Homotopy Pull-back Squares up to Localization

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ABSTRACT. We characterize the class of homotopy pull-back squares by means of elementary closure properties. The so called Puppe theorem which identifies the homotopy fiber of certain maps constructed as homotopy colimits is a straightforward consequence. Likewise we characterize the class of squares which are homotopy pull-backs "up to Bousfield localization". This yields a generalization of Puppe's theorem which allows us to identify the homotopy type of the localized homotopy fiber. When the localization functor is homological localization this is one of the key ingredients in the group completion theorem.

1. Introduction

In topology it is convenient to think about a continuous family of spaces as a map whose fibers constitute the family. The homotopy fiber of this map is then an important invariant of the family, but in general it is difficult to say anything about this invariant. However if the "transition functions" of the map preserve some property, it is often the case that the same property is inherited by the homotopy fiber. The classical example is given by the so called Puppe theorem [14] that says that if all the members of the family have the same homotopy type (the transition functions are weak equivalences), then the homotopy fiber has this homotopy type too. This is also the central idea in (generalized) Quillen's group completion theorem, as exposed by McDuff-Segal [12], Jardine [10], Tillman [16], see also Adams' book [1]. In their setting the members of the family have the same integral homology type (the transition functions are H$\mathbf{Z}$-isomorphisms). The statement asserts then that the homotopy fiber shares the same integral homology type as well. This was used to compute the homology of the group completion of certain topological monoids: a celebrated consequence is the Barratt–Priddy theorem [2] which identifies $B\Sigma_\infty^+ \times \mathbf{Z}$ with QS^0.

The aim of this paper is to generalize these results to the case when the members of a continuous family of spaces have the same homotopy type after Bousfield

localization with respect to a given map. It has been surprising to us that this statement follows from properly reformulating the classical Puppe theorem together with general properties of localizations of spaces.

To state our theorem it turns out that it is more appropriate to work in the category *Arrows* of maps of spaces (see Section 2.1) rather than in the category *Spaces* itself, where by a space we mean an unpointed simplicial set. Studying homotopy fibers of continuous families can then be translated into investigating properties of certain classes of morphisms in *Arrows*:

1.1. DEFINITION. A class $\mathcal{C}$ of morphisms in *Arrows* is called *distinguished* if:

(1) Weak equivalences belong to $\mathcal{C}$.
(2) Let $\phi : f \to g$ and $\psi : g \to h$ be morphisms. Assume that either ψ or ϕ is a weak equivalence. Then if two out of ϕ, ψ, $\psi\phi$ belong to $\mathcal{C}$, then so does the third.
(3) Let $\phi : f \to g$ and $\psi : g \to h$ be morphisms. If ψ and $\psi\phi$ belong to $\mathcal{C}$, then so does ϕ.
(4) If $F : I \to Arrows$ sends any morphism in I to a morphism in $\mathcal{C}$, then, for every $i \in I$, $F(i) \to \mathrm{hocolim}_I F$ belongs to $\mathcal{C}$.

At first it might seem pointless to consider such collections since for example the category *Spaces* has only one distinguished class that consists of all spaces. The key observation is that there are much more interesting distinguished collections in *Arrows*. For example a reformulation of Puppe's theorem now identifies the class of homotopy pull-backs.

6.2 THEOREM. *The collection of homotopy pull-backs is the smallest distinguished class of morphisms in Arrows.*

As in *Spaces*, Bousfield localization also exist in *Arrows*. Although it is very hard to identify these localization functors with respect to an arbitrary morphism, an explicit description can be given for L_ϕ when $\phi = (u, \mathrm{id}_{\Delta[0]})$ is a morphism between two maps collapsing a space to a point (see Section 2.1). In this case L_ϕ is written L_u and coincides with the "fiberwise" application of the Bousfield localization L_u of spaces (see Section 7). We say that a morphism $\psi : f \to g$ in *Arrows* is an L_u-homotopy pull-back if ψ induces L_u-equivalences of homotopy fibers of f and g (see Definition 8.1) or equivalently if $L_u\psi : L_uf \to L_ug$ is a homotopy pull-back (see Proposition 8.2). Our main theorem can be now stated as follows:

8.3 THEOREM. *The collection of L_u-homotopy pull-backs is the smallest distinguished class containing all L_u-equivalences.*

For example let us choose a map u for which L_u coincides with the localization with respect to a chosen homology theory. Let $F, G : I \to Spaces$ be functors and $\pi : F \to G$ be a natural transformation. Assume that for any morphism $\alpha \in I$, the commutative square:

$$
\begin{array}{ccc}
F(i) & \xrightarrow{F(\alpha)} & F(j) \\
{\scriptstyle \pi_i} \downarrow & & \downarrow {\scriptstyle \pi_j} \\
G(i) & \xrightarrow{G(\alpha)} & G(j)
\end{array}
$$

induces an homology isomorphism between the homotopy fibers of π_i and π_j (i.e. this square is an L_u-homotopy pull-back). Then, since L_u-homotopy pull-backs form a distinguished collection, according to condition (4) of Definition 1.1, the homotopy fibers of hocolim$_I\pi$ have the same homology type as the homotopy fibers of π_i for an appropriate i.

Acknowledgments: It was Emmanuel Dror Farjoun who suggested to view group completion as a fiberwise localization statement when we explained him the result in [**13**]. Unlike some other participants we were not able to prove our main theorem while hiking around Arolla. We thank the Göran Gustafssons Stiftelsethe for support and KTH in Stockholm for its hospitality.

2. The category of maps of spaces

In this section we deal with combinatorics and geometry of simplicial sets. We focus particularly on geometrical properties of push-outs and pull-backs.

2.1. The symbol *Arrows* denotes the category whose objects are maps in *Spaces* and morphisms are commutative squares. Explicitly a morphism $\phi : f \to g$ in *Arrows* is given by a pair $\phi = (\phi_0, \phi_1)$ of maps for which the following square commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\phi_0} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\phi_1} & B
\end{array}
$$

2.2. There are two forgetful functors $D, R : Arrows \to Spaces$ which assign to a map $f : X \to Y$ its domain $Df := X$ and its range $Rf := Y$. So for the morphism ϕ, $D\phi = \phi_0$ and $R\phi = \phi_1$, which should not be mistaken for the domain f and the range g of ϕ considered as a morphism in *Arrows*. A functor $F : I \to Arrows$ can be identified with a natural transformation $DF \to RF$ between functors with values in *Spaces*, denoted by πF. By the universal properties colim$_I F$ and lim$_I F$ are naturally isomorphic respectively to the maps colim$_I \pi F : $ colim$_I DF \to$ colim$_I RF$ and lim$_I \pi F : $ lim$_I DF \to$ lim$_I RF$.

2.3. A morphism $\phi : f \to g$ in *Arrows* is called a pull-back or a push-out if the corresponding square is so in *Spaces*. It is called a monomorphism if both $D\phi$ and $R\phi$ are so in *Spaces*.

Here is a list of basic properties of pull-backs and push-outs of spaces. One way of proving them is to show that they are true for sets and are preserved by functor categories, thus they remain valid for *Spaces*, *Arrows*, etc. The first property is the classical "two out of three" property, similar statements can be found for example in [**8**, Proposition 1.8].

2.4. LEMMA. *Let $\phi : f \to g$ and $\psi : g \to h$ be two morphisms in Arrows.*
(1) *Assume that ψ (respectively ϕ) is a pull-back (respectively a push-out). Then ϕ (respectively ψ) is a pull-back (respectively a push-out) if and only if $\psi\phi$ is so.*
(2) *Assume that ϕ is a pull-back and $R\phi : Rf \to Rg$ is an epimorphism. Then ψ is a pull-back if and only if $\psi\phi$ is so.*
(3) *Assume that ψ is a push-out and f, g, h are monomorphisms. Then $\psi\phi$ is a push-out if and only if ϕ is so.* □

The following examples show that the additional assumptions in points (2) and (3) above are essential.

2.5. EXAMPLE. Let S^0 denote the boundary of $\Delta[1]$. Here is a diagram in which the left square and the outer square are pull-backs, but the right square is not:

$$\begin{array}{ccccc} \emptyset & \longrightarrow & \Delta[0] & \xrightarrow{d^0} & S^0 \\ \downarrow & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1 s^0} \\ \Delta[0] & \xrightarrow{d^0} & S^0 & \xrightarrow{id} & S^0 \end{array}$$

Here is a diagram in which the right square and the outer square are push-outs, but the left square is not:

$$\begin{array}{ccccc} \Delta[0] & \longrightarrow & S^0 & \longrightarrow & \Delta[0] \\ \downarrow & & \downarrow & & \downarrow \\ \Delta[0] & \longrightarrow & \Delta[0] & \longrightarrow & \Delta[0] \end{array}$$

The next three properties are occurrences of push-out squares being at the same time pull-back squares. Consider a commutative diagram in *Arrows*:

(A)
$$\begin{array}{ccc} f & \xrightarrow{\phi} & g \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mu} \\ h & \xrightarrow{\psi} & k \end{array}$$

2.6. LEMMA.
(1) If $\phi : f \to g$ is a monomorphism and a push-out, then it is a pull-back.
(2) Assume that the square (A) is a push-out. If ϕ is a monomorphism, then so is ψ and this square is also a pull-back.
(3) Assume that the square (A) is a pull-back. If ψ is a pull-back (respectively a monomorphism), then so is ϕ. In particular if ψ is a push-out and a monomorphism, then ϕ is a monomorphism and a pull-back. □

Finally we state two different "cube theorems", named in analogy with Mather's theorem [**11**]. They say that sometimes push-outs do commute with pull-backs.

2.7. LEMMA. *Assume that in the square* (A) *the morphisms ϕ and ψ (respectively ϕ, ψ, π, and μ) are pull-backs. Then* (A) *is a pull-back (respectively a push-out) square if and only if the range square:*

$$\begin{array}{ccc} Rf & \xrightarrow{R\phi} & Rg \\ {\scriptstyle R\pi}\downarrow & & \downarrow{\scriptstyle R\mu} \\ Rh & \xrightarrow{R\psi} & Rk \end{array}$$

is a pull-back (respectively a push-out) of spaces. □

This statement, which will be referred to as the cube lemma, has the following extension to *Arrows*. We call it the hypercube lemma. Consider a commutative

cube in *Arrows*:

2.8. LEMMA. *In the above cube assume that the squares $(\bar{f}, \bar{h}, f, h)$, $(\bar{f}, \bar{g}, f, g)$, $(\bar{h}, \bar{k}, h, k)$, and $(\bar{g}, \bar{k}, g, k)$ are pull-backs and that the square (f, h, g, k) is a push-out. Then the square $(\bar{f}, \bar{h}, \bar{g}, \bar{k})$ is also a push-out.* $\qquad\square$

2.9. Fix a morphism $\phi : f \to g$ in *Arrows*. There are two natural operations one can perform. First, given a morphism $\tau \to g$ one can *pull it back* along ϕ and define $\phi^* \tau \to f$ to be the morphism that fits into the following pull-back square in *Arrows*:

$$\begin{array}{ccc} \phi^* \tau & \longrightarrow & \tau \\ \downarrow & & \downarrow \\ f & \overset{\phi}{\longrightarrow} & g \end{array}$$

In general $\phi^* \tau \to f$ is not a pull-back, but it is so whenever $\tau \to g$ is a pull-back.

Second, for any $\sigma \to f$, define the *push-forward* $\phi_* \sigma \to g$ to be a pull-back that fits into a commutative square in *Arrows* of the form:

$$\begin{array}{ccc} \sigma & \longrightarrow & \phi_* \sigma \\ \downarrow & & \downarrow \\ f & \overset{\phi}{\longrightarrow} & g \end{array}$$

and is initial with respect to this property. Explicitly $\phi_* \sigma \to g$ is given by the following pull-back square in *Spaces*:

$$\begin{array}{ccc} A & \longrightarrow & Dg \\ {\scriptstyle \phi_* \sigma}\downarrow & & \downarrow{\scriptstyle g} \\ R\sigma & \longrightarrow Rf \overset{R\phi}{\longrightarrow} & Rg \end{array}$$

This construction shows that the push-forward is functorial. Note that ϕ is a pull-back if and only if, for any pull-back $\sigma \to f$, the morphism $\sigma \to \phi_* \sigma$ is an isomorphism. By definition the push-forward $\phi_* \sigma \to g$ is always a pull-back.

Observe that there is a natural morphism $\sigma \to \phi^* \phi_* \sigma$, which is a pull-back if $\sigma \to f$ is so. For any pull-back $\tau \to g$, there is also a natural pull-back morphism $\phi_* \phi^* \tau \to \tau$.

2.10. PROPOSITION. *Assume that the range square of the square* (A) *is a pull-back of spaces. For any pull-back $\sigma \to h$, the induced morphism $\phi_* \pi^* \sigma \to \mu^* \psi_* \sigma$ is then an isomorphism.*

PROOF. To prove the proposition we need to show that the following square is a pull-back in *Arrows*:

$$
\begin{array}{ccc}
\phi_*\pi^*\sigma & \longrightarrow & g \\
\downarrow & & \downarrow{\scriptstyle \mu} \\
\psi_*\sigma & \longrightarrow & k
\end{array}
$$

Since the horizontal morphisms in this square are pull-backs, according to the cube Lemma 2.7, it is enough to show that this square is a pull-back on the range level. On the range level, this is the outer square of:
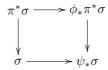
$$
\begin{array}{ccccc}
R(\pi^*\sigma) & \longrightarrow & Rf & \xrightarrow{R\phi} & Rg \\
\downarrow & & \downarrow{\scriptstyle R\pi} & & \downarrow{\scriptstyle R\mu} \\
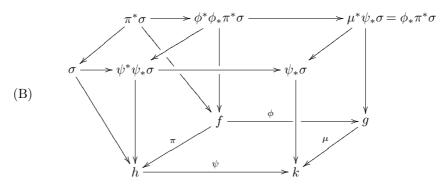R\sigma & \longrightarrow & Rh & \xrightarrow{R\psi} & Rk
\end{array}
$$

As the left and right squares of this diagram are pull-backs, then so is the outer one, proving the proposition. □

2.11. PROPOSITION. *Assume that the square* (A) *is a push-out and either ϕ is a monomorphism or π is a monomorphism and a pull-back. Then for any pull-back $\sigma \to h$ the following is a push-out square:*

$$
\begin{array}{ccc}
\pi^*\sigma & \longrightarrow & \phi_*\pi^*\sigma \\
\downarrow & & \downarrow \\
\sigma & \longrightarrow & \psi_*\sigma
\end{array}
$$

PROOF. By Lemma 2.6.(1) the square (A) is also a pull-back. Hence according to Proposition 2.10, the morphism $\phi_*\pi^*\sigma \to \mu^*\psi_*\sigma$ is an isomorphism. Consider next the following commutative diagram in *Arrows*:

(B)

The top right square $(\phi^*\phi_*\pi^*\sigma, \psi^*\psi_*\sigma, \phi_*\pi^*\sigma, \psi_*\sigma)$ is a push-out by the hypercube Lemma 2.8 since the square (f, h, g, k) is a push-out. Thus to prove the proposition it is enough to show that the top left square $(\pi^*\sigma, \sigma, \phi^*\phi_*\pi^*\sigma, \psi^*\psi_*\sigma)$ is also a push-out.

Assume that ϕ is a monomorphism. It follows that so is ψ. We claim that in this case $\sigma \to \psi^*\psi_*\sigma$ and $\pi^*\sigma \to \phi^*\phi_*\pi^*\sigma$ are isomorphisms. That will be proven

once we show that the following is a pull-back square:

$$
\begin{array}{ccc}
\sigma & \longrightarrow & \psi_*\sigma \\
\downarrow & & \downarrow \\
h & \xrightarrow{\ \psi\ } & k
\end{array}
$$

Since the vertical morphisms of this square are pull-backs, according to Lemma 2.7, it suffices to check that on the range level we have a pull-back of spaces:
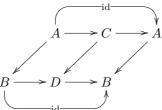
$$
\begin{array}{ccc}
R\sigma & \xrightarrow{\ \mathrm{id}\ } & R(\psi_*\sigma) \\
\downarrow & & \downarrow \\
Rh & \xrightarrow{\ R\psi\ } & Rk
\end{array}
$$

As $R\psi$ is a monomorphism, this is the case.

Assume now that π is a pull-back and a monomorphism. We use Lemma 2.7. The assumption on π implies that the morphisms $\pi^*\sigma \to \sigma$, $\phi^*\phi_*\pi^*\sigma \to \psi^*\psi_*\sigma$, and $\mu^*\psi_*\sigma \to \psi_*\sigma$ are also monomorphisms and pull-backs. Thus all the morphisms in the top left square of the diagram (B) are pull-backs. To see that this square is a push-out we need to prove that it is so on the range level. Let us look at the ranges of the top layer of the diagram (B). It is a commutative diagram of spaces of the form:

where the right square (C, D, A, B) is a push-out and the diagonal maps are monomorphisms. We can now use the "two out of three" Lemma 2.4 to conclude that the left square (A, B, C, D) is also a push-out. $\qquad\square$

3. Homotopy theory of maps

3.1. The category *Arrows* can be given a model category structure where:

- a morphism ϕ in *Arrows* is a weak equivalence (cofibration) if $D\phi$ and $R\phi$ are weak equivalences (cofibrations) in *Spaces*;
- a morphism $\phi : f \to g$ in *Arrows* is a fibration if both $R\phi$ and the map $Df \to \lim(Rf \xrightarrow{R\phi} Rg \xleftarrow{g} Dg)$, induced by $D\phi$ and f, are fibrations in *Spaces*.

3.2. The category *Arrows* also supports a canonical simplicial structure:

- for a space K and a map f, $f \otimes K := f \times id_K$;
- the mapping space $\mathrm{map}(f, g)$ is given by:

$$
\lim\big(\mathrm{map}(Df, Dg) \xrightarrow{g_*} \mathrm{map}(Df, Rg) \xleftarrow{f^*} \mathrm{map}(Rf, Rg)\big).
$$

The description of the mapping spaces is straightforward from the adjunction property [**15**, II.1.3]. This simplicial structure is compatible with the model category

structure defined above (the axiom SM7 is fulfilled), so that the category *Arrows* is actually a simplicial model category.

3.3. Let $F : I \to Arrows$ be a functor. By the universal properties, the morphisms $\mathrm{hocolim}_I F$ and $\mathrm{holim}_I F$ are respectively naturally isomorphic to the objects in $\mathrm{Ho}(Arrows)$ represented by $\mathrm{hocolim}_I \pi F : \mathrm{hocolim}_I DF \to \mathrm{hocolim}_I RF$ and $\mathrm{holim}_I \pi F : \mathrm{holim}_I DF \to \mathrm{holim}_I RF$.

4. Fiberwise decomposition

4.1. Let $f : X \to Y$ be a map and $\sigma : \Delta[n] \to Y$ be a simplex. Define $df(\sigma) \to \Delta[n]$ to be the map that fits into the following pull-back square in *Spaces*:

$$
\begin{array}{ccc}
df(\sigma) & \longrightarrow & X \\
\downarrow & & \downarrow f \\
\Delta[n] & \xrightarrow{\ \sigma\ } & Y
\end{array}
$$

These maps fit into a functor $df : Y \to Arrows$, indexed by the simplex category of Y (see [**6**, Definition 6.1]). The morphisms $\{df(\sigma) \to f\}_{\sigma \in Y}$, given by the above commutative squares, satisfy the universal property of the colimit and so $\mathrm{colim}_Y df = f$.

Functors of the form df are not arbitrary, they satisfy a certain homotopy invariance property.

4.2. DEFINITION. A functor $F : I \to Arrows$ (indexed by a small category) is called *pseudo-cofibrant* if the morphism $\mathrm{hocolim}_I F \to \mathrm{colim}_I F$ is an isomorphism in $\mathrm{Ho}(Arrows)$.

Although the next proposition already appeared in Dror Farjoun's book [**7**, p.183], we offer a proof illustrating the ideas of the present paper.

4.3. PROPOSITION. *For any $f : X \to Y$, the functor $df : Y \to Arrows$ is pseudo-cofibrant.*

PROOF. Let $\mathcal{S}$ be the class of spaces Y for which the proposition is true. To prove the proposition it is enough to show that $\mathcal{S}$ satisfies the following properties (which then imply that $\mathcal{S}$ consists of all spaces):

(1) $\Delta[n] \in \mathcal{S}$;
(2) $\coprod Y_i \in \mathcal{S}$ if $Y_i \in \mathcal{S}$;
(3) $\mathrm{colim}(Y_0 \leftarrow Y_1 \hookrightarrow Y_2) \in \mathcal{S}$ if $Y_i \in \mathcal{S}$ and $Y_1 \hookrightarrow Y_2$ is a cofibration.

Since $id : \Delta[n] \to \Delta[n]$ is the terminal object of the simplex category of $\Delta[n]$, statement (1) follows from cofinality properties of homotopy colimits ([**5**, Theorem XI.9.2] and [**6**, Theorem 30.5]).

Statement (2) is clear. To prove (3), set $Y := \mathrm{colim}(Y_0 \leftarrow Y_1 \hookrightarrow Y_2)$. Let $f : X \to Y$ be a map. Define $f_i : X_i \to Y_i$ to be the map that fits into the following pull-back square:

$$
\begin{array}{ccc}
X_i & \longrightarrow & X \\
f_i \downarrow & & \downarrow f \\
Y_i & \longrightarrow & Y
\end{array}
$$

These maps form a natural transformation between the following push-outs, where the indicated maps are cofibrations:

$$
\begin{array}{ccccccc}
X & = & \mathrm{colim} & (& X_0 \longleftarrow X_1 \hookrightarrow X_2 &) \\
f \downarrow & & & & f_0 \downarrow \quad f_1 \downarrow \quad\quad f_2 \downarrow & \\
Y & = & \mathrm{colim} & (& Y_0 \longleftarrow Y_1 \hookrightarrow Y_2 &)
\end{array}
$$

Let $Qf \to df\epsilon$ be a cofibrant replacement in $\mathrm{Fun}^b\big(N(Y), Arrows\big)$ ([**6**, Theorem 13.1]) of the composition of the diagram $df : Y \to Arrows$ with the forgetful functor $\epsilon : N(Y) \to Y$, $(\sigma_n \to \cdots \to \sigma_0) \mapsto \sigma_0$ ([**6**, Definition 6.6]). Since $N(Y_i) \to N(Y)$ is reduced ([**6**, Example 12.10] and [**6**, Proposition 5.1]), Qf restricted to $N(Y_i)$ is a cofibrant replacement in $\mathrm{Fun}^b\big(N(Y_i), Arrows\big)$ of the composition of df_i with the forgetful functor $N(Y_i) \to Y_i$. The spaces Y_i are assumed to be in $\mathcal{S}$ and hence the morphism $\mathrm{colim}_{N(Y_i)} Qf \to f_i$ is a weak equivalence. Property (3) follows now from the basic homotopy invariance of push-outs ([**6**, Proposition 2.5.(2)]). $\qquad\square$

5. Homotopy pull-backs

5.1. Recall that a morphism $f \to g$ in $Arrows$ is called a homotopy pull-back if, for some (equivalently any) weak equivalence $\psi : g \xrightarrow{\simeq} h$ with h a fibration in $Spaces$, the morphism $f \to \psi_* f$ is a weak equivalence. In other words the domain Df is weakly equivalent to the homotopy pull-back of the diagram $Rf \to Rg \leftarrow Dg$.

If $\phi : f \to g$ is a pull-back and either g or $R\phi : Rf \to Rg$ is a fibration, then ϕ is a homotopy pull-back.

A homotopy pull-back $\sigma \to f$ for which $R\sigma$ is contractible, is called a *homotopy fiber* of f. If $\sigma \to f$ and $\tau \to f$ are homotopy fibers of f such that the images of $R\sigma$ and $R\tau$ in Rf lie in the same connected component, then σ and τ are weakly equivalent.

5.2. Here is a list of some basic properties of homotopy pull-backs:

(1) Right properness: If $\phi : f \to g$ is a weak equivalence, then it is a homotopy pull-back.

(2) Fiber characterization: A morphism $\phi : f \to g$ is a homotopy pull-back if and only if it induces a weak equivalence of homotopy fibers, i.e. for any commutative square:

$$
\begin{array}{ccc}
\sigma & \xrightarrow{\pi} & \tau \\
\downarrow & & \downarrow \\
f & \xrightarrow{\phi} & g
\end{array}
$$

if $\sigma \to f$ and $\tau \to g$ are respectively homotopy fibers of f and g, then π is a weak equivalence.

(3) Two out of three: Let $\phi : f \to g$ and $\psi : g \to h$ be morphisms. Assume that ψ is a homotopy pull-back. Then ϕ is a homotopy pull-back if and only if $\psi\phi$ is so. Assume that ϕ is a homotopy pull-back and that the range $R\phi : Rf \to Rg$ induces an epimorphism on the sets of connected components. Then ψ is a homotopy pull-back if and only if $\psi\phi$ is so.

(4) Disjoint union: Let $\{f_i\}_{i\in I}$ and $\{g_j\}_{j\in J}$ be collections of maps, $h: I \to J$ a map of sets, and $\{\phi_i : f_i \to g_{h(i)}\}_{i\in I}$ a collection of homotopy pull-backs. Then the following induced morphism is also a homotopy pull-back:

$$\coprod_{i\in I} \phi_i : \coprod_{i\in I} f_i \to \coprod_{j\in J} g_j$$

Note that Example 2.5 also illustrates the failure of the full two out of three property for homotopy pull-backs.

A fibration f possesses the property that for any simplex $\sigma \in Rf$ the morphism $df(\sigma) \to f$ is a homotopy pull-back. Maps with that property play a crucial role in this paper.

5.3. DEFINITION. A map $f : X \to Y$ is called a *quasi-fibration* if for any morphism $\alpha : \sigma \to \tau$ in Y, the morphism $df(\alpha) : df(\sigma) \to df(\tau)$ is a weak equivalence.

Even if quasi-fibrations lack the global lifting properties enjoyed by fibrations, the local information given by the preimages of simplices still allows us to recover the homotopy fiber.

5.4. PROPOSITION. *A map f is a quasi-fibration if and only if the morphism $df(\sigma) \to f$ is a homotopy pull-back for any simplex $\sigma : \Delta[n] \to Rf$.*

PROOF. If the morphisms $df(\sigma) \to f$ are homotopy pull-backs, then they are homotopy fibers of f. Thus by the homotopy invariance of homotopy fibers, f is a quasi-fibration.

Assume that f is a quasi-fibration. Factor the morphism $df(\sigma) \to f$ as a composition $df(\sigma) \to p \to f$ where Rp is contractible and $p \to f$ is a fibration and a pull-back (hence a homotopy pull-back). It follows that p is a quasi-fibration and $df(\sigma) \to p$ is a pull-back. Since Rp is a contractible, according to [**6**, Lemma 27.8], the morphism $df(\sigma) \to \operatorname{hocolim}_{Rp} dp$ is an isomorphism in $\operatorname{Ho}(Arrows)$. Thus by Proposition 4.3, $df(\sigma) \to \operatorname{colim}_{Rp} dp = p$ is a weak equivalence and therefore a homotopy pull-back. We can conclude that the composition $df(\sigma) \to p \to f$ is also a homotopy pull-back. □

The last proposition combined with the fiber characterization of homotopy pull-backs (property 5.2.(2)) gives:

5.5. COROLLARY. *A morphism $\phi : f \to g$ between quasi-fibrations f and g is a homotopy pull-back if and only if, for any simplex $\sigma \in Rf$, the induced morphism $\phi(\sigma) : df(\sigma) \to dg((R\phi)\sigma)$ is a weak equivalence.* □

6. Distinguished collections

In this section we prove some fundamental properties of distinguished collections (see Definition 1.1 in the introduction). We start with a stronger form of condition (3).

6.1. PROPOSITION. *Let $\mathcal{C}$ be a distinguished class. If $\phi : f \to g$ and $\psi : g \to h$ belong to $\mathcal{C}$, then so does $\psi\phi$.*

PROOF. The homotopy colimit of the following diagram F in *Arrows*:

$$f \xrightarrow{\phi} g \xleftarrow{\operatorname{id}_g} g \xrightarrow{\psi} h$$

is homotopy equivalent to h. Moreover the morphism $f \to \mathrm{hocolim}F$ can be identified with the composition $\psi\phi$. Since all morphisms in this diagram belong to $\mathcal{C}$, by condition (4) in Definition 1.1 so does $\psi\phi$. □

We next characterize the collection of homotopy pull-backs.

6.2. THEOREM. *The collection of homotopy pull-backs is the smallest distinguished class of morphisms in Arrows.*

PROOF. We first show that homotopy pull-backs form a distinguished class. According to 5.2, the requirements (1), (2), and (3) of Definition 1.1 are satisfied. We need to prove that homotopy pull-backs satisfy also requirement (4). We first show a particular case:

6.3. LEMMA. *Let the following be a push-out square in Arrows:*

$$
\begin{array}{ccc}
f & \xrightarrow{\ \phi\ } & g \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mu} \\
h & \xrightarrow{\ \psi\ } & k
\end{array}
$$

Assume that ϕ and π are homotopy pull-backs and one of them is a monomorphism. Then μ and ψ are also homotopy pull-backs.

PROOF OF LEMMA 6.3. By making various factorizations we may assume that both morphisms ϕ and π are monomorphisms and the maps f, g, h are fibrations.

Choose a simplex in Rk. Since it is in the image of either $R\mu$ or $R\psi$, by symmetry, we can assume that it is of the form $\Delta[n] \xrightarrow{\sigma} Rh \xrightarrow{R\psi} Rk$. It follows that $dk(\sigma) = \psi_* dh(\sigma)$ and the following is a push-out square (see Proposition 2.11):

$$
\begin{array}{ccc}
\pi^* dh(\sigma) & \longrightarrow & \phi_* \pi^* dh(\sigma) \\
\downarrow & & \downarrow \\
dh(\sigma) & \longrightarrow & dk(\sigma) = \psi_* dh(\sigma)
\end{array}
$$

Since ϕ is a homotopy pull-back between fibrations, according to Proposition 4.3 and Corollary 5.5, the morphism $\pi^* dh(\sigma) \to \phi_* \pi^* dh(\sigma)$ is a weak equivalence. Same is therefore true for $dh(\sigma) \to dk(\sigma)$. We conclude that k is a quasi-fibration and ψ is a homotopy pull-back (Corollary 5.5). □

To prove in general that homotopy pull-backs satisfy requirement (4) of Definition 1.1, it would be enough to show that, for any bounded and cofibrant $F : K \to Arrows$ which sends morphisms in K to homotopy pull-backs, the morphism $F(\sigma) \to \mathrm{colim}_K F$ is a homotopy pull-back for any simplex $\sigma \in K$. Induction on the dimension of K seems to be the right strategy to do that. Unfortunately the notion of cofibrancy of bounded functors is to rigid for that: cofibrant functors are not preserved by restricting along maps of simplicial sets. To circumvent this problem we need to allow "more general cofibrant" diagrams. We are going to apply the idea of relative boundedness and cofibrancy introduced in [**6**, Sections 17, 19] to deal with such problems.

Fix a space L and denote by $\mathcal{S}_L$ the class of maps of the form $f : K \to L$ that satisfy the following property: If $F : K \to Arrows$ is any f-bounded and

f-cofibrant diagram which sends morphisms in K to homotopy pull-backs, then the morphism $F(\sigma) \to \mathrm{colim}_K F$ is a homotopy pull-back for any simplex $\sigma \in K$.

We will show that $\mathcal{S}_L$ satisfies the following properties, and thus consists of all maps with range L:

(1) All maps of the form $\Delta[n] \to L$ belong to $\mathcal{S}_L$.
(2) If a set of maps $K_i \to L$ belongs to $\mathcal{S}_L$ then so does $\coprod K_i \to L$.
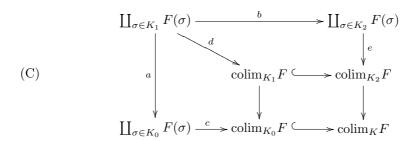(3) Let the following be a commutative diagram where the indicated map is a cofibration:

$$
\begin{array}{ccc}
K & = & \mathrm{colim}\ \left(\ K_0 \longleftarrow K_1 \lhook\joinrel\longrightarrow K_2\ \right) \\
f\downarrow & & \quad f_0\downarrow \qquad f_1\downarrow \qquad\qquad f_2\downarrow \\
L & = & \mathrm{colim}\ \left(\ L \xleftarrow{\ \mathrm{id}\ } L \xrightarrow{\ \mathrm{id}\ } L\ \right)
\end{array}
$$

If each f_i belongs to $\mathcal{S}_L$ then so does f.

Property (1) is clear, since $\Delta[n]$ has a terminal object. Property (2) is easily verified as a simplex of $\coprod K_i$ is a simplex of one of the spaces K_i. It remains to prove (3). Let $F : K \to \ Arrows$ be an f-bounded and f-cofibrant functor. Consider the following commutative diagram:

(C)
$$
\begin{array}{ccc}
\coprod_{\sigma \in K_1} F(\sigma) & \xrightarrow{\quad\quad b \quad\quad} & \coprod_{\sigma \in K_2} F(\sigma) \\
a\downarrow \quad \searrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
\quad\quad \mathrm{colim}_{K_1} F \lhook\joinrel\longrightarrow \mathrm{colim}_{K_2} F & & \\
\downarrow & \downarrow & \downarrow \\
\coprod_{\sigma \in K_0} F(\sigma) \xrightarrow{\ c\ } \mathrm{colim}_{K_0} F \lhook\joinrel\longrightarrow \mathrm{colim}_K F
\end{array}
$$

According to the disjoint union property 5.2.(4), the morphisms a and b are homotopy pull-backs. The functor F restricted along $f_i : K_i \to L$ is both f_i-bounded and f_i-cofibrant (see [**6**, Corollaries 17.5 (1), 19.6 (1)]). Thus by the inductive hypothesis c, d, and e are homotopy pull-backs. We can now apply the two out of three property 5.2(3) to see that both morphisms $\mathrm{colim}_{K_1} F \hookrightarrow \mathrm{colim}_{K_2} F$ and $\mathrm{colim}_{K_1} F \to \mathrm{colim}_{K_0} F$ are homotopy pull-backs too. By Lemma 6.3 we can conclude that all the morphisms in the diagram (C) are homotopy pull-backs.

We are left to show that homotopy pull-backs are contained in any distinguished class. For that it is enough to show that any pull-back $\phi : f \to g$ with g a fibration belongs to any distinguished class.

Assume first that ϕ coincides with $dg(\sigma) \to g$, for some $\sigma \in Rg$. Note that the functor $dg : Rg \to \ Arrows$ is pseudo-cofibrant (Proposition 4.3) and it takes all the morphisms in K to weak equivalences. As weak equivalences belong to any distinguished class, then so does $dg(\sigma) \to g$.

For a general pull-back $\phi : f \to g$, define I to be the Grothendieck construction $I := \mathrm{Gr}(\Delta[0] \leftarrow Rf \xrightarrow{R\phi} Rg)$ ([**6**, Section 38]). Define further $F : I \to \ Arrows$ to

be the functor given by the data (see [**6**, Section 40]):

(D)

$F_{\Delta[0]} : \Delta[0] \to Arrows$ is the constant functor with value f;

$F_{Rf} : Rf \to Arrows$ is df;

$F_{Rg} : Rg \to Arrows$ is dg;

$F_{Rf} \to F_{\Delta[0]}$ is given by the morphisms $df \to \mathrm{colim}_{Rf} df = f$;

$F_{Rf} \to F_{Rg}$ is induced by ϕ.

Again by Proposition 4.3, the functor F is pseudo-cofibrant. Moreover it takes any morphism in I to either a weak equivalence or a morphism of the form $df(\sigma) \to f$. Since such morphisms belong to any distinguished class we conclude that so does $\phi : f = F(\Delta[0]) \to \mathrm{colim}_I F = g$. $\qquad\square$

Note that we have not used condition (3) in Definition 1.1 while proving Theorem 6.2. This means that the collection of homotopy pull-backs can be characterized as the smallest class that satisfies only the three other requirements of the definition. The significance of the third condition is illustrated by:

6.4. COROLLARY. *Let $\mathcal{C}$ be a distinguished class. Then $\phi : f \to g$ belongs to $\mathcal{C}$ if and only if the morphism $\pi : \sigma \to \tau$ does so for any commutative square of the form:*

$$\begin{array}{ccc} \sigma & \xrightarrow{\;\pi\;} & \tau \\ \downarrow & & \downarrow \\ f & \xrightarrow{\;\phi\;} & g \end{array}$$

where $\sigma \to f$ and $\tau \to g$ are respectively homotopy fibers of f and g.

PROOF. Since $\sigma \to f$ and $\tau \to g$ are homotopy pull-backs they belong to any distinguished class by Theorem 6.2. Let us assume that ϕ is a member of $\mathcal{C}$. By Proposition 6.1 the morphism $\sigma \to g$ is in $\mathcal{C}$ and hence by condition (3) in Definition 1.1 so is π.

Let us prove now the converse. By making an appropriate factorization we can assume that f and g are fibrations. Let $F : I \to Arrows$ be the functor given by (D) in the proof of Theorem 6.2. This functor takes any morphism in I either to a morphism in $\mathcal{C}$ (by assumption) or to a homotopy pull-back, which is also in $\mathcal{C}$. Therefore $\phi : f = F(\Delta[0]) \to \mathrm{colim}_I F = g$ belongs to $\mathcal{C}$. $\qquad\square$

7. Fiberwise localization

7.1. Let ϕ be a morphism in $Arrows$. Recall that a map of spaces f is called ϕ-local if, for some (equivalently any) weak equivalence $f \simeq g$ with g fibrant, the map of spaces $\mathrm{map}(\phi, g)$ is a weak equivalence.

According to [**3**], [**4**], [**7**], and [**9**], there is a functor $L_\phi : Arrows \to Arrows$ and a natural transformation $f \to L_\phi f$ (called localization) such that:

- $L_\phi f$ is fibrant and ϕ-local;
- the map of spaces $\mathrm{map}(L_\phi f, g) \to \mathrm{map}(f, g)$ is a weak equivalence, for any fibrant and ϕ-local g.

We are going to refer to ϕ-local maps also as L_ϕ-local and to morphisms ψ for which $L_\phi \psi$ is a weak equivalence as L_ϕ-equivalences.

In general it is very difficult to understand the localization with respect to an arbitrary morphism ϕ. However when ϕ is a morphism of the form $(u, \mathrm{id}_{\Delta[0]})$ the next two propositions identify the localization with a very familiar object. Through out this section we are going to fix a map of spaces $u : A \to B$. The symbol L_u will be used to denote both the localization in *Spaces* with respect to u and the localization in *Arrows* with respect to the morphism $(u, \mathrm{id}_{\Delta[0]})$. We will show that the functor L_u in *Arrows* is the fiberwise version of L_u in *Spaces*.

We start by characterizing the L_u-local maps as those for which the homotopy fibers are L_u-local spaces.

7.2. PROPOSITION. *A map f is L_u-local in Arrows if and only if, for any homotopy fiber $\sigma \to f$, the space $D\sigma$ is L_u-local in Spaces, i.e. for any weak equivalence $D\sigma \simeq Z$ with Z fibrant, the map of spaces $\mathrm{map}(u, Z)$ is a weak equivalence.*

PROOF. Choose a weak equivalence $f \simeq g$ with g fibrant (Rg is fibrant and g is a fibration). By definition of the simplicial structure given in 3.2 we have the following cube of spaces :

where the top and bottom faces are pull-back squares and the labelled arrows are fibrations. Let $x \in Rg$ be a vertex. The fibers of a and b over the vertex x can be identified respectively with the mapping spaces $\mathrm{map}(A, g^{-1}(x))$ and $\mathrm{map}(B, g^{-1}(x))$. Thus $\mathrm{map}(\phi, g)$ is a weak equivalence if and only if, for any vertex $x \in Rg$, the map $\mathrm{map}(u, g^{-1}(x))$ is a weak equivalence.   □

In general local objects are not closed under homotopy colimits. However in the case of L_u we have:

7.3. COROLLARY. *Let $F : I \to Arrows$ be a pseudo-cofibrant functor. Assume that, for any i, the map $F(i)$ is L_u-local and, for any morphism α in I, the morphism $F(\alpha)$ is a homotopy pull-back. The map $\mathrm{colim}_I F$ is then L_u-local.*

PROOF. Since for any $i \in I$, $F(i) \to \mathrm{colim}_I F$ is a homotopy pull-back (see Theorem 6.2), any homotopy fiber of $\mathrm{colim}_I F$ is a homotopy fiber of some $F(i)$. As these are L_u-local spaces, the map $\mathrm{colim}_I F$ is L_u-local in *Arrows*.   □

Next we describe L_u-equivalences:

7.4. PROPOSITION. *A morphism ψ in Arrows is an L_u-equivalence if and only if firstly $R\psi$ is a weak equivalence and secondly for any commutative square*

$$
\begin{array}{ccc}
\sigma & \xrightarrow{\pi} & \tau \\
\downarrow & & \downarrow \\
f & \xrightarrow{\psi} & g
\end{array}
$$

if $\sigma \to f$ and $\tau \to g$ are homotopy fibers, then $D\pi$ is an L_u-equivalence in Spaces.

PROOF. Assume first that $\psi : f \to g$ is an L_u-equivalence. Note that for any fibrant space X, the map id_X is L_u-local by Proposition 7.2. It follows that $\mathrm{map}(\psi, \mathrm{id}_X) = \mathrm{map}(R\psi, X)$ is a weak equivalence of spaces for all fibrant X. Hence $R\psi$ is a weak equivalence.

If Z is a fibrant and L_u-local space, then the map $Z \to \Delta[0]$ is L_u-local in *Arrows*. Thus the map of spaces:

$$
\begin{array}{ccc}
\mathrm{map}(Dg, Z) & = & \mathrm{map}(g, Z \to \Delta[0]) \\
{\scriptstyle \mathrm{map}(D\psi, Z)}\downarrow & & \downarrow{\scriptstyle \mathrm{map}(\psi, Z \to \Delta[0])} \\
\mathrm{map}(Df, Z) & = & \mathrm{map}(f, Z \to \Delta[0])
\end{array}
$$

is a weak equivalence. This together with the fact that $R\psi$ is a weak equivalence implies that, for any commutative square:

$$
\begin{array}{ccc}
\sigma & \xrightarrow{\pi} & \tau \\
\downarrow & & \downarrow \\
f & \xrightarrow{\psi} & g
\end{array}
$$

where $\sigma \to f$ and $\tau \to g$ are homotopy fibers, $\mathrm{map}(D\pi, Z)$ is a weak equivalence of spaces. As this holds for any L_u-local space Z, the map $D\pi$ is an L_u-equivalence.

To prove the other implication consider the following commutative square:

$$
\begin{array}{ccc}
f & \xrightarrow{\psi} & g \\
\downarrow & & \downarrow \\
L_u f & \xrightarrow{L_u \psi} & L_u g
\end{array}
$$

The vertical morphisms are L_u-equivalences and we already know they induce weak equivalences on ranges and L_u-equivalences on homotopy fibers. Thus if ψ satisfies the two properties of the proposition, then so does $L_u\psi$. Since $L_u\psi$ is a morphism between L_u-local maps, i.e. maps whose homotopy fibers are L_u-local spaces (see Proposition 7.2), the morphism $L_u\psi$ is a weak equivalence. We can conclude that ψ is an L_u-equivalence. $\square$

8. L_u-homotopy pull-backs

In Theorem 6.2 we saw that any distinguished class containing all weak equivalences must also contain all homotopy pull-backs. The analogous statement for L_ϕ-equivalences should involve L_ϕ-homotopy pull-backs.

8.1. DEFINITION. A morphism $f \to g$ in *Arrows* is called an L_ϕ-*homotopy pull-back* if, for some (equivalently any) weak equivalence $\psi : g \xrightarrow{\simeq} h$ with h a fibration in *Spaces*, the morphism $f \to \psi_* f$ is an L_ϕ-equivalence.

There is a more amenable description of L_ϕ-homotopy pull-backs when ϕ is of the form $(u, \mathrm{id}_{\Delta[0]})$.

8.2. PROPOSITION. *If u is a map of spaces, then $\pi : f \to g$ is an L_u-homotopy pull-back if and only if $L_u\pi$ is a homotopy pull-back.*

PROOF. From the fiber characterization property 5.2.(2) we see that the morphism $L_u\pi : L_u f \to L_u g$ is a homotopy pull-back if and only if it induces a weak equivalence on homotopy fibers. According to Proposition 7.4 this is the case if and only if the morphism π induces L_u-equivalences on homotopy fibers. For any weak equivalence $\psi : g \xrightarrow{\simeq} h$ with h a fibration in *Spaces*, the morphism $\psi_* f \to h$ always induces an equivalence on homotopy fibers. By construction, f and $\psi_* f$ have the same range. Therefore, the morphism $f \to \psi_* f$ is an L_u-equivalence if and only if $L_u\pi$ is a homotopy pull-back. $\square$

Using this characterization, we can now prove our main result:

8.3. THEOREM. *Let u be a map of spaces. The collection of L_u-homotopy pull-backs is the smallest distinguished class containing all L_u-equivalences.*

PROOF. Consider a commutative square:

$$
\begin{array}{ccc}
f & \xrightarrow{\ \pi\ } & g \\
\downarrow & & \downarrow \\
L_u f & \xrightarrow{\ L_u\pi\ } & L_u g
\end{array}
$$

If π is an L_u-homotopy pull-back, then $L_u\pi$ is a homotopy pull-back and hence by Theorem 6.2 it belongs to any distinguished class. By condition (3) of Definition 1.1 it follows that π belongs to any distinguished class that contains L_u-equivalences.

To prove the theorem it remains to show that the collection in the statement is a distinguished class. It is clear that requirements (1), (2), and (3) of Definition 1.1 are satisfied. Let $F : I \to$ *Arrows* be a pseudo-cofibrant functor which takes morphisms in I to L_u-homotopy pull-backs. We need to show that, for any $i \in I$, $L_u F(i) \to L_u(\mathrm{colim}_I F)$ is a homotopy pull-back. The functor F fits into the following commutative square:

$$
\begin{array}{ccc}
H & \longrightarrow & G \\
\simeq \downarrow & & \downarrow \simeq \\
F & \longrightarrow & L_u F
\end{array}
$$

where H and G are pseudo-cofibrant and the indicated natural transformations are weak equivalences (see [**6**, Remark 16.3]). Choose an object $i \in I$ and consider the following commutative diagram, where the indicated arrows are weak equivalences:

$$
\begin{array}{ccccccc}
L_u F(i) & \xleftarrow{\ \simeq\ } & L_u H(i) & \xrightarrow{\ \simeq\ } & L_u G(i) & \xleftarrow{\ b\ } & G(i) \\
e\downarrow & & \downarrow & & \downarrow & & \downarrow a \\
L_u(\mathrm{colim}_I F) & \xleftarrow{\ \simeq\ } & L_u(\mathrm{colim}_I H) & \xrightarrow{\ d\ } & L_u(\mathrm{colim}_I G) & \xleftarrow{\ c\ } & \mathrm{colim}_I G
\end{array}
$$

As $G(\alpha)$ is a homotopy pull-back for any morphism $\alpha \in I$, by Theorem 6.2, the morphism a is a homotopy pull-back. Furthermore the values of G are L_u-local so b is a weak equivalence. It follows from Corollary 7.3 that $\mathrm{colim}_I G$ is also L_u-local and consequently the morphism c is a weak equivalence. Since L_u-equivalences are preserved by homotopy colimits, the morphism d is a weak equivalence. We can therefore conclude that e is a homotopy pull-back. $\qquad\square$

To illustrate this theorem we offer an application with a classical flavor. If for a map of spaces all the preimages of simplices have the same homotopy type up to L_u-localization, then the L_u-localization of the homotopy fiber shares the same homotopy type as well (the proof is identical as that of Proposition 5.4):

8.4. COROLLARY. *Let $f : X \to Y$ be a map over a connected space Y and denote by F its homotopy fiber. Assume that any morphism $\alpha : \sigma \to \tau$ in Y induces an L_u-homotopy pull-back $df(\sigma) \to df(\tau)$. Then the map $Ddf(\sigma) \to F$ is an L_u-equivalence for any simplex σ in Y.* $\qquad\square$

The particular case when the preimages of simplices are acyclic with respect to some generalized homology theory is due to E. Dror Farjoun [**7**, Corollary 9.B.3.2].

8.5. COROLLARY. *Let E be a generalized homology theory and $f : X \to Y$ be a map over a connected space Y such that $df(\sigma)$ is E-acyclic for any simplex $\sigma : \Delta[n] \to Y$. The homotopy fiber of f is then also E-acyclic.* $\qquad\square$

9. Concluding remark

According to Corollary 6.4 a distinguished collection $\mathcal{C}$ is determined by the class $\mathrm{T}(\mathcal{C})$ of maps $f : A \to B$ of spaces for which the morphism $(f, \mathrm{id}_{\Delta[0]})$ in *Arrows* belongs to $\mathcal{C}$. Members of $\mathrm{T}(\mathcal{C})$ are called *transition functions* for $\mathcal{C}$. For example the transition functions for homotopy pull-backs are weak equivalences and the transition functions for $\mathbf{HZ}$-homotopy pull-backs are $\mathbf{HZ}$-equivalences. More generally, if u is a map of spaces, then the transition functions for L_u-homotopy pull-backs are L_u-equivalences.

In these three examples, the class $\mathrm{T}(\mathcal{C})$ satisfies the following properties:

(A) $\mathrm{T}(\mathcal{C})$ contains weak equivalences.
(B) Let $f : A \to B$ and $g : B \to C$ be maps. If two out of f, g and gf are in $\mathrm{T}(\mathcal{C})$, then so is the third.
(C) Let $F : I \to Arrows$ be a functor. If, for any $i \in I$, $F(i)$ belongs to $\mathrm{T}(\mathcal{C})$, then so does $\mathrm{hocolim}_I F$.

On the other hand we could start with a class of maps $\mathcal{D}$ and define $\mathcal{D}$-homotopy pull-backs to be the collection $\mathcal{C}(\mathcal{D})$ of morphisms $\phi : f \to g$ in *Arrows* for which the maps induced on homotopy fibers of f and g belong to $\mathcal{D}$.

While writing this paper we have not found answers to the following questions:

9.1. QUESTION. Is it true that, for any distinguished collection $\mathcal{C}$, the class of transition functions $\mathrm{T}(\mathcal{C})$ satisfies the above three conditions (A), (B), and (C)?

Note that if $\mathrm{T}(\mathcal{C})$ satisfies the full "two out of three" condition (B), then $\mathcal{C}$ has the following "extended two out of three" condition: Let $\phi : f \to g$ and $\psi : g \to h$ be morphisms. Assume that $R\phi$ is an epimorphism on the set of connected components. Then ψ belongs to $\mathcal{C}$ if and only if $\psi\phi$ does. This should be compared with property (3) in Section 5.2. We do not know if any distinguished

collection satisfies such an "extended two out of three" condition. Maybe this extra requirement ought to be added to the definition of a distinguished collection. In this paper though we tried to avoid making any general connectivity assumption.

9.2. QUESTION. What are the necessary and sufficient requirements on a class of maps $\mathcal{D}$, so that $\mathcal{C}(\mathcal{D})$ is a distinguished collection?

Requirements (A), (B), and (C) above are possible candidates.

References

[1] J. F. Adams. *Infinite loop spaces*, volume 90. Princeton University Press, 1978. Annals of Mathematics Studies.
[2] M. Barratt and S. Priddy. On the homology of non-connected monoids and their associated groups. *Comment. Math. Helv.*, 47:1–14, 1972.
[3] A. K. Bousfield. Constructions of factorization systems in categories. *J. Pure Appl. Algebra*, 9(2):207–220, 1976/77.
[4] A. K. Bousfield. Localization and periodicity in unstable homotopy theory. *J. Amer. Math. Soc.*, 7(4):831–873, 1994.
[5] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.
[6] W. Chachólski and J. Scherer. Homotopy theory of diagrams. *Mem. Amer. Math. Soc.*, 155(736):x+90, 2002.
[7] E. Dror Farjoun. *Cellular spaces, null spaces and homotopy localization*, volume 1622 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.
[8] T. G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.
[9] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
[10] J. F. Jardine. The homotopical foundations of algebraic K-theory. In *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, pages 57–82. Amer. Math. Soc., Providence, RI, 1989.
[11] M. Mather and M. Walker. Commuting homotopy limits and colimits. *Math. Z.*, 175(1):77–80, 1980.
[12] D. McDuff and G. Segal. Homology fibrations and the "group-completion" theorem. *Invent. Math.*, 31(3):279–284, 1975/76.
[13] W. Pitsch and J. Scherer. Homology fibrations and "group-completion" revisited. *Homology Homotopy Appl.*, 6(1):153–166 (electronic), 2004.
[14] V. Puppe. A remark on "homotopy fibrations". *Manuscripta Math.*, 12:113–120, 1974.
[15] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.
[16] U. Tillmann. On the homotopy of the stable mapping class group. *Invent. Math.*, 130(2):257–275, 1997.

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