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INJECTIVE CLASSES OF MODULES

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We study classes of modules over a commutative ring which allow to do homological algebra relative to such a class. We classify those classes consisting of injective modules by certain subsets of ideals. When the ring is Noetherian the subsets are precisely the generization closed subsets of the spectrum of the ring.

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0. Introduction

Homological algebra is in part the art of replacing a module, or more generally a chain complex, by an equivalent chain complex whose entries are either injective or projective modules. What happens if one changes the meaning of the word injective? It is well known since the seminal work [5] of Eilenberg and Moore that each notion of injective (or projective) determines, and is determined by, a corresponding notion of exactness. This has been a very fruitful generization, leading for instance to the development of "pure homological algebra", a subject which started maybe with the work of Warfield [15], or the possibility to work with flat covers and

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cotorsion envelopes, as proved by Bican, El Bashir, and Enochs in [1] for modules, and generalized by Enochs and Estrada to quasi-coherent sheaves in [6].

In this paper we investigate which classes of modules are fit to do homological algebra. We call them injective classes and concentrate on those which consist of injective modules. Our main result is a characterization and a description of such classes in terms of ideals.

Theorem 3.7. *Let R be a commutative ring.*

(1) *A collection of injective modules is an injective class if and only if it is closed under retracts and products.*
(2) *Injective classes of modules are in one-to-one correspondence with saturated sets of ideals of R.*

When the ring is Noetherian saturated sets of ideals consist precisely of subsets of Spec(R) closed under generization, see Corollary 3.1. In Sec. 1 we set up the notation and recall classical examples of injective classes. In Sec. 2 we study "saturated sets" of ideals. We prove then our main result Theorem 3.1 in Sec. 3. This classification provides examples of injective classes with which one would like to do relative homological algebra. In particular, it is a straightforward consequence that the category $\mathrm{Ch}(R)^{\geq 0}$ of cochain complexes concentrated in degrees ≥ 0 is endowed with a model category structure where weak equivalences are detected by injective envelopes of modules of the form R/I where I belongs to a given saturated set of ideals, see Theorem 3.2.

It turns out that such subsets of ideals appear already in one form or another in areas related to module theory. They are closely related to *hereditary torsion theories* as defined by Golan [8], and thus to linear topologies on a ring as considered by Gabriel in [7]. A common feature of the classification of such objects is that it relies on understanding the cyclic modules they detect. As a consequence our classification by subsets of ideals is not unexpected and it could certainly be obtained by translating our point of view to that of torsion theories or Gabriel topologies. More generally one might use classification results of certain subcategories of the module categories of a ring as exposed in Krause [10] or Prest [14]. Nevertheless, on the one hand we believe that our methods give a nice and simple description of injective classes, even in the non-Noetherian case, to which a reader can handily refer to; and on the other hand, they are well adapted to our homotopically minded applications in [3], where we illustrate how classifying injective classes as a whole is not enough to be able to effectively replace an *unbounded* chain complex by an injective resolution for a particular choice of injective class.

1. Injective Classes of Modules

In this section we will recall the notion of injective classes and provide classical examples.

Definition 1.1. Let $\mathcal{I}$ be a collection of R-modules. A homomorphism $f : M \to N$ is an *$\mathcal{I}$-monomorphism* if, for any $W \in \mathcal{I}$, $f^* : \mathrm{Hom}_R(N,W) \to \mathrm{Hom}_R(M,W)$ is a surjection of sets. We say that R-Mod has *enough $\mathcal{I}$-injectives* if, for any object M, there is an $\mathcal{I}$-monomorphism $M \to W$ with $W \in \mathcal{I}$.

Denote by $\overline{\mathcal{I}}$ the class of retracts of arbitrary products of elements of $\mathcal{I}$. A morphism is an $\mathcal{I}$-monomorphism if and only if it is an $\overline{\mathcal{I}}$-monomorphism. We will thus require that $\mathcal{I}$ be closed under retracts and products so that $\mathcal{I} = \overline{\mathcal{I}}$ and we define the following.

Definition 1.2. A collection $\mathcal{I}$ of R-modules is an *injective class* if it is closed under retracts and products and R-Mod has enough $\mathcal{I}$-injectives.

It should be pointed out that general products have considerably more retracts than direct sums, see Proposition 3.1. It is clear that a composite of $\mathcal{I}$-monomorphisms is again an $\mathcal{I}$-monomorphism. We say that a morphism f has a retraction, if there is a morphism r such that $rf = \mathrm{id}$. Any morphism that has a retraction is an $\mathcal{I}$-monomorphism, for any class $\mathcal{I}$, and in fact these are the only morphisms that are $\mathcal{I}$-monomorphisms for every class $\mathcal{I}$. Observe also that $\mathcal{I}$-monomorphisms are preserved under base change: if $f : M \to N$ is an $\mathcal{I}$-monomorphism, then, by the universal property of a push-out, so is its push-out along any morphism $M \to M'$. By the same argument an arbitrary coproduct of $\mathcal{I}$-monomorphisms is also an $\mathcal{I}$-monomorphism. In general however, limits and products of $\mathcal{I}$-monomorphisms fail to be $\mathcal{I}$-monomorphisms (see Example 1.2).

Example 1.1. If $\mathcal{I}$ consists of all R-modules, then $\mathcal{I}$-monomorphisms are morphisms $f : M \to N$ that have retractions (there is $r : N \to M$ such that $rf = \mathrm{id}_M$). It is clear that there are enough $\mathcal{I}$-injectives. This is the biggest injective class.

The collection $\mathcal{I}$ of all injective modules is an injective class and $\mathcal{I}$-monomorphisms are the ordinary monomorphisms.

The use of adjoint functors to provide new injective classes has proved to be fruitful and goes back at least to Eilenberg–Moore [5]. The following proposition is just a reformulation of their Theorem 2.1.

Proposition 1.1. *Let $l : S$-Mod $\leftrightarrows R$-Mod $: r$ be a pair of functors such that l is left adjoint to r. Let $\mathcal{I}$ be a collection of R-modules.*

(1) *A morphism f in S-Mod is an $r(\mathcal{I})$-monomorphism if and only if lf is an $\mathcal{I}$-monomorphism in R-Mod.*
(2) *If $lM \rightarrowtail W$ is an $\mathcal{I}$-monomorphism in R-Mod, then its adjoint $M \to rW$ is an $r(\mathcal{I})$-monomorphism in S-Mod.*
(3) *If there are enough $\mathcal{I}$-injectives in R-Mod, then there are enough $r(\mathcal{I})$-injectives in S-Mod.*
(4) *If $\mathcal{I}$ is an injective class in R-Mod, then the collection of retracts of objects of the form $r(W)$, for $W \in \mathcal{I}$, is an injective class in S-Mod.*

At the end of the section we now turn to some classical examples. Other injective classes will be described and studied more thoroughly in the next two sections.

Example 1.2. Let $S \to R$ be a ring homomorphism. The forgetful functor R-Mod $\to$ S-Mod is left adjoint to $\mathrm{Hom}_S(R,-) : S\text{-Mod} \to R\text{-Mod}$. We can then apply Proposition 1.1 to the injective classes in R-Mod given in Example 1.1 to get new injective classes in S-Mod. Hence, the collection of R-modules which are retracts of modules of the form $\mathrm{Hom}_S(R,N)$, for all S-modules N, is an injective class of R-modules. This class was originally considered by Hochschild in his foundational paper [9]. Similarly, the collection of R-modules which are retracts of modules of the same form $\mathrm{Hom}_S(R,N)$, but for all injective S-modules N, is also an injective class of R-modules.

Assume now that S is a commutative ring and $S \to R$ is an S-algebra (i.e. the image of this ring homomorphism lies in the center of R). The forgetful functor R-Mod $\to$ S-Mod is right adjoint to $R \otimes_S - : S\text{-Mod} \to R\text{-Mod}$. Thus, again by Proposition 1.1, both the collection of S-linear summands of R-modules and the collection of S-linear summands of all injective R-modules form injective classes of S-modules. A monomorphism relative to the first collection is a homomorphism f for which $f \otimes_S R$ is a split monomorphism. A monomorphism relative to the second collection is a homomorphism f for which $f \otimes_S R$ is a monomorphism. In general, monomorphisms with respect to these classes are not preserved by infinite products. Consider the rational numbers as an algebra over the integers $\mathbf{Z} \to \mathbf{Q}$. Let $\mathcal{I}$ be the class of abelian groups consisting of $\mathbf{Q}$-vector spaces. Then, for any prime number p, $\mathbf{Z}/p \to 0$ is an $\mathcal{I}$-monomorphism. However the product $\prod_p \mathbf{Z}/p \to 0$ is not an $\mathcal{I}$-monomorphism since, for instance, the diagonal map gives an inclusion $\mathbf{Z} \hookrightarrow \prod_p \mathbf{Z}/p$.

2. Saturated Sets of Ideals in a Commutative Ring

Let R be a commutative ring. Our aim is to describe injective classes in R-Mod which consist in injective modules. In Sec. 3 we are going to enumerate them by certain sets of ideals in R called saturated. In this section we discuss such sets of ideals.

Definition 2.1. We say that a set of ideals $\mathcal{L}$ in R is *saturated* if it consists of proper ideals and is closed under the following operations:

(1) an intersection of ideals in $\mathcal{L}$ belongs to $\mathcal{L}$;
(2) if $I \in \mathcal{L}$, then, for any $r \in R\backslash I$, $(I : r) = \{s \in R \mid sr \in I\}$ also belongs to $\mathcal{L}$;
(3) if a proper ideal J has the property that, for any $r \in R\backslash J$, there is $I \in \mathcal{L}$ such that $(J : r) \subset I$, then $J \in \mathcal{L}$.

Example 2.1. In the ring of integers $\mathbf{Z}$ the class of ideals (p^n) for a fixed prime p and all $n \geq 1$ forms a saturated class. In the ring of dual numbers $k[x]/(x^2)$ over a field k the maximal ideal (x) forms a saturated class by itself.

Conditions (2) and (3) of Definition 2.1 can be phrased in terms of submodules and homomorphisms if we identify an ideal I with the cyclic module R/I (a module generated by one element whose annihilator is given by I). A set of ideals $\mathcal{L}$ satisfies (2) if the annihilator of any non-trivial and cyclic submodule of R/I belongs to $\mathcal{L}$. Condition (3) is equivalent to the following: if, for any non-trivial and cyclic submodule of R/J, there is a non-zero homomorphism to R/I for some $I \in \mathcal{L}$, then $J \in \mathcal{L}$.

Definition 2.2. Let $\mathcal{L}$ be a given set of ideals in R. The *saturated set of ideals generated by* $\mathcal{L}$ is the smallest saturated set of ideals containing $\mathcal{L}$. We write simply $\mathrm{Sat}(I)$ to denote the saturated set of ideals generated by the collection consisting of one ideal I.

Note that the intersection of saturated sets of ideals is again saturated. In particular $\mathrm{Sat}(\mathcal{L})$ is the intersection of all saturated sets of ideals containing $\mathcal{L}$.

Proposition 2.1. *Let I be an ideal in R. The set* $\mathrm{Sat}(I)$ *consists of intersections of ideals J with the following property: for any $r \in R\backslash J$, there is $s \in R\backslash I$ such that $(J : r) \subset (I : s)$.*

Proof. Let S be the set of such intersections. The inclusion $S \subset \mathrm{Sat}(I)$ is a consequence of the fact that $\mathrm{Sat}(I)$ satisfies condition (3) of Definition 2.1. To prove the statement we need to show that S is saturated. By definition S is closed under intersections.

Let $J \in S$ and $t \in R\backslash J$. By definition of S and since for any $r \in R\backslash(J : t)$, the element rt is not in J, there is $s \in R\backslash I$, so that $((J : t) : r) = (J : rt) \subset (I : s)$. This proves that condition (2) of Definition 2.1 holds for S.

Finally to prove that S satisfies condition (3) of Definition 2.1, let J be a proper ideal such that, for any $r \in R\backslash J$, $(J : r)$ is included in some ideal $J_1 \in S$. By definition of S, there is $s \in R\backslash I$ so that $(J : r) \subset J_1 = (J_1 : 1) \subset (I : s)$. This means that $J \in S$. □

Proposition 2.2. *With the inclusion relation the saturated sets of ideals in R form a complete lattice.*

Proof. Let $\{\mathcal{L}_s\}_{s\in S}$ be a collection of saturated sets of ideals in R. The supremum among the saturated sets which are contained in $\mathcal{L}_s$ for any $s \in S$ is given by the intersection $\bigcap_{s\in S} L_s$. The infimum among saturated sets containing $\mathcal{L}_s$ for any $s \in S$ is given by $\mathrm{Sat}(\bigcup_{s\in S} \mathcal{L}_s)$. □

Ultimately we would like to be able to describe the lattice of saturated sets of ideals in terms of properties of the ring R. For example, for a Noetherian R, this lattice can be identified, as we show next, with the inclusion lattice of subsets of the prime spectrum $\mathrm{Spec}(R)$ of R closed under generization. Recall that a set of primes S in R is *closed under generization* if, for $p \in S$, and q is a prime ideal contained in p, then $q \in S$. Arbitrary intersections and unions of such subsets are also closed under generization. It follows that, with the inclusion relation, the collection of subsets of $\mathrm{Spec}(R)$ closed under generization is a complete lattice. The Noetherian assumption is used here to ensure that any ideal has at least one associated prime ideal.

Lemma 2.1. *Let R be a Noetherian ring.*

(1) *For a set of prime ideals $\wp$ in R, $\mathrm{Sat}(\wp)$ consists of ideals whose associated primes are sub-ideals of ideals in $\wp$.*
(2) *If $\mathcal{L}$ is a saturated set of ideals in R, then $\mathcal{L} = \mathrm{Sat}(\mathrm{Spec}(R) \cap \mathcal{L})$.*

Proof. (1) Set $\mathcal{L}$ to be the collection of ideals whose associated primes are sub-ideals of ideals in $\wp$. We first show that $\mathcal{L}$ is saturated. This is basically a consequence of two facts: all associated primes to an ideal I are of the form $(I : r)$ for some $r \in R\backslash I$, and maximal ideals of the form $(I : r)$, for $r \in R\backslash I$, are primes associated to I. The first fact implies that $\mathcal{L}$ satisfies condition (3) of Definition 2.1. Condition (2) is a consequence of the fact that, for $r \in R\backslash I$, the associated primes of $(I : r)$ are among the associated primes of I. It remains to show that $\mathcal{L}$ is closed under intersections (condition (1)). Let $\{I_s\}$ be a family of ideals in $\mathcal{L}$. An associated prime p to $\bigcap_s I_s$ is of the form $(\bigcap_s I_s : r)$, for some $r \in R\backslash \bigcap_s I_s$. Now $p = (\bigcap_s I_s : r)$ is contained in $(I_s : r)$, for any s such that $r \notin I_s$ and $(I_s : r)$ is a subideal of an associated prime to I_s, which by definition is a subideal of an ideal in $\wp$. The ideal $\bigcap_s I_s$ is thus a member of $\mathcal{L}$. Since $\wp \subset \mathcal{L}$ and $\mathcal{L}$ is saturated, we have $\mathrm{Sat}(\wp) \subset \mathcal{L}$.

To show the opposite inclusion $\mathcal{L} \subset \mathrm{Sat}(\wp)$, let $I \in \mathcal{L}$. We are going to use condition (3) of Definition 2.1 to show that $I \in \mathrm{Sat}(\wp)$. For any $r \in R\backslash I$, the ideal $(I : r)$ is a subideal of an associated prime p which, by definition of $\mathcal{L}$, is a subideal of an ideal in $\wp \subset \mathrm{Sat}(\wp)$. Condition (3) implies then that $I \in \mathrm{Sat}(\wp)$.

(2) The inclusion $\mathrm{Sat}(\mathrm{Spec}(R) \cap \mathcal{L}) \subset \mathcal{L}$ follows from the fact that $\mathcal{L}$ is saturated. Let $I \in \mathcal{L}$. For any $r \in R\backslash I$, $(I : r)$ is a sub-ideal of an associated prime $p = (I : s)$ to I. As $p \in \mathcal{L}$ (condition (2) of Definition 2.1), $p \in \mathrm{Sat}(\mathrm{Spec}(R) \cap \mathcal{L})$. We can then use condition (3) of Definition 2.1 to conclude that $I \in \mathrm{Sat}(\mathrm{Spec}(R) \cap \mathcal{L})$. ☐

Definition 2.3. For any collection $\mathcal{L}$ of ideals in R, the *spectral part* $\mathrm{Sp}(\mathcal{L})$ of $\mathcal{L}$ is the intersection $\mathit{Spec}(R) \cap \mathcal{L}$ of this collection with the prime spectrum of R.

Proposition 2.3. *Assume that R is Noetherian. Then the association $\mathcal{L} \mapsto \mathrm{Sp}(\mathcal{L})$ is an order preserving isomorphism between the inclusion lattice of saturated sets of ideals in R and the inclusion lattice of subsets of $\mathrm{Spec}(R)$ which are closed under generization.*

Proof. We first prove that, if $\mathcal{L}$ is saturated, then $\mathrm{Sp}(\mathcal{L})$ is closed under generization so the association Sp is well defined. To see this, it is enough to observe that if $q \in \mathcal{L}$ then, for any prime $p \subset q$, $p \in \mathcal{L}$. This follows from condition (3) of Definition 2.1 and the fact that, for a prime ideal p, $(p : r) = p$, for any $r \in R\backslash p$.

Let S be a set of prime ideals in R closed under generization. According to Lemma 2.1, $\mathrm{Sat}(S)$ consists of those ideals whose associated primes are sub-ideals of primes that belong to S. As S is closed under generization, $\mathrm{Sat}(S)$ consists simply of those ideals whose associated primes do belong to S.

As both Sp and Sat are order preserving, to show that Sp is an isomorphism of lattices, it is enough to prove that, for a saturated set of ideals $\mathcal{L}$ and a subset S of $\mathrm{Spec}(R)$ closed under generization, $\mathrm{Sat}(\mathrm{Sp}(\mathcal{L}))=\mathcal{L}$ and $\mathrm{Sp}(\mathrm{Sat}(S))=S$. First, by definition $\mathrm{Sat}(\mathrm{sp}(\mathcal{L}))=\mathrm{Sat}(\mathrm{Spec}(R)\cap\mathcal{L})$. We can now use Lemma 2.1(2) to conclude that $\mathrm{Sat}(\mathrm{sp}(\mathcal{L}))=\mathcal{L}$.

Second, the definition of Sp says that $p \in \mathrm{Sp}(\mathrm{Sat}(S))$ if and only $p \in \mathrm{Sat}(S)$. This is equivalent to p having an associated prime that belongs to S. However p has only p as an associated prime and thus $p \in \mathrm{Sp}(\mathrm{Sat}(S))$ if and only if $p \in S$. □

When the ring R is not Noetherian, there can be considerably more saturated sets of ideals in R than subsets of $\mathrm{Spec}(R)$ closed under generization.

Example 2.2. Let k be a field and $R = k[X_1, X_2, \ldots]/(X_1^2, X_2^2, \ldots)$. This ring and its Bousfield classes have been studied recently by Dwyer and Palmieri in [4]. It is also very similar to Neeman's original example of a ring with a single prime ideal but uncountably many Bousfield classes, [13]. We are going to show that this ring has uncountably many saturated sets of ideals.

Any element in R can be written uniquely as a k-linear combination of monomials of the form $X_{i_1}X_{i_2}\cdots X_{i_k}$. For any subset $S \subset \mathbf{N}$, we denote the ideal $(X_i)_{i\in S}$ by I_S. Recall that the symmetric difference of two subsets $S, T \subset \mathbf{N}$ is given by $S\Delta T := (S \cup T)\backslash(S \cap T)$. We claim that if the symmetric difference of S and T is infinite then $\mathrm{Sat}(I_S)$ and $\mathrm{Sat}(I_T)$ are different saturated sets of ideals in R.

Notice first that for $k \notin S$ the ideal $(I_S : x_k)$ is $I_{S\cup\{k\}}$. This implies that if S and T differ by only finitely many natural numbers, then $\mathrm{Sat}(I_S) = \mathrm{Sat}(I_T)$.

Let $f \in R\backslash I_S$. This element f is a sum of monomials that involves only finitely many variables X_i. Let F be the finite set of indexes of these variables. We first show that $(I_S : f) \subset I_{S\cup F}$. Let $g \in (I_S : f)$. We can write $g = g_1+g_2$, where g_1 is a sum of monomials for which the index of at least one of the variables belongs to $S\cup F$ and g_2 is a sum of monomials that are products of variables whose indexes do not belong to $S\cup F$. Thus, since $g_1 \in I_{S\cup F}$, the element g belongs to $I_{S\cup F}$ if and only if g_2 does. As the variables occurring in g_2 are different from the variables occurring in f, the products of monomials of f and monomials of g_2 are linearly independent over k. However fg_2 can be written as $fg - fg_1$, where $fg \in I_S$. Thus fg_2 can be written as a k-linear combination of monomials that are products of variables for which at least one index belongs to $S \cup F$. Since monomials are linearly independent over

k, if $g_2 \neq 0$, the monomials of f are products of variables where at least one index is in S. That would however imply that $f \in I_S$, which contradicts our assumption that $f \in R\backslash I_S$. It follows that $g_2 = 0$ and hence $g = g_1 \in I_{S\cup F}$.

Assume now that the symmetric difference of S and T is infinite. This happens if either $S\backslash T$ or $T\backslash S$ is infinite. Assume that $S\backslash T$ is infinite. In this case we claim that $I_S \notin \mathrm{Sat}(I_T)$. Otherwise, according to Proposition 2.1, there would be some $f \in R\backslash I_T$, for which $I_S \subset (I_T : f)$. This is however impossible, since we have just proved that $(I_T : f) \subset I_{T\cup F}$ for some finite F.

To show that the ring R has uncountably many saturated sets of ideals, it is enough to exhibit an uncountable family of subsets of $\mathbf{N}$ for which the symmetric difference of any two is infinite. Let us choose a partition of $\mathbf{N}$ by countably many infinite subsets $A_1, A_2, \ldots$ For example A_1 could be the subset of odd numbers, A_2 the subset of those even numbers which are not divisible by 4, and A_n the subset of numbers divisible by 2^{n-1} but not by 2^n. For any subset X of $\mathbf{N}$ define S_X to be the subset of those numbers which belong to A_x for some $x \in X$. In other words $S_X = \bigcup_{x\in X} A_x$. Since any A_n is infinite, the symmetric difference between any two subsets of the form S_X must be infinite. Finally, as the set of subsets of $\mathbf{N}$ is uncountable, we have exhibited uncountably many saturated sets of ideals.

3. Injective Subclasses of Injective Modules

Let R be a commutative ring. In this section we are going to classify injective classes of R-modules that consists of injectives. We show that there is a bijection between such classes and the collection of saturated sets of ideals in R. In particular we will show that there is a set of such injective classes.

For an R-module M, let $E(M)$ be the injective envelope of M. Here are some basic properties of the injective envelope of a cyclic module.

Lemma 3.1. *Let I be an ideal in R, $E(R/I)$ the injective envelope of R/I, M an R-module, and $f : M \to N$ an R-module homomorphism.*

(1) $\mathrm{Hom}_R(M, E(R/I)) = 0$ *if and only if, for any $m \in M$ and any $r \in R\backslash I$,* $\mathrm{ann}(m) \not\subset (I : r)$.
(2) $\mathrm{Hom}_R(f, E(R/I))$ *is an epimorphism if and only if* $\mathrm{Hom}_R(\ker f, E(R/I)) = 0$.
(3) $\mathrm{Hom}_R(f, E(R/I))$ *is an monomorphism if and only if the set of homomorphisms* $\mathrm{Hom}_R(\operatorname{coker} f, E(R/I)) = 0$.

Proof. (1) If $\mathrm{ann}(m) \subset (I : r)$, for some $m \in M$ and $r \in R\backslash I$, then the composition of the surjection $R/\mathrm{ann}(m) \to R/(I : r)$, the multiplication by $r : R/(I : r) \to R/I$, and the inclusion $R/I \subset E(R/I)$ can be extended to a non-trivial homomorphism $M \to E(R/I)$. Consequently $\mathrm{ann}(m) \not\subset (I : r)$ for any $m \in M$ and any $r \in R\backslash I$.

Assume now that there is a non-zero homomorphism $f : M \to E(R/I)$. Then for some $m \in M$, $0 \neq f(m) \in R/I \subset E(R/I)$. Let $r \in R \backslash I$ be such that $[r] = f(m)$. It is then clear that $\mathrm{ann}(m) \subset (I : r)$.

(2) and (3) Since $E(R/I)$ is injective, we have the following isomorphisms of abelian groups:

$$\ker(\mathrm{Hom}_R(f, E(R/I))) \cong \mathrm{Hom}_R(\mathrm{coker}(f), E(R/I)),$$

$$\mathrm{coker}(\mathrm{Hom}_R(f, E(R/I))) \cong \mathrm{Hom}_R(\ker(f), E(R/I)).$$

Statements (2) and (3) clearly follow. □

Definition 3.1. For a set of ideals $\mathcal{L}$ in R, we denote by $E(\mathcal{L})$ the collection of retracts of products of injective envelopes $E(R/I)$ where $I \in \mathcal{L}$.

The following proposition tells us that $E(\mathcal{L})$ is the smallest injective class generated by the set $\mathcal{L}$.

Proposition 3.1. *Let $\mathcal{L}$ be a set of ideals in R.*

(1) *An R-module homomorphism $f : M \to N$ is an $E(\mathcal{L})$-monomorphism if and only if* $\mathrm{Hom}_R(\ker f, E(R/I)) = 0$ *for any $I \in \mathcal{L}$.*
(2) *The collection $E(\mathcal{L})$ is an injective class.*

Proof. (1) This follows from Lemma 3.1(2).

(2) We need to show that R-Mod has enough $E(\mathcal{L})$-injectives. Let M be an R-module. Let S be the set of elements $m \in M$ such that there is a non-trivial homomorphism $R/\mathrm{ann}(m) \to E(R/I)$, for some $I \in \mathcal{L}$. For any $m \in S$ let us choose an ideal $I_m \in \mathcal{L}$ and a non-trivial homomorphism $\phi_m : M \to E(R/I_m)$ such that $\phi(m) \neq 0$. The existence of such a homomorphism is guaranteed by that of a non-trivial homomorphism $R/\mathrm{ann}(m) \to E(R/I_m)$ and the injectivity of $E(R/I_m)$. We claim that the following homomorphism is an $E(\mathcal{L})$-monomorphism:

$$\prod_{m\in S} \phi_m : M \to \prod_{m\in S} E(R/I_m).$$

The kernel K of this homomorphism consists of elements m for which there is no non-trivial homomorphisms from $R/\mathrm{ann}(m)$ to $E(R/I)$ for any $I \in \mathcal{L}$. This means that K is in the kernel of any homomorphism $f : M \to E(R/I)$, where $I \in \mathcal{L}$. Using now the injectivity of $E(R/I)$, we see that any such homomorphism f factors through $\prod_{m\in S} \phi_m$. By definition, this means that $\prod_{m\in S} \phi_m$ is an $E(\mathcal{L})$-monomorphism. □

Definition 3.2. For a collection of R-modules $\mathcal{I}$, define $L(\mathcal{I})$ to be the set of ideals of the form $\mathrm{ann}(x) = \{r \in R \,|\, rx = 0\}$ for some non-zero element $x \in W \in \mathcal{I}$.

Thus $L(\mathcal{I})$ consists of those proper ideals I for which there is an inclusion $R/I \subset W$ for some $W \in \mathcal{I}$.

Lemma 3.2. *If $\mathcal{I}$ is a collection of injective modules which is closed under products, then $L(\mathcal{I})$ is saturated.*

Proof. We need to check the three conditions of Definition 2.1. Let $\{I_s\}_{s\in S}$ be a set of ideals in $L(\mathcal{I})$. For any s choose $W_s \in \mathcal{I}$ such that $R/I_s \subset W_s$. Then $R/(\bigcap_{s\in S} I_s)$ is a submodule of $\prod_{s\in S} R/I_s$ and hence a submodule of $\prod_{s\in S} W_s$. As $\mathcal{I}$ is assumed to be closed under products, $\bigcap_{s\in S} I_s \in L(\mathcal{I})$.

If R/I is a submodule of some $W \in \mathcal{I}$, then so is $R/(I : r)$, as $R/(I : r)$ is a submodule of R/I generated by r. Condition (2) of Definition 2.1 is then satisfied.

To prove condition (3), assume J has the property that, for any $r \in R\backslash J$, there is $I_r \in L(\mathcal{I})$ such that $(J : r) \subset I_r$. Let $W_r \in \mathcal{I}$ be a module for which $R/I_r \subset W_r$. Set $\phi_r : R/J \to W_r$ to be a homomorphism which restricted to the submodule $R/(J : r) \subset R/J$, generated by r, is given by the composition $R/(J : r) \to R/I_r \subset W_r$, where the first homomorphism is given by the quotient induced by the inclusion $(J : r) \subset I_r$. Such a homomorphism ϕ_r exists since W_r is an injective module. The product $\prod_{r\in R\backslash J} \phi_r : R/J \to \prod_{r\in R\backslash J} W_r$ is then a monomorphism. As $\mathcal{I}$ is closed under products, it follows that $J \in L(\mathcal{I})$. □

Proposition 3.2. (1) *If $\mathcal{L}$ is saturated, then $\mathcal{L} = L(E(\mathcal{L}))$.*

(2) *If $\mathcal{I}$ is a collection of injective modules which is closed under retracts and products, then $\mathcal{I} = E(L(\mathcal{I}))$.*

Proof. (1) The inclusion $\mathcal{L} \subset L(E(\mathcal{L}))$ follows from the fact that, for any ideal I, R/I is a submodule of its envelope $E(R/I)$.

To show the other inclusion $L(E(\mathcal{L})) \subset \mathcal{L}$ we need the assumption that $\mathcal{L}$ is saturated. Let $J \in L(E(\mathcal{L}))$. This means that $J = \operatorname{ann}((x_s)_{s\in S})$ for some element $(x_s)_{s\in S}$ in $\prod_{s\in S} E(R/I_s)$, where $I_s \in \mathcal{L}$. Thus $J = \bigcap_{s\in S} \operatorname{ann}(x_s)$. Since $\mathcal{L}$ is saturated, to show that $J \in \mathcal{L}$, it is enough to prove that if $I \in \mathcal{L}$, then $\operatorname{ann}(x) \in \mathcal{L}$ for any non-trivial $x \in E(R/I)$. Consider an envelope $R/I \subset E(R/I)$. For any $r \notin \operatorname{ann}(x)$, the intersection of R/I with the submodule of $E(R/I)$ generated by rx is non-trivial. Thus there is $t \notin \operatorname{ann}(rx)$, so that $trx \in R/I$. It is then clear that $(\operatorname{ann}(x) : r) = \operatorname{ann}(rx) \subset \operatorname{ann}(trx) = (I : trx)$. As this happens for any $r \notin \operatorname{ann}(x)$, we can use condition (3) in Definition 2.1 to conclude that $\operatorname{ann}(x) \in \mathcal{L}$.

(2) If $I \in L(\mathcal{I})$, then $R/I \subset W$ for some $W \in \mathcal{I}$. As $\mathcal{I}$ consists of injective modules, then $E(R/I) \subset W$ and consequently $E(R/I)$ is a retract of W. Since $\mathcal{I}$ is closed under retracts, $E(R/I)$ belongs to $\mathcal{I}$. It is then clear that $E(L(\mathcal{I})) \subset \mathcal{I}$.

Let $W \in \mathcal{I}$. For any $0 \neq w \in W$, let $\phi_w : W \to E(R/\operatorname{ann}(w))$ be a homomorphism which restricted to the submodule generated by w is given by the inclusion in its injective envelope $R/\operatorname{ann}(w) \subset E(R/\operatorname{ann}(w))$. The existence of such a homomorphism is guaranteed by the injectivity of $E(R/\operatorname{ann}(w))$. The product $\prod \phi_w : W \to \prod E(R/\operatorname{ann}(w))$ is then a monomorphism. As W is injective, it is a retract of $\prod E(R/\operatorname{ann}(w))$. However $E(R/\operatorname{ann}(w)) \in E(L(\mathcal{I}))$, for any w. This implies that $W \in E(L(\mathcal{I}))$ and shows the inclusion $\mathcal{I} \subset E(L(\mathcal{I}))$. □

We are now ready to show that there is a bijective correspondence between injective classes consisting of injective modules and saturated classes of ideals.

Theorem 3.1. (1) *A collection of injective modules is an injective class if and only if it is closed under retracts and products.*
(2) *The following operations are inverses bijections*:

$$\begin{tikzcd} \textit{Injective classes of } R\textit{-modules consisting of injective } R\textit{-modules} \arrow[d, bend right, "L"'] \\ \textit{Saturated sets of ideals in } R \arrow[u, bend right, "E"'] \end{tikzcd}$$

Proof. Any injective class is by definition closed under product and retracts. If $\mathcal{I}$ is closed under products and retracts, and it consists of injective modules, then, according to Proposition 3.2(2), $\mathcal{I} = E(L(\mathcal{I}))$. We can then use Proposition 3.1(2) to conclude that $\mathcal{I}$ is an injective class. This proves statement (1). Statement (2) is then a direct consequence of Proposition 3.2. □

Here is the kind of consequences we are interested in. One could construct by hand explicit factorizations, but we prefer to simply refer to the general work of Bousfield [2, Sec. 4.4] to prove that the category $\mathrm{Ch}(R)^{\geq 0}$ consisting of cochain complexes concentrated in non-negative degrees is endowed with a model structure determined by a saturated set of ideals.

Theorem 3.2. *Let $\mathcal{L}$ be a saturated set of ideals in R. The following choice of weak equivalences, cofibrations and fibrations in* $\mathrm{Ch}(R)^{\geq 0}$ *satisfies the axioms of a model category*:

- $f : X \to Y$ *is a weak equivalence if* $f^* : \mathrm{Hom}_R(Y, E(R/I)) \to \mathrm{Hom}_R(X, E(R/I))$ *is a quasi-isomorphism of complexes of abelian groups for any $I \in \mathcal{L}$.*
- $f : X \to Y$ *is a cofibration if $f^i : X^i \to Y^i$ is an $E(\mathcal{L})$-monomorphism for all $i > n$.*
- $f : X \to Y$ *is called a $\mathcal{W}$-fibration if $f^i : X^i \to Y^i$ has a section and its kernel belongs to $E(\mathcal{L})$ for all $i \geq n$.*

We end this section with a discussion of the Noetherian case. If R is Noetherian, then we have seen (see Lemma 2.1) that a saturated set of ideals in R is determined by the prime ideals it contains. Thus to describe an arbitrary injective class consisting of injective R-modules, we can start with a set of prime ideals $\wp$.

Lemma 3.3. *Let R be a Noetherian ring and $\wp$ be a set of prime ideals.*

(1) $E(\wp) = E(\mathrm{Sat}(\wp))$.
(2) *$f : M \to N$ is an $E(\wp)$-monomorphism if and only if, for any prime $p \in \wp$, $f_p : M_p \to N_p$ is a monomorphism.*

Proof. (1) The inclusion $E(\wp) \subset E(\mathrm{Sat}(\wp))$ is clear. To show the opposite inclusion, it is enough to show that $E(R/I) \in E(\wp)$, for any $I \in \mathrm{Sat}(\wp)$. Associated primes of such an ideal I are sub-ideals of ideals in $\wp$ (see Lemma 2.1(1)). For any $r \in R\backslash I$, we can then choose a prime $p_r \in \wp$ such that $(I : r) \subset p_r$. For any $r \in R\backslash I$, let $\phi_r : R/I \to E(R/p_r)$ be a homomorphism whose restriction to the submodule $R/(I : r)$, generated by r, is given by the composition of the quotient $R/(I : r) \to R/p_r$ and an inclusion $R/p_r \subset E(R/p_r)$. The existence of such a homomorphism is guaranteed by the injectivity of $E(R/p_r)$. Next, the homomorphism $\prod_{r\in R\backslash I} \phi_r : R/I \to \prod_{r\in R\backslash I} E(R/p_r)$ is a monomorphism. It follows that $E(R/I)$ is a retract of $\prod_{r\in R\backslash I} E(R/p_r)$ and hence belongs to $E(\wp)$.

(2) The homomorphism f is an $E(\wp)$-monomorphism if and only if for any prime $p \in \wp$ $\mathrm{Hom}(\ker(f), E(R/p))$ is zero, for any prime $p \in \wp$ (see Proposition 3.1(1)). Recall that $E(R/p)$ is an R_p-module. Thus $\mathrm{Hom}(\ker(f), E(R/p)) = 0$ if and only if $\mathrm{Hom}(\ker(f)_p, E(R/p)) = 0$. It follows that if f_p is a monomorphism ($\ker(f)_p = 0$), for any $p \in \wp$, then f is an $E(\wp)$-monomorphism. To show the reverse implication, let f be an $E(\wp)$-monomorphism and $p \in \wp$. If $\ker(f)_p$ were non-trivial, there would be an inclusion $R/I \subset \ker(f)$, for some $I \subset p$. The composition of the projection $R/I \to R/p$ and the inclusion $R/p \subset E(R/p)$ could be then extended over $\ker(f)$ to produce a non-trivial element in $\mathrm{Hom}(\ker(f), E(R/p))$, a contradiction. □

Corollary 3.1. *Let R be a Noetherian ring. The association $S \mapsto E(S)$ is a bijection between the collection of subsets of* $\mathrm{Spec}(R)$ *closed under generization and injective classes of R-modules consisting of injective R-modules.*

Proof. This is a consequence of Theorem 3.1(2) and Proposition 2.3. □

Remark 3.1. In [12] Neeman classified the smashing subcategories in the derived category of a Noetherian ring R, compare also with Krause's classification of thick subcategories of R-modules via subsets of $\mathrm{Spec}(R)$, [11]. Smashing subcategories are in one-to-one correspondence with the subsets of $\mathrm{Spec}(R)$ closed under specialization (the complements of the subsets closed under generization). Arbitrary localizing categories correspond to arbitrary subsets of the spectrum. The reason why we only see smashing ones is the following. The localizing subcategory we consider is that of $\mathcal{I}$-acyclic complexes, i.e. those complexes X such that $\mathrm{Hom}_R(X, W)$ is acyclic for any $W \in \mathcal{I}$. The main result in [12] tells us that this class is determined by the prime ideals q for which $k(q)$ is $\mathcal{I}$-acyclic, i.e. for which $\mathrm{Hom}_R(k(q), E(R/p)) = 0$ for all prime ideals p with $E(R/p) \in \mathcal{I}$. But if there is a non-zero homomorphism $k(q) \to E(R/p)$, then there is also one from $k(q')$ for any $q' \subset q$ (coming from the projection $R/q' \to R/q$). This shows that the subset $\{q \in \mathrm{Spec}(R) \,|\, k(q) \text{ is } \mathcal{I}\text{−acyclic}\}$ is closed under specialization.

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