, \infty)$, and
- convergence in $L^2(\Omega)$.

$$Z(t,x) = L^p(\Omega; \mathcal{H}_T) - \lim_{n\to\infty} \Lambda_n(t-\cdot, x-*)\sigma(u_n(\cdot,*)).$$

We show that

$$\sup_{n\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(\|Du_n(t,x)\|^p_{\mathcal{H}_T}) < +\infty.$$

Hence, we get that $u(t,x) \in \mathbb{D}^{1,p}$.

It remains to show that the Malliavin derivative satisfies Equation (2). For this, we consider

$$U(t,x) = Z(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)\sigma'(u(s,z))U(s,z)M(ds,dz) + \int_0^t ds \int_{\mathbb{R}^d} \Lambda(s,dz)b'(u(t-s,x-z))U(t-s,x-z).$$

- The above equation has a unique solution in $L^p(\Omega; \mathcal{H}_T)$ (Sanz-Solé (2005)).
- We conclude by showing

$$U(t,x) = L^p(\Omega; \mathcal{H}_T) - \lim_{n\to\infty} Du_n(t,x).$$

For the study of the Malliavin matrix we go back to the 3−dimensional stochastic wave equation.

Theorem (Q-S and Sanz-Solé (JFA 2004))

Assume that

- $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$, *for some* $\sigma_0 > 0$,
- *there exists* $\eta \in (0, \frac{1}{2})$ *such that*

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(dx) \mathcal{F}^{-1} \left(\frac{1}{(1 + |\xi|^2)^\eta} \right) (x - y) < \infty.$$

Then, $\|Du(t,x)\|_{\mathcal{H}_T} > 0$, *a.s.*

- $E(\|Du(t,x)\|_{\mathcal{H}_T}^{-p}) < +\infty$, for some $p > 0$,
- regularisation: $\Lambda_{\epsilon^{-1}} = \psi_{\epsilon^{-1}} * S_3$, for $\epsilon > 0$.

- Using that $E(Y) = \int_0^\infty P\{Y > z\}dz$, it turns out that we have to find the size in ϵ of

$$P\{\|Du(t,x)\|^2_{\mathcal{H}_T} < \epsilon\},$$

for *small* $\epsilon > 0$.

- Let $\epsilon_1, \delta > 0$ be such that, for all $\epsilon \in (0, \epsilon_1]$, $t - \epsilon^\delta > 0$.
- We decompose

$$\begin{aligned} P\{\|Du(t,x)\|^2_{\mathcal{H}_T} < \epsilon\} &\leq P\{\big|\|D_{t-\cdot,*}u(t,x)\|^2_{\mathcal{H}_{\epsilon^\delta}} - \|Z_{t-\cdot,*}(t,x)\|^2_{\mathcal{H}_{\epsilon^\delta}}\big| \geq \epsilon\} \\ &\quad + P\{\|\Lambda_{\epsilon^{-1}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}} < 6\epsilon\} \\ &\quad + P\{\|Z_{t-\cdot,*}(t,x) - \Lambda_{\epsilon^{-1}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}} \geq \epsilon\}. \end{aligned}$$

We have

$$
\begin{aligned}
\|\Lambda_{\epsilon^{-1}}(r, x-*)\sigma(u(t-r,*))\|_{\mathcal{H}}^2 &\geq \sigma_0^2 \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}\Lambda_{\epsilon^{-1}}(r)(\xi)|^2 \\
&\geq \sigma_0^2 \left(\frac{1}{2} \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}S_3(r)(\xi)|^2 - \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}S_3(r)(\xi)|^2 |\mathcal{F}\psi_{\epsilon^{-1}}(\xi) - 1|^2 \right)
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}S_3(r)(\xi)|^2 |\mathcal{F}\psi_{\epsilon^{-1}}(\xi) - 1|^2 \leq 4\pi \int_{\mathbb{R}^3} \mu(d\xi)|\xi||\mathcal{F}S_3(r)(\xi)|^2.
$$

We need bounds of integrals of the Fourier transform of S_3:

$$\mathcal{F}S_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|},\ \xi \in \mathbb{R}^d.$$

For this, we consider the hypothesis

$(\overline{H}_\eta)$ $\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma(dx)\mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x-y) < \infty$,

(H_η) $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty$.

- $(\overline{H}_\eta) \Rightarrow (H_\eta)$ (Lévêque (2001)),
- recently, it has been proved that $(H_\eta) \Rightarrow (\overline{H}_\eta)$, (Sanz-Solé (2005)).

For instance, under $(\overline{H}_\eta)$ we have that,

- for all $t \in (0,1)$, (Lévêque (2001))

$$C_1 t^3 \leq \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F} S_d(s)(\xi)|^2,$$

- for all $t \in [0,T]$,

$$\int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi) |\xi| |\mathcal{F} S_d(s)(\xi)|^2 \leq C_2 t^{2-2\eta}.$$

In progress (with Nualart): to show that indeed it is enough to assume that

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty.$$

Smoothness of the density

Recall that we consider

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x).$$

Theorem (Q-S and Sanz-Solé (Bernoulli 2004))

Assume that

- σ *and* b *are* $\mathcal{C}^\infty$ *functions with bounded derivatives of any order greater than or equal to one,*
- $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$, *for some* $\sigma_0 > 0$,
- *there exists* $\eta \in (0, \frac{1}{2})$ *such that*

$$\sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(dx)\mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x-y) < \infty.$$

Then, $u(t,x)$ *has a density which is a* $\mathcal{C}^\infty$ *function.*

The proof is based on the criterion of the Malliavin calculus

Proposition

Let $F = (F^1, \ldots, F^m)$ *be a random vector satisfying the assumptions*

- $F^j \in \mathbb{D}^\infty$, *for any* $j = 1, \ldots, m$,
- *the Malliavin matrix* γ_F *is invertible, a.s. and*

$$(\det \gamma_F)^{-1} \in \cap_{p \geq 1} L^p(\Omega).$$

Then, the law of F *has an infinitely differentiable density with respect to the Lebesgue measure on* $\mathbb{R}^m$.

Regularity in the Malliavin sense:

Theorem (Q-S and Sanz-Solé (Bernoulli 2004))

Assume Hypothesis D. Then $u(t,x) \in \mathbb{D}^\infty$ and $D^N u(t,x)$ satisfies some stochastic evolution equation in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$ with some initial condition $\{Z^N(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$, $N \geq 1$.

- We may consider more general SPDEs,
- D^N is a closed operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}_T^{\otimes N})$,
- same regularisation of Λ: $\Lambda_n = \psi_n * \Lambda$, $n \geq 1$.
- the $L^p(\Omega; \mathcal{H}_T^{\otimes N})-$random variable $Z^N(t,x)$ is defined inductively,
- convergences of order p: uniform bound for the $p-$norm and convergence for $p = 2$.

$$
\begin{aligned}
D^N u(t,x) &= Z^N(t,x) \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)[\Delta^N(\sigma, u(s,z)) + D^N u(s,z)\sigma'(u(s,z))]M(ds,dz) \\
&+ \int_0^t ds \int_{\mathbb{R}^d} \Lambda(t-s, dz)[\Delta^N(b, u(s,x-z)) + D^N u(s,x-z)b'(u(s,x-z)
\end{aligned}
$$

$$
\Delta^N(\sigma, u(s,z)) = D^N \sigma(u(s,z)) - \sigma'(u(s,z))D^N u(s,z).
$$

Malliavin matrix: 3−dimensional stochastic wave equation.

Theorem (Q-S and Sanz-Solé (Bernoulli 2004))

Assume that

- σ *and* b *are* $\mathcal{C}^1$ *functions with bounded Lipschitz continuous derivatives,*
- $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$, *for some* $\sigma_0 > 0$,
- *there exists* $\eta \in (0, \frac{1}{2})$ *such that*

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(dx) \mathcal{F}^{-1} \left(\frac{1}{(1 + |\xi|^2)^\eta} \right) (x - y) < \infty.$$

Then, for any $p > 0$,

$$E(\|Du(t,x)\|_{\mathcal{H}_T}^{-p}) < \infty.$$

Notice that for the existence we proved the above boundedness only for some positive p.

Main ingredients for the proof:

- Recall that the condition on Γ is equivalent to

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < +\infty,$$

- regularisation of the fundamental solution S_3:

$$\Lambda_{\epsilon^{-\nu}} = \psi_{\epsilon^{-\nu}} * S_3,$$

  with $\epsilon, \nu > 0$,
- bounds of integrals of the Fourier transform of S_3, as for the existence of density.

References

- **Dalang**, R. C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. Electron. J. Probab. 4 (1999), no. 6.
- **Márquez-Carreras**, D., **Mellouk**, M. and **Sarrà**, M. On stochastic partial differential equations with spatially correlated noise: smoothness of the law. Stochastic Process. Appl. 93 (2001), no. 2, 269-284.
- **Quer-Sardanyons**, L. and **Sanz-Solé**, M. Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation. J. Funct. Anal. 206 (2004), no. 1, 1-32.
- **Quer-Sardanyons**, L. and **Sanz-Solé**, M. A stochastic wave equation in dimension 3: smoothness of the law. Bernoulli 10 (2004), no. 1, 165-186.
- **Sanz-Solé**, M. Malliavin Calculus, with Applications to Stochastic Partial Differential Equations. EPFL Press. Fundamental Sciences, Mathematics, Taylor and Francis 2005.