THE 2-TORSION IN THE SECOND HOMOLOGY OF THE GENUS 3 MAPPING CLASS GROUP

WOLFGANG PITSCH

Abstract. This work is NOT to be used as reference. First, because as C.F. Bödigheimer and M. Korkmaz pointed to us the computation of the $\mathbb{Z}_2$ factor that remained undecided in M. Korkmaz and A. Stipsicz, *The second homology groups of mapping class groups of orientable surfaces.* Math. Proc. Camb. Phil. Soc., was shown to exist by Skasai, see hi Theorem 4.9 and Corollary 4.10 in *Lagrangian mapping class groups from a group homological point of view.* Algebr. Geom. Topol. 12 (2012), no. 1, 267–291. Second, because one could obtain this result by gathering old results in the literature, first by noticing as Korkmaz kindly reminded me, that D. Johnson, in *Homeomorphisms of a surface which act trivially on homology* Porc. AMS Volume 75, Number 1, 1979. proved that the quotient of the Torelli group $\mathcal{T}_g/[\mathcal{T}_g, \mathcal{M}_g]$ is trivial for $g \geq 3$, the five term exact sequence then implies that the $\mathbb{Z}_2$ factor in Stein's computation of $H_2(Sp(6,\mathbf{Z});\mathbf{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$ (see his *The Schur Multipliers of* $Sp_6(\mathbf{Z}), Spin_8(\mathbf{Z}), Spin_7(\mathbf{Z})$, *and* $F_4(\mathbf{Z})$. Math. Ann. 215 (1975), 173–193.), detects the undecided $\mathbb{Z}_2$ factor in $H_2(\mathbf{M}_3;\mathbb{Z})$.

1. Introduction

Denote by $\Sigma_{g,n}^r$ an oriented surface of genus g with n boundary components and r punctures and by $\mathcal{M}_{g,n}^r$ its mapping class group, that is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n}^r$ that are the identity on the boundary and fix the punctures. Also denote by $\mathbf{Z}_2$ the mod 2 reduction of $\mathbf{Z}$. In a famous paper [1] Harer computed the second homology group of mapping class group for $g \geq 5$. Then, using a presentation of $\mathcal{M}_{g,1}$ given by Wajnryb in [7] we showed in [4] that Harer's computations could be obtained from Hopf's formula and extended to $g \geq 4$, yielding moreover an explicit generator for this group. Later on in [3] Korkmaz and Stipsicz pushed this computations to encompass the remaining genus $g = 2, 3$ and $n \geq 2, r \geq 1$. Notice that for $g = 2$ Benson and Cohen had computed the Poincaré series of $H_*(\mathcal{M}_2;\mathbf{Z}_2)$. Unfortunately a small gap remained after Korkmaz and Stipsicz computations: they

Date: 26 October, 2013.

2010 *Mathematics Subject Classification.* Primary 20J06, Secondary 57M60.

Key words and phrases. Mapping class group, lantern relation.

The author was partially supported by FEDER/MEC grant MTM2010-20622.

showed that $H_2(\mathcal{M}_3;\mathbf{Z}) \simeq \mathbf{Z} \oplus A$ and $H_2(\mathcal{M}_{3,1};\mathbf{Z}) \simeq \mathbf{Z} \oplus B$, where $0 \leq B \leq A \leq \mathbf{Z}_2$. The purpose of this note is to close this gap and to finally prove:

Theorem 1.1. *We have* $H_2(\mathcal{M}_3;\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}/2$ *and* $H_2(\mathcal{M}_{3,1};\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}/2$.

As observed by Korkmaz and Stipsicz the computations using Hopf's formula show that the homomorphism induced by capping-off the boundary component $H_2(\mathcal{M}_{3,1};\mathbf{Z}) \to H_2(\mathcal{M}_3;\mathbf{Z})$ is surjective and hence, from our computation, an isomorphism.

2. Proof of the Theorem

Denote by $\mathcal{T}_3$ or $\mathcal{T}_{3,1}$ accordingly the Torelli groups, that is the kernel of the surjective map from the mapping class group onto the symplectic group with integer entries $Sp(6,\mathbf{Z})$. By computations of Stein [6] we know that $H_2(\mathrm{Sp}(6;\mathbf{Z});\mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}/2$. Since mapping class groups of genus $g \geq 3$ are perfect, the 5 term exact sequence in low dimensional homology gives a small exact sequence:

$$H_2(\mathcal{M}_*;\mathbf{Z}) \longrightarrow H_2(\mathrm{Sp}(6;\mathbf{Z});\mathbf{Z}) \longrightarrow (H_1(\mathcal{T}_*;\mathbf{Z}))_{\mathrm{Sp}(6;\mathbf{Z})} \longrightarrow 0$$

Where $*$ denotes either g or $g,1$ and in view of Stein's result all we have to do is to show that $H_1(\mathcal{T}_*;\mathbf{Z}))_{\mathrm{Sp}(6;\mathbf{Z})} \simeq \mathcal{T}_*/[\mathcal{T}_*,\mathcal{M}_*] = 0$.

By Johnson's fundamental result the Torelli group $\mathcal{T}_3$ and $\mathcal{T}_{3,1}$ are generated by twists along bounding pairs of genus 1, that is by mapping classes of the form $T_\alpha T_\beta^{-1}$, where α and β are two simple closed curves that are not isotopic, homologous, not homologous to 0 and such that the complement of $\{\alpha,\beta\}$ has a component of genus 1. In particular capping-off the boundary component induces a surjective map $H_1(\mathcal{T}_{g,1};\mathbf{Z}) \to H_1(\mathcal{T}_g;\mathbf{Z})$ and since taking coinvariants is a right-exact functor it is enough to prove that

$$H_1(\mathcal{T}_{g,1};\mathbf{Z})_{\mathrm{Sp}(6;\mathbf{Z})} \simeq \mathcal{T}_{g,1}/[\mathcal{T}_{g,1},\mathcal{M}_{g,1}] = 0.$$

The mapping class group acts transitively on bounding pairs of genus 1, hence the group $\mathcal{T}_{g,1}/[\mathcal{T}_{g,1},\mathcal{M}_{g,1}]$ is monogenic generated by the class of any bounding pair map. Also, given a bounding pair $\{\alpha,\beta\}$ there exists a mapping class, say ϕ, that exchanges α and β, hence in $\mathcal{T}_{g,1}/[\mathcal{T}_{g,1},\mathcal{M}_{g,1}]$:

$$T_\alpha {T_\beta}^{-1} = \phi T_\alpha {T_\beta}^{-1}\phi^{-1} = T_{\phi(\alpha)}{T_{\phi(\beta)}}^{-1} = T_\beta {T_\alpha}^{-1} = (T_\alpha T_\beta^{-1})^{-1},$$

and $\mathcal{T}_{g,1}/[\mathcal{T}_{g,1},\mathcal{M}_{g,1}]$ is at most $\mathbf{Z}_2$.

In [2] Johnson computed the abelianization of the Torelli group $\mathcal{T}_{g,1}$, it fits into a short exact sequence:

$$0 \longrightarrow B^2_{3,1} \longrightarrow H_1(\mathcal{T}_{g,1};\mathbf{Z}) \longrightarrow \Lambda^3 H \longrightarrow 0,$$

where:

(1) H stands for the homology group $\mathrm{H}_1(\Sigma_{3,1};\mathbf{Z})$,
(2) $B^2_{3,1}$ is the Boolean algebra of polynomials of degree ≤ 2 generated by the elements $\overline{x} \in H$ and subject to the relations:
   - $\overline{x+y} = \overline{x} + \overline{y} + x \cdot y$, where $x \cdot y$ is the mod 2 intersection number,
   - $\overline{x}^2 = \overline{x}$.

Notice that this is a short exact sequence of $\mathrm{Sp}(6;\mathbf{Z})$-modules, where the action on the quotient is simply given by the third exterior power of the action on homology and the action of the kernel is given on generators by $\phi(\overline{x}) = \overline{\phi(x)}$ extended in the obvious way. Finally, this kernel is a $\mathbf{Z}_2$-vector space and is the image in $H_1(\mathcal{T}_{g,1};\mathbf{Z})$ of the Johnson subgroup $\mathcal{K}_{g,1}$, the subgroup generated by twists along bounding simple closed curves.

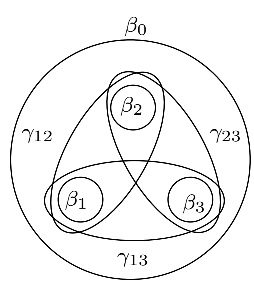
Lemma 2.1. *The image of $B^2_{3,1}$ in $H_1(\mathcal{T}_{g,1};\mathbf{Z})_{\mathrm{Sp}(6;\mathbf{Z})}$ is trivial.*

Proof. This image is a quotient of $(B^2_{3,1})_{\mathrm{Sp}(6;\mathbf{Z})}$, so it suffices to prove that this group is trivial. First notice that since the mod 2 symplectic form is non-degenerated any non-zero element in H can be completed into a symplectic basis, and in particular $\mathrm{Sp}(6;\mathbf{Z})$ acts transitively on the non-zero elements in H. Let a, b be two elements such that $a \cdot b = 1$, then if τ_a denotes the transvection along a, we have $\tau_a(\overline{b}) = \overline{a+b} = \overline{a} + \overline{b} + 1$, and in $(B^2_{3,1})_{\mathrm{Sp}(6;\mathbf{Z})}$ this gives that $\overline{a} = 1$, hence in $(B^2_{3,1})_{\mathrm{Sp}(6;\mathbf{Z})}$ all monomials of degree 1 or 2 are in fact constants and this group is at most $\mathbf{Z}_2$. Finally, let $c \in H$ be such that $a \cdot c = 0 = b \cdot c$. In $(B^2_{3,1})_{\mathrm{Sp}(6;\mathbf{Z})}$ we have:

$$\begin{aligned} 1 &= \overline{a}\overline{b} &&= \tau_{b+c}(\overline{a}\overline{b}) \\ &= \overline{a+b+c}\;\overline{b} &&= (\overline{a} + \overline{b} + \overline{c} + 1)\overline{b} \\ &= \overline{a}\overline{b} + \overline{b}^2 + \overline{c}\overline{b} + \overline{b} &&= 1+1+1+1 \\ &= 0. \end{aligned}$$

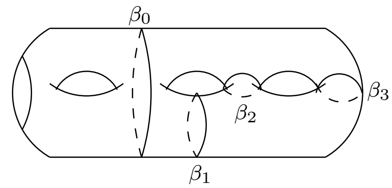
□

To conclude we apply the lantern relation:



$$T_{\beta_0} T_{\beta_1} T_{\beta_2} T_{\beta_3} = T_{\gamma_{12}} T_{\gamma_{13}} T_{\gamma_{23}}$$

to the following curves (we only draw the four boundary curves).



The lantern relation shows that the twist around β_0, a bounding simple closed curve of genus 2 can be written as the product of three twists around bounding pairs of genus 1:

$$T_{\beta_0} = T_{\gamma_{12}} T_{\beta_3}^{-1} T_{\gamma_{13}} T_{\beta_2}^{-1} T_{\gamma_{23}} T_{\beta_1}^{-1}$$

If t denotes the generator of $H_1(\mathcal{T}_{g,1}; \mathbf{Z})_{\mathrm{Sp}(6;\mathbf{Z})}$, which is of order 2, then this equation becomes $0 = t^3$, and this finishes the proof.

References

[1] J. L. Harer, *The second homology group of the mapping class group of an orientable surface.*, Invent. Math. (2) 72 (1982), 221–239.

[2] D. L. Johnson, *The structure of the Torelli group. III. The abelianization of $\mathcal{T}$*, Topology 24 (1985), 127–144.

[3] M. Korkmaz and A. Stipsicz, *The second homology groups of mapping class groups of orientable surfaces.* Math. Proc. Camb. Phil. Soc. 134 (2003), 479–488.

[4] W. Pitsch, *Un calcul élémentaire de $H_2(\mathcal{M}_{g,1}; \mathbf{Z})$ pour $g \geq 4$.* C. R. Acad. Sci. Paris Sér. I Math. (8) 29 (1999), 667-670.

[5] T. Sakasai, *Lagrangian mapping class groups from a group homological point of view.* Algebr. Geom. Topol. 12 (2012), no. 1, 267291.

[6] M. R. Stein, *The Schur Multipliers of $Sp_6(\mathbf{Z}), Spin_8(\mathbf{Z}), Spin_7(\mathbf{Z})$, and $F_4(\mathbf{Z})$.* Math. Ann. 215 (1975), 173–193.

[7] B. Wajnryb, *An elementary approach to the mapping class group of a surface.* Geom. Topol. 3 (1999), 405–466.

Universitat Autònoma de Barcelona, Departament de Matemàtiques. E-08193 Bellaterra, Spain

E-mail address: `pitsch@mat.uab.es`