Torsiones de Reidemeister

J. Porti

Universitat Autònoma de Barcelona

XVII Encuentro de Topología

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K. REIDEMEISTER Abh. Math. Semin. Hamb. Univ. **11**, 102-109, 1935.

Homotopieringe und Linsenräume.

Von KURT REIDEMEISTER in Marburg.

Im folgenden soll mittels der Homotopiegruppen eine neue Invariante für Linsenräume[1]) abgeleitet werden, die die Klassifikation dieser Räume gestattet.

Reidemeister-Franz-De Rham 1935

W. FRANZ J. Reine Angew. Math. **173**, 245-274, 1935.

Über die Torsion einer Überdeckung.

Von *Wolfgang Franz* in Marburg.

Im folgenden wird jeder Überdeckung [1][2] eines topologischen Komplexes mit einem (kommutativen) Körper, deren sämtliche Bettischen Zahlen Null sind, eine Invariante zugeordnet. Auf Grund ihrer Bedeutung für die Überdeckung und den Komplex,

Reidemeister-Franz-De Rham 1935

G. DE RHAM Rec. Math. (Mat. Sbornik) N.S., **1(43)**:5 (1936), 737-743.

Sur les nouveaux invariants topologiques de M. Reidemeister

Georges de Rham[1] (Lausanne)

$$S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$t \cdot (z_1, z_2) = (e^{\frac{2\pi}{p} i} z_1, e^{\frac{2\pi q}{p} i} z_2), \qquad p, q \in \mathbf{N} \text{ coprimos.}$$

$$L(p,q) = S^3 \Big/ (z_1, z_2) \sim t \cdot (z_1, z_2)$$

$$\pi_1(L(p,q)) = \mathbf{Z}/p\mathbf{Z}$$

- Pregunta: ¿Para qué valores de q y q', $L(p,q) \cong L(p,q')$?

Espacios lenticulares

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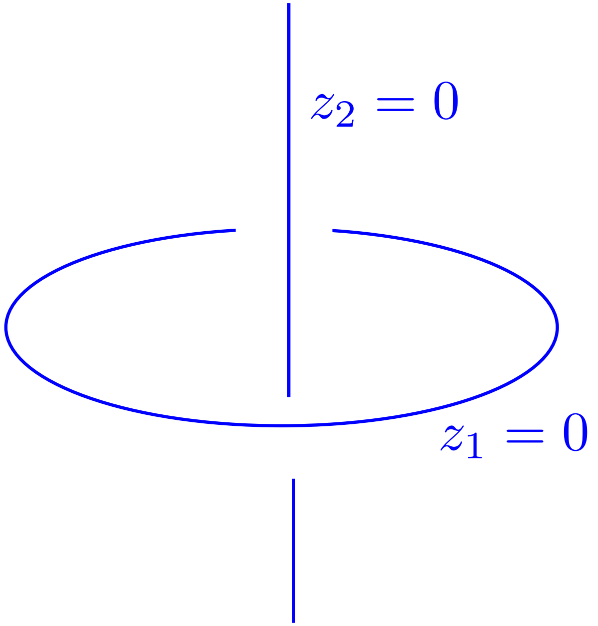

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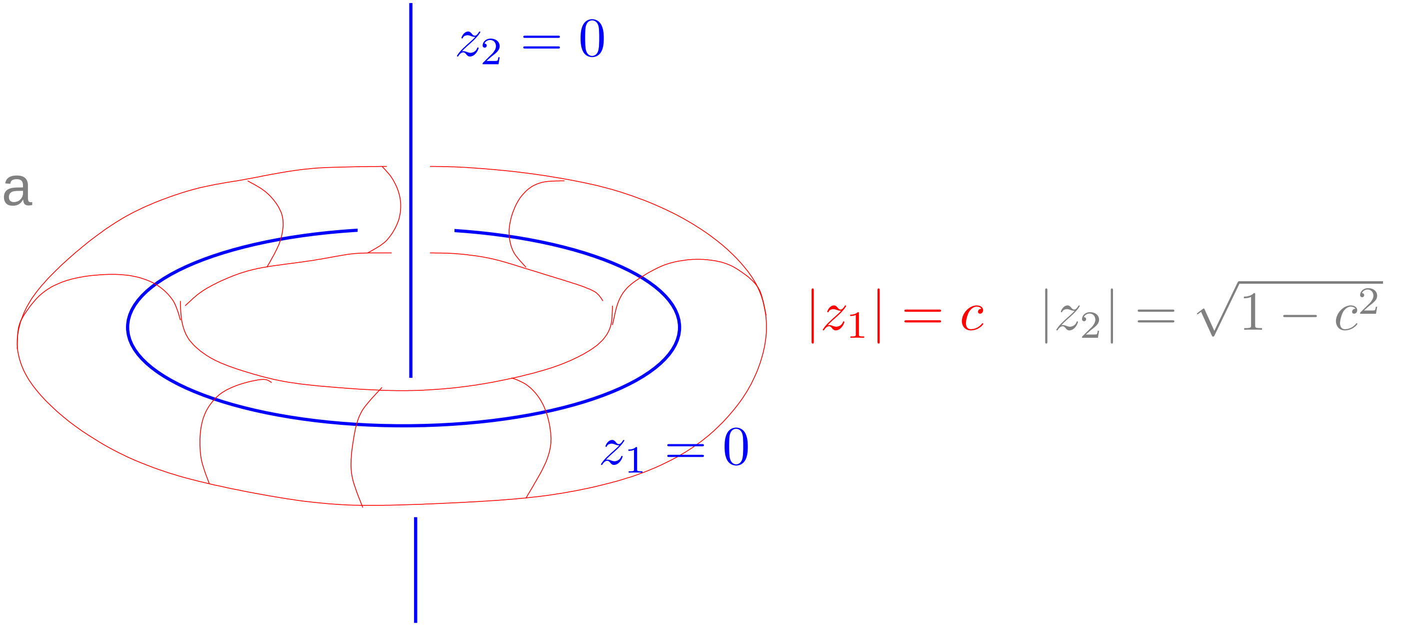

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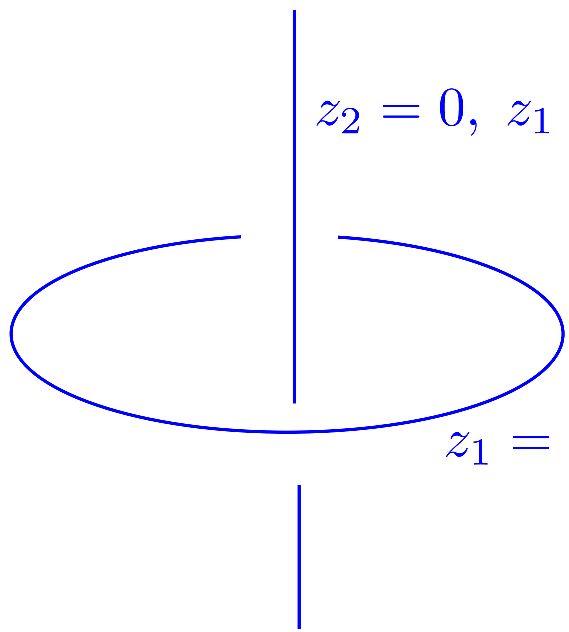

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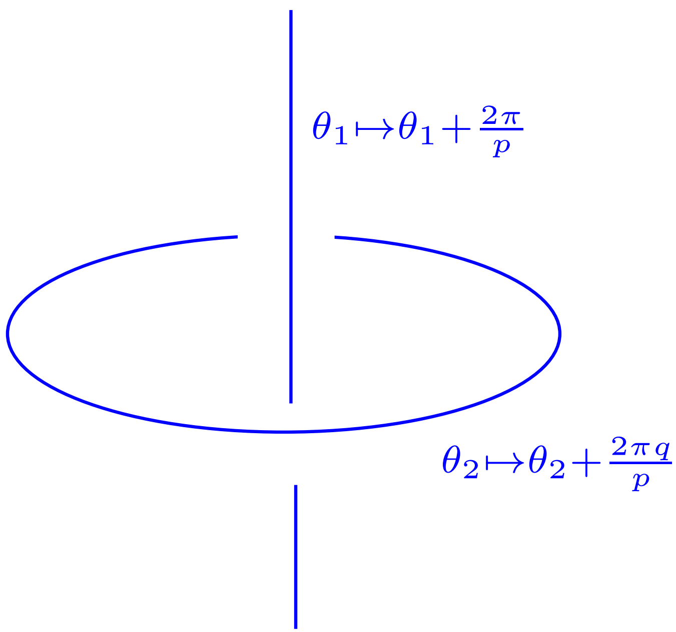
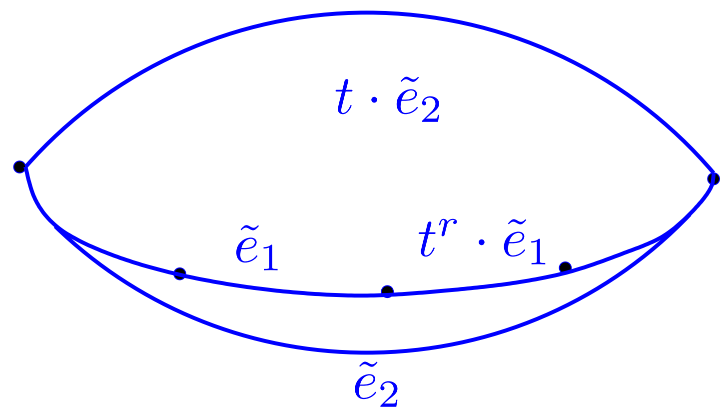


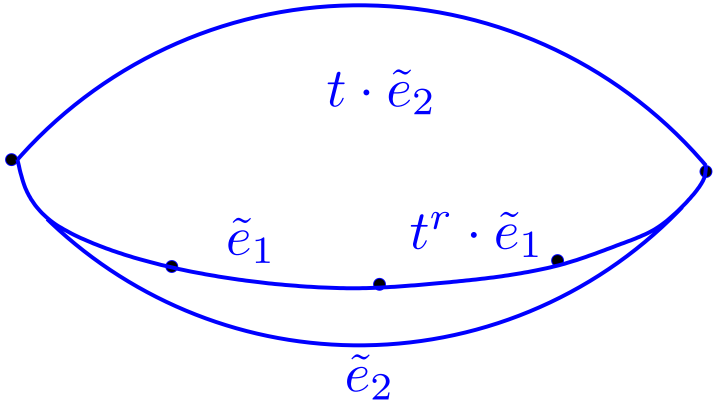


La lente es un dominio fundamental de t

$$rq = 1 \mod p$$

Torsión de un espacio lenticular

$$S^3 = \{z \in \mathbf{C}^2 \mid |z| = 1\},\ t \cdot (z_1, z_2) = (e^{\frac{2\pi}{p} i} z_1, e^{\frac{2\pi q}{p} i} z_2),$$
$$L(p,q) = S^3/(z_1, z_2) \sim t \cdot (z_1, z_2)$$



$$\begin{cases} \partial \tilde{e}_3 = (1-t)\tilde{e}_2 \\ \partial \tilde{e}_2 = (1 + t + \cdots + t^{p-1})\tilde{e}_1 \\ \partial \tilde{e}_1 = (t^r - 1)\tilde{e}_0 \end{cases}$$

$$t \to 1, \qquad \begin{cases} \partial e_3 = 0 \\ \partial e_2 = p e_1 \\ \partial e_1 = 0 \end{cases} \qquad H_i^{CW}(L(p,q), \mathbf{Z}) = \begin{cases} \mathbf{Z} & i = 3 \\ \mathbf{0} & i = 2 \\ \mathbf{Z}/p\mathbf{Z} & i = 1 \\ \mathbf{Z} & i = 0 \end{cases}$$

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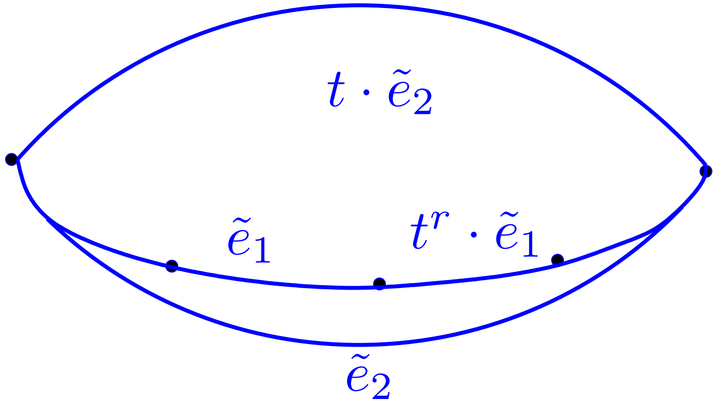

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$t \mapsto \xi,\ \xi^p = 1,\ \xi \neq 1,$

$$\begin{cases} \partial \tilde{e}_3 = (1-\xi)\tilde{e}_2 \\ \partial \tilde{e}_2 = 0 \\ \partial \tilde{e}_1 = (\xi^r - 1)\tilde{e}_0 \end{cases}$$

$$H_*(L(p,q), \xi) = 0$$

Torsión de un espacio lenticular

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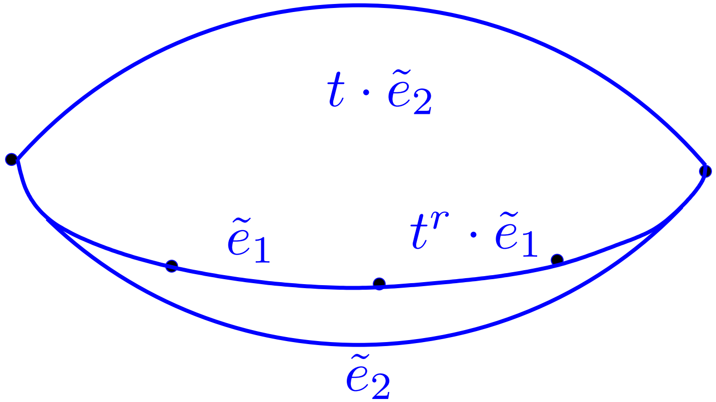

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$H_*(L(p,q), \xi) = 0$

Def. (Reidemeister): $\tau(L(p,q), \xi) := |(1-\xi)(1-\xi^r)|$

$$\{\tau(L(p,q),\xi)\}_{\xi^p=1, \xi\neq 1} = \{\tau(L(p,q'),\xi)\}_{\xi^p=1, \xi\neq 1} \Leftrightarrow q' = \pm q^{\pm 1} \mod p$$

Torsión de un CW-complejo

K CW-complejo comp., $\rho : \pi_1 K \to GL(V)$, V=F-esp. vect. dim finita

Def: $C_*(K, \rho) := V \otimes_\rho C_*^{CW}(\tilde{K}, \mathbf{Z})$

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- $B_i := Im(\partial : C_{i+1}(K,\rho) \to C_i(K,\rho))$

  Si $H_*(K,\rho) = 0$ (ρ es acíclica), entonces

  $$0 \to B_i \to C_i(K,\rho) \stackrel{\partial}{\to} B_{i-1} \to 0 \text{ es exacta.}$$

  b_i F-base de B_i, $\Longrightarrow b_i \sqcup \tilde{b}_{i-1}$ F-base de $C_i(K,\rho)$.

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Def: $$\boxed{\tau(K,\rho) := \prod_{i=0}^{n} \det(b_i \sqcup \tilde{b}_{i-1} \mid c_i)^{(-1)^{i+1}} \in F^*/\{\pm \det \rho(\pi_1 K)\}}$$

- $\tau(K,\rho)$ invariante combinatorio (homeos celulares y subdivisión) y de la clase de conjugación de ρ

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- Si $H_*(K,\rho) \neq 0$, definimos $\tau(K,\rho,h_*)$ para h_* = F-base de $H_*(K,\rho)$.

  Hay que elegir una base h_* *natural*!!

Invariante combinatorio (no topològico)

ANNALS OF MATHEMATICS
Vol. 74, No. 3, November, 1961
Printed in Japan

TWO COMPLEXES WHICH ARE HOMEOMORPHIC BUT COMBINATORIALLY DISTINCT

BY JOHN MILNOR[1]

(Received March 14, 1961)

Let L_q denote the 3-dimensional lens manifold of type $(7, q)$, suitably triangulated (see § 1), and let σ^n denote an n-simplex. A finite simplicial complex X_q is obtained from the product $L_q \times \sigma^n$ by adjoining a cone over the boundary $L_q \times \partial\sigma^n$. The dimension of X_q is $n + 3$.

THEOREM 1. *For $n + 3 \geqq 6$ the complex X_1 is homeomorphic to X_2.*

THEOREM 2. *No finite cell subdivision of the simplicial complex X_1 is isomorphic to a cell subdivision of X_2. In particular there is no piecewise linear homeomorphism from X_1 to X_2.*

The proof of Theorem 1 will be based on a recent result of B. Mazur. For the special case $n = 3$ (which is somewhat more difficult) the proof will make use of theorems of A. Haefliger and J. Stallings.

The proof of Theorem 2 will be based on the concept of "torsion" as defined by Reidemeister, Franz, and de Rham.

- $R =$ anillo, dominio factorial (tiene mcd = máximo comun divisor)
  D = R-módulo finitamente presentado con matriz de presentación P
  $\#$ relaciones de P (posiblemente 0) $\geq \#$ generadores.

  Def: $\mathrm{orden}_R(D) :=$ mcd$\{$menores de mayor rango de $P\}$

  Ejemplo: $\mathrm{orden}_{\mathbf{Z}}(\bigoplus_i \mathbf{Z}/n_i\mathbf{Z}) = \prod_i n_i$

Dominio factorial

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Teorema (Turaev 1986)

$\rho : \pi_1 K \to U_R$=unidades de R, $\quad i : R \hookrightarrow F_R$ cuerpo de fracciones

Si $H_*(K; i \circ \rho) = 0$, entonces

$$\tau(K; i \circ \rho) = \prod_{i=0}^{n} \mathrm{orden}_R(H_i(K;\rho))^{(-1)^{i+1}} \in F_R^*/\pm U_R$$

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Ej. M^3 variedad cerrada orientable t.q. $H_1(M^3; \mathbf{Q}) = 0$, entonces

$$\tau(M^3, triv) = \mathrm{orden}_{\mathbf{Z}}\, H_1(M, \mathbf{Z}) = \pm\, \mathrm{card}\, H_1(M, \mathbf{Z})$$

- $K \subset S^3$ nudo, $\qquad \mathrm{Alex}_K(t) \in \mathbf{Z}[t^{\pm 1}]$
  $M = S^3 \setminus \mathcal{N}(K)$, $\qquad \mathrm{ab} : \pi_1(M) \to H_1(M) \cong \mathbf{Z} = \langle t \mid \rangle \subset \mathbf{Z}[t^{\pm 1}]$.

Teorema (Milnor 1962):

$$\tau(M, \mathrm{ab}) = \pm\, \mathrm{Alex}_K(t)/(t-1) \in \mathbf{Z}(t)^*/\{\pm t^k\}$$

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Demostración (de Turaev):

$\tilde{M} \to M$ cubierta cíclica infinita

$$(1 \to \pi_1 \widetilde{M} \to \pi_1 M \to \mathbf{Z} \to 1)$$

$$H_*(M, ab) = H_*(M, \mathbf{Z}[t^{\pm 1}]) \cong H_*(\tilde{M}; \mathbf{Z}),$$ (como $\mathbf{Z}[t^{\pm 1}]$-módulos)

$$\mathrm{orden}_{\mathbf{Z}[t^{\pm 1}]} H_0(\widetilde{M}, \mathbf{Z}) = \mathrm{orden}_{\mathbf{Z}[t^{\pm 1}]}(\mathbf{Z}) = t - 1$$

$$\mathrm{orden}_{\mathbf{Z}[t^{\pm 1}]} H_1(\widetilde{M}, \mathbf{Z}) = \mathrm{Alex}_K(t)$$

Y aplicamos $\tau(M; ab) = \prod_{i=0}^{n} \mathrm{orden}_{\mathbf{Z}[t^{\pm 1}]}(H_i(M; ab))^{(-1)^{i+1}}$

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- Fórmula de Fox (1956): $M_n \to S^3$ cubierta cíclica ramificada de orden n
  ($M_n \setminus \tilde{K} \to S^3 \setminus K$ cubierta cíclica de orden n)

$$\mathrm{card}(H_1(M_n, \mathbf{Z})) = \prod_{k=1}^{n} |\, \mathrm{Alex}_K(e^{\frac{2\pi k}{n} i})|$$

Idea (Turaev 1986):

$$C_*(M_n \setminus \tilde{K}; \mathbf{C}) = C_*(S^3 \setminus K, \mathbf{C}[t]/(t^n - 1)) = \bigoplus_{k=1}^{n} C_*(S^3 \setminus K, \mathbf{C}[t]/(t - \xi^k))$$

- Torsión de Whitehead J.H.C. Whitehead (1950)

$$K_1(\mathbf{Z}[\pi_1(K)]) = \varinjlim_n GL(n, \mathbf{Z}[\pi_1(K)])/\{\text{matrices elementales}\}$$

$\tau^{Wh}(K) \in K_1(\mathbf{Z}[\pi_1(K)])/\pi_1(K)$, inv. homotopía simple

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- Torsion l^2 (W. Lück-M. Rothemberg 1991).
  $\pi_1(K) \curvearrowright l^2(\mathbf{C}[\pi_1(K)])$ (determinante de Fuglede-Kadison).
  Lück-Schick (1999): M^3 hiperbólica $\Rightarrow \tau^{l^2}(M^3) = cte\,\mathrm{Vol}(M^3)$

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- Polinomios de Alexander "torcidos" (JW-L-W-K-CF 1990's)
  $\phi : \pi_1 M^3 \twoheadrightarrow \mathbf{Z} = \langle t \mid \rangle$, $\qquad \rho : \pi_1(M^3) \to SL(n, F)$

$$\mathrm{Alex}_{M^3,\rho}(t) := \tau(M, \rho \otimes \phi) \in F(t)^*/\{\pm t^n\}$$

Friedl-Vidusi (2008): con el grado de $\mathrm{Alex}_{M^3,\rho}(t)$ para toda $\rho : \pi_1(M^3) \to \mathbf{C}[G]$, $\pi_1(M^3) \twoheadrightarrow G$ finito se puede determinar si ϕ està inducido por una fibración $M^3 \to S^1$.

Otras torsiones

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- Polinomios de Alexander "torcidos" (JW-L-W-K-CF 1990's)
  $\phi : \pi_1 M^3 \twoheadrightarrow \mathbf{Z} = \langle t \mid \rangle$, $\qquad \rho : \pi_1(M^3) \to SL(n, F)$

  $$\mathrm{Alex}_{M^3,\rho}(t) := \tau(M, \rho \otimes \phi) \in F(t)^* / \{\pm t^n\}$$

  Friedl-Vidusi (2008): con el grado de $\mathrm{Alex}_{M^3,\rho}(t)$ para toda $\rho : \pi_1(M^3) \to \mathbf{C}[G]$, $\pi_1(M^3) \twoheadrightarrow G$ finito se puede determinar si ϕ està inducido por una fibración $M^3 \to S^1$.
- Torsion analítica (D.B.Ray-I.M.Singer 1971)

Torsión analítica de Ray-Singer

- $K = M^n$ lisa y compacta, $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$, $H^*(M, \rho) = 0$

- Ray-Singer (1971): Definición de $\tau(M, \rho)$
que pueda adaptarse a formas diferenciales
(cohomología de de Rham)

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$\delta_i : C^i(K;\rho) \to C^{i+1}(K,\rho)$,
$\delta_{i-1}^t : C^i(K;\rho) \to C^{i-1}(K,\rho)$ (utilizamos la $\mathbf{R}$-base para transponer)
$\Delta_i^{comb}(\rho) := \delta_i^t \circ \delta_i + \delta_{i-1} \circ \delta_{i-1}^t$
$\tau(K,\rho)^2 = \prod_{k=0}^n (\det \Delta_k^{comb}(\rho))^{k(-1)^k}$

- $\log|\tau(K,\rho)| = \frac{1}{2} \sum_{k=0}^{n} (-1)^k k \, \log(\det(\Delta_k^{comb}(\rho)))$

- Idea: Definir la torsion a partir del "*determinante*" del laplaciano en k-formas $\Delta^k : \Omega^k(M;V_\rho) \to \Omega^k(M;V_\rho)$

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- $\Delta^k : \Omega^k(M; V_\rho) \to \Omega^k(M; V_\rho)$ laplaciano en k-formas
- $\mathrm{Spec}(\Delta^k) = \{\lambda \in \mathbf{R} \mid \exists \omega \in \Omega^k(M; V_\rho), \Delta^k \omega = \lambda\omega\}$
  $\mathrm{Spec}(\Delta^k) > 0$ es discreto.

$$\zeta_k(s) = \sum_{\lambda \in \mathrm{Spec}(\Delta^k)} \lambda^{-s} \qquad \text{para } s \in \mathbf{C},\ Re(s) \gg 0$$

- $\zeta(s)$ se extiende meromórficamente a $s = 0$.

$\zeta_k'(s) = \sum -\lambda^{-s} \log\lambda$, $\Rightarrow \zeta_k'(0) = -\sum \log\lambda = -\log\det\Delta^k$ **(FORMAL!)**

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Def (Ray-Singer 1971)

Torsion analítica: $\log RS(M, \rho) := \frac{1}{2} \sum_k (-1)^{k+1} k \zeta_k'(0)$

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Teorema de Cheeger Müller

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  $\log RS(M, \rho) := \frac{1}{2} \sum_k (-1)^k k \xi_k'(0)$

Teorema (Cheeger-Müller)

La torsión analítica coincide con la combinatoria:

$$RS(M, \rho) = |\tau(M, \rho)|^{-1}$$

- Probado por Jeff Cheeger y Werner Müller
  para $\rho : \pi_1 M^n \to SO(m, \mathbf{R})$ (1978)
- Probado por W. Müller para $SL(m, \mathbf{R})$ (1993).

Fried y la función Zeta de Ruelle

- M^n hiperbólica cerrada, $n \geq 3$, $\rho : \pi_1(M^n) \to SO(m)$ acíclica.
- Función Zeta de Ruelle:

$$R_\rho(s) = \prod_\gamma \det(I - \rho(\gamma)e^{-sl(\gamma)}), \qquad s \in \mathbf{C},\ \mathrm{Re}(s) > n-1.$$

se multiplica sobre todas las geodésicas cerradas simples $\gamma \subset M^n$

Teorema (D. Fried 1986):

$R_\rho(s)$ se extiende meromórficamente a $\mathbf{C}$ y

$$|R_\rho(0)^{(-1)^n}| = \tau(M^n, \rho)^2,$$

Variedades hiperbólicas tridimensionales

- M^3 hiperbólica orientable, $M^3 \cong \mathbf{H}^3/\Gamma$
  $\Gamma < \mathrm{Isom}^+(\mathbf{H}^3) = PSL_2(\mathbf{C}) = SL_2(\mathbf{C})/\{\pm Id\}$
  $hol : \pi_1 M^3 \cong \Gamma < PSL_2(\mathbf{C})$ se levanta a $\widetilde{hol} : \pi_1 M^3 \to SL_2(\mathbf{C})$.

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- $\sigma_N : SL_2(\mathbf{C}) \to SL_N(\mathbf{C})$ irreducible
  $\mathbf{C}^N \cong \{p(x,y) \in \mathbf{C}[x,y]$ homogéneo y $\deg(p(x,y)) = N-1\}$
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Teorema (P. - P. Menal Ferrer 2010)

$K \subset S^3$ nudo hiperbólico, $2N > 1$, $H^*(S^3 \setminus K, \rho_{2N}) = 0$

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Teorema (W. Müller 2010) M^3 cerrada.

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Müller: M^3 cerrada. $\lim_{N\to\infty} \frac{\log|\tau(M^3,\rho_N)|}{N^2} = \frac{\mathrm{Vol}(M^3)}{4\pi}$

- Ingredientes de la demostración de Müller :

– $R_N(s) = \prod_\gamma \det(I - \rho(\gamma)e^{-sl(\gamma)}), \qquad s \in \mathbf{C},\ \mathrm{Re}(s) > n-1.$

– $R_N(0)^{-1} = |\tau(M^3,\rho_N)|^2$ (aunque ρ_N no sea ortogonal)

– Ecuación funcional

$$\tau(M^3,\rho_{2N}) = \tau(M^3,\rho_4)\prod_{k=3}^{N}\frac{1}{|R_{2k}(k)|}\exp\left(\frac{1}{\pi}\mathrm{Vol}(M^3)(N(N+1)-6)\right)$$

Idea de la demostración

Müller: M^3 cerrada. $\lim_{N\to\infty} \frac{\log|\tau(M^3,\rho_N)|}{N^2} = \frac{\mathrm{Vol}(M^3)}{4\pi}$

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- Para $S^3 \setminus K$, $\mathrm{Spec}(\Delta_k)$ no es discreto
- Thurston: $S^3 \setminus K$ se aproxima por variedades hiperbólicas cerradas, obtenidas por cirugía de Dehn en K ($(S^3 \setminus \mathcal{N}(K)) \cup_{T^2} S^1 \times D^2$).

Clave: controlar la geometría de las geodésicas de estas variedades y aplicar la ecuación funcional de Müller

¿Relación con la conjetura del volumen?

$K \subset S^3$ nudo hiperbólico: $\lim\limits_{N\to\infty} \frac{\log|\tau(S^3\setminus K, \rho_{2N})|}{N^2} = \frac{\mathrm{Vol}(S^3\setminus K)}{\pi}$

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Conjetura del Volumen (Kashaev 1997, Murakami-Murakami 2001)

$K \subset S^3$ nudo hiperbólico

$$\lim_{n\to\infty} \frac{\log|J_K(n)(e^{\frac{2\pi i}{n}})|}{n} = \frac{1}{2\pi} \mathrm{Vol}(S^3 \setminus K)$$

$J_K(n)(t) \in \mathbf{Z}[t^{\pm 1}]$ = polinomio de Jones coloreado con la representación irreducible de $\mathfrak{sl}_2$ de dim n y normalizado $J_0(n) = 1$.

¿Relación con la conjetura del volumen?

$K \subset S^3$ nudo hiperbólico: $\lim_{N\to\infty} \frac{\log|\tau(S^3\backslash K, \rho_{2N})|}{N^2} = \frac{\mathrm{Vol}(S^3\backslash K)}{\pi}$

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- ¿Se puede interpretar $J_K(n)(t)$ como torsión(es) de Reidemeister?