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On the isolated points in the space of groups ☆

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Abstract

We investigate the isolated points in the space of finitely generated groups. We give a workable characterization of isolated groups and study their hereditary properties. Various examples of groups are shown to yield isolated groups. We also discuss a connection between isolated groups and solvability of the word problem.

0. Introduction

At the end of his celebrated paper “Polynomial growth and expanding maps” [Gro81], Gromov sketched what could be a topology on a set of groups. His ideas led to the construction by Grigorchuk of the space of marked groups [Gri84], where points are finitely generated groups with m marked generators. This “space of marked groups of rank m” $\mathcal{G}_m$ is a totally discontinuous compact metrizable space.

One of the main interests of $\mathcal{G}_m$ is to find properties of groups that are reflected in its topology. Various elementary observations in these directions are made in [CG05]: for instance, the

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class of nilpotent finitely generated groups is open, while the class of solvable finitely generated groups is not; the class of finitely generated orderable groups is closed, etc. Deeper results can be found about the closure of free groups (see [CG05] and the references therein) and the closure of hyperbolic groups [Cha00]. The study of the neighborhood of the first Grigorchuk group has also proved to be fruitful [Gri84] in the context of growth of finitely generated groups.

An example of an open question about $\mathcal{G}_m$ ($m \geqslant 2$) is the following: does there exist a surjective continuous invariant: $\mathcal{G}_m \to [0,1]$? On the other hand, it is known that there exists no real-valued injective measurable invariant [Cha00].

The aim of this paper is the study of the isolated points in $\mathcal{G}_m$, which we call *isolated groups*. It turns out that these groups have already occurred in a few papers [Neu73,Man82], without the topological point of view. They are introduced by B.H. Neumann [Neu73] as “groups with finite absolute presentation.” It follows from a result of Simmons [Sim73] that they have solvable word problem, see the discussion in Section 3. However the only examples quoted in the literature are finite groups and finitely presented simple groups; we provide here examples showing that the class of isolated groups is considerably larger. Let us now describe the paper. In Section 1 we construct the space of finitely generated groups.

Here is an elementary but useful result about this topology:

Lemma 1. *Consider two marked groups $G_1 \in \mathcal{G}_{m_1}$, $G_2 \in \mathcal{G}_{m_2}$. Suppose that they are isomorphic. Then there are clopen (= closed open) neighborhoods V_i, $i = 1, 2$, of G_i in $\mathcal{G}_{m_i}$ and a homeomorphism $\varphi : V_1 \to V_2$ mapping G_1 to G_2 and preserving isomorphism classes, i.e. such that, for every $H \in V_1$, $\varphi(H)$ is isomorphic to H (as abstract groups).*

This allows us to speak about the “space of finitely generated groups”[1] rather than “space of marked groups” whenever we study local topological properties. In particular, to be isolated is an algebraic property of the group, i.e. independent of the marking. This bears out the terminology of “isolated groups.”

In Section 2, we proceed with the characterization of isolated groups. In a group G, we call a subset F a *discriminating subset* if every non-trivial normal group of G contains an element of F, and we call G *finitely discriminable* if it has a *finite* discriminating subset. Finitely discriminable groups are introduced as “semi-monolithic groups” in [Man82]. Here is an algebraic characterization of isolated groups (compare with [Man82, Proposition 2(a)]).

Proposition 2. *A group G is isolated if and only if the two following properties are satisfied*:

(i) *G finitely presented*;
(ii) *G is finitely discriminable.*

Since the class of finitely presented groups is well understood in many respects, we are often led to study finite discriminability.

[1] This must *not* be viewed as the set of isomorphism classes of finitely generated groups, even locally. Indeed, an infinite family of isomorphic groups may accumulate at the neighborhood of a given point (e.g., groups isomorphic to $\mathbf{Z}$ at the neighborhood of $\mathbf{Z}^2$ [CG05, Example 2.4(c)]), and we certainly do not identify them. What we call “space of finitely generated groups” might be viewed as the disjoint union of all $\mathcal{G}_n$, but this definition is somewhat arbitrary; and is not needed since we only consider local properties.

Proposition 3. *The subspace of finitely discriminable groups is dense in the space of finitely generated groups.*

In other words, the subspace of finitely discriminable groups is dense in $\mathcal{G}_m$ for all $m \geqslant 1$.

In Section 3, we discuss the connection with the word problem. We call a finitely generated group G recursively discriminable if there exists in G a recursively enumerable discriminating subset (see Section 3 for a more precise definition if necessary). Proposition 2 has the following analog [Sim73, Theorem B].

Theorem 4. *A finitely generated group has solvable word problem if and only if it is both recursively presentable and recursively discriminable.*

This is a conceptual generalization of a well-known theorem by Kuznetsov stating that a recursively presentable simple group has solvable word problem.

Corollary 5. *An isolated group has solvable word problem.*

The existence of certain pathological examples of finitely presented groups due to Miller III [Mil81] has the following consequence.

Proposition 6. *The class of groups with solvable word problem is not dense in the space of finitely generated groups. In particular, the class of isolated groups is not dense.*

In Section 4 we explore the hereditary properties of finitely discriminable groups, and thus isolated groups. For instance, we prove

Theorem 7. *The class of finitely discriminable* (*respectively isolated*) *groups is stable under*:

(1) *extensions of groups*;
(2) *taking overgroups of finite index, i.e. if H is finitely discriminable* (*respectively isolated*) *and of finite index in G then so is G.*

The proof of Theorem 7 is less immediate than one might expect. For instance, it involves the classification of finitely discriminable abelian groups (Lemma 4.1). We state a much more general result in Theorem 4.4 (see also Section 5.6 for the case of wreath products).

Finally in Section 5 we provide examples of isolated groups. First note that the most common infinite finitely generated groups are not finitely discriminable. For instance, if G is an infinite residually finite group, then for every finite subset $F \subset G - \{1\}$ there exists a normal subgroup of finite index N satisfying $N \cap F = \emptyset$. This prevents G from being finitely discriminable. On the other hand, Champetier [Cha00] has proved that the closure in $\mathcal{G}_n$ $(n \geqslant 2)$ of the set of non-elementary hyperbolic groups, is a Cantor set and therefore contains no isolated point. We leave for record

Proposition 8. *Infinite residually finite groups and infinite hyperbolic groups are not finitely discriminable.*

On the other hand, the simplest examples of isolated groups are finite groups. There are also finitely presented simple groups. But the class of isolated groups is considerably larger, in view of the following result, proved in Section 5.2:

Theorem 9. *Every finitely generated group is a quotient of an isolated group.*

This shows in particular that the lattice of normal subgroups of an isolated group can be arbitrarily complicated; for instance, in general it does not satisfy the descending/ascending chain condition.

Proposition 10. *There exists an isolated group that is 3-solvable and non-Hopfian.*[2]

This is in a certain sense optimal, since it is known that finitely generated groups that are either nilpotent or metabelian (2-solvable) are residually finite [Hal59], and thus cannot be isolated unless they are finite. The example we provide to prove Proposition 10 (see Section 5.4) is a group that had been introduced by Abels [Abe79] as the first example of a non-residually finite (actually non-Hopfian) finitely presented solvable group. A variation on this example provides (recall that a countable group G has Kazhdan's property (T) if every isometric action of G on a Hilbert space has a fixed point):

Proposition 11. *There exists an infinite isolated group with Kazhdan's property* (*T*).

We provide some other examples. One of them (see Section 5.3) is Houghton's group, which is an extension of the group of finitely supported permutations of a countable set, by $\mathbf{Z}^2$. In particular, this group is elementary amenable but non-virtually solvable.

Another one (see Section 5.7) is a group exhibited by Grigorchuk [Gri98], which is the first known finitely presented amenable group that is not elementary amenable. This is an ascending HNN extension of the famous "first Grigorchuk group," which has intermediate growth and is torsion; the latter is certainly not isolated since it is infinitely presented and is not finitely discriminable since it is residually finite. The fact that this group is isolated contradicts a conjecture by Stepin in [Gri98, §1], stating that every amenable finitely generated group can be approximated by elementary amenable ones.

Finally (see Section 5.8), some lattices in non-linear simple Lie groups provide examples of isolated groups that are extensions with infinite residually finite quotient and finite central kernel.

Throughout this article we use the following notation. If x and y are elements in a group G then

$$[x, y] = xyx^{-1}y^{-1}, \qquad x^y = y^{-1}xy.$$

Similarly if N and K are subgroups of G, then $[N, K]$ stands for the subgroup generated by $\{[n, k] \mid n \in N,\ k \in K\}$. Finally in any group G, we denote by $Z(G)$ the center, and more generally by $C_G(X)$ the centralizer of a subset $X \subset G$.

[2] A group G is non-Hopfian if there exists an epimorphism $G \to G$ with non-trivial kernel.

1. The space of finitely generated groups

Let G be a group. We denote by $\mathcal{P}(G)$ the set of subsets of G and by $\mathcal{G}(G)$ the set of normal subgroups of G. We endow $\mathcal{P}(G)$ with the product topology through the natural bijection with $\{0,1\}^G$. Hence $\mathcal{P}(G)$ is a compact and totally discontinuous space.

Limits in $\mathcal{P}(G)$ have the following simple description, whose proof is straightforward and omitted.

Lemma 1.1. *The net (A_i) converges to A in $\mathcal{P}(G)$ if and only if $A = \liminf A_i = \limsup A_i$, where $\liminf A_i = \bigcup_i \bigcap_{j\geqslant i} A_j$ and $\limsup A_i = \bigcap_i \bigcup_{j\geqslant i} A_i$.*

Since, for every net (N_i) of normal subgroups, $\liminf N_i$ is also a normal subgroup, the following proposition follows immediately from the lemma.

Proposition 1.2. *The subset $\mathcal{G}(G)$ is closed in $\mathcal{P}(G)$.*

The space $\mathcal{G}(G)$ can be identified with the space of quotients of G, which we also call, by abuse of notation, $\mathcal{G}(G)$ (in the sequel it will always be clear when we consider an element of $\mathcal{G}$ as a normal subgroup or as a quotient of G). It is endowed with a natural order: $H_1 \preccurlyeq H_2$ if the corresponding normal subgroups N_1, N_2 satisfy $N_1 \supset N_2$.

The topology of $\mathcal{G}(G)$ has the following basis:

$$(\Omega_{r_1,\dots,r_k,s_1,\dots,s_\ell}), \quad r_1,\dots,r_k,s_1,\dots,s_l \in G,$$

where $\Omega_{r_1,\dots,r_k,s_1,\dots,s_\ell}$ is the set of quotients of G in which each $r_i = 1$ and each $s_j \neq 1$. These are open and closed subsets.

If F_m is a free group of rank m with a given freely generating family, then $\mathcal{G}(F_m)$ is usually called the *space of marked groups on m generators* and we denote it by $\mathcal{G}_m$. An element in $\mathcal{G}_m$ can be viewed as a pair $(G,\mathcal{T})$ where G is an m-generated group and $\mathcal{T}$ is a generating m-tuple.

If G, H are any groups, every homomorphism $f : G \to H$ induces a continuous map $f^* : \mathcal{G}(H) \to \mathcal{G}(G)$, which is injective if f is surjective. The main features of the spaces $\mathcal{G}(G)$ is summarized in the following lemma, which is essentially known (see [Cha00, Lemme 2.2 and Proposition 3.1]).

Lemma 1.3.

(1) *Let G be a group and H a quotient of G; denote by p the quotient map $G \to H$. Then the embedding $p^* : \mathcal{G}(H) \to \mathcal{G}(G)$ is a closed homeomorphism onto its image, which we identify with $\mathcal{G}(H)$. Moreover, the following are equivalent:*
  (i) *$\mathcal{G}(H)$ is open in $\mathcal{G}(G)$.*
  (ii) *H is contained in the interior of $\mathcal{G}(H)$ in $\mathcal{G}(G)$ (in other words: H has a neighborhood in $\mathcal{G}(G)$ consisting of quotients of itself).*
  (iii) $\mathrm{Ker}(p)$ *is finitely generated as a normal subgroup of G.*

(2) *If, in addition, G is a finitely presented group, then these are also equivalent to*
  (iv) *H is finitely presented.*

(3) *Let G_1, G_2 be finitely presented groups, and consider quotients $H_i \in \mathcal{G}(G_i)$, $i = 1, 2$. Suppose that H_1 and H_2 are isomorphic groups. Then there exist finitely presented intermediate*

quotients $H_i \preccurlyeq K_i \preccurlyeq G_i$, $i = 1, 2$, and an isomorphism $\phi : K_1 \to K_2$, such that ϕ^* maps the point H_2 to H_1.

Note that Lemma 1 is an immediate consequence of Lemma 1.3.

Proof. (1) is straightforward and left to the reader.

(2) follows from (1) and the fact that if G is a finitely presented group and N a normal subgroup, then G/N is finitely presented if and only if N is finitely generated as a normal subgroup [Rot95, Lemma 11.84].

To prove (3), fix an isomorphism $\alpha : H_1 \to H_2$ and identify H_1 and H_2 to a single group H through ϕ. Take a finitely generated subgroup F_0 of the fibre product $G_1 \times_H G_2$ mapping onto both G_1 and G_2, and take a free group F of finite rank mapping onto F_0. Then the following diagram commutes.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G_2 \arrow[d] \\ G_1 \arrow[r] & H \end{tikzcd}$$

For $i = 1, 2$, let N_i denote the kernel of $F \to G_i$. Then F/N_1N_2 is a finitely presented quotient of both G_1 and G_2, having H as a quotient. If we do not longer identify H_1 and H_2, then F/N_1N_2 can be viewed as a quotient K_i of G_i, and the obvious isomorphism ϕ between K_1 and K_2 (induced by the identity of F) induces α. $\square$

It follows that every local topological consideration makes sense in a "space of finitely generated groups." Roughly speaking the later looks like a topological space. Its elements are finitely generated groups (and therefore do not make up a set). If G is a finitely generated group, then a neighborhood of G is given by $\mathcal{G}(H)$, where H is a finitely presented group endowed with a given homomorphism onto G. For instance, if $\mathcal{C}$ is an isomorphism-closed class of groups (as all the classes of groups we consider in the paper), then we can discuss whether $\mathcal{C}$ is open, whether it is dense.

2. Isolated groups

Proposition-Definition 2.1. *Let G be a group. We say that G is* finitely discriminable *if it satisfies the following equivalent conditions.*

(i) *The trivial normal subgroup $\{1\}$ is isolated in $\mathcal{G}(G)$.*
(ii) *The group G has finitely many minimal normal subgroups and any non-trivial normal subgroup contains a minimal one.*
(iii) *There exists a finite* discriminating subset *in G: this is a finite subset $F \subset G - \{1\}$ such that any non-trivial normal subgroup of G contains at least one element of F.*

Proof of the equivalences. (ii) $\Rightarrow$ (iii): Define F by taking a non-trivial element in each minimal normal subgroup.

(iii) $\Rightarrow$ (ii): Every non-trivial normal subgroup contains the normal subgroup generated by an element of F.

(iii) $\Rightarrow$ (i): Since F is finite, the set of normal subgroups with empty intersection with F is open in $\mathcal{G}(G)$; by assumption this set is reduced to $\{\{1\}\}$.

(i) $\Rightarrow$ (iii): We contrapose. For every finite subset $F \subset G - \{1\}$, consider a non-trivial normal subgroup N_F having empty intersection with F. Then $N_F \to \{1\}$ when F becomes large (that is, tends to $G - \{1\}$ in $\mathcal{P}(G)$). $\square$

Proposition 2.2. *A group G is isolated if and only if it is both finitely presentable and finitely discriminable.*

Proof. An isolated group is finitely presented: this follows from the implication (ii) $\Rightarrow$ (iv) of Lemma 1.3.

Suppose that G is finitely generated but not finitely discriminable. Then G is not isolated in $\mathcal{G}(G)$ (viewed as the set of quotients of G), hence is not isolated in the space of finitely generated groups.

Conversely suppose that G satisfies the two conditions. Since G is finitely presented, by the implication (iv) $\Rightarrow$ (i) of Lemma 1.3, $\mathcal{G}(G)$ is a neighborhood of G in the space of finitely generated groups. Since G is finitely discriminable, it is isolated in $\mathcal{G}(G)$. Hence G is isolated. $\square$

Proposition 2.2 allows us to split the study of isolated groups into the study of finitely discriminable groups and finitely presented groups. These studies are in many respects independent, and it is sometimes useful to drop the finite presentability assumption when we have to find examples. However, these properties also have striking similarities, for instance:

Proposition 2.3. *The classes of finitely discriminable and finitely presentable groups are both dense in the space of finitely generated groups.*

Proof. The case of finitely presentable groups is an observation by Champetier [Cha00, Lemme 2.2] (it suffices to approximate every finitely generated group by truncated presentations).

Let us deal with finite discriminability. If G is finite then it is finitely discriminable. Otherwise, for every finite subset F of $G - \{1\}$, consider a maximal normal subgroup N_F among those with empty intersection with F. Then G/N_F is finitely discriminated by the image of F, while the net (G/N_F) converges to G when F becomes large. $\square$

Remark 2.4. The proof above shows that more generally, every group G is approximable by finitely discriminable quotients in $\mathcal{G}(G)$.

In contrast, we show in the next paragraph that being isolated is not a dense property.

3. Isolated groups and the word problem

Roughly speaking, a sequence of words in a free group F of finite rank is *recursive* if it can be computed by a finite algorithm; more precisely by an ideal computer, namely a Turing machine. See Rotman's book [Rot95, Chapter 12] for a precise definition. A subset $X \subset F$ is *recursively enumerable* if it is the image of a recursive sequence.

Proposition-Definition 3.1. *Let G be a finitely generated group. We call a sequence (g_n) in G recursive if it satisfies one of the two equivalent properties*:

(1) *There exists a free group of finite rank F, an epimorphism $p : F \to G$, and a recursive sequence (h_n) in F such that $p(h_n) = g_n$ for all n.*
(2) *For every free group of finite rank F and every epimorphism $p : F \to G$, there exists a recursive sequence (h_n) in F such that $p(h_n) = g_n$ for all n.*

Proof. We have to justify that the two conditions are equivalent. Note that (2) is a priori stronger. But if (1) is satisfied, then, using Tietze transformations to pass from a generating subset of G to another, we obtain that (2) is satisfied. $\square$

Definition 3.2. A finitely generated group G is *recursively discriminable* if there exists a recursively enumerable discriminating subset: there exists a recursive sequence (g_n) in $G - \{1\}$ such that every normal subgroup $N \neq 1$ of G contains some g_n.

Remark 3.3. A related notion, namely that of terminal groups, is introduced by A. Mann in [Man82, Definition 2]. A finitely generated group $G = F/N$, with F free of finite rank, is *terminal* if $F - N$ is recursively enumerable. Clearly this implies that G is recursively discriminable, but the converse is false: indeed, there are only countably many terminal groups, while there are $2^{\aleph_0}$ non-isomorphic finitely generated simple groups [LS77, Chapter IV, Theorem 3.5]. Nevertheless, there exist terminal groups with unsolvable word problem [Man82, Proposition 1].

Let G be a finitely generated group, and write $G = F/N$ with F a free group of finite rank. Recall that G is recursively presentable if and only if N is recursively enumerable, and that G has solvable word problem if and only if both N and $F - N$ are recursively enumerable; that is, N is recursive.

The following theorem was originally proved by Simmons [Sim73]. We offer here a much more concise proof of this result. It can be viewed as a conceptual generalization of a well-known theorem of Kuznetsov [LS77, Chapter IV, Theorem 3.6], which states that a recursively presentable simple group has solvable word problem.

Theorem 3.4. *Let $G = F/N$ be a finitely generated group as above. Then G has solvable word problem if and only if it is both recursively presentable and recursively discriminable.*

Proof. The conditions are clearly necessary. Conversely, suppose that they are satisfied, and let us show that F/N has solvable word problem. Consider a recursive discriminating sequence (g_n) in $F - N$.

Let x belong to F, and let N_x be the normal subgroup generated by N and x; it is recursively enumerable. Set $W_x = \{y^{-1}g_n \mid y \in N_x,\ n \in \mathbf{N}\}$. Then W_x is recursively enumerable. Observe that $1 \in W_x$ if and only if $x \notin N$. Indeed, if $1 \in W_x$, then $g_n = y$ for some $y \in N_x$, and since $g_n \notin N$ this implies that $x \notin N$. Conversely if $x \notin N$, then N_x projects to a non-trivial subgroup of F/N, and hence contains one of the g_n's. So the algorithm is the following: enumerate both N and W_x: either x appears in N, or 1 appears in W_x and in this case $x \notin N$. $\square$

Our initial motivation in proving Theorem 3.4 is the following corollary.

Corollary 3.5. *An isolated group has solvable word problem.*

There exists an alternative short proof of the corollary using model theory (see the proof of the analogous assertion in [Rip82]). More precisely, groups with solvable word problem are characterized by Rips [Rip82] as isolated groups for some topology on $\mathcal{G}_n$ stronger than the topology studied here, making the corollary obvious.

Since the class of isolated groups is the intersection of the classes of finitely presented and finitely generated finitely discriminable groups and since these classes are both dense, it is natural to ask whether the class of isolated groups is itself dense. This question was the starting point of our study. However it has a negative answer.

Proposition 3.6. *The class of finitely generated groups with solvable word problem is not dense.*

Corollary 3.7. *The class of isolated groups is not dense.*

Proof of Proposition 3.6. C.F. Miller III has proved [Mil81] that there exists a non-trivial finitely presented group G such that the only quotient of G having solvable word problem is $\{1\}$ (and moreover G is SQ-universal, i.e. every countable group embeds in some quotient of G). Thus, using Lemma 1.3, G is not approximable by isolated groups. $\square$

This leaves many questions open.

Question 1. Is every finitely generated group with solvable word problem a limit of isolated groups?

Question 2. Is every word hyperbolic group a limit of isolated groups?

Note that a word hyperbolic group has solvable word problem. The following stronger question is open: is every word hyperbolic group residually finite?

Question 3. Is every finitely generated solvable group a limit of isolated groups?

Note that there exist finitely presented solvable groups with unsolvable word problem [Kar81]; this suggests a negative answer.

Let G be a group. An *equation* (respectively *inequation*) over G is an expression "$m = 1$" (respectively "$m \neq 1$") where $m = m(x_1, \ldots, x_n)$ is an element of the free product of G with the free group over unknowns $x_1, \ldots, x_n$. A solution of an (in)equation in G is an n-tuple $(g_1, \ldots, g_n)$ of G such that $m(g_1, \ldots, g_n) = 1$ (respectively $m(g_1, \ldots, g_n) \neq 1$). A system of equations and inequations over a group G is *coherent* if it has solution in some overgroup of G. A group Ω is *existentially closed* if every coherent finite system of equations and inequations over Ω has a solution in Ω.

Using free products with amalgamation, one can prove [Sco51] that every group G embeds in an existentially closed group, which can be chosen countable if G is so. The *skeleton* of a group G is the class of finitely generated groups embedding in G. Skeletons of existentially closed groups have been extensively studied (see [HS88]). What follows is not new but merely transcribed in the language of the space of finitely generated groups. Our aim is to prove that this language is relevant in this context.

The link with isolated groups is given by the following observation [Neu73, Lemma 2.4]: a given isolated group embeds in every existentially closed group. This is contained in the following more general result.

Proposition 3.8. *If Ω is an existentially closed group, then its skeleton is dense in the space of finitely generated groups.*

Proof. Fix an existentially closed group Ω. Let F be a free group of rank n, and $g_1, \dots, g_n$ be generators. Choose elements $m_1, \dots, m_d, \mu_1, \dots, \mu_\delta$ in F. Consider the set $\mathcal{S}$ of quotients of F in which $m_1, \dots, m_d = 1$ and $\mu_1, \dots, \mu_\delta \neq 1$. Subsets of this type make up a basis of open (and closed) subsets in $\mathcal{G}(F)$. To say that such a subset $\mathcal{S}$ is non-empty means that the system of equations and inequations

$$\begin{cases} m_1, \dots, m_d = 1, \\ \mu_1, \dots, \mu_\delta \neq 1 \end{cases}$$

is coherent. If this is the case, then it has a solution $(s_1, \dots, s_n)$ in Ω. Thus some group in $\mathcal{S}$, namely the subgroup of Ω generated by $(s_1, \dots, s_n)$, embeds in Ω. This proves that the skeleton of Ω is dense. $\square$

Propositions 3.6 and 3.8 together prove that every existentially closed group contains a finitely generated subgroup with unsolvable word problem. On the other hand, Macintyre [Mac72] (see also [LS77, Chapter IV, Theorem 8.5]) has proved that there exist two existentially closed groups Ω_1, Ω_2 such that the intersection of the skeletons of the two is reduced to the set of groups with solvable word problem. Moreover, Boone and Higman [BH74] (see also [LS77, Chapter IV, Theorem 7.4]) have proved that every group with solvable word problem embeds in a simple subgroup of a finitely presented group; this easily implies ([Neu73] or [LS77, Chapter IV, Theorem 8.4]) that a finitely generated group with solvable word problem embeds in every existentially closed group. We leave open the following question.

Question 4. Does every finitely generated group with solvable word problem embed into an isolated group?

Note that the stronger well-known question whether every finitely generated group with solvable word problem embeds in a finitely presented simple group is open.

4. Hereditary constructions for isolated groups

The two characterizing properties for isolated points, finite presentation and finite discriminability, are quite different in nature. The hereditary problem for finite presentation is classical, and in most cases is well understood. Therefore our results mostly deal with the hereditary problem for finite discriminability.

The analysis of extensions of isolated groups rests on the analysis of their center. As the center of a finitely discriminable group is itself finitely discriminable (see Lemma 4.2 below), the analysis of the center fits into the more general problem of understanding finitely discriminable abelian groups.

A group G is finitely cogenerated if it has a finite subset F having non-empty intersection with every non-trivial subgroup of G.

The class of finitely cogenerated groups is clearly contained in the class of finitely discriminable groups, but is much smaller; for instance, it is closed under taking subgroups, and therefore a finitely cogenerated group is necessarily torsion. However in restriction to abelian groups, the two classes obviously coincide.

We denote by C_{p^∞} the p-primary Prüfer group (also called quasi-cyclic): this is the direct limit of cyclic groups of order p^n when $n \to \infty$; it can directly be constructed as the quotient group $\mathbf{Z}[1/p]/\mathbf{Z}$.

In [Yah62] Yahya characterizes finitely cogenerated (i.e. finitely discriminable) abelian groups.

Lemma 4.1. *For an abelian group G, the following are equivalent*:

(i) *G is finitely discriminable.*
(ii) *G is artinian*: *every descending sequence of subgroups stabilizes.*
(iii) *The three following conditions are satisfied*:
  (1) *G is a torsion group.*
  (2) *Its p-torsion $\{x \in G \mid px = 0\}$ is finite for all primes p.*
  (3) *G has non-trivial p-torsion for only finitely many primes p.*
(iv) *G is a finite direct sum of finite cyclic groups and Prüfer groups.*

Lemma 4.1 is useful even when we focus on finitely generated groups, in view of the following fact.

Lemma 4.2. *If G is a finitely discriminable group, then its center is finitely discriminable.*

Proof. This immediately follows from the fact that every subgroup of the center of G is normal in G. □

The converse of Lemma 4.2 is true in the case of nilpotent groups, and more generally hypercentral groups. Recall that, in a group G, the transfinite ascending central series (Z_α) is defined as follows: Z_1 is the center of G, $Z_{\alpha+1}$ is the preimage in G of the center of G/Z_α, and $Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$ if λ is a limit ordinal. The group G is *hypercentral* if $Z_\alpha = G$ for some α.

Corollary 4.3. *Let G be a hypercentral group. Then G is finitely discriminable if and only if its center $Z(G)$ is so.*

Proof. As noticed before, the "if" part is straightforward. The converse implication follows immediately from the known fact that if G is hypercentral, then any normal subgroup of G intersects the center $Z(G)$ non-trivially. Let us recall the argument. Suppose that a normal subgroup $N \neq 1$ has trivial intersection with the center. Let α be the smallest ordinal such that Z_α contains a non-trivial element x of N. Clearly, α is a successor. Let M be the normal subgroup generated by x. Then $M \subset N \cap Z_\alpha$. On the other hand, $M \cap Z_{\alpha-1} = 1$, and since $M \subset Z_\alpha$, by definition of Z_α we have $[G, Z_\alpha] \subset Z_{\alpha-1}$. Hence $[G, M] \subset Z_{\alpha-1} \cap M = 1$, so that M is central, a contradiction. □

We now study hereditary properties of finitely discriminable groups. It is convenient to extend some of the definitions above. If G is a group, we define a G-group as a group H endowed with an action of G (when H is abelian it is usually called a G-module). For instance, normal subgroups and quotients of G are naturally G-groups. We call a G-group finitely discriminable (or G-finitely discriminable) if $\{1\}$ is isolated among normal G-subgroups of H. (Proposition 2.1 has an obvious analog in this context.)

Theorem 4.4. *Consider an extension of groups*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

Denote by W the kernel of the natural homomorphism $Q \to \mathrm{Out}(K)$.

(1) *G is finitely discriminable if and only if K and $C_G(K)$ are both G-finitely discriminable.*
(2) *Suppose that K and W are both G-finitely discriminable. Suppose moreover that*
   ($*$) *$Z(K)$ contains no infinite simple G-submodule G-isomorphic to a normal subgroup of Q contained in W.*

   Then G is finitely discriminable.

Remark 4.5.

(1) Assumption ($*$) is satisfied when $Z(K)$ or W does not contain any infinite-dimensional vector space over a prime field ($\mathbf{F}_p$ or $\mathbf{Q}$). In particular, if $Z(K)$ is artinian it is satisfied.
(2) In Assumption ($*$), we can replace "infinite" by "with infinite endomorphism ring," which is a more natural hypothesis in view of the subsequent proof.
(3) Assumption ($*$) cannot be dropped: see the example in Remark 4.10.

Proof of Theorem 4.4(1). The conditions are clearly necessary. Conversely suppose that they are satisfied. Let N_i be a net of normal subgroups of G tending to 1. Then $N_i \cap K \to 1$; this implies that eventually $N_i \cap K = 1$. Thus eventually $N_i \subset C_G(K)$. But similarly eventually $N_i \cap C_G(K) = 1$. Accordingly eventually $N_i = 1$. $\square$

Before proving (2), we need some preliminary results. Consider an extension of groups:

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1.$$

For any normal subgroup $H \lhd Q$, denote by $\mathcal{I}(H)$ the set of normal subgroups of G that are sent isomorphically onto H by the projection π.

Lemma 4.6. *Let M be a normal subgroup in G such that $M \cap K$ is trivial and denote by $\pi(M)$ its image in Q. Then*:

(1) *Any group in $\mathcal{I}(\pi(M))$ is isomorphic as a G-group to $\pi(M)$ via π and in particular is naturally a Q-group.*
(2) *If $\pi(M)$ is minimal in Q then all groups in $\mathcal{I}(\pi(M))$ are minimal in G.*
(3) *The set $\mathcal{I}(\pi(M))$ is in one-to-one correspondence with the set of G-equivariant group homomorphisms $\mathrm{Hom}_G(\pi(M), Z(K))$.*

Proof. Points (1) and (2) are easy exercises; we concentrate on the proof of the third point.

For any element $H \in \mathcal{I}(\pi(M))$, denote by σ_H the inverse homomorphism to $\pi : H \to \pi(M)$. There is an obvious one-to-one correspondence between the set $\mathcal{I}(\pi(M))$, and the set of homomorphisms $\{\sigma_H \mid H \in \mathcal{I}(\pi(M))\}$ given by the map $H \mapsto \sigma_H$ and its inverse $\sigma_H \mapsto \mathrm{Image}(\sigma_H)$. We use these maps to identify these two sets.

We claim that there is a faithful transitive action of the abelian group $\mathrm{Hom}_G(\pi(M), Z(K))$ on the latter set. For $\phi \in \mathrm{Hom}_G(\pi(M), Z(K))$ and $\sigma_H \in \mathcal{I}(\pi(M))$ we define

$$\begin{aligned}\phi \cdot \sigma_H : \pi(M) &\longrightarrow G,\\ x &\longmapsto \sigma_H(x)\phi(x).\end{aligned}$$

We first prove that $\phi \cdot \sigma_H$ is indeed a group homomorphism. If $x, y \in \pi(M)$, then

$$\begin{aligned}(\phi \cdot \sigma_H)(xy) &= \sigma_H(xy)\phi(xy)\\ &= \sigma_H(x)\sigma_H(y)\phi(x)\phi(y)\\ &= \sigma_H(x)\phi(x)\big[\phi(x)^{-1}, \sigma_H(y)\big]\sigma_H(y)\phi(y).\end{aligned}$$

Since H and $Z(K)$ are both normal subgroups of G, we have $[\phi(x)^{-1}, \sigma_H(y)] \in [Z(K), H] \subset Z(K) \cap H = \{1\}$, so that

$$(\phi \cdot \sigma_H)(xy) = (\phi \cdot \sigma_H)(x)(\phi \cdot \sigma_H)(y).$$

As σ_H and ϕ are G-equivariant homomorphisms, it is immediate that $\phi \cdot \sigma_H$ is G-equivariant; it is also immediate that it defines an action of $\mathrm{Hom}_G(\pi(M), Z(K))$ on $\mathcal{I}(\pi(M))$.

To see that the action is transitive, we consider an element σ_H in $\mathcal{I}(\pi(M))$ and we define the "transition map" from σ_M to σ_H:

$$\begin{aligned}\phi_H : \pi(M) &\longrightarrow G,\\ x &\longmapsto \sigma_M(x)^{-1}\sigma_H(x).\end{aligned}$$

We claim that the map ϕ_H is a G-equivariant homomorphism and has values in $Z(K)$.

Indeed, by construction $\pi \circ \phi_H(x) = 1$ for all x in $\pi(M)$, thus the image of ϕ_H is contained in K. More precisely $\phi_H(\pi(M)) \subset HM \cap K$. Since H and M are normal subgroups and have trivial intersection with K, they are contained in the centralizer of K in G. It follows that $\phi_H(\pi(M))$ is contained in the center of K. The fact that ϕ_H is a G-equivariant homomorphism follows now from direct computation.

Finally the stabilizer of σ_M is trivial, as $\phi \cdot \sigma_M = \sigma_M$ if and only if for all $x \in \pi(M)$ $\sigma_M(x)\phi(x) = \sigma_M(x)$, that is, ϕ is identically 1. This ends the proof of Lemma 4.6. □

Proof of Theorem 4.4(2). Denote by $L_1, \dots, L_d$ the minimal normal subgroups of G contained in K. Let N be a normal subgroup of G. If $N \cap K \neq 1$, then N contains some L_i.

Let us assume now that $N \cap K = 1$. Then N is contained in the centralizer $C_G(K)$. On the other hand, one can check that the image of $C_G(K)$ in Q is W. Denote by $Q_1, \dots, Q_n$ the minimal normal subgroups of Q contained in W. Then the image of N in Q contains some Q_i. It follows from Lemma 4.6(2) that $\pi^{-1}(Q_i) \cap N$ is a minimal normal subgroup of G (belonging to $\mathcal{I}(Q_i)$). Thus it remains to prove that $\mathcal{I}(Q_i)$ is finite for every i. By Lemma 4.6(3), if non-empty, this set is in one-to-one correspondence with $\mathrm{Hom}_G(Q_i, Z(K))$; let us show that this set is finite. We discuss the possible cases.

- If Q_i is non-abelian, then, since it is characteristically simple, it is perfect, and therefore $\mathrm{Hom}_G(Q_i, Z(K)) \subset \mathrm{Hom}(Q_i, Z(K)) = \{1\}$.
- If Q_i is infinite abelian, then by the assumption $(*)$, we know that the center $Z(K)$ does not contain any G-submodule isomorphic to Q_i, and therefore $\mathrm{Hom}_G(Q_i, Z(K)) = \{1\}$.
- Suppose that Q_i is finite abelian. Let V be the sum of all G-submodules of $Z(K)$ isomorphic to Q_i. Since Q_i is a simple G-module, one can check that V, as a G-module, is a direct sum $\bigoplus_{j\in J} V_j$ of submodules V_j isomorphic as G-modules to Q_i. Since $Z(K)$ is G-finitely discriminable, the index set J must necessarily be finite. Therefore $\mathrm{Hom}_G(Q_i, Z(K)) = \mathrm{Hom}_G(Q_i, \bigoplus_{j\in I} V_j) \simeq \prod_{j\in J} \mathrm{End}_G(Q_i)$, which is finite. □

Corollary 4.7. *The classes of finitely discriminable groups and of isolated groups are closed under extensions.*

Proof. Since K is finitely discriminable, it is G-finitely discriminable, and since Q is finitely discriminable, its normal subgroup W must be Q-finitely discriminable. We can therefore apply Theorem 4.4(2), noting that since K is finitely discriminable, its center is artinian (Lemmas 4.1 and 4.2). It follows that the assumption $(*)$ is satisfied: indeed, if a subgroup of $Z(K)$ is a simple G-submodule, then it must be contained in the p-torsion of $Z(K)$ for some prime p and therefore is finite.

The second assertion is obtained by combining this with the fact that the class of finitely presented groups is closed under extensions. □

Corollary 4.8. *Consider an extension of groups*

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

Suppose that K is G-finitely discriminable, and that the natural homomorphism $Q \to \mathrm{Out}(K)$ is injective. Then G is finitely discriminable.

Proof. Note that since the kernel W of the natural homomorphism $Q \to \mathrm{Out}(K)$ is trivial, the assumption $(*)$ of Theorem 4.4 is trivially satisfied. □

Corollary 4.9. *To be finitely discriminable and to be isolated are properties inherited by overgroups of finite index.*

Proof. Let G be a group, and H a finitely discriminable subgroup of finite index. Let N be a subgroup of finite index of H which is normal in G. Then N is H-finitely discriminable and therefore G-finitely discriminable. Since G/N is finite, it is clear that all assumptions of Theorem 4.4(2), are satisfied, so that G is finitely discriminable. □

Remark 4.10. In contrast, the finite discriminable property does not pass to subgroups of finite index, as the following example shows. Consider the wreath product $\Gamma = \mathbf{Z} \wr \mathbf{Z}$. In [Hal59] P. Hall has constructed that there exists a simple faithful $\mathbf{Z}\Gamma$-module V whose underlying abelian group is an infinite-dimensional $\mathbf{Q}$-vector space. Then there is an obvious action of $\Gamma \times \mathbf{Z}/2\mathbf{Z}$ on the direct sum $V \oplus V$, where the cyclic group permutes the two copies. Consider then the semi-direct product $G = (V \oplus V) \rtimes (\Gamma \times \mathbf{Z}/2\mathbf{Z})$. Its subgroup of index two $(V \oplus V) \rtimes \Gamma$ is not finitely discriminable as for each rational r we have a different minimal subgroup $V_r = \{(v, rv) \in$

$V \oplus V \mid v \in V\}$. Nevertheless, G is finitely discriminable since every non-trivial normal subgroup of G contains one of its two minimal normal subgroups V_1 and V_{-1}.

Remark that G is not finitely presented. Indeed, $\mathbf{Z} \wr \mathbf{Z}$ is not a quotient of any finitely presented solvable group [Bau61,BS80]. However, we conjecture that there exists an example of an isolated group having a non-isolated finite index subgroup.

5. Examples of isolated groups

5.1. Elementary class

Obvious examples of isolated groups are finitely presented simple groups. According to Corollary 4.7, to get new examples of isolated groups it suffices to consider the class of groups that can be obtained from these by successively taking extensions of groups. The class of groups we get is fairly well understood, for by definition any such group has a composition series of finite length, and by a theorem of Wielandt [Wie39] this is precisely the class of finitely presented groups that contain finitely many subnormal subgroups. More generally we have

Proposition 5.1. *Any group with finitely many normal subgroups is finitely discriminable. In particular, finitely presented groups with finitely many normal subgroups are isolated.*

Remark 5.2. A finitely presented group G has finitely many normal subgroups if and only if every quotient of G is isolated. The condition is clearly necessary. Conversely if G has infinitely many normal subgroups, then, by compactness of $\mathcal{G}(G)$, there exists an accumulation point, and hence G has a non-isolated quotient.

Note, in contrast, that the Prüfer group C_{p^∞} has all its quotients finitely discriminable but has infinitely many normal subgroups.

5.2. Quotients of isolated groups

The inclusion above is strict in a strong sense:

Theorem 5.3. *Every finitely generated group is a quotient of an isolated group.*

The theorem easily follows from the following lemma, which is probably known but for which we found no reference.

Lemma 5.4. *There exists an isolated group K such that* $\mathrm{Out}(K)$ *contains a non-abelian free group.*

Proof of Theorem 5.3. Clearly it suffices to deal with a free group F_n. Consider K as in Lemma 5.4. Then $\mathrm{Out}(K)$ contains a free group of rank n. Lift it to $\mathrm{Aut}(K)$ and consider the semi-direct product $G = K \rtimes F_n$ given by this action. By Corollary 4.8, G is isolated. □

We prove later Lemma 5.4 (see Proposition 5.11).

5.3. Houghton groups

These groups were first introduced by Houghton [Hou78] in his study of the relationship between ends and the cohomology of a group. They were then studied by K.S. Brown in connection with the so-called *FP* cohomological properties [Bro87].

Fix an integer $n \geqslant 1$, let $\mathbf{N}$ denote the set of positive integers and let $S = \mathbf{N} \times \{1, \dots, n\}$ denote a disjoint union of n copies of $\mathbf{N}$. Let H_n be the subgroup of all permutations g of S such that on each copy of $\mathbf{N}$, g is eventually a translation. More precisely, $g \in H_n$ if there is an n-tuple $(m_1, \dots, m_n) \in \mathbf{Z}^n$ such that for each $i \in \{1, \dots, n\}$ one has $g(x,i) = (x + m_i, i)$ for all sufficiently large $x \in \mathbf{N}$. The map $g \mapsto (m_1, \dots, m_n)$ is a homomorphism $\phi : H_n \to \mathbf{Z}^n$ whose image is the subgroup $\{(m_1, \dots, m_n) \in \mathbf{Z}^n \mid \sum m_i = 0\}$, of rank $n-1$. The kernel of ϕ is the infinite symmetric group, consisting of all permutations of S with finite support. It coincides with the commutator subgroup of H_n for $n \geqslant 3$, while for $n = 1$ and $n = 2$, the commutator subgroup is the infinite alternating group $\mathrm{Alt}(S)$. In all cases the second commutator is the infinite alternating group, which is a locally finite, infinite simple group.

Proposition 5.5. *For every $n \geqslant 1$, the group H_n is finitely discriminable.*

Proof. There is an extension

$$1 \longrightarrow \mathrm{Alt}(S) \longrightarrow H_n \longrightarrow H_n / \mathrm{Alt}(S) \longrightarrow 1.$$

Since $\mathrm{Alt}(S)$ has trivial centralizer in the full group of permutations of S, the assumption of Corollary 4.8 is satisfied. Actually $\mathrm{Alt}(S)$ is the unique minimal normal subgroup of H_n and is contained in all other non-trivial normal subgroups. □

The group H_1 is the infinite symmetric group and hence is not finitely generated. The group H_2 is finitely generated, but not finitely presented; indeed, it is a classical example of a non-residually finite group that is a limit of finite groups [Ste96,VG97]. For $n \geqslant 3$, it is a result of K.S. Brown [Bro87] that H_n is finitely presented. Therefore by Proposition 5.5 it is isolated.

5.4. Abels groups

Fix an integer $n \geqslant 2$ and a prime p. Denote by $A_n \subset \mathrm{GL}_n(\mathbf{Z}[1/p])$ the subgroup of upper triangular matrices a such that $a_{11} = a_{nn} = 1$ and such that the other diagonal coefficients are positive. For example,

$$A_2 = \begin{pmatrix} 1 & \mathbf{Z}[1/p] \\ 0 & 1 \end{pmatrix}; \qquad A_3 = \begin{pmatrix} 1 & \mathbf{Z}[1/p] & \mathbf{Z}[1/p] \\ 0 & p^{\mathbf{Z}} & \mathbf{Z}[1/p] \\ 0 & 0 & 1 \end{pmatrix}.$$

The group A_3, introduced by Hall in [Hal61] is not finitely presented [Gro78]. Abels [Abe79] introduced A_4 and showed that it is finitely presented, a result that was subsequently extended for $n \geqslant 4$, see [Bro87] for a proof and a discussion of homological properties of these groups.

The center of A_n consists of unipotent matrices with a single possibly non-trivial element in the upper right corner. It is clearly isomorphic to $\mathbf{Z}[1/p]$. The conjugation by the diagonal matrix $\mathrm{Diag}(p, 1, \dots, 1)$ provides an automorphism of A_n which induces the multiplication by p on the center. Consider the canonical copy Z of $\mathbf{Z}$ contained in the center through its identification with

$\mathbf{Z}[1/p]$. Then the quotient A_n/Z is non-Hopfian (see [Abe79] for details). As noticed before, we have the following proposition.

Proposition 5.6. *The groups A_n and A_n/Z are finitely presented for all $n \geqslant 4$.*

We now turn to finite discriminability. The group A_n itself is certainly not finitely discriminable because its center $\mathbf{Z}[1/p]$ is not an artinian abelian group (or because it is residually finite). In contrast, the center of A_n/Z is a Prüfer group C_{p^∞}.

Proposition 5.7. *The groups A_n/Z are finitely discriminable for $n \geqslant 2$. In particular, for $n \geqslant 4$ these groups are infinite, solvable* (3*-solvable when $n = 4$), and isolated.*

Before giving the proof of the proposition, let us give some of its consequences. First notice that this allows to obtain a kind of converse of Lemma 4.2.

Corollary 5.8. *Every finitely discriminable abelian group is isomorphic to the center of an isolated group.*

Proof. In this proof, let us denote the Abels group by $A_{n,p}$ to make explicit the dependence on p. Let G be a finitely discriminable abelian group. By Lemma 4.1, G is isomorphic to $F \times \prod_{i=1}^{n} C_{p_i^\infty}$, where F is a finite abelian group, and $p_1, \ldots, p_n$ are prime. Then G is isomorphic to the center of $F \times \prod_{i=1}^{n} A_{4,p_i}/Z(A_{4,p_i})$, which is an isolated group by Proposition 5.7 and Corollary 4.7. $\square$

Corollary 5.9. *The Hopfian property is not dense in the space of finitely generated groups.*

The non-Hopfian property is clearly not dense since finite groups are isolated Hopfian. The Hopfian property is not open [ABL05,Sta05]: the residually finite (metabelian) groups $\mathbf{Z} \wr \mathbf{Z}$ and $\mathbf{Z}[1/6] \rtimes_{3/2} \mathbf{Z}$ are approximable by non-Hopfian Baumslag–Solitar groups. On the other hand, let us allow ourselves a little digression:

Proposition 5.10. *The Hopfian property is not closed in the space of finitely generated groups. More precisely, there exists a finitely generated solvable group that is approximable by finite groups, but is non-Hopfian.*

Proof. For $n \geqslant 3$, consider the group B_n defined in the same way as A_n, but over the ring $\mathbf{F}_p[t, t^{-1}]$ rather than $\mathbf{Z}[1/p]$. It is easily checked to be finitely generated, and its center can be identified with $\mathbf{F}_p[t, t^{-1}]$, mapping on the upper right entry. Similarly to the case of A_n, the group $B_n/\mathbf{F}_p[t]$ is non-Hopfian. On the one hand, write $\mathbf{F}_p[t]$ as the union of an increasing sequence of finite additive subgroups H_k. Then B_n/H_k converges to the non-Hopfian group $B_n/\mathbf{F}_p[t]$ when $k \to \infty$. On the other hand, as a finitely generated linear group, B_n is residually finite, and therefore so is B_n/H_k. So the non-Hopfian finitely generated group $B_n/\mathbf{F}_p[t]$ is approximable by finite groups. $\square$

Before proving Proposition 5.7, let us describe a variation of Abels' group introduced in [Cor05a]. Consider integers n_1, n_2, n_3, n_4 satisfying $n_1, n_4 \geqslant 1$ and $n_2, n_3 \geqslant 3$. Consider the group H of upper triangular matrices by blocks (n_1, n_2, n_3, n_4) of the form

$$\begin{pmatrix} I_{n_1} & (*)_{12} & (*)_{13} & (*)_{14} \\ 0 & (**)_{22} & (*)_{23} & (*)_{24} \\ 0 & 0 & (**)_{33} & (*)_{34} \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix},$$

where $(*)$ denote any matrices and $(**)_{ii}$ denote matrices in SL_{n_i}, $i = 2, 3$.

Set $\Gamma = H(\mathbf{Z}[1/p])$, and let Z be the subgroup consisting of matrices of the form $I + A$ where the only non-zero block of A is the block $(*)_{14}$ and has integer coefficients; note that Z is a free abelian group of rank $n_1 n_4$, and is central in Γ.

It is proved in [Cor05a] that Γ and Γ/Z have Kazhdan's property (T), are finitely presentable and that the group $\mathrm{GL}_{n_1}(\mathbf{Z})$ embeds in $\mathrm{Out}(\Gamma/Z)$.

Proposition 5.11. *The group Γ/Z is isolated, has Kazhdan's property* (*T*)*, and its outer automorphism group contains a non-abelian free subgroup if $n_1 \geqslant 2$.*

It only remains to prove that Γ/Z is finitely discriminable. We now prove it together with Proposition 5.7.

Proof of Propositions 5.7 and 5.11. Let us make a more general construction.

Consider integers $m_1, m_2, m_3 \geqslant 1$, and $n = m_1 + m_2 + m_3$. Consider the subgroup U of GL_n given by upper unipotent by (m_1, m_2, m_3)-blocks matrices, that is, matrices of the form:

$$\begin{pmatrix} I_{m_1} & A_{12} & A_{13} \\ 0 & I_{m_2} & A_{23} \\ 0 & 0 & I_{m_3} \end{pmatrix}.$$

Let V denote the subgroup of U consisting of matrices with $A_{12} = 0$ and $A_{23} = 0$.

Lemma 5.12. *The centralizer C of U modulo V in GL_n is reduced to U_s, the group generated by U and scalar matrices.*

Proof. First compute the normalizer N of U in GL_n. Denote by E the vector space of rank n. Since the fixed points of U is the subspace E_1 generated by the m_1 first coefficients, it must be invariant under N. Since the fixed point of U on E/E_1 is the subspace E_2 generated by the $m_1 + m_2$ first coefficients, it must also be invariant under N. We thus obtain that N is the group of upper triangular matrices under this decomposition by blocks.

Let us now show that $C = U_s$. Since $U \subset C$, it suffices to show that $C \cap D = S$, where D is the group of diagonal by blocks matrices and S is the group of scalar matrices.

If we take

$$A = \begin{pmatrix} I_{m_1} & A_{12} & A_{13} \\ 0 & I_{m_2} & A_{23} \\ 0 & 0 & I_{m_3} \end{pmatrix} \in U \quad \text{and} \quad D = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix} \in C \cap D,$$

then

$$[D, A] = \begin{pmatrix} I_{m_1} & D_1 A_{12} {D_2}^{-1} - A_{12} & (\cdots) \\ 0 & I_{m_2} & D_2 A_{23} {D_3}^{-1} - A_{23} \\ 0 & 0 & I_{m_3} \end{pmatrix}$$

must belong to V. Thus $D_1 A_{12} = A_{12} D_2$ and $D_2 A_{23} = A_{23} D_3$ for all A_{12}, A_{23}. This easily implies that there exists a scalar λ such that $D_i = \lambda I_{m_i}$ for each $i = 1, 2, 3$. $\square$

Denote now by G the group of upper triangular by (m_1, m_2, m_3)-blocks matrices with $A_{11} = I_{m_1}$ and $A_{33} = I_{m_3}$. Note that V is central in G.

Lemma 5.13. *Let R be a commutative ring. Let H be a subgroup of $G(R)$ containing $U(R)$, and let Z be any subgroup of $V(R)$ satisfying the following assumption: for every $x \in V(R) - \{0\}$, there exists $\alpha \in R$ such that $\alpha x \notin Z$. Then H/Z is finitely discriminable if and only if the abelian group $V(R)/Z$ is finitely discriminable.*

Proof. By Lemma 4.2, the condition is necessary since $V(R)/Z$ is central in H.

Conversely, we have an extension

$$1 \longrightarrow U(R)/Z \longrightarrow H/Z \longrightarrow H/U(R) \longrightarrow 1.$$

By Lemma 5.12, the centralizer of $U(R)/Z$ in H/Z is contained in $U(R)/Z$, and therefore the natural homomorphism $H/U(R) \to \mathrm{Out}(U(R)/Z)$ is injective. Now $U(R)/Z$ is nilpotent, and by Corollary 4.3 it is finitely discriminable if and only if its center is so. Thus it suffices to prove that the center of $U(R)/Z$ is $V(R)/Z$.

Suppose that a matrix

$$A = \begin{pmatrix} I_{m_1} & A_{12} & A_{13} \\ 0 & I_{m_2} & A_{23} \\ 0 & 0 & I_{m_3} \end{pmatrix}$$

is central in $U(R)/Z$. By an immediate computation it must satisfy, for all B_{12}, B_{23}, the property $A_{12}B_{23} - B_{12}A_{13} \in Z$. If $A \notin V(R)$, we can choose B_{12}, B_{23} so that $x = A_{12}B_{23} - B_{12}A_{13} \neq 0$. Choose α as in the assumption of the lemma. Then $A_{12}(\alpha B_{23}) - (\alpha B_{12})A_{13} = \alpha x \notin Z$, a contradiction. Thus the center of $U(R)/Z$ is $V(R)/Z$. $\square$

In the case of Abels' group A_n, we have $R = \mathbf{Z}[1/p]$, $m_1 = m_3 = 1$, $m_2 = n - 2$, and $V(R)/Z$ is isomorphic to the Prüfer group C_{p^∞}. The group H is given by matrices in $G(\mathbf{Z}[1/p])$ whose diagonal block 22 is upper triangular with powers of p on the diagonal. The assumption of Lemma 5.13 is always satisfied, with $\alpha = p^{-k}$ for some k. This proves Proposition 5.7.

In the case of the variant of Abels' group Γ, we have $R = \mathbf{Z}[1/p]$, $m_1 = n_1$, $m_2 = n_2 + n_3$, $m_3 = n_4$, and $V(R)/Z$ is isomorphic to $(C_{p^\infty})^{n_1 n_4}$. The group H is given by matrices in $G(\mathbf{Z}[1/p])$ whose diagonal block 22 is upper triangular by blocks (n_2, n_3) with the two diagonal sub-blocks of determinant one. The assumption of Lemma 5.13 is also always satisfied, again with $\alpha = p^{-k}$ for some k. This proves Proposition 5.11. $\square$

5.5. Thompson groups

The results we mention on these groups can be found in [CFP96]. Let F be the set of piecewise linear increasing homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2. This turns out to be a finitely presented group. There is a natural morphism $F \to \mathbf{Z}^2$, mapping f to (m,n), where the slope of f at 0 is 2^m and the slope at 1 is 2^n. The kernel F_0 is a simple (infinitely generated) group. The corresponding extension satisfies the assumptions of Corollary 4.8: more precisely, F_0 is contained in every non-trivial normal subgroup of F. In particular

Proposition 5.14. *Thompson's group F is isolated.*

Observe the analogy with Houghton's group H_3 mentioned above.

Consider now S^1 as the interval $[0,1]$ with the endpoints identified. Thompson's group T is defined as the group of piecewise linear homeomorphisms from S^1 to itself that map images of dyadic rational numbers to images of dyadic rational numbers and that are differentiable except at finitely many images of rational dyadic numbers and on the intervals of differentiability the derivatives are power of 2. The group T is finitely presented and simple, and in particular is isolated. We use the groups F and T in the next paragraph.

5.6. Wreath products

If G and W are two groups, the *standard wreath product* $W \wr G$ is the semi-direct product $W^{(G)} \rtimes G$, where $W^{(G)}$ denotes the direct sum of copies of W indexed by G, on which G acts via the action on the labels by left multiplication.

A wreath product $W \wr G$ of two finitely presented groups is finitely presented only in trivial cases, namely when G is finite or $W = 1$ (see [Bau61]). Nevertheless if we consider *permutational wreath products*, then positive results on finite presentation do exist. Let G and W be groups and X a G-set, which we suppose transitive to simplify, so that we can write $X = G/H$. The *permutational wreath product* $W \wr_X G$ is the semi-direct product $W^{(X)} \rtimes G$, where $W^{(X)}$ denotes the direct sum of X copies of W, on which G acts via the natural action on the labels. The following theorem is shown in [Cor05b].

Theorem 5.15. *Let G and $W \neq 1$ be groups, and $X = G/H$ a transitive G-set. The group $W \wr_X G$ is finitely presented if and only if*:

(i) *both W and G are finitely presented*;
(ii) *H is finitely generated*;
(iii) *the product action of G on $X \times X$ has finitely many orbits* (*equivalently, the double coset space $H \backslash G/H$ is finite*).

Proposition 5.16. *Keep the notation as above. Suppose that $X = G/H$ is a faithful transitive G-set, and that $W \neq 1$ is finitely discriminable and has trivial center. Then the wreath product $W \wr_X G$ is finitely discriminable.*

Proof. Consider the extension

$$1 \longrightarrow W^{(X)} \longrightarrow W \wr_X G \longrightarrow G \longrightarrow 1.$$

Since $W \neq 1$, the natural morphism $G \to \mathrm{Out}(W^{(X)})$ is injective. So, to apply Corollary 4.8, it suffices to show that $W^{(X)}$ is a finitely discriminable G-group. Let N be a normal subgroup of G contained in $W^{(X)}$. For $x \in X$, denote by W_x the xth copy of W in $W^{(X)}$. If $N \cap W_x = 1$, for some x, then by transitivity of the G-action on X, we have $N \cap W_x = 1$ for all x and therefore N centralizes all W_x. Since W has trivial center, this implies $N = 1$. Therefore if $N \neq 1$, then $N \cap W_x$ contains a minimal normal subgroup M of W. It follows that N contains $M^{(X)}$, and thus $W^{(X)}$ is a finitely discriminable G-group. □

To deal with the case when W has non-trivial center we need some further assumptions.

Proposition 5.17. *Keep the notation as above. Suppose that $X = G/H$ is a faithful G-set, and that $W \neq 1$ is finitely discriminable. Suppose moreover that the two following conditions are satisfied.*

(i) *For every $n \geqslant 0$ and all distinct $y, x_1, \ldots, x_n \in X$, there exists $g \in G$ such that $gy \neq y$ and $gx_i = x_i$ for all $i = 1, \ldots, n$.*
(ii) *The action of G on X^2 has finitely many orbits.*

Then the wreath product $W \wr_X G$ is finitely discriminable.

Proof. Arguing as in the proof of Proposition 5.16, we are reduced to deal with a subgroup $N \neq 1$ contained in $Z(W)^{(X)}$. Let Z_p denote the p-torsion in $Z(W)$. Clearly, for some p, ${Z_p}^{(X)} \cap N \neq 1$, and replacing N by $N \cap {Z_p}^{(X)}$ we can suppose N contained in ${Z_p}^{(X)}$.

Consider a non-trivial element w of N, and denote its support as $\{y, x_1, \ldots, x_n\}$. Using the assumption on the G action on X, there exists $g \in G$ fixing all x_i's and mapping y to some $y' \in X - \{y\}$. Then $[g, w]$ has support reduced to $\{y, y'\}$. Consider a fixed finite family of elements (u_i, v_i) in X^2 with an element in each G-orbit. There exists i and $h \in G$ mapping (y, y') to (u_i, v_i). Thus $h[g, w]h^{-1}$ belongs to N and has support $\{u_i, v_i\}$. We obtain that if we take elements with support some $\{u_i, v_i\}$ and with values in elements of prime order in the center of W, along with a finite discriminant subset in one copy of W, we obtain a finite discriminating subset of $W \wr_X G$. □

Let us now give examples of (G, X) satisfying the conditions of both Theorem 5.15 and Proposition 5.17 (observe that the choice of the non-trivial isolated group W plays no role there). Trivial examples are those when X is finite.

- *Houghton groups*. For $n \geqslant 1$, the group H_n described in Section 5.3 acts on $X = \mathbf{N} \times \{1, \ldots, n\}$. The action contains the groups of finitely supported permutations and therefore Assumption (iii) of Theorem 5.15 and Assumptions (i) and (ii) of Proposition 5.17 are satisfied. The stabilizer of a point is also isomorphic to H_n. Accordingly, for $n \geqslant 3$, the group H_n is finitely presented, the stabilizers are finitely generated, so that all assumptions are fulfilled.
- *Thompson groups*. Thompson's group F (respectively T) acts on $X = \mathbf{Z}[1/2] \cap]0, 1[$ (respectively $X = \mathbf{Z}[1/2]/\mathbf{Z}$). This action is transitive on ordered (respectively cyclically or-

dered) n-tuples for all n. The stabilizer of a point is isomorphic to $F \times F$ (respectively F) and is therefore finitely generated. Thus all the assumptions of Theorem 5.15 and Proposition 5.17 are satisfied.

5.7. *Grigorchuk's finitely presented amenable group*

Let us consider the group $\widetilde{\Gamma}$ constructed by Grigorchuk in [Gri98] and given by the presentation

$$\langle a, b, c, d, t \mid a^2 = b^2 = bcd = (ad)^4 = (adacac)^4 = 1,\ a^t = aca,\ b^t = d,\ c^t = b,\ d^t = c \rangle.$$

This group was provided as an example of a finitely presented amenable group that is not elementary amenable [Gri98]. The group $\widetilde{\Gamma}$ is an ascending HNN-extension over the first Grigorchuk group $\Gamma = \langle a, b, c, d \rangle$, introduced in [Gri80], which, among other remarkable properties, is the first known example of a group with intermediate growth [Gri84].

Proposition 5.18. *The group $\widetilde{\Gamma}$ is isolated.*

Proof. Sapir and Wise [SW02] have proved that every proper quotient of $\widetilde{\Gamma}$ is metabelian. Therefore every normal subgroup $N \neq 1$ contains the second derived subgroup of $\widetilde{\Gamma}$, which is not the trivial group since $\widetilde{\Gamma}$ itself is not metabelian. □

This result contradicts a conjecture of Stepin, appearing in [Gri98, §1] too, which states that the class of elementary amenable finitely generated groups is dense in that of amenable finitely generated groups.

5.8. *Deligne's central extension*

Deligne [Del78] has shown that there exists a central extension

$$1 \longrightarrow Z \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1,$$

where Γ is a subgroup of finite index in $\mathrm{PSp}_{2n}(\mathbf{Z})$, the group $\widetilde{\Gamma}$ is its preimage in the universal covering $\widetilde{\mathrm{Sp}_{2n}(\mathbf{R})}$ of $\mathrm{PSp}_{2n}(\mathbf{R})$, and Z is infinite cyclic, that satisfies the following remarkable property: every finite index subgroup of $\widetilde{\Gamma}$ contains Z. By the Kazhdan–Margulis theorem [Zim84, Theorem 8.12] every non-trivial normal subgroup of Γ has finite index. It follows that every normal subgroup of $\widetilde{\Gamma}$ either is contained in Z or contains Z (and has finite index in $\widetilde{\Gamma}$). Thus $Z - \{1\}$ is a discriminating subset for $\widetilde{\Gamma}$. This is infinite, but becomes finite after taking a proper quotient of the center. Thus we obtain:

Proposition 5.19. *For every $n \geqslant 2$, the group $\widetilde{\Gamma}/nZ$ is isolated.*

This shows that a (finite central) extension of an infinite residually finite group can be isolated; moreover, they are lattices in the non-linear simple Lie group with finite center $\widetilde{\mathrm{Sp}_{2n}(\mathbf{R})}/nZ$. Other similar examples appear in [Rag84], with Γ a cocompact lattice in $\mathrm{SO}(2,n)$ for $n \geqslant 3$. Erschler [Ers04] provides examples where Γ is the first Grigorchuk group; these are examples of finitely generated, finitely discriminable groups with intermediate growth.

6. Further developments

Let K be a compact space. By induction on ordinals, define $I_0(K)$ as the set of isolated points of K, and for $\alpha > 0$, define $I_\alpha(K)$ as the set of isolated points in $K - \bigcup_{\beta<\alpha} I_\beta(K)$; call $I_\alpha(K)$ the set of α-isolated points of K. Let $\mathrm{Cond}(K)$ denote the condensation points, that is the complement of the union $\bigcup_\alpha I_\alpha(K)$.

If K is metrizable, then the sequence $(I_\alpha(K))$ breaks off after a countable number of steps. In the case of the space of finitely generated groups (in which case we simply write $\mathbf{I}_\alpha$ and **Cond**), what is this number?

It might be interesting to study $\mathbf{I}_\alpha$ for small values of α. For instance, it is easy to check that $\mathbf{Z}^n \in \mathbf{I}_n$ for all n.

The study of **Cond** is also of interest. It is characterized by: a finitely generated group G is in **Cond** if and only if every neighborhood of G is uncountable. By the results of Champetier [Cha00] **Cond** contains all non-elementary hyperbolic groups. It can be showed that it also contains the wreath product $\mathbf{Z} \wr \mathbf{Z}$ and the first Grigorchuk group Γ.

A related variant of these definitions is the following: consider a group G (not necessarily finitely generated). We say that G is in the class $\mathbf{II}_\alpha$ if $G \in I_\alpha(\mathcal{G}(G))$. This holds for at most one α; otherwise say that G belongs to the class **ICond**. The new I stands for “Inner” or “Intrinsic.” Note that, if we restrict to finitely generated groups, we have $\mathbf{ICond} \subset \mathbf{Cond}$, and every group in $\mathbf{I}_\alpha$ belongs to $\mathbf{II}_\beta$ for some $\beta \leqslant \alpha$. Note also that the two classes coincide in restriction to finitely presented groups, but, for instance, the first Grigorchuk group Γ belongs to $\mathbf{II}_1$.

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