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Trivial cocycles and invariants of homology 3-spheres

Wolfgang Pitsch [1]

Universitat Autònoma de Barcelona, Departament de Matemàtiques, E-08193 Bellaterra, Spain

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Abstract

We study the relationship between trivial cocycles on the Torelli group and invariants of oriented integral homology 3-spheres. We apply this study to give a new purely algebraic construction of the Casson invariant. As a by-product we get a new 2-torsion cohomology class in the second integral cohomology of the Torelli group.

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0. Introduction

If one tries to understand 3-manifolds by "cut and paste" techniques one faces two different paths: either one can concentrate the difficulties in the pieces and have "simple" glueing maps (see Kneser's prime decomposition or Thurston decomposition into geometric pieces) or one can concentrate the difficulty into the glueing maps and get "simple" pieces. In the latter path one finds the theory of Heegaard splittings, the pieces are handlebodies and the glueing problems are encompassed within the mapping class groups of oriented surfaces, $\mathcal{M}_{g,1}$. It is natural then to try to construct invariants of 3-manifolds out of the algebraic properties of these groups. For general 3-manifolds this strategy has been adopted for instance by Birman in [1] but is usually hopelessly difficult, for the structure of $\mathcal{M}_{g,1}$ is quite involved. In this paper we will concentrate

E-mail address: pitsch@mat.uab.es.

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on the subclass of integral homology 3-spheres. In this case the group that controls the glueings is the Torelli group, $\mathcal{T}_{g,1}$. This approach has been that of Morita in [16] where he constructed a function out of the Torelli group that he showed to coincide point-wise with the Casson invariant. Although this did not succeed into a new construction of this important invariant it was the starting point for a fruitful exploration the interplay between the Casson invariant and algebraic properties of the Torelli group [16–18]. Notice however that later Perron in [21] proved that Morita's function restricted to a suitable subgroup of the Torelli group is an invariant, independently from the existence of the Casson invariant.

In this paper we will give a general framework to construct invariants of homology spheres in a purely algebraic setting as functions out of the full Torelli group. We will show that the algebraic problems boil down to low-dimensional cohomological problems. As an example we will give a construction of an invariant of homology spheres and by proving the "surgery formulas" we will show that it coincides with the Casson invariant.

Denote by $\mathcal{V}(3)$ the set of diffeomorphism classes of compact, closed and oriented smooth 3-manifolds and by $\mathcal{S}(3) \subset \mathcal{V}(3)$ the subset of homology spheres, that is diffeomorphisms classes that have the same integral homology as the standard 3-sphere $\mathbf{S}^3$. Let Σ_g denote an oriented surface of genus g standardly embedded in the oriented 3-sphere $\mathbf{S}^3$. In particular Σ_g separates $\mathbf{S}^3$ into two genus g handlebodies $\mathbf{S}^3 = \mathcal{H}_g \cup -\mathcal{H}_g$ with opposite induced orientation. Denote by $\mathcal{M}_{g,1}$ the mapping class group of Σ_g, that is the group of orientation-preserving diffeomorphisms of Σ_g which are the identity on a small fixed disc modulo isotopies which fix that small disc pointwise. The embedding $\Sigma_g \hookrightarrow \mathbf{S}^3$ determines three natural subgroups of $\mathcal{M}_{g,1}$, namely the subgroup $\mathcal{B}_{g,1}$ of mapping classes that are restrictions of diffeomorphisms of the first handlebody $\mathcal{H}_g$, the subgroup $\mathcal{A}_{g,1}$ of mapping classes that are restrictions of diffeomorphisms of the second handlebody $-\mathcal{H}_g$ and their intersection $\mathcal{AB}_{g,1}$.

From the theory of Heegaard splittings we learn that any element in $\mathcal{V}(3)$ can be obtained by cutting $\mathbf{S}^3$ along Σ_g for some g and glueing back the two handlebodies by some element $\phi \in \mathcal{M}_{g,1}$. The lack of injectivity of this construction is controlled by the subgroups $\mathcal{B}_{g,1}$ and $\mathcal{A}_{g,1}$. More precisely there is a natural injective stabilization map $\mathcal{M}_{g,1} \hookrightarrow \mathcal{M}_{g+1,1}$, which is compatible with the definitions of the above subgroups and one gets a well-defined bijective map:

$$\lim_{g\to\infty} \mathcal{A}_{g,1}\backslash\mathcal{M}_{g,1}/\mathcal{B}_{g,1} \xrightarrow{\sim} \mathcal{V}(3),$$
$$\phi \longmapsto \mathbf{S}^3_\phi = \mathcal{H}_g \cup_\phi -\mathcal{H}_g.$$

Thus any problem on 3-dimensional manifolds can be translated into a problem on the mapping class group. In particular any invariant $F : \mathcal{V}(3) \to \mathbf{Z}$ can be viewed as a compatible family of functions on the mapping class groups $\mathcal{M}_{g,1}$ which are constant on double cosets.

If we restrict our study to $\mathcal{S}(3)$ the situation becomes more tractable. First we can restrict our attention to those mapping classes that act trivially on the homology of the underlying surface. Recall that the Torelli group $\mathcal{T}_{g,1}$ is defined as the kernel of the natural map $\mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}(H_1(\Sigma_g; Z))$. The above bijection induces a new bijection [16]:

$$\lim_{g\to\infty} \mathcal{A}_{g,1}\backslash\mathcal{T}_{g,1}/\mathcal{B}_{g,1} \xrightarrow{\sim} \mathcal{S}(3),$$
$$\phi \longmapsto \mathbf{S}^3_\phi = \mathcal{H}_g \cup_\phi -\mathcal{H}_g,$$

where $\mathcal{A}_{g,1}\backslash\mathcal{T}_{g,1}/\mathcal{B}_{g,1}$ stands for those mapping classes in $\mathcal{M}_{g,1}$ that contain an element of the Torelli group. Denote by $\mathcal{TB}_{g,1}$ (resp. $\mathcal{TA}_{g,1}$) the group $\mathcal{T}_{g,1}\cap\mathcal{B}_{g,1}$ (resp. $\mathcal{T}_{g,1}\cap\mathcal{A}_{g,1}$). The induced equivalence relation on the Torelli group has an intrinsic description, which will be proven in Section 1:

Theorem 1. *Two elements $\phi,\psi\in\mathcal{T}_{g,1}$ belong to the same double coset in $\mathcal{B}_{g,1}\backslash\mathcal{M}_{g,1}/\mathcal{A}_{g,1}$ if and only if there exist maps $\xi_b\in\mathcal{TB}_{g,1}$, $\xi_a\in\mathcal{TA}_{g,1}$ and $\mu\in\mathcal{AB}_{g,1}$ such that*

$$\phi=\mu\xi_a\psi\xi_b\mu^{-1}.$$

The conjugacy part of this equivalence relation is the key tool of our study. Consider an integral-valuated invariant of homology spheres $F:\mathcal{S}(3)\to\mathbf{Z}$. By the above bijection and Theorem 1 we can view F as a family of compatible functions F_g (i.e. $F_{g+1}|_{\mathcal{T}_{g,1}}=F_g$) that are constant on the double coset classes $\mathcal{TA}_{g,1}\backslash\mathcal{T}_{g,1}/\mathcal{TB}_{g,1}$ and invariant under conjugation by $\mathcal{AB}_{g,1}$. To any such family of functions we associate a family of *trivialized* 2-cocycles on the Torelli groups $C_g(\phi,\psi)=F_g(\phi)+F_g(\psi)-F_g(\phi\psi)$. It turns out that these functions are not trivial unless F_g is itself trivial. Since $\mathcal{T}_{g,1}$ is not perfect there is a difference between trivialized cocycles and trivial cocycles. One might wonder what conditions we should impose on a family (C_g) of trivial 2-cocycles on the Torelli groups such that from their trivializations one can extract a family of compatible trivializations (F_g) that reassemble into an invariant of homology spheres $F:\mathcal{S}(3)\to\mathbf{Z}$. Notice that the maps F_g are necessarily $\mathcal{AB}_{g,1}$-invariant trivializations of the cocycles.

The cocycles C_g inherit the following properties of the maps F_g:

(1) The cocycles C_g are be compatible $C_{g+1}|_{\mathcal{T}_{g,1}\times\mathcal{T}_{g,1}}=C_g$.
(2) The cocycles C_g are 0 on $\mathcal{TA}_{g,1}\times\mathcal{T}_{g,1}\cup\mathcal{T}_{g,1}\times\mathcal{TB}_{g,1}$.
(3) The cocycles C_g are invariant under conjugation by $\mathcal{AB}_{g,1}$.

Then, the existence of an $\mathcal{AB}_{g,1}$-invariant trivialization of the cocycle C_g is controlled by a cohomology class, the torsor:

$$\rho(C_g)\in\mathrm{H}^1\big(\mathcal{AB}_{g,1};\textstyle\bigwedge^3\mathrm{H}_1(\Sigma_g;\mathbf{Z})\big).$$

As we will see, in our case, the torsor class can be viewed as an homomorphism

$$\rho(C_g):\mathcal{TAB}_{g,1}\otimes\textstyle\bigwedge^3\mathrm{H}_1(\Sigma_g;\mathbf{Z})\to\mathbf{Z}.$$

Here $\mathcal{TAB}_{g,1}=\mathcal{T}_{g,1}\cap\mathcal{AB}_{g,1}$.

The three conditions above and the nullity of the torsor turn out to be not only necessary but also sufficient:

Theorem 2. *A family of cocycles $(C_g)_{g\geqslant 3}$ on the Torelli groups $\mathcal{T}_{g,1}$, $g\geqslant 3$, satisfying conditions* (1)–(3) *provides a compatible family of trivializations $F_g:\mathcal{T}_{g,1}\to\mathbf{Z}$ that reassemble into an invariant of homology spheres*

$$\lim_{g\to\infty}F_g:\mathcal{S}(3)\to\mathbf{Z}$$

if and only if the following two conditions hold:

(i) *The associated cohomology classes* $[C_g] \in \mathrm{H}^2(\mathcal{T}_{g,1}; \mathbf{Z})$ *are trivial.*
(ii) *The associated torsors* $\rho(C_g) \in \mathrm{Hom}(\mathcal{T}\mathcal{AB}_{g,1} \otimes \bigwedge^3 \mathrm{H}_1(\Sigma_g), \mathbf{Z})$ *are trivial.*

In this case the maps F_g *are the unique* $\mathcal{AB}_{g,1}$*-invariant trivializations of the cocycles* C_g.

Obviously, constructing (trivial) 2-cocycles directly on the Torelli group is still a difficult problem but instead one could try to pull-back known 2-cocycles defined on homomorphic images of the Torelli group. We successfully apply this strategy to the Johnson homomorphism $\tau : \mathcal{T}_{g,1} \to \bigwedge^3 \mathrm{H}_1(\Sigma_g; \mathbf{Z})$.

Theorem 3. *The unique* 2*-cocycles on* $\bigwedge^3 \mathrm{H}_1(\Sigma_g; \mathbf{Z})$ *whose pull-back along the Johnson homomorphism satisfy conditions* (1)–(3) *are of the form* nJ_g, $n \in \mathbf{Z}$, *for an explicit* 2*-cocycle* J_g. *Moreover:*

(1) *The pull-backs of the cocycles* $2J_g$ *and the associated torsors* $\rho(2J_g)$ *are trivial.*
(2) *The associated invariant is equal to the Casson invariant.*

Moreover:

Theorem 4.

(1) *The pull-backs of the cocycles* J_g *on the Torelli groups are not trivial and define stable* 2*-torsion cohomology classes*

$$[J_g] \in \mathrm{H}^2(\mathcal{T}_{g,1}; \mathbf{Z}).$$

(2) *Viewing the Rohlin invariant as a family of classes* $R_g \in \mathrm{H}^1(\mathcal{T}_{g,1}; \mathbf{Z}/2\mathbf{Z})$, *we have*

$$\beta_{\mathbf{Z}}(R_g) = [J_g],$$

where $\beta_{\mathbf{Z}}$ *stands for the integral Bockstein operation.*

We wish to point out that this construction of an invariant as a compatible family of trivializations of the pull-backs of cocycles $2J_g$ is independent of the construction of the Casson invariant. It is also possible to show independently of previous computations, but the work of Casson, that the invariant associated to the cocycles $-2J_g$ is the Casson invariant, but it is much shorter to use the fundamental results of Morita [18] to identify them.

Here is the plan of this work. In Section 1 we turn back to the definition of the groups $\mathcal{A}_{g,1}$, $\mathcal{B}_{g,1}$, $\mathcal{AB}_{g,1}$, we describe their actions on the first homology and homotopy groups of the underlying surface and we prove Theorem 1. In Section 2 we study the relationship between trivial cocycles on the Torelli groups and invariants of homology spheres. In particular we prove Theorem 2 up to a technical Lemma which is delayed until Section 4. In Section 3 we apply our results to give a purely algebraic construction of the Casson invariant and we prove Theorems 3 and 4. Finally, in Section 4 we cope with the proof of the technical Lemma.

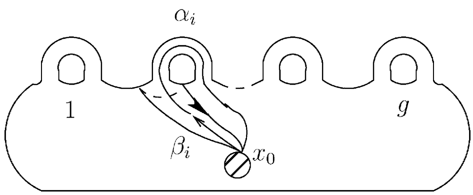

Fig. 1. Model of Σ_g.

General conventions

The properties of the genus 1 and 2 mapping class groups and their subgroups is very peculiar. Since the injectivity of the stabilization map $\mathcal{M}_{g,1} \hookrightarrow \mathcal{M}_{g+1,1}$ implies that in our case it is enough to consider large enough values of g we will always assume that $g \geqslant 3$. All invariants considered will take the value 0 on the standard oriented sphere $\mathbf{S}^3$. If γ denotes a simple closed curve on the surface Σ_g, we will denote by T_γ the right-hand Dehn twist about γ.

To make notations lighter we will denote by H the group $\mathrm{H}_1(\Sigma_{g,1};\mathbf{Z})$ and by $(\bigwedge^3 H)^*$ the group $\mathrm{Hom}(\bigwedge^3 H, \mathbf{Z})$.

1. Heegaard splittings of homology spheres

1.1. The mapping class group and some of its subgroups

For convenience we fix a model of our genus g surface Σ_g as in Fig. 1. We denote by $\Sigma_{g,1}$ the complement of the interior of a small disc embedded in Σ_g. We fix a base point on the boundary of $\Sigma_{g,1}$. The (isotopy class of) the curves α_i, β_i, $1 \leqslant i \leqslant g$, are free generators of the free group $\pi_1(\Sigma_{g,1}, x_0)$. The first homology group of the surface $\mathrm{H}_1(\Sigma_g;\mathbf{Z}) \simeq H$ is endowed via Poincaré duality with a natural symplectic intersection form $\omega: \bigwedge^2 H \to \mathbf{Z}$. The homology classes a_i, b_i of the above curves freely generate the abelian group $H \simeq \mathbf{Z}^{2g}$ and define two transverse Lagrangians A and B in H.

Denote by $\mathrm{Diff}^+(\Sigma_g, \mathrm{rel}.D^2)$ the group of orientation preserving diffeomorphisms of Σ_g that are the identity on the fixed small disc, endowed with the compact-open topology. The mapping class group $\mathcal{M}_{g,1}$ is the group of connected components $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}^+(\Sigma_g, \mathrm{rel}.D^2))$.

The natural action of the mapping class group $\mathcal{M}_{g,1}$ on H clearly preserves the intersection form and we have a short exact sequence where the kernel is known as the Torelli group $\mathcal{T}_{g,1}$:

$$1 \longrightarrow \mathcal{T}_{g,1} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}\,\omega \longrightarrow 1.$$

Recall that our surface is standardly embedded in the oriented 3-sphere $\mathbf{S}^3$. As such it determines two embedded handlebodies $\mathbf{S}^3 = \mathcal{H}_g \cup -\mathcal{H}_g$. By the *inner handlebody* $\mathcal{H}_g$ we will mean the one that is visible in Fig. 1 and by the *outer handlebody* $-\mathcal{H}_g$ we will mean the complementary handlebody. They are naturally pointed by $x_0 \in \mathcal{H}_g \cap -\mathcal{H}_g$.

From these we get three natural subgroups of $\mathcal{M}_{g,1}$. First, the subgroup of those mapping classes that are restrictions of diffeomorphisms of the inner handlebody $\mathcal{H}_g$ which we call $\mathcal{B}_{g,1} \subset \mathcal{M}_{g,1}$. Second, the subgroup of those that are restrictions of the outer handlebody $\mathcal{A}_{g,1} \subset \mathcal{M}_{g,1}$. Finally, their intersection $\mathcal{AB}_{g,1}$, which may be identified to subgroup of mapping classes that are restrictions of diffeomorphisms of the whole sphere that leave our embedded surface invariant. We denote the groups $\mathcal{T}_{g,1} \cap \mathcal{A}_{g,1}$, $\mathcal{T}_{g,1} \cap \mathcal{B}_{g,1}$ and $\mathcal{T}_{g,1} \cap \mathcal{AB}_{g,1}$ respectively by $\mathcal{TA}_{g,1}$, $\mathcal{TB}_{g,1}$ and $\mathcal{TAB}_{g,1}$.

Remark. In this article we deal mostly with mapping class groups relative to a boundary component. Most references, in particular those dealing with the subgroups $\mathcal{A}_{g,1}$, $\mathcal{B}_{g,1}$, $\mathcal{AB}_{g,1}$ [4,13,22] are written for closed surfaces but lifting their results to the boundary case is not difficult. Indeed, taking isotopy class of diffeomorphisms of the closed surface relative to the base point x_0 gives us another mapping class group usually denoted by $\mathcal{M}_{g,*}$. The natural "forgetfull" operation induces a surjective map $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$. Its kernel is generated by a Dehn twist around a curve parallel to the boundary and we get a short exact sequence:

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,*} \longrightarrow 1.$$

The mapping class group of the "inner" handlebody $\mathcal{H}_g$ relative to the base point can be identified with a subgroup $\mathcal{B}_{g,*} \subset \mathcal{M}_{g,*}$, where the inclusion is induced by restricting mapping classes to the boundary. Since the aforementioned Dehn twist extends naturally to the handlebody $\mathcal{H}_g$ the preimage of $\mathcal{B}_{g,1} \subset \mathcal{M}_{g,1}$ of $\mathcal{B}_{g,*}$ can be identified as the mapping class group of the handlebody $\mathcal{H}_g$ relative to a small ball B^3 such that $B^3 \cap \Sigma_g$ is our distinguished small disk D^2 and we get a commutative diagram with vertical arrows induced by restricting to the boundary:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \mathcal{M}_{g,*} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathcal{B}_{g,1} & \longrightarrow & \mathcal{B}_{g,*} & \longrightarrow & 1. \end{array}$$

Similar identifications hold for the groups $\mathcal{A}_{g,1}$ and $\mathcal{AB}_{g,1}$.

1.2. Homology and homotopy actions

According to Griffith [4], the subgroup $\mathcal{B}_{g,1}$ (resp. $\mathcal{A}_{g,1}$) is characterised by the fact that its action on $\pi_1(\Sigma_g, x_0)$ preserves the normal subgroup generated by the curves $\beta_1, \dots, \beta_g$ (resp. $\alpha_1, \dots, \alpha_g$). As a consequence the action on homology of $\mathcal{B}_{g,1}$ (resp. $\mathcal{A}_{g,1}$) preserves the Lagrangian B (resp. A).

If one writes the matrices of the symplectic group $\mathrm{Sp}\,\omega$ as blocks according to the decomposition $H = A \oplus B$, then the image of $\mathcal{B}_{g,1} \to \mathrm{Sp}\,\omega$ is contained in the subgroup $\mathrm{Sp}_B\,\omega$ of matrices of the form: $\left(\begin{smallmatrix} G_1 & 0 \\ M & G_2 \end{smallmatrix}\right)$.

Such matrices are symplectic if and only if $G_2 = {}^tG_1^{-1}$ and ${}^tG_1 M$ is symmetric and we have an isomorphism:

$$\mathrm{Sp}_B\,\omega \xrightarrow{\sim} \mathrm{GL}_g(\mathbf{Z}) \ltimes S_g(\mathbf{Z}),$$
$$\begin{pmatrix} G & 0 \\ M & {}^tG^{-1} \end{pmatrix} \longmapsto \left(G, {}^tGM\right).$$

Here $S_g(\mathbf{Z})$ denotes the symmetric group of $g \times g$ matrices over the integers; the composition on the semi-direct product is given by the rule $(G, S)(H, T) = (GH, {}^tHSH + T)$. Checking on generators (see Suzuki [22]) of $\mathcal{B}_{g,1}$ we get:

Lemma 1. *There is a short exact sequence of groups:*

$$1 \longrightarrow \mathcal{TB}_{g,1} \longrightarrow \mathcal{B}_{g,1} \longrightarrow \mathrm{GL}_g(\mathbf{Z}) \ltimes S_g(\mathbf{Z}) \longrightarrow 1.$$

An analogous statement holds for $\mathcal{A}_{g,1}$ replacing the lagrangian B by A.

We also recall a result due to Luft [13, Corollary 2.1].

Lemma 2. *The natural homomorphism $\mathcal{B}_{g,1} \to \mathrm{Aut}\,\pi_1(\mathcal{H}_g, x_0)$ is onto.*

Again an analogous statement holds for $\mathcal{A}_{g,1}$. If we restrict our attention to $\mathcal{AB}_{g,1}$ then the natural homomorphism $\mathcal{AB}_{g,1} \to \mathrm{Aut}\,\pi_1(\mathcal{H}_g, x_0)$ is still an automorphism for the elements of $\mathcal{B}_{g,1}$ that hit the generators of the automorphism group in Luft's paper are readily seen to live in fact in $\mathcal{AB}_{g,1}$ (for a geometric description of these generators see [13] or [22] and for an algebraic description see Section 4). As for the previous lemma, checking on generators we get:

Lemma 3. *There is a short exact sequence of groups:*

$$1 \longrightarrow \mathcal{TAB}_{g,1} \longrightarrow \mathcal{AB}_{g,1} \longrightarrow \mathrm{GL}_g(\mathbf{Z}) \longrightarrow 1.$$

1.3. Heegaard splittings of homology spheres

It is well known that by glueing two handlebodies with opposite orientations along a diffeomorphism of their boundary one can construct all oriented compact 3 manifolds. Choose a map $\iota_g \in \mathcal{M}_{g,1}$ such that $S^3 = \mathcal{H}_g \cup_{i_g} -\mathcal{H}_g$. If we twist this glueing by an arbitrary map in $\mathcal{T}_{g,1}$ we get back a new homology sphere $S^3_\phi = \mathcal{H}_g \cup_{i_g\phi} -\mathcal{H}_g$ and in fact we can get all homology spheres by letting g vary [16]. More precisely, consider the following equivalence relation on $\mathcal{T}_{g,1}$:

$$\phi \sim \psi \quad \Leftrightarrow \quad \exists \zeta_a \in \mathcal{A}_{g,1}\ \exists \zeta_b \in \mathcal{B}_{g,1} \quad \text{such that} \quad \zeta_a \phi \zeta_b = \psi. \tag{$*$}$$

Moreover define the *stabilization map* on the mapping class group as follows. Glue one of the boundary components of a two-holed torus on the boundary of $\Sigma_{g,1}$ to get $\Sigma_{g+1,1}$. Extending an element of $\mathcal{M}_{g,1}$ by the identity over the torus yields an injective homomorphism $\mathcal{M}_{g,1} \hookrightarrow \mathcal{M}_{g+1,1}$, this is the stabilization map. This map is compatible with the action on homology and is compatible with the definition of the above two subgroups $\mathcal{A}_{g,1}$ and $\mathcal{B}_{g,1}$. In particular the equivalence relation $(*)$ is compatible with the stabilisation map. It is also possible to choose the map i_g to be compatible with the stabilization map $i_{g+1}|_{\Sigma_{g,1}} = i_g$ and we have the following precise version of Heegaard splittings of integral homology spheres (see [16] for a proof):

Theorem 5. *The following map is well defined and is bijective:*

$$\lim_{g\to\infty} \mathcal{T}_{g,1}/\sim \longrightarrow \mathcal{S}(3),$$
$$\phi \longmapsto S^3_\phi = \mathcal{H}_g \cup_\phi -\mathcal{H}_g.$$

From a group-theoretical point of view the equivalence relation $(*)$ is quite unsatisfactory, for it looks like, but is not, a double coset relation on the Torelli group. In fact it is the composite of a double coset relation in the Torelli group and a conjugacy-induced equivalence relation:

Lemma 4. *Two maps $\phi, \psi \in \mathcal{T}_{g,1}$ are equivalent if and only if there exist a map $\mu \in \mathcal{AB}_{g,1}$ and two maps $\xi_a \in \mathcal{TA}_{g,1}$ and $\xi_b \in \mathcal{TB}_{g,1}$ such that $\phi = \mu\xi_a\psi\xi_b\mu^{-1}$.*

Proof. The "if" part of the theorem is trivial. Conversely, assume that $\psi = \xi_a\phi\xi_b$, where $\psi, \phi \in \mathcal{T}_{g,1}$. Projecting this equality on $\mathrm{Sp}\,\omega$ we get $Id = \mathrm{H}_1(\xi_a)\mathrm{H}_1(\xi_b)$. According to Lemma 1 the matrix $\mathrm{H}_1(\xi_b)$ is of the form

$$\begin{pmatrix} G & 0 \\ M & {}^tG^{-1} \end{pmatrix}.$$

Similarly, $\mathrm{H}_1(\xi_a)$ is of the form

$$\begin{pmatrix} H & N \\ 0 & {}^tH^{-1} \end{pmatrix}.$$

Therefore:

$$Id = \begin{pmatrix} H & N \\ 0 & {}^tH^{-1} \end{pmatrix}\begin{pmatrix} G & 0 \\ M & {}^tG^{-1} \end{pmatrix} = \begin{pmatrix} HG + NM & N\,{}^tG^{-1} \\ {}^tH^{-1}M & {}^tH^{-1}G^{-1} \end{pmatrix}.$$

Thus:

$$N = 0 = M \quad \text{and} \quad G = H^{-1}.$$

In particular $\mathrm{H}_1(\xi_a), \mathrm{H}_1(\xi_b) \in \mathrm{GL}(g, \mathbf{Z})$ and $\mathrm{H}_1(\xi_a) = \mathrm{H}_1(\xi_b)^{-1}$. By Lemma 3 we can choose a map $\mu \in \mathcal{AB}_{g,1}$ such that $\mathrm{H}_1(\mu) = \mathrm{H}_1(\xi_a)$, and we get

$$\psi = \mu \circ \left(\mu^{-1}\xi_a\right)\phi(\xi_b\mu) \circ \mu^{-1}.$$

By construction $(\mu^{-1}\xi_a) \in \mathcal{TA}_{g,1}$ and $(\xi_b\mu) \in \mathcal{TB}_{g,1}$. □

2. Trivial cocycles and invariants

2.1. The Johnson homomorphism

Computing the action of the Torelli group on the second nilpotent quotient of $\pi_1(\Sigma_{g,1}, x_0)$ Johnson defines a morphism of groups known as the first Johnson homomorphism:

$$\tau : \mathcal{T}_{g,1} \longrightarrow \bigwedge^3 H.$$

Notice that the mapping class group $\mathcal{M}_{g,1}$ acts naturally by conjugation on $\mathcal{T}_{g,1}$ and acts also on $\bigwedge^3 H$ via its natural action on homology. In [10–12] Johnson proves that

Proposition 1. *The map τ is $\mathcal{M}_{g,1}$-equivariant with respect to the above actions. Up to finite dimensional $\mathbf{Z}/2\mathbf{Z}$-vector space $\bigwedge^3 H$ is the abelianization of the Torelli group: any homomorphism $\mathcal{T}_{g,1} \to A$ where A is an abelian group without 2-torsion factors uniquely through τ.*

2.2. From invariants to trivial cocycles

Consider an integer-valuated invariant of homology spheres $F : \mathcal{S}(3) \to \mathbf{Z}$. Precomposing with the canonical maps $\mathcal{T}_{g,1} \to \lim_{g\to\infty} \mathcal{T}_{g,1}/\sim \to \mathcal{S}(3)$ we get a family of maps $F_g : \mathcal{T}_{g,1} \to \mathbf{Z}$. Since the stabilization maps are injective the map F_g determines by restriction all maps $F_{g'}$ for $g' < g$. Therefore, as stated in the introduction, we avoid the peculiarities of the first Torelli groups by restricting ourselves to $g \geqslant 3$. We also consider the associated trivial cocycles, which measure the failure of the maps F_g to be homomorphisms of groups

$$\begin{aligned} C_g : \mathcal{T}_{g,1} \times \mathcal{T}_{g,1} &\longrightarrow \mathbf{Z}, \\ (\phi, \psi) &\longmapsto F_g(\phi) + F_g(\psi) - F_g(\phi\psi). \end{aligned}$$

Since F is an invariant the cocycles C_g inherit the following properties:

(1) The cocycles C_g are compatible, i.e. the following diagram of maps commutes:

$$\begin{array}{ccc} \mathcal{T}_{g,1} \times \mathcal{T}_{g,1} & \hookrightarrow & \mathcal{T}_{g+1,1} \times \mathcal{T}_{g+1,1} \\ & \searrow^{C_g} & \downarrow C_{g+1} \\ & & \mathbf{Z}. \end{array}$$

(2) The cocycles C_g are invariant under conjugation by elements in $\mathcal{AB}_{g,1}$: $C_g(\phi - \phi^{-1}, \phi - \phi^{-1}) = C_g(-,-)$.

(3) If $\phi \in \mathcal{TA}_{g,1}$ or $\psi \in \mathcal{TB}_{g,1}$ then $C_g(\phi, \psi) = 0$.

Proposition 2. *The cocycle C_g is constantly equal to 0 if and only if F_g is the zero map.*

Proof. If C_g is 0 then F_g is a morphism of groups and therefore factors via τ:

$$\begin{array}{ccc} \mathcal{T}_{g,1} & & \\ \downarrow \tau & \searrow^{F_g} & \\ \bigwedge^3 H & \xrightarrow{\overline{F}_g} & \mathbf{Z}. \end{array}$$

The morphism $\overline{F}_g$ is then $\mathrm{GL}_g(\mathbf{Z}) = \mathcal{AB}_{g,1}/\mathcal{TAB}_{g,1}$-invariant. As $-Id \in \mathrm{GL}_g(\mathbf{Z})$ acts as $-Id$ on $\bigwedge^3 H$, we get that $\overline{F}_g = 0$. □

Any two trivializations of a given trivial cocycle differ by a homomorphism of groups and by the same argument we get:

Proposition 3. *Any family of trivial cocycles satisfying* (1)–(3) *corresponds to at most one invariant of homology spheres.*

2.3. From trivial cocycles to invariants

Conversely, what are the conditions for a family of trivial 2-cocycles C_g on $\mathcal{T}_{g,1}$ satisfying properties (1)–(3) to actually provide an invariant?

Firstly we need to check the existence of an $\mathcal{AB}_{g,1}$-invariant trivialization of each C_g. This is a cohomological problem.

Denote by $\mathcal{Q}_{C_g}$ the set of all trivializations of the cocycle C_g:

$$\mathcal{Q}_{C_g} = \{q : \mathcal{T}_{g,1} \to \mathbf{Z} \mid q(\phi) + q(\psi) - q(\phi\psi) = C_g(\phi, \psi)\}.$$

Recall that any two trivializations of a given 2-cocycle differ by an element of the group $\mathrm{Hom}(\mathcal{T}_{g,1}, \mathbf{Z}) = (\bigwedge^3 H)^*$. As the cocycle C_g is invariant under conjugation by $\mathcal{AB}_{g,1}$ this latter group acts on $\mathcal{Q}_{C_g}$ via its conjugation action on the Torelli group. Explicitly if $\phi \in \mathcal{AB}_{g,1}$ and $q \in \mathcal{Q}_{C_g}$ then $\phi \cdot q(\eta) = q(\phi^{-1}\eta\phi)$. This action confers the set $\mathcal{Q}_{C_g}$ the structure of an affine set over the abelian group $(\bigwedge^3 H)^*$. Choose an arbitrary element $q \in \mathcal{Q}_{C_g}$ and define a map as follows

$$\begin{aligned} \rho_q : \mathcal{AB}_{g,1} &\longrightarrow \left(\textstyle\bigwedge^3 H\right)^*, \\ \phi &\longmapsto \phi \cdot q - q. \end{aligned}$$

A direct computation shows that ρ_q is a derivation, i.e. $\rho_q(\phi\psi) = \phi \cdot \rho_q(\psi) + \rho_q(\psi)$, and that the difference $\rho_q - \rho_{q'}$ for two elements in $\mathcal{Q}_{C_g}$ is a principal derivation. Therefore we have a well-defined cohomology class

$$\rho(C_g) \in \mathrm{H}^1\left(\mathcal{AB}_{g,1}; \left(\textstyle\bigwedge^3 H\right)^*\right)$$

called the *torsor* of the cocycle C_g.

By construction, if the action of $\mathcal{AB}_{g,1}$ on $\mathcal{Q}_{C_g}$ has a fixed point, the class $\rho(C_g)$ is trivial. Conversely, if $\rho(C_g)$ is trivial, then for any $q \in \mathcal{Q}_{C_g}$ the map ρ_q is a principal derivation: there exists $m_q \in \mathrm{Hom}(\mathcal{T}_{g,1}, \mathbf{Z})$ such that

$$\forall \phi \in \mathcal{AB}_{g,1}, \quad \rho_q(\phi) = \phi \cdot m_q - m_q.$$

In particular the element $q - m_q \in \mathcal{Q}_{C_g}$ is fixed under the action of $\mathcal{AB}_{g,1}$. So we have proved:

Proposition 4. *The natural action of $\mathcal{AB}_{g,1}$ on $\mathcal{Q}_{C_g}$ admits a fixed point if and only if the associated torsor $\rho(C_g)$ is trivial.*

Proposition 5. *The torsor class $\rho(C_g)$ is naturally an element of the group* $\mathrm{Hom}(\mathrm{H}_1(\mathcal{TAB}_{g,1}) \otimes \bigwedge^3 H, \mathbf{Z})^{\mathcal{AB}_{g,1}}$, *where $\mathcal{TAB}_{g,1} \otimes \bigwedge^3 H$ is endowed with the diagonal action and $\mathbf{Z}$ with the trivial one.*

Proof. By Lemma 3 we have an exact sequence

$$1 \longrightarrow \mathcal{TAB}_{g,1} \longrightarrow \mathcal{AB}_{g,1} \longrightarrow \mathrm{GL}_g(\mathbf{Z}) \longrightarrow 1.$$

From this we get the Lyndon–Hochschild–Serre exact sequence in cohomology [8, Theorem 2]:

$$0 \to \mathrm{H}^1(\mathrm{GL}_g(\mathbf{Z}); (\textstyle\bigwedge^3 H)^*) \longrightarrow \mathrm{H}^1(\mathcal{AB}_{g,1}; (\textstyle\bigwedge^3 H)^*) \longrightarrow \mathrm{H}^1(\mathcal{TAB}_{g,1}; (\textstyle\bigwedge^3 H)^*)^{\mathrm{GL}_g(\mathbf{Z})}.$$

First we show that $\mathrm{H}^1(\mathrm{GL}_g(\mathbf{Z}); (\bigwedge^3 H)^*) = 0$.

Let $f : \mathrm{GL}_g(\mathbf{Z}) \longrightarrow \bigwedge^3 H$ be any crossed morphism. As $-Id \in \mathrm{GL}_g(\mathbf{Z})$ acts as $-Id$ on $(\bigwedge^3 H)^*$ and is central, for all $S \in \mathrm{GL}_g(\mathbf{Z})$ we have $f(-Id \circ S) = f(-Id) - f(S) = f(S) + S \cdot f(-Id)$. In particular, $\forall S \in \mathrm{GL}_g(\mathbf{Z})$, $2f(S) = f(-Id) - S \cdot f(-Id)$. Using the standard generators (elementary matrices) E_{ij}, defined by $E_{ij}(a_k) = a_k + \delta_{jk} a_i$ one shows that $f(-Id)$ is divisible by 2, so f itself is a principal derivation.

We are left with the exact sequence:

$$0 \to \mathrm{H}^1\big(\mathcal{AB}_{g,1}; \big(\textstyle\bigwedge^3 H\big)^*\big) \xrightarrow{i_*} \mathrm{H}^1\big(\mathcal{TAB}_{g,1}; \big(\textstyle\bigwedge^3 H\big)^*\big)^{\mathrm{GL}(g,\mathbf{Z})}.$$

As $\mathcal{T}_{g,1}$ and therefore $\mathcal{TAB}_{g,1}$, act trivially on the group $(\bigwedge^3 H)^*$, which is free abelian, the universal coefficients theorem [2, Chap. III.1 Ex. 3] and the classical adjunction properties of Hom-groups give us a canonical $\mathcal{AB}_{g,1}$-equivariant isomorphism

$$\mathrm{H}^1\big(\mathcal{TAB}_{g,1}; \big(\textstyle\bigwedge^3 H\big)^*\big) \simeq \mathrm{Hom}\big(\mathrm{H}_1(\mathcal{TAB}_{g,1}) \otimes \textstyle\bigwedge^3 H, \mathbf{Z}\big). \qquad \square$$

Arguing as in Proposition 2 one checks that the $\mathcal{AB}_{g,1}$-invariant trivialization of C_g, if it exists, is unique. If we have fixed points q_g for all g, by unicity, we have that q_{g+1} restricted to $\mathcal{T}_{g,1}$ is equal to q_g. Therefore we have a well-defined map

$$q = \lim_{g \to \infty} q_g : \lim_{g \to \infty} \mathcal{T}_{g,1} \longrightarrow \mathbf{Z}.$$

This is the only candidate to be an invariant of homology spheres. For this map to be an invariant, since it is already $\mathcal{AB}_{g,1}$-invariant, we only have to prove that it is constant on the double cosets $\mathcal{TA}_{g,1} \backslash \mathcal{T}_{g,1} / \mathcal{TB}_{g,1}$.

From property (3) of our cocycle we get that $\forall \phi \in \mathcal{T}_{g,1}$, $\forall \psi_a \in \mathcal{TA}_{g,1}$ and $\forall \psi_b \in \mathcal{TB}_{g,1}$:

$$q_g(\phi) - q_g(\phi \psi_b) = -q_g(\psi_b),$$
$$q_g(\phi) - q_g(\psi_a \phi) = -q_g(\psi_a).$$

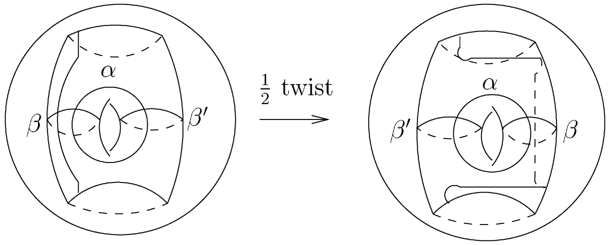

Fig. 2. Regular neighbourhood of $\beta \cup \alpha \cup \beta'$ and half twist.

Theorem 6. *For each $g \geqslant 3$ the induced homomorphisms*

$$q_g : \mathcal{TB}_{g,1} \to \mathbf{Z} \quad \textit{and} \quad q_g : \mathcal{TA}_{g,1} \to \mathbf{Z}$$

are trivial.

Proof. We only give the proof for the morphism $q_g : \mathcal{TB}_{g,1} \to \mathbf{Z}$, the other case is similar.

Denote by $\mathcal{L}_{g,1}$ the kernel of the map $\mathcal{B}_{g,1} \twoheadrightarrow \operatorname{Aut}\pi_1(\mathcal{H}_g)$. This was identified by Luft [13] as the "Twist group" of the handlebody $\mathcal{H}_g$.

Consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{L} \cap \mathcal{TB}_{g,1} & \longrightarrow & \mathcal{TB}_{g,1} & \longrightarrow & IA & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{L}_{g,1} & \longrightarrow & \mathcal{B}_{g,1} & \longrightarrow & \operatorname{Aut}\pi_1(\mathcal{H}_g) & \longrightarrow & 1. \end{array}$$

Here IA stands for the kernel of the natural map $\operatorname{Aut}(\mathcal{H}_g) \to \mathrm{GL}_g(\mathbf{Z})$. In Section 4 we prove the following result:

Proposition 6. *The group $\mathcal{L} \cap \mathcal{TB}_{g,1}$ is generated by maps of the form $T_\beta T_{\beta'}^{-1}$, where β and β' are two homologous non-isotopic and disjoint simple closed curves on $\Sigma_{g,1}$ such that each one bounds a properly embedded disc in $\mathcal{H}_g$.*

Corollary 1. *The morphism $q_g : \mathcal{L} \cap \mathcal{TB}_{g,1} \to \mathbf{Z}$ is trivial.*

Proof. By Proposition 6 it is enough to prove that q_g vanishes on the aforementioned maps $T_\beta T_{\beta'}^{-1}$. As our embedding of Σ_g into $\mathbf{S}^3$ is standard, there exists a simple closed curve, $\alpha \subset \Sigma_{g,1}$ which bounds a properly embedded disc on $-\mathcal{H}_g$ (the outer handlebody) and which intersects each of the curves β and β' in exactly one point. Consider a regular neighbourhood of the union $\delta \cup d \cup \delta'$, where δ, δ', d are disks bounded by β, β', α respectively. It is a 3-ball, whose intersection with the surface looks like in Fig. 2.

There is a half twist map ψ inside this ball that exchanges the curves β and β'. This half twist map ψ belongs to $\mathcal{AB}_{g,1}$ since it is a self diffeomorphism of the depicted 3-ball and can be extended by the identity outside this ball. In particular, since q_g is $\mathcal{AB}_{g,1}$-invariant:

$$\begin{aligned} q_g\big(T_\beta T_{\beta'}^{-1}\big) &= q_g\big(\psi T_\beta T_{\beta'}^{-1}\psi^{-1}\big)\\ &= q_g\big(T_{\psi(\beta)}T_{\psi(\beta')}^{-1}\big)\\ &= q_g\big(T_{\beta'}T_\beta^{-1}\big)\\ &= -q_g\big(T_\beta T_{\beta'}^{-1}\big) \end{aligned}$$

and therefore $q_g|_{\mathcal{L}\cap\mathcal{TB}_{g,1}} = 0$. □

As a consequence q_g factors through IA. As the action on the fundamental group of the inner handlebody $\mathcal{H}_g$ induces a surjective map $\mathcal{AB}_{g,1} \to \operatorname{Aut}\pi_1(\mathcal{H}_g)$ we can even view q_g as an $\operatorname{Aut}\pi_1(\mathcal{H}_g)$-invariant map $q_g : IA \to \mathbf{Z}$. Let $\alpha_1, \dots, \alpha_g$ denote the generators of $\pi_1(\mathcal{H}_g)$ (this can be identified with the curves in Fig. 1). According to Magnus [14], [15, Theorem N4, p. 168], the group IA is normally generated as a subgroup of $\operatorname{Aut}\pi_1(\mathcal{H}_g)$ by the automorphism K_{12} given by $K_{12}(\alpha_1) = \alpha_2\alpha_1\alpha_2^{-1}$ and $K_{12}(\alpha_i) = \alpha_i$ for $i \geqslant 2$. By invariance, q_g is determined by its value on K_{12}. An easy computation shows that if we denote by σ_2 the automorphism given by $\sigma_2(\alpha_2) = \alpha_2^{-1}$ and $\sigma_2(\alpha_i) = \alpha_i$ for $i \neq 2$ then $\sigma_2 K_{12}\sigma_2^{-1} = K_{12}^{-1}$. In particular $q_g(K_{12}) = q_g(\sigma_2 K_{12}\sigma_2^{-1}) = -q_g(K_{12})$ and q_g is must be trivial. □

We summarize the discussion of this section in the following

Theorem 7. *A family of cocycles $(C_g)_{g\geqslant 3}$ on the Torelli groups $\mathcal{T}_{g,1}$, $g \geqslant 3$, satisfying conditions* (1)–(3) *provides a compatible family of trivializations $F_g : \mathcal{T}_{g,1} \to \mathbf{Z}$ that reassemble into an invariant of homology spheres*

$$\lim_{g\to\infty} F_g : \mathcal{S}(3) \to \mathbf{Z}$$

if and only if the following two conditions hold:

(i) *The associated cohomology classes $[C_g] \in \mathrm{H}^2(\mathcal{T}_{g,1}; \mathbf{Z})$ are trivial.*
(ii) *The associated torsors $\rho(C_g) \in \mathrm{H}^1(\mathcal{AB}_{g,1}, (\bigwedge^3 \mathrm{H})^*)$ are trivial.*

In this case the maps F_g are the unique $\mathcal{AB}_{g,1}$-invariant trivializations of the cocycles C_g.

3. Application to the Casson invariant

If one is interested in invariants that come from pull-backing cocycles defined on abelian groups without 2-torsion, in view of Proposition 1 it is enough to study the case where the abelian group is $\bigwedge^3 H$ and the homomorphism is τ.

Recall that we have a decomposition $H = A \oplus B$, this induces the decomposition $\bigwedge^3 H = \bigwedge^3 A \oplus B \wedge (\bigwedge^2 A) \oplus A \wedge (\bigwedge^2 B) \oplus \bigwedge^3 B$. Set $W_A = \bigwedge^3 A$, $W_B = \bigwedge^3 B$ and $W_{AB} = B \wedge (\bigwedge^2 A) \oplus A \wedge (\bigwedge^2 B)$. The Johnson homomorphism computes the action of the Torelli group on the second nilpotent quotient of the fundamental group of $\Sigma_{g,1}$. Computing on specific elements one can check that (see [16]):

Lemma 5. *The image of $\mathcal{TA}_{g,1}$ under τ in $\bigwedge^3 H$ is $W_A \oplus W_{AB}$, the image of $\mathcal{TB}_{g,1}$ is $W_{AB} \oplus W_B$.*

For each g, the intersection form on homology induces a bilinear form $\omega : A \otimes B \to \mathbf{Z}$. This in turn induces bilinear forms $J_g : W_A \otimes W_B \to \mathbf{Z}$ and ${}^t J_g : W_B \otimes W_A \to \mathbf{Z}$ that we extend by 0 to degenerate bilinear forms on $\bigwedge^3 H = W_A \oplus W_{AB} \oplus W_B$. Written as matrices according to the decomposition $\bigwedge^3 H = W_A \oplus W_{AB} \oplus W_B$ these are:

$$J_g := \begin{pmatrix} 0 & 0 & Id \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}^t J_g := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Id & 0 & 0 \end{pmatrix}.$$

Notice that bilinear forms are naturally 2-cocycles on abelian groups.

Proposition 7. *For each $g \geqslant 3$, the cocycle J_g is the unique cocycle (up to a multiplicative constant) on $\bigwedge^3 H$ which pull-back on the Torelli group $\mathcal{T}_{g,1}$ satisfies conditions* (2) *and* (3). *Moreover once we have fixed a common multiplicative constant the family of pull-backed cocycles satisfies also* (1).

Proof. Fix an integer $n \in \mathbf{Z}$. It is obvious from the definition and from Lemma 5 that the family (nJ_g) satisfies (1), (2) and (3).

Let B denote an arbitrary cocycle on $\bigwedge^3 H$ which pull-back on $\mathcal{T}_{g,1}$ satisfies (2) and (3).

Write each element $w \in \bigwedge^3 H$ as $w_a + w_{ab} + w_b$ according to the decomposition $W_A \oplus W_{AB} \oplus W_B$. The cocycle relation together with condition (3) and Lemma 5 imply that

$$\forall v, w \in \bigwedge^3 H, \quad B(v, w) = B(v_b, w_a).$$

We first prove that B is bilinear. For the linearity on the first variable compute

$$\begin{aligned} B(u+v, w) &= B(u_b + v_b, w_a) \\ &= B(v_b, w_a) + B(u_b, v_b + w_a) - B(u_b, v_b) \\ &= B(u_b, w_a) + B(v_b, w_a) \\ &= B(u, w) + B(v, w). \end{aligned}$$

A similar proof holds for the linearity on the second variable.

By the equivariance properties of τ, the subgroup $\mathcal{AB}_{g,1} \subset \mathcal{M}_{g,1}$ acts on $\bigwedge^3 H$ via the projection $\mathcal{AB}_{g,1} \xrightarrow{\mathrm{H}_1} \mathrm{GL}_g(\mathbf{Z})$. It is well known that the only $\mathrm{GL}_g(\mathbf{Z})$-invariant bilinear forms on $\bigwedge^3(A \oplus B)$ are the pairing J_g and the dual pairing ${}^t J_g$, so that $B = nJ + m^t J$ for some integers n and m. As condition (3) implies that $\tau(\mathcal{TB}_{g,1}) = W_B \oplus W_{AB}$ has to be in the kernel of B, evaluating on the elements of W_B yields that $m = 0$. $\square$

We would like to apply Theorem 7 to the family (J_g) or to one of its multiples. First we must check the triviality of the cocycles.

Proposition 8. *For all $g \geqslant 3$, the pull-back of the 2-cocycle $2J_g$ is trivial.*

Proof. From Johnson (see Proposition 1) we know that the abelianization of $\mathcal{T}_{g,1}$ is equal to $\bigwedge^3 H \oplus V$ where V is a $\mathbf{Z}/2\mathbf{Z}$-vector space. By the universal coefficients theorem $\mathrm{H}^2(\mathcal{T}_{g,1}, \mathbf{Z}) = \mathrm{Hom}(\mathrm{H}_2(\mathcal{T}_{g,1}, \mathbf{Z}), \mathbf{Z}) \oplus \mathrm{Ext}^1(\mathrm{H}_1(\mathcal{T}_{g,1}, \mathbf{Z}), \mathbf{Z})$. The first factor is torsion free and the second factor is isomorphic to $\mathrm{Ext}^1(V, \mathbf{Z})$ which is a $\mathbf{Z}/2\mathbf{Z}$ vector space.

By naturality we have a commutative diagram for cohomology groups with trivial coefficients:

$$\begin{array}{ccc} \mathrm{H}^2(\bigwedge^3 H; \mathbf{Q}) & \xrightarrow{\tau_{\mathbf{Q}}^*} & \mathrm{H}^2(\mathcal{T}_{g,1}; \mathbf{Q}) \\ \uparrow & & \uparrow \\ \mathrm{H}^2(\bigwedge^3 H; \mathbf{Z}) & \xrightarrow{\tau^*} & \mathrm{H}^2(\mathcal{T}_{g,1}; \mathbf{Z}). \end{array}$$

Denote $H \otimes \mathbf{Q}$ by $H_{\mathbf{Q}}$. The inclusion $H \hookrightarrow H_{\mathbf{Q}}$ induces an isomorphism in rational cohomology, and it will be more convenient for our purposes to use the latter group. As in the rest of the argument we only use cohomology with trivial rational coefficients we drop the mention of the coefficients.

Notice that there is a canonical map $\bigwedge^3 H_{\mathbf{Q}} \to H_{\mathbf{Q}}$ given by $u \wedge v \wedge w \mapsto \omega(u,v)w + \omega(v,w)u + \omega(w,u)v$. It is classical that this is an $\mathrm{Sp}\,\omega_{\mathbf{Q}}$ split equivariant map and that if we denote its kernel by $U_{\mathbf{Q}}$, then this yields a canonical decomposition into irreducible representations of the rational symplectic group $\bigwedge^3 H_{\mathbf{Q}} \simeq U_{\mathbf{Q}} \oplus H_{\mathbf{Q}}$.

By the Küneth formula $\mathrm{H}^2(\bigwedge^3 H_{\mathbf{Q}}) \simeq H^2(U_{\mathbf{Q}}) \oplus \mathrm{H}^1(U_{\mathbf{Q}}) \otimes \mathrm{H}^1(H_{\mathbf{Q}}) \oplus \mathrm{H}^2(H_{\mathbf{Q}})$.

As $U_{\mathbf{Q}}$ is a torsion free abelian group $\bigwedge^2 \mathrm{H}^1(U_{\mathbf{Q}}) = \bigwedge^2 \mathrm{Hom}(U_{\mathbf{Q}}, \mathbf{Q}) \simeq \mathrm{H}^2(U_{\mathbf{Q}})$, where the last isomorphism is given by cup product and all these identifications are compatible with the action of the symplectic group.

By construction the cohomology class J_g is the $Gl_g(\mathbf{Q})$-orbit of the cup product $(a_1 \wedge a_2 \wedge a_3)^* \cup (b_1 \wedge b_2 \wedge b_3)^*$ and clearly both $(a_1 \wedge a_2 \wedge a_3)^*$ and $(b_1 \wedge b_2 \wedge b_3)^*$ belong to $\mathrm{H}^1(U_{\mathbf{Q}})$. In [12, Paragraph 6], D. Johnson identifies the map $\mathrm{H}_1(\mathcal{T}_{g,1}) \to \mathrm{H}_1(\mathcal{T}_g)$ with the projection $\bigwedge^3 H_{\mathbf{Q}} \to U_{\mathbf{Q}}$. Here $\mathcal{T}_g$ denotes the kernel of the action of the mapping class group of the closed surface Σ_g on its first integral homology group (i.e. we forget about the fixed disc in our definition).

In particular we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(\mathcal{T}_g) & \longrightarrow & \mathrm{H}^2(\mathcal{T}_{g,1}) \\ \uparrow & & \uparrow \\ \mathrm{H}^2(U_{\mathbf{Q}}) & \hookrightarrow & \mathrm{H}^2(\bigwedge^3 H_{\mathbf{Q}}). \end{array}$$

R. Hain [6, Paragraph 14 and Theorems 10.1 and 10.2] has computed the kernel of the map $\mathrm{H}^2(U_{\mathbf{Q}}) \to \mathrm{H}^2(\mathcal{T}_g)$. He proved in particular that the map is 0 when $g = 3$ and by stability of our cocycle and the stability of the decomposition of the two cohomology groups as symplectic modules this implies that the pull-back of J_g in $\mathrm{H}^2(\mathcal{T}_g; \mathbf{Q})$ and therefore its pull-back in $\mathrm{H}^2(\mathcal{T}_{g,1}; \mathbf{Q})$ are 0 for all $g \geqslant 3$.

Coming back to integral coefficients we conclude that $\tau^*(J_g)$ is annihilated by multiplication by 2 so that the pull-back of the 2-cocycle $2J_g$ is trivial. $\square$

We will come back to the homology class of the pull-back of the cocycle J_g (see Proposition 10). To avoid an unnecessarily heavy notation from now on we will also denote by J_g the pull-back of the cocycle J_g along the morphism τ. To see if there is an invariant associated to the family $(2J_g)_{g\geqslant 3}$ we have to check the triviality of the associated torsors:

Proposition 9. *For each $g \geqslant 3$, the torsor $[\rho(2J_g)]$ is trivial.*

Proof. By construction the torsor class $\rho(2J_g): \mathcal{TAB}_{g,1} \otimes \bigwedge^3 H \to \mathbf{Z}$ (cf. Propositions 4–5) may be described as follows. Fix an arbitrary coboundary $q \in \mathcal{Q}_{2J_g}$. For each tensor $f \otimes l \in \mathrm{H}_1(\mathcal{TAB}_{g,1}) \otimes \bigwedge^3 H$, choose arbitrary lifts of $\phi \in \mathcal{TAB}(1)_{g,1}$ and $\lambda \in \mathcal{T}_{g,1}$, then $\rho(2J_g) \times (f \otimes l) = q(\phi\lambda\phi^{-1}) - q(\lambda)$. As $2J_g$ is a coboundary of q we get:

$$\begin{aligned}
\rho(2J_g)(f \otimes l) &= q\big(\phi\lambda\phi^{-1}\big) - q(\lambda) \\
&= q\big(\phi\lambda\phi^{-1}\lambda^{-1}\big) \\
&= 2J_g\big(\tau(\phi), \tau(\lambda)\big) - 2J_g\big(\tau(\lambda), \tau(\phi)\big) \\
&= 0 \quad \text{by condition (2).}
\end{aligned}$$
$\square$

Applying Theorem 7 we get

Theorem 8. *The $\mathcal{AB}_{g,1}$-invariant trivializations of the pull-backs of the cocycles $2J_g$ reassemble into an invariant of homology spheres $F: \mathcal{S}(3) \to \mathbf{Z}$. Up to a multiplicative constant this is trivialization of a 2-cocycle defined on an abelian group without 2-torsion.*

We have now to identify the invariant, say λ that we have just constructed with the $-$ the Casson invariant (the reason for the -1 sign comes from the choice of the form of a trivial 2-cocycle). There are two ways to do this, the longest consists to show from the properties of the cocycle $2J_g$ that λ that we have constructed is a *good Casson number* (see for instance [5]). This identifies λ with the Casson invariant up to a multiplicative constant, which is determined to be 1 by choosing an explicit pair of maps $\phi_a \in \mathcal{TA}_{g,1}$ and $\phi_b \in \mathcal{TB}_{g,1}$ such that $-\lambda(\phi_b\phi_a) = 2J_g(\tau(\phi_b, \phi_a)$ and $Casson(S^3_{\phi_b\phi_a})$ coincide. We leave this straightforward but a bit long path to the interested reader. The shortest path amounts to use the work of Morita [18, Theorem 4.3], who showed precisely that the Casson invariant has as associated 2-cocycle our $-2J_g$. As a corollary we get in particular:

Corollary 2. *The Casson invariant is the unique integral-valuated invariant of oriented homology 3-spheres that comes from the trivialization of a 2-cocycle defined on an abelian group without 2-torsion.*

We also get back to one of the main results from [16]:

Corollary 3. *Denote by $\mathcal{K}_{g,1}$ the kernel of the Johnson homomorphism τ (see [11] for geometric properties of this group). Then the Casson invariant restricted to $\mathcal{K}_{g,1}$ is a homomorphism of groups.*

Now that we know that our invariant F constructed out of the cocycles $2J_g$ is the opposite of the Casson invariant we proceed by showing that one cannot get rid of the factor 2. Denote the Casson invariant by λ and by λ_g if one views it as a function on $\mathcal{T}_{g,1}$.

Proposition 10. *The pull-back of the cocycle J_g on the Torelli group defines a non-trivial cohomology class $[J_g] \in \mathrm{H}^2(\mathcal{T}_{g,1};\mathbf{Z})$ of order two. Moreover this class is stable in the sense that the image of the class $[J_{g+1}]$ under the stabilisation map $\mathrm{H}^2(\mathcal{T}_{g+1,1};\mathbf{Z}) \to \mathrm{H}^2(\mathcal{T}_{g,1};\mathbf{Z})$ is $[J_g]$.*

Proof. If the pull-backs were trivial then the proof of Proposition 9 would carry on and provide us with an invariant $F_g : \mathcal{T}_{g,1} \to \mathbf{Z}$ associated to the family (J_g). Then by the unicity of invariants associated to cocycles, Proposition 3, the invariant $2F_g$ would be the invariant associated to $2J_g$ so we would have $2F_g = -\lambda_g$. Now, the Poincaré sphere has a Heegaard splitting of genus 2 and therefore by stabilization, it has a Heegaard splitting of every genus $g \geqslant 3$. The Casson invariant of the Poincaré sphere is 1 and therefore all functions λ_g take the value 1 and thus are not divisible by 2. $\square$

It is known that the mod 2 reduction of the Casson invariant is the Rohlin invariant, which might be viewed as a homomorphism $R_g : \mathcal{T}_{g,1} \to \mathbf{Z}/2\mathbf{Z}$ or equivalently as a cohomology class $R_g \in \mathrm{H}^1(\mathcal{T}_{g,1};\mathbf{Z}/2\mathbf{Z}) \simeq \mathrm{Hom}(\mathcal{T}_{g,1},\mathbf{Z}/2\mathbf{Z})$. By definition of the Bockstein homomorphism $\beta_{\mathbf{Z}}$ associated to the exact sequence $1 \longrightarrow \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 1$, we have:

Proposition 11. *For $g \geqslant 3$, the image of the class R_g under the integral Bockstein $\beta_{\mathbf{Z}} : \mathrm{H}^1(\mathcal{T}_{g,1};\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}^2(\mathcal{T}_{g,1};\mathbf{Z})$ is the non-trivial class $[J_g]$.*

4. Generators for the Luft–Torelli group

In this section we finally prove Proposition 6. Before we need to recall some known facts on Dehn twists and maps in $\mathcal{TB}_{g,1}$.

In [13] Luft identified the kernel $\mathcal{L}_{g,1}$ of the map

$$\mathcal{B}_{g,1} \longrightarrow \mathrm{Aut}\,\pi_1(\mathcal{H}_g) \longrightarrow 1$$

with the so-called "Twist group": the subgroup of $\mathcal{M}_{g,1}$ generated by Dehn twists around simple closed curves that are contractible in $\mathcal{H}_g$.

In analogy with the generators of the Torelli group defined by Johnson [12], we define a Contractible Bounding Pair (CBP for short) to be a pair of two disjoint and non-isotopic homologous curves β, β' on $\Sigma_{g,1}$ such that neither β nor β' is null-homologous and such that each one bounds a properly embedded disk in $\mathcal{H}_g$. A typical pair is given in Fig. 2.

A Contractible Bounding Simple Closed Curve (CBSCC for short) is a non-contractible simple closed curve δ on $\Sigma_{g,1}$ such that $\Sigma_{g,1} \setminus \delta$ has two connected components and that bounds a properly embedded disk in $\mathcal{H}_g$. For instance, a curve parallel to the boundary of $\Sigma_{g,1}$ is a CBSCC.

Combining the cited papers of Luft and Johnson we get that if β, β' is a CBP then the map $T_\beta T_{\beta'}^{-1}$ belongs to $\mathcal{L}_{g,1} \cap \mathcal{TB}_{g,1}$. We call such a map a *CBP-twist*; we also call the intersection group the Luft–Torelli group and we denote it by $\mathcal{LTB}_{g,1}$. In [9], Johnson proved that opposite twists along Bounding Pairs generate the Torelli group for $g \geqslant 3$. In this section we prove an analogous theorem for the Luft–Torelli group:

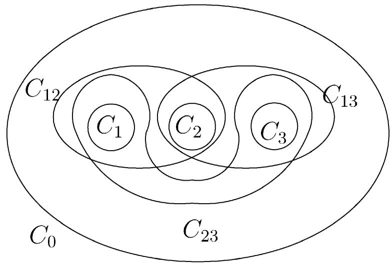

Fig. 3. Lantern configuration.

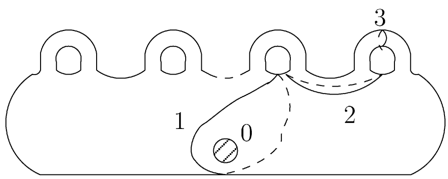

Fig. 4. Lantern relation for δ.

Theorem 9. *The Luft–Torelli group $\mathcal{LTB}_{g,1}$ is generated by CBP-twists.*

4.1. Reduction to the closed case

The reduction to the closed case as many other results in this section are based on the following Lantern Relation, originally due to Dehn [3] and later rediscovered by Johnson [9].

Lemma 6 *(Lantern Relation). Consider a* 2*-sphere with* 4 *holes. Let the boundary components be C_0, C_1, C_2, C_3 and for $1 \leqslant i < j \leqslant 3$ denote by C_{ij} a simple curve encircling C_i and C_j (see Fig.* 3*). Then the following relation between Dehn twists holds:*

$$T_{C_0}T_{C_1}T_{C_2}T_{C_3} = T_{C_{12}}T_{C_{13}}T_{C_{23}}.$$

Notice that once the four boundary circles are ordered the remaining curves and thus the Lantern Relation are determined.

Recall from Section 1 that the kernel of the map $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$ is an infinite cyclic group generated by a Dehn twist along a curve ∂ parallel to the boundary and that this is a CBSCC. In particular the kernel is contained in $\mathcal{TB}_{g,1}$. Moreover the action of this Dehn twist on the homology of the surface and also on the first homotopy group of the handlebody $\mathcal{H}_g$ is trivial. As a consequence we have a short exact sequence:

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{LTB}_{g,1} \longrightarrow \mathcal{LTB}_{g,*} \longrightarrow 1.$$

The three curves depicted in Fig. 4 plus the boundary curve δ define a "Lantern" i.e. a 4-holed sphere on the surface Σ_g. Applying the lantern relation of Johnson (see Johnson [9]) one gets:

Lemma 7. *The Dehn twist along δ can be written as a product of CBP-twists.*

As $\mathcal{LTB}_{g,1}$ is generated by lifts of generators of $\mathcal{LTB}_{g,*}$, plus the twist T_δ and since any CBP-twist in $\mathcal{LTB}_{g,*}$ naturally lifts to a CBP-twist in $\mathcal{LTB}_{g,1}$ Theorem 9 follows from:

Proposition 12. *The group $\mathcal{LTB}_{g,*}$ is generated by CBP-twists.*

4.2. Strategy of the proof of Proposition 12

Recall from Section 1.2 that we have two short exact sequences

$$1 \longrightarrow \mathcal{L}_{g,*} \longrightarrow \mathcal{B}_{g,*} \longrightarrow \operatorname{Aut}\pi_1(\mathcal{H}_{g,*}) \longrightarrow 1,$$

$$1 \longrightarrow \mathcal{TB}_{g,*} \longrightarrow \mathcal{B}_{g,*} \longrightarrow \mathrm{GL}_g(\mathbf{Z}) \ltimes \mathcal{S}_g(\mathbf{Z}) \longrightarrow 1.$$

The map $\mathcal{B}_{g,*} \to \mathrm{GL}_g(\mathbf{Z})$ can be identified with the map given by the natural action of $\mathcal{B}_{g,1}$ on the first homology of $\mathcal{H}_{g,*}$. Therefore we have a short exact sequence:

$$1 \longrightarrow \mathcal{LTB}_{g,*} \longrightarrow \mathcal{B}_{g,*} \longrightarrow \operatorname{Aut}\pi_1(\mathcal{H}_{g,*}) \ltimes \mathcal{S}_g(\mathbf{Z}) \longrightarrow 1.$$

In [19] Nielsen gave an explicit finite presentation with four generators and 17 relations of the automorphism group of a free group on g generators (see also [15, Chapter 3]), denote this presentation by

$$\langle x_1, \ldots, x_4 \mid r_1, \ldots, r_{17} \rangle.$$

The group $\mathcal{S}_g$ is free on $\frac{g(g+1)}{2}$ generators, so we can find a presentation of the form

$$\Big\langle t_1, \ldots, t_{\frac{g(g+1)}{2}} \;\Big|\; [t_i, t_j] \Big\rangle,$$

where $[t_i, t_j] = t_i^{-1} t_j^{-1} t_i t_j$ and $1 \leqslant i, j \leqslant \frac{g(g+1)}{2}$. Let the action of $\operatorname{Aut}\pi_1(\mathcal{H}_g)$ on $\mathcal{S}_g(\mathbf{Z})$ be given by expressions of the form: $x_i(t_j) = w_{ij}$ where w_{ij} is a word in the alphabet t_k. Then a presentation of the semi-direct product $\operatorname{Aut}\pi_1(\mathcal{H}_{g,1}) \ltimes \mathcal{S}_g(\mathbf{Z})$ is given by

$$\Big\langle x_1, \ldots, x_4, t_1, \ldots, t_n \;\Big|\; r_1, \ldots, r_{17}, [t_i, t_j], x_k^{-1} t_l x_k w_{kl}^{-1} \Big\rangle,$$

where $1 \leqslant k \leqslant 4$ and $1 \leqslant i, j, l \leqslant \frac{g(g+1)}{2}$.

Assume that we have lifts of the generators $\tilde{x_i}, \tilde{t_j}$ and that these lifts moreover generate $\mathcal{B}_{g,*}$. Then a set of normal generators of the group $\mathcal{LTB}_{g,*}$ is given by the lifts of the relations $\tilde{r_i}, [\tilde{t_i}, \tilde{t_j}], \tilde{x}_k^{-1} \tilde{t_l} \tilde{x}_k \tilde{w}_{kl}^{-1}$ as words in the "lifted" alphabet.

In the mapping class group one has the well-known relation for Dehn twists $\phi T_\gamma \phi^{-1} = T_{\phi(\gamma)}$. In particular as the image of a CBP by an element $\phi \in \mathcal{B}_{g,*}$ is again a CBP, the group generated by the CBP-twists is normal in $\mathcal{B}_{g,*}$. Therefore to prove Theorem 9 it is enough to prove

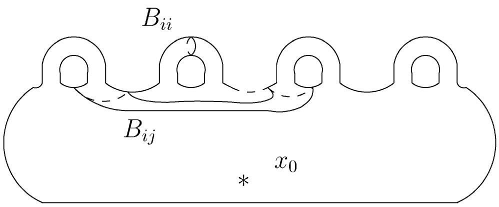

Fig. 5. Curves for the generating twists.

Proposition 13. *Under the above hypothesis the lifts*

$$\tilde{r_i}, [\tilde{t_i}, \tilde{t_j}], \tilde{x}_k^{-1}\tilde{t}_l\tilde{x_k}\tilde{w}_{kl}^{-1} \in \mathcal{LTB}_{g,*}$$

are products of CBP-twists.

It is a classical result of Nielsen [20] that a base-preserving mapping class is determined by its action on the fundamental group of the underlying surface. Geometric descriptions of generators for $\mathcal{B}_{g,*}$ were given for instance by Suzuki [22] or Luft [13]. From their work we learn that we need one Dehn twist and 4 particular generators. Here we enlarge the list of Dehn twists to hit each one of the $\frac{g(g+1)}{2}$ generators needed for the group $\mathcal{S}_g(\mathbf{Z})$.

4.2.1. Twist generators

We consider the $\frac{g(g+1)}{2}$ curves of Fig. 5. The curve B_{ij} for $i < j$ goes around the right foot of handles i passes in front of handles k for $i < k < j$ and goes around the left foot of handle j, the curve B_{ii} is a meridian of handle i.

The corresponding Dehn twists will be denoted respectively by T_{ij} and T_{ii}. Notice that by construction the homology class of B_{ij} is $b_{ij} = b_i - b_j$ and that of B_{ii} is $b_{ii} = b_i$.

This twists belong to the Twist group $\mathcal{L}_{g,*}$ and so act trivially on the homotopy group $\pi_1(\mathcal{H}_g)$. In particular the image of T_{ij} in $\operatorname{Aut}\pi_1(\mathcal{H}_g) \ltimes \mathcal{S}_g(\mathbf{Z})$ is $(0, t_{b_{ij}})$ where $t_{b_{ij}}$ denotes the transvection along the homology class b_{ij}. It is easily verified that these transvections freely generate the group $\mathcal{S}_g(\mathbf{Z})$.

4.2.2. Non-twist generators

We will keep the names given to these maps by Suzuki in [22] but label them according to the generator of $\operatorname{Aut}(\pi_1(\mathcal{H}_g))$ they hit (see [15, Corollary N1, p. 164]). All elements of the basis of $\pi_1(\Sigma_g)$ that do not appear in the description of the action of a map fixed under the action. We denote by σ_i the commutator $\alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i$.

(1) Cyclic translation of handles, Q. Action on homotopy:

$$\alpha_i \mapsto \alpha_{i+1},$$
$$\beta_i \mapsto \beta_{i+1}.$$

Indices are counted mod g.

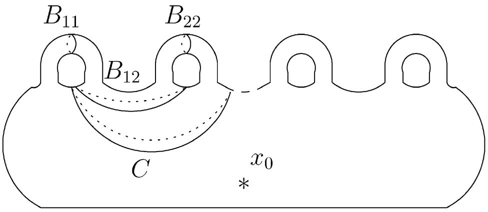

Fig. 6. Luft map.

(2) Twist of knob 1 σ. Action on homotopy:

$$\alpha_1 \mapsto \alpha_1^{-1}\sigma_1^{-1},$$
$$\beta_1 \mapsto \sigma_1\beta_1^{-1}.$$

(3) Interchange of knobs 1 and 2, P. Action on homotopy:

$$\alpha_1 \mapsto \sigma_1^{-1}\alpha_2\sigma_1,$$
$$\alpha_2 \mapsto \alpha_1,$$
$$\beta_1 \mapsto \sigma_1^{-1}\beta_2\sigma_1,$$
$$\beta_2 \mapsto \beta_1.$$

(4) Luft map U. This is a half twist that interchanges the curves B_{22} and B_{12}, the boundaries of the two-holed torus which is the support of this map are B_{11} and the curve C depicted in Fig. 6.
Action on homotopy:

$$\alpha_1 \mapsto \alpha_1\alpha_2,$$
$$\beta_1 \mapsto \beta_1,$$
$$\alpha_2 \mapsto \alpha_2^{-1}\beta_2^{-1}\alpha_2^{-1}\beta_2\alpha_2,$$
$$\beta_2 \mapsto \alpha_2^{-1}\beta_2^{-1}\alpha_1^{-1}\beta_1\alpha_1\alpha_2.$$

4.3. Proof of Proposition 13

According to our strategy of proof we have to lift the relators r_i, $[t_{ij}, t_{kl}]$ and $r_i t_{kl} r_i^{-1} w_{ij}^{-1}$ to $\mathcal{B}_{g,1}$ and show that these lifts are products of CBP-twists. We will deal successively with the twists relators $[t_{ij}, t_{kl}]$, the action relators $x_i t_{kl} x_i^{-1} w_{ij}^{-1}$ and finally with the non-twist relators r_i.

Our main tool for recognizing elements that are products of CBP-twists is

Lemma 8. *Let $\phi \in \mathcal{TB}_{g,*}$ be a map. Assume that there exist g disjoint disks D_i properly embedded in the inner handlebody so that $\mathcal{H}_g \setminus \bigcup_{i=1}^{g} D_i$ is a three ball and such that $\phi(D_i) = D_i$ for $1 \leqslant i \leqslant g$. Then ϕ is a product of CBP-twists.*

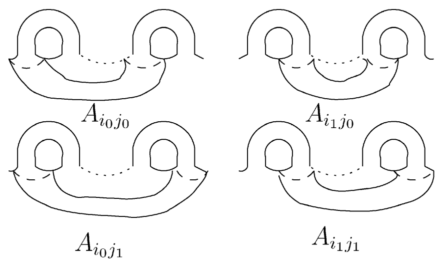

Fig. 7. Curves for the generators of the pure braid group.

Proof. Since ϕ acts trivially in homology it cannot reverse the orientations of the boundaries of the discs and therefore we may assume that ϕ fixes the disc D_i pointwise. In particular ϕ is in the image of the mapping class group relative to the boundary $\mathcal{M}_{0,2g+1}$ of the $2g+1$-holed 3-ball that is of the complementary of a small neighbourhood of the discs D_i. More precisely it is in the kernel of action on homology of this group. This mapping class group is well known to be isomorphic to the framed pure braid group on $2g$ strands: $\mathbf{Z}^{2g} \times P_{2g}$, where P_{2g} denotes the pure braid group on $2g$ strands.

Without loss of generality we may assume that the discs D_i are our preferred discs B_{ii}. Denote the left foot of the ith handle by i_0 and the right foot by i_1.

Then the group $\mathcal{M}_{0,2g}$ is generated by the following Dehn twists:

(1) Twists along the curves $A_{i_0i_0}$ (resp. $A_{i_1i_1}$) which go to the left (resp. right) foot.
(2) Twists along the curves $A_{i_0i_1}$ that enclose the two feet of the ith handle.
(3) Twists the curves $A_{i_0j_0}, A_{i_0j_1}, A_{i_1j_0}, A_{i_1j_1}$ for $i<j$ (see Fig. 7).

Notice that the twist around $A_{i_0j_1}$ is our preferred Dehn twist T_{ij}.

If we project compute the action of homology of these twists we get a surjective map $\mathbf{Z}^{2g} \times P_{2g} \to \mathcal{S}_g(\mathbf{Z})$ and in terms of the images $\overline{A}_{i_sj_t}$ of the above generators a complete set of generators is given by:

(1) $\overline{A}_{i_0i_1}=1$, $\overline{A}_{i_0j_1}=\overline{A}_{i_1j_0}$, $\overline{A}_{i_1j_1}=\overline{A}_{i_0j_0}$, $\overline{A}_{i_0j_0}=\overline{A}^2_{i_0i_0}\overline{A}^{-1}_{i_1j_0}\overline{A}^2_{j_0j_0}$.
(2) $[\overline{A}_{i_0j_1}, \overline{A}_{k_0l_1}]=1$, $[\overline{A}_{i_0j_1}, \overline{A}_{k_0k_0}]=1$, $[\overline{A}_{i_0i_0}, \overline{A}_{j_0j_0}]=1$.

Notice that the first series of relations simply reduce the number of relators and the second series is the standard presentation of the free abelian group on the remaining relators.

The kernel of the map $\mathcal{M}_{0,2g} \to \mathrm{Aut}(H)$ is therefore normally generated by the lifts of the above relations that are not relations in $\mathcal{M}_{0,2g}$. We now prove that these lifts as maps in $\mathcal{TB}_{g,1}$ are products of CBP-twists.

There are only three relations in the above list that do not lift obviously to either a relation or a product of CBP-twists.

(1) Relation $\overline{A}_{i_0i_1}$. The twists $A_{i_0i_1}$ are non-trivial mapping classes. For $i=1$, applying the Lantern Relation determined by the four curves in Fig. 8 we can express $A_{1_01_1}$ as a product of CBP-twists. Similar computations hold for the $g-1$ other cases.
(2) The relation $\overline{A}_{i_0j_0}=\overline{A}^2_{i_0i_0}\overline{A}^{-1}_{i_1j_0}\overline{A}^2_{j_0j_0}$. We may find a contractible simple closed curve $A'_{i_0j_1}$ that encloses the foots $A_{i_0i_0}, A_{i_1i_0}, A_{j_0j_0}$ defining a Lantern and does not intersect the curves

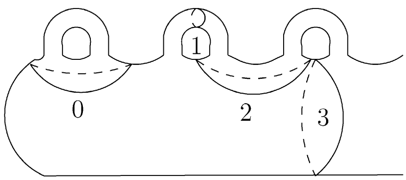

Fig. 8. Lantern for the Dehn twist $A_{1_0 1_1}$.

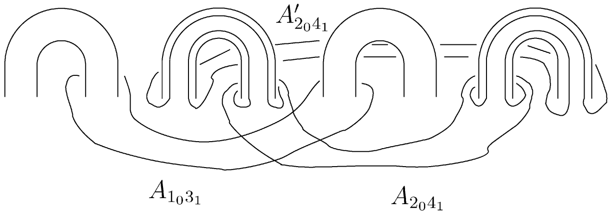

Fig. 9. Curves for lifting $[\overline{A}_{1_0 3_1}, \overline{A}_{2_0 3_1}]$.

$A_{i_0 j_0}$ and $A_{i_1 j_0}$. Applying the Lantern relation one finds:

$$T_{A_{i_0 j_0}} T_{A_{i_1 j_0}} T^{-2}_{A_{i_0 i_0}} T^{-2}_{A_{j_0 j_0}} = T^{-1}_{A_{i_0 i_1}} T_{A'_{i_0 j_0}} T^{-1}_{A_{j_0 j_0}} T^{-1}_{A_{i_0 i_0}} T_{A_{i_1 i_1}}.$$

And this is a product of CBP-twists.

(3) The relation $[\overline{A}_{i_0 j_1}, \overline{A}_{k_0 l_1}] = 1$ when the curves $A_{i_0 j_1}$ and $A_{k_0 l_1}$ intersect non-trivially. By definition of the curves this happens if and only if $i < k < j < l$.
The relation lifts to $T^{-1}_{T_{A_{i_0 j_1}}(A_{k_0 l_1})} T_{A_{k_0 l_1}}$. As the homology classes of curves $A_{i_0 j_1}$ and $A_{k_0 l_1}$ both belong to the Lagrangian B, one checks that $T_{A_{i_0 j_1}}(A_{k_0 l_1})$ is homologous to $A_{k_0 l_1}$. In particular the lift is almost a CBP-twists except that the underlying curves intersect. It is enough to find a third curve $A'_{k_0 l_1}$, disjoint from $A_{i_0 j_1}$ and $A_{k_0 l_1}$, contractible in the inner handlebody and homologous to $A_{k_0 l_1}$, for then it will be also disjoint from $T_{A_{i_0 j_1}}(A_{k_0 l_1})$ and we will have

$$T^{-1}_{T_{A_{i_0 j_1}}(A_{k_0 l_1})} T_{A_{k_0 l_1}} = T^{-1}_{T_{A_{i_0 j_1}}(A_{k_0 l_1})} T_{A'_{k_0 l_1}} T^{-1}_{A'_{k_0 l_1}} T_{A_{k_0 l_1}},$$

a product of CBP-twists. This is done by using curves that go "through the handles," see Fig. 9 for the case $(i_0, j_1 k_0, l_1) = (1, 3, 2, 4)$. □

4.3.1. Lifts of twist relators

One checks that he lifts of the relators $[t_{ij}, t_{kl}]$ all leave the curves B_{ii}, $1 \leqslant i \leqslant g$, invariant and we may apply Lemma 8.

4.3.2. Lifts of action relators

Recall that the generators t_{ij} lift to Dehn twists around the curves B_{ij} and that the lift of the action of the generators Q, σ, P, U is conjugation in $\mathcal{B}_{g,*}$ by the corresponding map. For each of the following relations one checks directly that the lifts leave the curves B_{ii} invariant and therefore we may apply in each case Lemma 8.

(1) Action of Q. The relations to lift are all of the form $Q(t_{ij}) = t_{i+1 j+1}$ (indices mod g).
(2) Action of σ. Relations are of the form $\sigma(t_{1i}) = t_{11}^{-2} t_{1j} t_{jj}^{-2}$ for $1 < i$, $\sigma(t_{ij}) = t_{ij}$ for $1 < i \leqslant j$ and $\sigma(t_{11}) = t_{11}$.
(3) Action of P. Relations are $P(t_{11}) = t_{22}$, $P(t_{22}) = t_{11}$, $P(t_{1i}) = t_{2i}$ for $i \geqslant 3$, $P(t_{2i}) = t_{1i}$ for $i \geqslant 3$. All other generators are fixed.
(4) Action of U. Relations are $U(t_{22}) = t_{12}$, $U(t_{12}) = t_{22}$, $U(t_{2i}) = t_{1i} t_{12} t_{11}^{-1}$ for $i \geqslant 3$. All other generators are fixed.

4.3.3. Lifts of non-twists relators

Instead of using a case-by-case check we use the following rephrasing of a result of Hirose (see [7, Theorem B_*] and the description of generators therein):

Proposition 14. *The kernel of the map $\mathcal{AB}_{g,*} \to \operatorname{Aut}\pi_1(\mathcal{H}_g)$ is generated by maps which have the following property:*

There exist g properly embedded discs $D_1, \ldots, D_g$ in $\mathcal{H}_g$ such that $\mathcal{H}_g \setminus (D_1 \cup \cdots \cup D_g)$ is a 3-ball and such that the map fixes the boundaries of the discs (up to isotopy).

In view of Lemma 8 the maps described in the above proposition are all products of CBP-twists.

Consider any relation r among the generators of $\operatorname{Aut}\pi_1(\mathcal{H}_g)$. Since our lifts of the generators of $\operatorname{Aut}\pi_1(\mathcal{H}_g)$ all belong to $\mathcal{AB}_{g,*}$ the lift $\widetilde{r}$ of r belongs to the kernel of the map $\mathcal{AB}_{g,*} \to \operatorname{Aut}\pi_1(\mathcal{H}_g)$. Therefore by the above Proposition 14 and by Lemma 8, the lift $\widetilde{r}$ is a product of CBP-twists.

This ends the proof of Proposition 13.

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