WREATH PRODUCTS AND REPRESENTATIONS OF $p$-LOCAL FINITE GROUPS

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Abstract. Given two finite $p$-local finite groups and a fusion preserving morphism between their Sylow subgroups, we study the question of extending it to a continuous map between the classifying spaces. The results depend on the construction of the wreath product of $p$-local finite groups which is also used to study $p$-local permutation representations.

1. Introduction

The concept of a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ was introduced in [7] by Broto, Levi and Oliver and a short exposition is given in §2. It consists of a finite $p$-group $S$ and two categories $\mathcal{F}$ and $\mathcal{L}$ whose objects are subgroups of $S$. This structure is suitable for studying $p$-completed classifying spaces of finite groups whose Sylow $p$-subgroup is $S$. Every finite group has an associated $p$-local finite group [7, Proposition 1.3, page 786] but the converse is not true.

In this paper we study maps between classifying spaces of $p$-local finite groups. Suppose that $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are $p$-local finite groups. Given a group homomorphism $\rho: S \to S'$ it is natural to ask if $B\rho: BS \to BS'$ can be extended, up to homotopy, to a map $\tilde{f}: |L| \wedge p \to |L'| \wedge p$ such that the following square is homotopy commutative where $\Theta$ and $\Theta'$ are the natural maps described in §2.

Recall that given fusion systems $\mathcal{F}$ and $\mathcal{F}'$ on $S$ and $S'$ respectively, a homomorphism $\psi: S \to S'$ is called fusion preserving if for every $\varphi \in \mathcal{F}(P, Q)$ there exists some $\varphi' \in \mathcal{F}'(\psi(P), \psi(Q))$ such that $\psi \circ \varphi = \varphi' \circ \psi$. Ragnarsson shows in [19] that stably, namely in the homotopy category of spectra, $\tilde{f}$ in the diagram above exists if and only if $\rho$ is fusion preserving. Unstably this is unknown.

The content of Theorem 1.3 below is that $\tilde{f}$ exists provided the target $L'$ is replaced with its wreath product with some symmetric group $\Sigma_n$, a construction which we now describe.

Let $X$ be a space, then $G \leq \Sigma_n$ acts on $X^n$ by permuting the factors. The wreath product of $X$ with $G$, denoted $X \wr G$, is the homotopy orbit space $(X^n)_{hG}$ (see Definition 3.4). This construction is equipped with a map $\Delta: X \to X \wr G$ which factors through the diagonal map $X \to X^n$. For example, we prove in 3.6
below that if \( X \) is an Eilenberg-MacLane space \( K(H, 1) \) then there is a homotopy equivalence \( X \circ G \simeq K(H \circ G, 1) \) such that \( \Delta \) is induced by the diagonal inclusion \( H \leq H \circ G \).

1.1. Theorem. Fix a \( p \)-local finite group \((S, \mathcal{F}, \mathcal{L})\) where \( S \neq 1 \). Let \( K \) be a subgroup of \( \Sigma_n \) and let \( S' \) be a Sylow \( p \)-subgroup of \( S \circ K \). Then there exists a \( p \)-local finite group \((S', \mathcal{F}', \mathcal{L}')\) which is equipped with a homotopy equivalence \( |\mathcal{L}| \circ K \simeq |\mathcal{L}'| \) such that the composition

\[
BS' \xrightarrow{\text{Bincl}} B(S \circ K) \simeq (BS) \circ K \xrightarrow{\Theta K} |\mathcal{L}| \circ K \simeq |\mathcal{L}'|
\]

is homotopic to the natural map \( \Theta': BS' \to |\mathcal{L}'| \). Moreover, \((S', \mathcal{F}', \mathcal{L}')\) satisfying these properties is unique up to an isomorphism of \( p \)-local finite groups.

In Remark 5.3 we show that when Theorem 1.1 is applied to a \( p \)-local finite group \((S, \mathcal{F}, \mathcal{L})\) of a finite group \( G \) then \((S', \mathcal{F}', \mathcal{L}')\) is the \( p \)-local finite group of \( G \circ K \).

We prove Theorem 1.1 in \( \S 5 \) which is highly technical, however, the remainder of the paper is completely independent of it.

1.2. Definition. We call the \( p \)-local finite group \((S', \mathcal{F}', \mathcal{L}')\) in the theorem above the \( \text{wreath product} \) of \((S, \mathcal{F}, \mathcal{L})\) with \( K \) and denote its fusion system and linking system by \( \mathcal{F} \circ K \) and \( \mathcal{L} \circ K \) respectively. Let \( \Delta: |\mathcal{L}| \to |\mathcal{L}'| \circ K \simeq |\mathcal{L}'| \) denote the diagonal inclusion followed by the homotopy equivalence in Theorem 1.1.

If \( S = 1 \) we cannot apply Theorem 1.1, but in this case \(|\mathcal{L}| = * \) and we choose \((S', \mathcal{F}', \mathcal{L}')\) to be the \( p \)-local finite group associated to \( K \) and the map \( \Delta: |\mathcal{L}| \to |\mathcal{L}'| \) is any map \(* \to |\mathcal{L}'|\).

1.3. Theorem. Let \((S, \mathcal{F}, \mathcal{L})\) and \((S', \mathcal{F}', \mathcal{L}')\) be \( p \)-local finite groups and suppose that \( \rho: S \to S' \) is a fusion preserving homomorphism. Then there exists some \( m \geq 0 \) and a map \( \hat{j}: |\mathcal{L}'|_{\rho=1} \to \mathcal{L}' \circ \Sigma_{\rho=1} |\mathcal{L}'|_{\rho=1} \) such that the diagram below commutes up to homotopy

\[
\begin{array}{ccc}
BS & \xrightarrow{\eta \circ \Theta} & |\mathcal{L}'|_{\rho=1} \\
\downarrow B_\rho & & \downarrow \Delta_ho \\
BS' & \xrightarrow{\eta \circ \Theta'} & |\mathcal{L}'|_{\rho=1} \circ \Sigma_{\rho=1} |\mathcal{L}'|_{\rho=1}
\end{array}
\]

A permutation representation of a finite group \( G \) is a homomorphism \( \rho: G \to \Sigma_n \). The rank of \( \rho \) is \( n \). In this paper we shall call \( \rho \) simply a “representation”. Clearly \( G \) acts on itself by left (or right) translations giving rise to Cayley’s embedding

\[
\text{reg}_G: G \to \Sigma_{|G|}
\]

which is called the \textit{regular permutation representation} of \( G \).

Two representations \( \rho_1, \rho_2: G \to \Sigma_n \) are \textit{equivalent} if they are conjugate in \( \Sigma_n \), that is, if they differ by an inner automorphism of \( \Sigma_n \). The set of equivalence classes of representations of \( G \) of rank \( n \) is denoted \( \text{Rep}_n(G) \). The inclusions of subgroups \( \Sigma_\alpha \times \Sigma_\beta \leq \Sigma_{\alpha+\beta} \) and \( \Sigma_\alpha \times \Sigma_\gamma \leq \Sigma_{\alpha+\gamma} \) obtained by taking the disjoint union and the product of the sets \([n] = \{1, \ldots, n\} \) and \([m] = \{1, \ldots, m\} \) give rise to commutative, associative and unital binary operations + and \( \times \) on the set \( \prod_{n \geq 0} \text{Rep}_n(G) \). We shall write \( k \cdot \rho \) for the \( k \)-fold sum \( \rho + \cdots + \rho \).
A classical result which goes back to Hurewicz states that the classifying space functor induces a bijection

$$\text{Rep}_n(G) \approx [BG, BS\Sigma_n], \quad (\rho \mapsto B\rho).$$

When the target is $p$-completed, a theorem of Dwyer and Zabrodsky [12] shows that there is also a bijection $\text{Rep}_n(P) \approx [BP, (BS\Sigma_n)^\wedge]_p$ when $P$ is a $p$-group. Therefore, given a map $f: |\mathcal{L}| \to (BS\Sigma_n)^\wedge$, $f$ admits a representation $\rho: S \to \Sigma_n$, unique up to equivalence, which renders the following square homotopy commutative

$$\begin{align*}
BS & \xrightarrow{\Theta} |\mathcal{L}| \\
\downarrow B\rho & \quad \downarrow f \\
BS\Sigma_n & \xrightarrow{n} (BS\Sigma_n)^\wedge.
\end{align*}$$

1.4. Definition. A permutation representation of a $p$-local finite group $(S, F, \mathcal{L})$ is a homotopy class of maps $f: |\mathcal{L}| \to (BS\Sigma_n)^\wedge$. We say that $f$ is $S$-regular if $n = m \cdot |S|$ for some $m \geq 0$ and $\rho$ in the diagram above is equivalent to $m \cdot \text{reg}_S$.

We shall deduce from Theorem 1.3 the following result which is a $p$-local form of Cayley’s theorem. Recall from [6, Definition 2.2] that a map $f: X \to Y$ of spaces is a homotopy monomorphism at $p$ if $H^*(X; \mathbb{F}_p)$ is a finitely generated module over $H^*(Y; \mathbb{F}_p)$ via $f^*$.

1.5. Theorem. Every $p$-local finite group $(S, F, \mathcal{L})$ admits an $S$-regular permutation representation $f: |\mathcal{L}| \to (BS\Sigma_n)^\wedge$ which is a homotopy monomorphism at $p$.

The reason we didn’t define permutation representations as maps $|\mathcal{L}| \to BS\Sigma_n$ (without $p$-completing the target) is that in general there is little hope to expect to find “interesting” such maps. For example, the nerve of the linking system of the Solomon $p$-local finite group, constructed by Levi and Oliver in [14], was shown to be simply connected in [10] and therefore [21, Theorem 8.1.11] implies that $[\mathcal{L}_{\text{Sol}}, BS\Sigma_n] = \ast$. In particular, the restriction of any $f: |\mathcal{L}_{\text{Sol}}| \to BS\Sigma_n$ to $BS$ via $\Theta$ is induced by the trivial representation $\rho: S \to \Sigma_n$.

Let $F$ be a fusion system on $S$. A representation $\rho: S \to \Sigma_n$ is called $F$-invariant if for every $P \leq S$ and every $\varphi \in F(P, S)$ the representations $\rho|_P$ and $\rho \circ \varphi$ of $P$ are equivalent. Let $\text{Rep}_n(F)$ denote the set of all the equivalence classes of the $F$-invariant representations of $S$ of rank $n$. The inclusions $\Sigma_m \times \Sigma_n \leq \Sigma_{m+n}$ and $\Sigma_m \times \Sigma_n \leq \Sigma_{mn}$ render the sets $\prod_{n\geq 0} \text{Rep}_n(F)$ with commutative, associative and unital binary operations $+$ and $\times$ such that $+$ is distributive over $\times$.

More generally, the set of representations at $p$ of rank $n$ of a space $X$ is $\text{Rep}_n(X) = [X, (BS\Sigma_n)^\wedge]_p$. Since $(BS\Sigma_m)^\wedge \times (BS\Sigma_n)^\wedge \approx (B(\Sigma_m \times \Sigma_n))^\wedge$ (see [3, Theorem 1.7.2]), the maps $(B(\Sigma_m \times \Sigma_n))^\wedge_0 \to (BS\Sigma_{m+n})^\wedge_0$ and $(B(\Sigma_m \times \Sigma_n))^\wedge_0 \to (BS\Sigma_{mn})^\wedge_0$ induced by the inclusions equip $\prod_{n\geq 0} \text{Rep}_n(X)$ with commutative and associative binary operations $+$ and $\times$ such that $+$ is distributive over $\times$.

Given $(S, F, \mathcal{L})$ we let $\text{Rep}_n(\mathcal{L})$ denote $\text{Rep}_n(|\mathcal{L}|)$.

1.6. Definition. The ring $\text{Rep}(\mathcal{L})$ of the virtual permutation representations of a $p$-local finite group $(S, F, \mathcal{L})$ is the Grothendieck group completion of the commutative monoid $(\prod_{n\geq 0} \text{Rep}_n(\mathcal{L}), +)$. 

3
The ring $\text{Rep}(F)$ of the virtual $F$-invariant representations of $S$ of a saturated fusion system $F$ on $S$ is the Grothendieck group completion of the commutative monoid $(\bigoplus_{n \geq 0} \text{Rep}_n(F), +)$.

Clearly $\text{Rep}(F)$ is a subring of $\text{Rep}(S)$. In §8 we will construct a ring homomorphism $\Phi: \text{Rep}(L) \to \text{Rep}(F)$ which sends a map $f: |L| \to (B\Sigma_n)^F_p$ to the representation $\rho: S \to \Sigma_n$ such that $f \circ \Theta = \eta \circ B\rho$ as in Definition 1.4. We shall also see that $\text{reg}_S: S \to \Sigma_{|S|}$ generates an ideal $\text{Rep}^\text{reg}(F)$ in $\text{Rep}(F)$ whose underlying group is isomorphic to $\mathbb{Z}$.

The idea behind the next definition is that if $H$ is a subgroup of index $n$ in a finite group $G$ then $\text{reg}_G|_H \cong n \cdot \text{reg}_H$. Therefore the image of the restriction map $\text{Rep}(G) \to \text{Rep}(H)$ intersects $\text{Rep}^\text{reg}(H) := \{k \cdot \text{reg}_H\}_{k \in \mathbb{Z}}$ in a subgroup of index divisible by $n$.

1.7. Definition. The lower index of $S$ in $L$ denoted $\text{Lind}(L; S)$ is the index of $\text{Im}(\Phi) \cap \text{Rep}^\text{reg}(F)$ in $\text{Rep}^\text{reg}(F)$.

We will prove in Lemma 8.5 that $\text{Lind}(L; S)$ is a $p$-power. We conjecture that it is always equal to 1. A partial result is the theorem below.

1.8. Theorem. Let $(S, F, L)$ be a $p$-local finite group. Then $\text{Lind}(L; S) = 1$ if either

(a) $(S, F, L)$ is associated with a finite group.
(b) $(S, F, L)$ is one of the exotic examples in [20] or in [7] or in [8].

2. Preliminaries on $p$-local finite groups

We start with the notion of a saturated fusion system which is due to Puig [17] (see also [7]).

2.1. Definition. A fusion system $F$ on a finite $p$-group $S$ is a category whose objects are the subgroups of $S$ and the set of morphisms $F(P, Q)$ between two subgroups $P$, $Q$, satisfies the following conditions:

(a) $F(P, Q)$ consists of group monomorphisms and contains the set $\text{Hom}_S(P, Q)$ of all the homomorphisms $c_s: P \to Q$ which are induced by conjugation by elements $s \in S$.
(b) Every morphism in $F$ factors as an isomorphism in $F$ followed by an inclusion.

In a fusion system $F$ over a $p$-group $S$, we say that two subgroups $P, Q \leq S$ are $F$-conjugate if there is an isomorphism between them in $F$. Let $\text{Syl}_p(G)$ the set of the Sylow $p$-subgroups of a group $G$. Given $P \leq G$ and $g \in G$, $c_g \in \text{Hom}(P, G)$ is the monomorphism $c_g(x) = gxg^{-1}$. We write $\text{Out}_F(P) = \text{Aut}_F(P)/\text{Inn}(P)$.

2.2. Definition. Let $F$ be a fusion system on a $p$-group $S$. A subgroup $P \leq S$ is fully centralized in $F$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $F$-conjugate to $P$. A subgroup $P \leq S$ is fully normalized in $F$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $F$-conjugate to $P$.

A fusion system $F$ on $S$ is saturated if:

(I) Each fully normalized subgroup $P \leq S$ is fully centralized and $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_F(P))$. 


(II) For $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$ set
\[ N_{\varphi} = \{g \in N_S(P) | \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi(P)) \}. \]

If $\varphi(P)$ is fully centralized then there is $\tilde{\varphi} \in \mathcal{F}(N_{\varphi}, S)$ such that $\tilde{\varphi}|_P = \varphi$.

2.3. Definition. Let $\mathcal{F}$ be a fusion system on a $p$-group $S$. A subgroup $P \leq S$ is $\mathcal{F}$-centric if $P$ and all its $\mathcal{F}$-conjugates contain their $S$-centralizers. A subgroup $P \leq S$ is $\mathcal{F}$-radical if $\text{Out}_F(P)$ has no non-trivial normal $p$-subgroup.

2.4. Definition. [7] Let $\mathcal{F}$ be a fusion system on a $p$-group $S$. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor $\pi: \mathcal{L} \to \mathcal{F}^c$ and monomorphisms $P \xrightarrow{\delta_P} \text{Aut}_\mathcal{L}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(A) $\pi$ is the identity on objects. For each pair of objects $P, Q \in \mathcal{L}$, the action of $Z(P)$ on $\mathcal{L}(P,Q)$ via precomposition and $\delta_P: P \to \text{Aut}_\mathcal{L}(P)$ is free and $\pi$ induces a bijection $\mathcal{L}(P,Q)/Z(P) \xrightarrow{\simeq} \mathcal{F}(P,Q)$.

(B) If $P \leq S$ is $\mathcal{F}$-centric then $\pi(\delta_P(g)) = c_g \in \text{Aut}_\mathcal{F}(P)$ for all $g \in P$.

(C) For each $f \in \mathcal{L}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:
\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\delta_P(g) \downarrow & & \downarrow \delta_Q(\pi(f)(g)) \\
P & \xrightarrow{f} & Q
\end{array}
\]

A $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ consists of a saturated fusion systems $\mathcal{F}$ on $S$ together with an associated linking system.

2.5. Remark. For $P, Q \leq S$, let $N_S(P, Q)$ denote the set of the elements $s \in S$ such that $sPs^{-1} \leq Q$. In [7, Proposition 1.11] it is shown that $(S, \mathcal{F}, \mathcal{L})$ can be equipped with injections $\delta_{P,Q}: N_S(P, Q) \to \mathcal{L}(P,Q)$ where $P, Q \leq S$ are $\mathcal{F}$-centric such that $\delta_{P,Q}$ extends the monomorphisms $\delta_P: P \to \text{Aut}_\mathcal{L}(P)$. We denote $\delta_{P,Q}(s)$ by $\hat{s} \in \mathcal{L}(P,Q)$. The construction of the $\delta_{P,Q}$'s has the property that $\hat{s}_1 \circ \hat{s}_2 = \hat{s}_{1 \circ s_2}$. Also, if $P \leq Q$ we write $\iota_P^Q$ for $\delta_{P,Q}(1)$. This gives a choice of lifts in $\mathcal{L}$ for the inclusion of $\mathcal{F}$-centric subgroups in $\mathcal{F}$. This choice is “compatible” in the sense that $\iota_{Q}^{P} \circ \iota_{P}^{Q} = \iota_{Q}^{P}$.

2.6. Remark. Every morphism in $\mathcal{L}$ is both a monomorphism and an epimorphism (but not necessarily an isomorphism). This is shown in [7, remarks after Lemma 1.10] and [4, Corollary 3.10]. We shall use this fact repeatedly throughout.

The orbit category of a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ is denoted by $\mathcal{O}(\mathcal{F})$. This is the category whose objects are the subgroups of $S$ and whose morphisms are
\[ \mathcal{O}(\mathcal{F})(P, Q) = \text{Rep}_F(P,Q) \xrightarrow{\text{def}} \text{Inn}(Q) \setminus \mathcal{F}(P,Q). \]

Also, $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of $S$.

2.7. Proposition. [7, Proposition 2.2] Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. There exists a functor $\tilde{\mathcal{B}}: \mathcal{O}(\mathcal{F}^c) \to \text{Top}$ which is isomorphic in the homotopy category of spaces to the functor $P \mapsto BP$, and such that there is a homotopy equivalence
\[ \text{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{\mathcal{B}} \xrightarrow{\simeq} |\mathcal{L}|. \]
2.8. **Notation.** For a finite group $G$, let $BG$ denote the category with one object $\bullet_G$ and $G$ as its set of automorphisms. For an $F$-centric $P \leq S$ the monomorphism $\delta_P$ gives rise to a functor $BP \to \mathcal{L}$ which, by abuse of notation, we denote by $\delta_P$. For $P = S$, upon taking nerves of categories, we obtain a map
\[ \Theta: BS \to |\mathcal{L}| \]
and we write $\Theta|_{\mathcal{B}Q}$ for $\Theta \circ \text{Bind}^S_{\mathcal{B}Q}$.

If $Q$ is $F$-centric, then the natural isomorphism of functors in Proposition 2.7 shows that $\Theta|_{\mathcal{B}Q}$ is homotopic to $\mathcal{B}Q \simeq B(Q) \to \text{hocolim}_{t \in \mathcal{F}} \tilde{B} = |\mathcal{L}|$. Therefore, for any $F$-centric $Q \leq S$ and any morphism $\rho: Q \to S$ in $\mathcal{F}$ we have $\Theta \circ B\rho \simeq \Theta|_{\mathcal{B}Q}$. In particular, $\Theta|_{\mathcal{B}Q} \circ B\psi \simeq \Theta|_{\mathcal{B}Q}$ for any $\psi \in \text{Iso}_F(Q, Q')$. It follows from Alperin's fusion theorem for saturated fusion systems [7, Theorem A.10] that:

2.9. **Proposition.** For any $Q, Q' \leq S$ and any $\rho \in \mathcal{F}(Q, Q')$ there is a homotopy equivalence $\Theta|_{\mathcal{B}Q'} \circ B\rho \simeq \Theta|_{\mathcal{B}Q}$.

2.10. **Notation.** Given a map $f: X \to Y$ of spaces, let $\text{map}^f(X, Y)$ denote the path component of $f$ in $\text{map}(X, Y)$. By convention $f$ is the basepoint of this space.

The following proposition on mapping spaces will be needed in §7.

2.11. **Proposition.** Fix a $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ and let $P$ be a finite $p$-group. Given a homomorphism $\rho: P \to S$, set $Q = \rho(P) \leq S$. Then:

(a) There is a homotopy equivalence
\[ \text{map}^\rho(\rho^*\mathcal{B}P, |\mathcal{L}|_p^\wedge) \simeq \text{map}^{\rho \circ \Theta|_{\mathcal{B}Q}}(\mathcal{B}Q, |\mathcal{L}|_p^\wedge), \]
and this space is the $p$-completed classifying space of a $p$-local finite group.

(b) After $p$-completion, the map
\[ \text{map}^{\Theta|_{\mathcal{B}Q}}(\mathcal{B}Q, |\mathcal{L}|_p^\wedge) \hto \text{map}^{\rho \circ \Theta|_{\mathcal{B}Q}}(\mathcal{B}Q, |\mathcal{L}|_p^\wedge). \]
induces a split surjection on homotopy groups.

**Proof.** (a) First of all, we can choose a fully centralized subgroup $Q' \leq S$ in $\mathcal{F}$ and an isomorphism $\psi: Q \to Q'$ in $\mathcal{F}$. Let $\rho': P \to S$ denote the composition $P \xrightarrow{\rho} Q \xrightarrow{\psi} Q' \leq S$. By Proposition 2.9 observe that
\[ (1) \quad \Theta|_{\mathcal{B}Q} \simeq \Theta|_{\mathcal{B}Q'} \circ B\rho'. \]
Hence, $\Theta \circ B\rho \simeq \Theta \circ B\rho'$. It follows from [7, Theorem 6.3] that there are homotopy equivalences
\[ \text{map}^{\rho \circ \Theta|_{\mathcal{B}Q}}(\mathcal{B}P, |\mathcal{L}|_p^\wedge) \simeq \text{map}^{\rho \circ \Theta|_{\mathcal{B}Q}}(\mathcal{B}P, |\mathcal{L}|_p^\wedge) \simeq \text{map}^{\rho \circ \Theta|_{\mathcal{B}Q'}}(\mathcal{B}Q', |\mathcal{L}|_p^\wedge) \simeq \text{map}^{\rho \circ \Theta|_{\mathcal{B}Q}}(\mathcal{B}Q, |\mathcal{L}|_p^\wedge) \]
where the first equivalence is implied by equation (1) and the third one follows since $B\psi: \mathcal{B}Q \to \mathcal{B}Q'$ is a homotopy equivalence. Also by [7, Theorem 6.3], this space is homotopy equivalent to the classifying space of a $p$-local finite group $|\mathcal{C}_L(Q')|_p^\wedge$.

(b) We can assume from (1), by replacing $Q$ with $Q'$ if necessary, that $Q$ is fully centralised in $\mathcal{F}$. In [7, pp. 822] a functor
\[ \Gamma: C_L(Q) \times \mathcal{B}Q \to \mathcal{L} \]
is constructed where $C_L(Q)$ is the centraliser linking system [7, Definition 2.4] of $Q$ in $\mathcal{F}$. By $p$-completing the geometric realisation of $\Gamma$ and taking adjoints we obtain
a commutative square in which the bottom row is a homotopy equivalence by [7, Theorem 6.3]

\[ |C_\mathcal{L}(Q)| \xrightarrow{|\Gamma|^\#} \text{map}^{\Theta|\mathcal{L}|}(BQ, |\mathcal{L}|) \]
\[ \phi \quad \downarrow \quad \eta_s \]
\[ |C_\mathcal{L}(Q)|_p \xrightarrow{((|\Gamma|^\#)_p} \text{map}^{p\Theta|\mathcal{L}|}(BQ, |\mathcal{L}|_p). \]

Since \(|C_\mathcal{L}(Q)|\) is \(p\)-good by [7, Proposition 1.12], upon \(p\)-completion of the diagram (2), we see that the vertical arrow on the left becomes an equivalence and therefore the composition \((\eta_s)_p^\# \circ (|\Gamma|^\#)_p^\#\) is a homotopy equivalence. In particular \((\eta_s)_p^\#\) is split surjective on homotopy groups.

We end this section with a description of the product of \(p\)-local finite groups.

2.12. Let \(\mathcal{F}_i\) be a saturated fusion system on a finite \(p\)-group \(S_i\) for \(i = 1, \ldots, n\). Define \(S = \prod_{i=1}^n S_i\) and consider the product category \(\prod_{i=1}^n \mathcal{F}_i\). Its objects are the subgroups of \(S\) of the form \(\prod\overline{P}_i\), where \(P_i \leq S_i\), and morphisms have the form \(\prod\overline{P}_i \xrightarrow{\prod \overline{\phi}_i} \prod\overline{Q}_i\), where \(\overline{\phi}_i \in \mathcal{F}_i(P_i, Q_i)\).

2.13. Notation. For \(P \leq S = \prod_{i=1}^n S_i\), we denote by \(P^{(i)}\) the image of \(P\) under the projection \(p^{(i)}: S \to S_i\). Clearly \(P \leq \prod_{i=1}^n P^{(i)}\).

Let \(\mathcal{F}\) be the fusion system on \(S\) generated by \(\prod_i \mathcal{F}_i\). Thus, every morphism \(\varphi \in \mathcal{F}(P, Q)\) is given by the restriction of a morphism \(\prod_i P^{(i)} \xrightarrow{\prod \overline{\phi}_i} \prod_i Q^{(i)}\) in \(\prod_i \mathcal{F}_i\). The \(\overline{\phi}_i\)’s are unique in the sense that they are completely determined by \(\varphi\) because \(p^{(i)}|_P: P \to P^{(i)}\) are by definition surjective and \(p^{(i)}|_Q \circ \varphi = \varphi_i \circ p^{(i)}|_P\).

We see that \(\varphi \mapsto (\varphi_i)_{i=1}^n\) induces an inclusion \(\mathcal{F}(P, Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})\). In particular, \(\prod_i \mathcal{F}_i\) is a full subcategory of \(\mathcal{F}\).

We shall write \(\times_{i=1}^n \mathcal{F}_i\) for the fusion system \(\mathcal{F}\) just defined and we call it the product fusion system of the \(\mathcal{F}_i\)’s.

2.14. Lemma. With the notation above, \((S, \mathcal{F})\) is a saturated fusion system. If \(P \leq S\) is \(\mathcal{F}\)-centric then all the groups \(P^{(i)}\) are \(\mathcal{F}_i\)-centric for \(i = 1, \ldots, n\).

The assignment \(P \mapsto \prod_i P^{(i)}\) and the inclusions \(\mathcal{F}(P, Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})\) give rise to a functor \(r: \mathcal{F}^c \to \prod_i \mathcal{F}_i^c\) which is a retract of the inclusion \(\prod_i \mathcal{F}_i \subseteq \mathcal{F}^c\).

Proof. In [7, Lemma 1.5] it is proven that \(\mathcal{F} = \times_i \mathcal{F}_i\) is a saturated fusion system on \(S\).

The assignments \(P \mapsto \prod_i P^{(i)}\) and \(\varphi \mapsto \prod_i \varphi_i\) give rise to a functor \(r: \mathcal{F} \to \prod_i \mathcal{F}_i\) which by inspection is a retract to the inclusion \(j: \prod_i \mathcal{F}_i \to \mathcal{F}\). It remains to show that \(j\) and \(r\) restrict to \(\prod_i \mathcal{F}_i^c\) and \(\mathcal{F}^c\).

Observe that \(C_S(P) = \prod_i C_{S_i}(P^{(i)})\) for any \(P \leq S\). If \(P\) is \(\mathcal{F}\)-centric then

\[ \prod_{i=1}^n C_{S_i}(P^{(i)}) = C_S(P) \leq \prod_{i=1}^n P^{(i)}. \]

Therefore \(C_{S_i}(P^{(i)}) \leq P^{(i)}\) for all \(i\). Now, if \(Q_i\) are \(\mathcal{F}_i\)-conjugate to \(P^{(i)}\) via isomorphisms \(\varphi_i \in \mathcal{F}_i(P^{(i)}, Q_i)\) then \((\varphi_1 \times \ldots \times \varphi_n)|_P\) is an \(\mathcal{F}\)-isomorphism onto some \(Q \leq S\) such that \(Q^{(i)} = Q_i\). By definition \(Q\) is also \(\mathcal{F}\)-centric and applying (1) to \(Q\) we obtain that \(C_S(Q) \leq Q_i\) for all \(i\). We deduce that \(P^{(i)}\) are \(\mathcal{F}_i\)-centric.
Assume now that $P_i \leq S_i$ are $\mathcal{F}_i$-centric for all $i = 1, \ldots, n$. Then $P = \prod_i P_i$ is
$\mathcal{F}$-centric because if $Q$ is $\mathcal{F}$-conjugate to $P$ then it has the form $\prod_i Q_i$ where $Q_i$
are $\mathcal{F}_i$-conjugate to $P_i$ and therefore $C_{S}(Q) = \prod_i C_{S_i}(Q_i) \leq Q$. □

While the construction of the product of saturated fusion systems appears in [7],
we were not able to find a construction of the product of $p$-local finite groups in
the literature.

2.15. **Definition.** Let $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be $p$-local finite groups for $i = 1, \ldots, n$. Their
product $\times_{i=1}^n (S_i, \mathcal{F}_i, \mathcal{L}_i)$ is the $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ where
$S = \prod_{i=1}^n S_i$ and $\mathcal{F} = \times_{i=1}^n \mathcal{F}_i$. The centric linking system $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$
is defined as the following pullback of small categories where $r$ is defined in Lemma 2.14

$$
\begin{array}{ccc}
\times_{i=1}^n \mathcal{L}_i & \xrightarrow{r_c} & \prod_{i=1}^n \mathcal{L}_i \\
\pi & & \downarrow \Pi_{i=1}^n \pi_i \\
(\times_{i=1}^n \mathcal{F}_i)^c & \xrightarrow{r} & \prod_{i=1}^n \mathcal{F}_i.
\end{array}
$$

The functor $\pi: \mathcal{L} \to \mathcal{F}$ is defined by the pullback and the monomorphisms $\delta_P: P \to
\text{Aut}_\mathcal{L}(P)$ are defined by the compositions

$$P \leq \prod_i P^{(i)} \xrightarrow{\Pi_i \delta_P^{(i)}} \prod_i \text{Aut}_\mathcal{L}_i(P^{(i)}).
$$

We need to prove that axioms (A)-(C) of Definition 2.4 hold.

**Proof.** We first note that for any $\mathcal{F}$-centric subgroups $P, Q \leq S$ the set $\mathcal{L}(P, Q)$
is the pullback in

$$
\begin{array}{ccc}
\mathcal{L}(P, Q) & \xrightarrow{r} & \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)}) \\
\pi & & \downarrow \Pi_i \pi
\end{array}
$$

We start by proving that the monomorphisms $\delta_P$ are well-defined. That is, given
$g \in \{g_i\} \in P \leq S$ where $P$ is $\mathcal{F}$-centric, $\prod_i \delta_P^{(i)}(g_i) \in \text{Aut}_\mathcal{L}(P)$. The pullback
diagram (1) shows that it is enough to check that $\prod_i \pi_i(\delta_P^{(i)}(g_i)) \in r((\times_{i=1}^n \mathcal{F}_i)^c)$. It
follows from the fact that $\pi_i(\delta_P^{(i)}(g_i)) = c_{g_P} \in \text{Aut}_\mathcal{F}_i(P^{(i)})$ and $r(c_g) = \prod_i c_{g_P}$. This
also shows that axiom (B) holds since $\pi(\delta_P(g)) = \prod_i \pi_i(\delta_P^{(i)}(g_i))_{|P} = c_{g_P} \circ_P$. We continue to prove
that $(S, \mathcal{F}, \mathcal{L})$ satisfies axioms (A) and (C). It follows from the
definition that $\pi$ is the identity on objects. Observe that $\prod_i C_{S_i}(P^{(i)})$ acts
transitively and freely on the fibre of the right-hand arrow in (1) because axiom (A)
holds in $(S_i, \mathcal{F}_i, \mathcal{L}_i)$. Now, axiom (A) for $(S, \mathcal{F}, \mathcal{L})$ follows from the fact that
$C_S(P) = \prod_i C_{S_i}(P^{(i)})$ and that diagram (1) is a pullback square so the fibres of
the vertical arrows are isomorphic.

Finally, axiom (C) for $(S, \mathcal{F}, \mathcal{L})$ follows by applying axiom (C) to each component of
a morphism $f \in \mathcal{L}(P, Q)$ and each $g \in P \leq \prod_i P^{(i)}$. □

2.16. **Remark.** A choice of compatible lifts for inclusion $\{i_{P_i}^Q\}$ in every $\mathcal{L}_i$ (see 2.5)
gives rise to a choice $\{i_P^Q\}$ of compatible lifts for the inclusions in $(S, \mathcal{F}, \mathcal{L})$ where
$i_P^Q = (i_{P^{(i)}}^Q)_{i=1}^n$.  

8
2.17. Proposition. Given $p$-local finite groups $(S_i, F_i, \mathcal{L}_i)$ for $i = 1, \ldots, n$, the category $\prod_i \mathcal{L}_i$ is a full subcategory of $\times_i \mathcal{L}_i$ and the inclusion $j: \prod_i \mathcal{L}_i \to \times_i \mathcal{L}_i$ induces a homotopy equivalence on nerves. In particular, $\prod_{i=1}^n |\mathcal{L}_i| \simeq |\times_{i=1}^n \mathcal{L}_i|$.

Proof. Set $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$. The category $\prod_i \mathcal{L}_i$ is a full subcategory of $\times_i \mathcal{L}_i$. The assignment $P \mapsto \prod_i P(i)$ and the inclusion $\mathcal{L}(P, Q) \subseteq \prod_{i=1}^n \mathcal{L}_i(P(i), Q(i))$ give rise to a functor $r_{\mathcal{L}}: \mathcal{L} \to \prod_{i=1}^n \mathcal{L}_i$, see the pullback diagram in Definition 2.15 which is a retract to the inclusion $j$ by Lemma 2.14. Also there is a natural transformation $\text{Id} \to j \circ r$ which is defined on an object $P \in \mathcal{L}$ by $i_P(P): P \to r(P) = \prod_{i=1}^n P(i)$ (see Remark 2.16). This shows that $|r|$ is a homotopy inverse to $|j|: \prod_i |\mathcal{L}_i| \to |\mathcal{L}|$. □

2.18. Remark. Given a $p$-local finite group $(S, F, \mathcal{L})$, Definition 2.15 allows us to consider its $n$-fold product with itself denoted $(S^{\times n}, F^{\times n}, \mathcal{L}^{\times n})$. By construction, the action of the symmetric group $\Sigma_n$ on $S^{\times n}$ extends to an action on the fusion system $F^{\times n}$ and the linking system $\mathcal{L}^{\times n}$ by permuting the factors. Moreover, the functor $\pi: \mathcal{L}^{\times n} \to F^{\times n}$ and the distinguished monomorphisms $\delta_P: P \to \text{Aut}_{\mathcal{L}^{\times n}}(P)$ for every $F^{\times n}$-centric $P \leq S^{\times n}$ are $\Sigma_n$-equivariant from the construction in Definition 2.15. Therefore, also the inclusion $BS^{\times n} \xrightarrow{\delta_{BS^n}} B\text{Aut}_{\mathcal{L}^{\times n}}(S^{\times n}) \to L^{\times n}$ is $\Sigma_n$-equivariant and so is the induced map $\Theta: BS^{\times n} \to |L^{\times n}| \simeq |L|^{\times n}$.

The choice of $i_P$ in $L^{\times n}$ made in Remark 2.16 is easily seen to be invariant under the action of $\Sigma_n$ as well.

Finally, the functor $j$ and the homotopy equivalence in Proposition 2.17 are also equivariant with respect to the action of $\Sigma_n$ by permuting coordinates.

3. The wreath product of spaces

Let $G$ be a finite group and $X$ a $G$-space. The Borel construction $X_{hG}$ is the orbit space of $EG \times X$ where $EG$ is a contractible space on which $G$ acts freely on the right. Recall from 2.8 that $BG$ is the small category with one object and $G$ as a morphism set. Then $X$ can be viewed as a functor $X: BG \to \text{Top}$ and the Borel construction is a model for $\text{hocolim}_{BG}X$. There is a natural map $X_{hG} \to X/G$ to the orbit space of $X$ induced by the map $EG \to *$.

A standard model for $EG$ is given by the nerve of the category $\mathcal{E}G$ whose object set is $G$ and there exists a unique morphism between any two objects. This construction is natural so that if $H \leq G$ then $EH$ is an $H$-subspace of $EG$. Moreover, the identity element of $G$ renders $EG$ with a natural choice of a basepoint (which is not invariant under $G$.) This basepoint provides an augmentation map $\kappa(X): X \to X_{hG}$ which fits into a fibration sequence

$$X \xrightarrow{\kappa(X)} X_{hG} \to BG.$$  

A fixed point $x \in X$ corresponds to a $G$-map $* \to X$ and gives rise to a section $s: BG \to X_{hG}$ for this fibration.

If $N \triangleleft G$ then $EG \times_N X$ is a model for $X_{hN}$ on which $G/N$ acts freely in a natural way. As a consequence we obtain a composite homotopy equivalence

$$X_{hN})_{hG/N} \xrightarrow{\sim} (EG \times_N X)_{hG/N} \xrightarrow{\sim} (EG \times_N X)/\Omega \simeq EG \times_G X = X_{hG}.$$

Moreover, note that $(EG \times_N X)/\Omega = EG \times_G X = X_{hG}$ and that the composition in the bottom row of the following commutative diagram is by inspection equal to
the map \( \kappa: X \to X_{hG} \)

\[
X \xrightarrow{\kappa} X_{hN} = EN \times_N X \xrightarrow{\kappa} (EN \times_N X)_{hG/N} \xrightarrow{\cong} (EN \times_N X) / \sim = (EN \times_N X)_{hG/N}.
\]

This shows that

\[
(3.3) \quad X \xrightarrow{\kappa} X_{hN} \xrightarrow{\kappa} (X_{hN})_{hG/N} \xrightarrow{\cong} X_{hG} \quad \text{is equal to} \quad X \xrightarrow{\kappa} X_{hG}.
\]

### 3.4. Definition

The wreath product of a space \( X \) with a subgroup \( G \) of \( \Sigma_k \) is the space

\[
X \wr G := (X^k)_{hG}
\]

where \( G \) acts by permuting the factors of \( X^k \). The diagonal map \( \Delta_X: X \to X^k \) and \( \kappa: X^k \to X \wr G \) give rise to a natural map

\[
\Delta(X): X \to X \wr G.
\]

We shall use a left normed notation for iteration of the wreath product construction. That is, by convention, \( X \wr G_1 \wr G_2 \cdots \wr G_n \) denotes \((\cdots ((X \wr G_1) \wr G_2) \cdots) \wr G_n \).

### 3.5. Proposition

Given permutation groups \( G_i \leq \Sigma_k \), where \( i = 1, \ldots, n \), there is a homotopy equivalence

\[
\alpha_n: X \wr G_1 \wr G_2 \cdots \wr G_n \xrightarrow{\cong} X \wr (G_1 \wr G_2 \cdots \wr G_n)
\]

which is natural in \( X \). Moreover, the composition

\[
X \xrightarrow{\Delta} X \wr G_1 \xrightarrow{\Delta} ((X \wr G_1) \wr G_2) \xrightarrow{\Delta} \cdots \xrightarrow{\Delta} X \wr G_1 \wr G_2 \cdots \wr G_n \xrightarrow{\alpha_n} X \wr (G_1 \wr G_2 \cdots \wr G_n)
\]

is homotopic to \( \Delta: X \to X \wr (G_1 \wr G_2 \cdots \wr G_n) \).

**Proof.** We start with \( n = 2 \). Define \( G = G_1 \wr G_2 \) and set \( N = G_1^{k_2} \). Since \( EN = (EG_1)^{k_2} \), we obtain a homeomorphism

\[
(((X^{k_1})_{hG_1})^{k_2})_{hN} \cong (X^{k_1 k_2})_{hN}
\]

which is \( \Sigma_{k_2} \)-equivariant and where \( N \) acts on \( \prod_{k_3} X^{k_1} \) via \( k_2 \) copies of the action of \( G_1 \) on \( X^{k_1} \). Clearly \( G/N \cong G_2 \leq \Sigma_{k_2} \) acts on this space by permuting the factors and the homotopy equivalence \( \alpha_2: X \wr G_1 \wr G_2 \cong X \wr (G_1 \wr G_2) \) is defined with the aid of \( (3.2) \) by

\[
(((X^{k_1})_{hG_1}^{k_2})_{hG_2} = (((X^{k_1 k_2})_{hG_1})_{hG_2} \xrightarrow{\cong} (X^{k_1 k_2})_{hG}).
\]

Furthermore the triangle below commutes by \( (3.3) \)

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta(x)} & X \wr G_1 \wr G_2 \\
\downarrow{\Delta(x)} & & \downarrow{\alpha_2} \\
X \wr (G_1 \wr G_2) & & \\
\end{array}
\]

We define \( \alpha_n \) for \( n \geq 2 \) inductively by the composition

\[
X \wr G_1 \cdots \wr G_n \xrightarrow{\alpha_{n-1} \wr G_n} X \wr (G_1 \cdots \wr G_{n-1}) \wr G_n \xrightarrow{\alpha_n} X \wr (G_1 \cdots \wr G_n).
\]
Consider the following commutative diagram where the triangle on the left commutes by induction hypothesis

\[
\begin{array}{ccc}
\alpha & \rightarrow & G_n \\
\downarrow & & \downarrow \\
\alpha_n & \rightarrow & G_n \\
\end{array}
\]

\[
\begin{array}{ccc}
\Delta & \rightarrow & G_n \\
\downarrow & & \downarrow \\
\triangle & \rightarrow & G_n \\
\end{array}
\]

The property of \(\alpha_n\) stated in the proposition follows from (1) applied to the composition at the bottom row of this diagram. \(\square\)

3.6. **Remark.** Clearly \(\Sigma_k\) fixes all the points in the image of the diagonal map \(X \rightarrow X^k\). If \(X \neq \emptyset\), then the fibre sequence (3.1) \(X^k \rightarrow X \vee G \rightarrow BG\) splits for any \(G \leq \Sigma_k\) and the long exact sequence in homotopy groups gives rise to isomorphisms

\[\pi_1(X \vee G) \cong (\pi_1X) \vee G\] and

\[\pi_i(X \vee G) \cong (\pi_iX)^k\] for all \(i \geq 2\).

Moreover, \(\kappa: X^k \rightarrow X \vee G\) induces inclusions \(\prod_{i} \pi_* X \leq \pi_*(X \vee G)\) on which \(G \leq \pi_1(X \vee G)\) acts by permuting the factors.

In particular, if \(X = BH\) for a discrete group \(H\), there is a homotopy equivalence \((BH) \vee G \simeq B(\{H\} G)\) and \(\Delta: BH \rightarrow (BH) \vee G \simeq B(\{H\} G)\) is homotopic to the map induced by the diagonal inclusion \(H \leq \{H\} G\).

Let \(Y\) be a \(G\)-space. For any space \(X\), the evaluation map \(X \times \text{map}(X,Y) \xrightarrow{\text{ev}_G} Y\) is clearly \(G\)-equivariant. Therefore it gives rise to a map \(\text{ev}_{hG}: X \times \text{map}(X,Y)_{hG} \rightarrow Y_{hG}\) whose adjoint is denoted

\[(\text{ev}_{hG})^\#: \text{map}(X,Y)_{hG} \rightarrow \text{map}(X,Y_{hG})\]

If the component map\(^f(X,Y)\) of some \(f: X \rightarrow Y\) is invariant under the \(G\)-action then inspection of the adjunction shows that \((\text{ev}_{hG})^\#: \text{map}(X,Y)_{hG} \rightarrow \text{map}(X,Y_{hG})\). Moreover, the composite

\[(3.7) \quad \text{map}(X,Y)_{hG} \xrightarrow{\kappa} \text{map}(X,Y)_{hG} \xrightarrow{(\text{ev}_{hG})^\#} \text{map}(X,Y_{hG})\]

coincides with the natural map induced by \(Y_{hG} \xrightarrow{\kappa(Y)} Y_{hG}\) when applying \(\text{map}(X, -)\).

3.8. **Proposition.** Fix a map \(f: A \rightarrow X\) and \(G \leq \Sigma_k\). Denote the adjoint of

\[A \times \text{map}(A,X) \vee G = A \times \text{map}(A,X_{hG}) \xrightarrow{\text{ev}_{hG}} (X_{hG})_{hG} = X \vee G\]

by \(\gamma: \text{map}(A,X) \vee G \rightarrow \text{map}(A,X_{hG})\). Then:

(a) \text{The triangle}

\[
\begin{array}{ccc}
\text{map}(A,X) & \xrightarrow{\Delta} & \text{map}(A,\Delta(X)) \\
\downarrow & & \downarrow \\
\text{map}(A,X \vee G) & \xrightarrow{\gamma} & \text{map}(A,X \vee G) \\
\end{array}
\]

is commutative.

(b) If the natural map \(BG \rightarrow \text{map}(A,BG)\) into the the space of the constant maps induces a homotopy equivalence then \(\gamma\) is a homotopy equivalence.
Proof. (a) Note that \( \prod_k \text{map}^f(A, X) = \text{map}^{\Delta \times o f}(A, X^k) \) and that this component is invariant under the action of \( G \leq \Sigma_k \). The commutativity of the triangle follows from (3.7) and Definition 3.4.

(b) Consider the following ladder in which the rows are fibre sequences and \( \pi_* \) is induced by \( X \to *. \)

\[
\begin{array}{c}
\text{map}^f(A, X)^k \\
\text{incl}
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\gamma
\end{array} \quad \begin{array}{c}
\text{map}^f(A, X \wr G) \\
\pi_*
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\text{map}^f(A, BG)
\end{array}
\]

It commutes because the right hand square commutes as a consequence of the commutativity of the following square and adjunction

\[
\begin{array}{c}
A \times \text{map}^{\Delta \times o f}(A, X^k)_{hG}
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\text{ev}_{hG}
\end{array} \quad \begin{array}{c}
A \times \text{map}(A, *)_{hG}
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\pi_{hG} = BG
\end{array}
\]

Now, \( F \) is a union of path components of \( \text{map}(A, X^k) \) because it is the fibre of the fibration \( \text{map}(A, X \wr G) \to \text{map}(A, BG) \) over the component of the constant map. Moreover, \( F \) clearly contains the component \( \text{map}^{\Delta \times o f}(A, X^k) \) and inspection of \( \gamma \) shows that the map between the fibres is simply the inclusion. Comparison of the long exact sequences in homotopy of the fibre sequences in (1) shows that \( F \) is connected, whence \( F = \text{map}^f(A, X)^k. \) Application of the five lemma to the exact sequences in homotopy now yields the result.

3.9. Remark. The hypothesis on \( A \) in part (b) of Proposition 3.8 is satisfied by all classifying spaces \( BK \) of finite groups since \( \text{map}^c(BK, BG) \cong BG \).

4. Killing homotopy groups

The aim of this section is to study the effect on homotopy groups of the map \( X \xrightarrow{\Delta(X)} X \wr \Sigma_n \xrightarrow{\eta} (X \wr \Sigma_n)_p \) where \( \Delta(X) \) was defined in the last section and \( \eta \) is the \( p \)-completion map.

4.1. Proposition. Let \( X \) be a pointed space. Then the kernel of \( \pi_* X \to \pi_*(X^\wedge_p) \) contains all the elements whose order is prime to \( p \).

Proof. Let \( [\Theta] \in \pi_*(X) \) be an element of order \( k \) prime to \( p \). Then the map \( \Theta: S^n \to X \) factors through the Moore space \( M(\mathbb{Z}/k, n) \), which is a nilpotent space with the same mod \( p \) homology of a point. It follows that \( \eta \circ \Theta: S^n \to X^\wedge_p \) factors through \( M(\mathbb{Z}/k, n)_p \simeq * \) (see [3, Ch. VI.5]), and therefore is nullhomotopic.

An element of exponent \( n \) in a group \( G \) is an element whose order divides \( n \). For the proof of the next result, recall that for any space, \( \pi_1(X) \) acts on the groups \( \pi_* X \), see e.g. [21, Corollary 7.3.4] or [23, Ch. III]. We write \( \alpha^n \) for the action of \( \omega \in \pi_1 X \) on \( \alpha \in \pi_n X \).

4.2. Lemma. Fix an integer \( n \geq 3 \) and a pointed space \( X \). Then the kernel of \( \pi_* X \xrightarrow{\Delta(X)} \pi_*(X \wr \Sigma_n) \xrightarrow{\eta_*} \pi_*((X \wr \Sigma_n)_p) \) contain all the elements of exponent \( n \) in \( \pi_* X \).
\begin{proof}
We recall from Remark 3.6 that
\[ \pi_1(X \wr \Sigma_n) = (\pi_1 X) \wr \Sigma_n \]
\[ \pi_i(X \wr \Sigma_n) = \oplus_n \pi_i X \quad \text{for } i \geq 2. \]
Furthermore, \( \kappa: \prod_n X \to X \wr \Sigma_n \) induces the inclusion \( \prod_n \pi_\ast X \leq \pi_\ast (X \wr \Sigma_n) \).
Since \( \map \kappa \) is defined as the composition \( \prod \pi \ast X \).
\end{proof}

The section \( s: B\Sigma_n \to X \wr \Sigma_n \) defined by the fixed point \( (\ast, \ldots, \ast) \in X^k \) induces the inclusion \( \Sigma_n \leq \pi_1(X \wr \Sigma_n) \) which acts by permuting the factors of \( \pi_\ast (X^k) \leq \pi_\ast (X \wr \Sigma_n) \).

Since \( n \geq 3 \) we can choose elements \( \omega_k \in \Sigma_n \) whose order is prime to \( p \) and \( \omega_k(1) = k \) for all \( k = 1, \ldots, n \). Indeed, if \( p > 2 \) we can choose the involutions \( \omega_k = (1, k) \). If \( p = 2 \) we can choose \( \omega_k \) to be 3-cycles (note that \( n \geq 3 \)). In both cases we choose \( \omega_1 \) as identity.

For every \( k = 1, \ldots, n \) let \( j_k: X \to \prod_n X \) denote the inclusion into the \( k \)th factor. Note that \( \map \kappa_{\ast} \) is defined as the composition \( \prod \pi \ast X \).

By inspection of the action of \( \omega_k \in \pi_1(X \wr \Sigma_n) \), it follows that for any \( \theta \in \pi_\ast X \), \( (\kappa \circ j_k)_{\ast}(\theta) = ((\kappa \circ j_1)_{\ast}(\theta))^{\omega_k} \in \pi_\ast (X \wr \Sigma_n) \).
Now fix some \( \theta \in \pi_\ast X \) of exponent \( n \).

Since \( \Delta(X) \) is defined as the composition \( \prod X \overset{\map \kappa_{\ast}}{\longrightarrow} \prod_n X \overset{\kappa}{\longrightarrow} X \wr \Sigma_n \), we have
\[ \Delta(X)_{\ast}(\theta) = \prod_{k=1}^{n} (\kappa \circ j_k)_{\ast}(\theta) = \prod_{k=1}^{n} ((\kappa \circ j_1)_{\ast}(\theta))^{\omega_k}. \]

Now consider the \( p \)-completion map \( X \wr \Sigma_n \overset{\gamma_p}{\longrightarrow} (X \wr \Sigma_n)_p \) and note that it maps \( \omega_k \) to the trivial element by Proposition 4.1. By applying \( \map \eta_{\ast} \) and using the naturality of the action of the fundamental group we see that
\[ (\eta \circ \Delta(X))_{\ast}(\theta) = \prod_{k=1}^{n} \eta_{\ast}((\kappa \circ j_1)_{\ast}(\theta))^{\omega_k} = \prod_{k=1}^{n} \eta_{\ast}((\kappa \circ j_1)_{\ast}(\theta))^{\eta_{\ast}(\omega_k)} = \eta_{\ast}((\kappa \circ j_1)_{\ast}(\theta^n)) = 0. \]
\end{proof}

4.3. Lemma. Fix a map \( f: X \to Y \) and assume that every element of \( \pi_1 \map f(X, Y) \) has exponent \( k \) for some \( k \geq 3 \). Assume further that \( \map \eta_{\ast} \Delta(Y) \circ f(X, (Y \wr \Sigma_k)_p) \) is \( p \)-complete. Then the induced homomorphism
\[ \pi_1 \map f(X, Y) \overset{\map \eta_{\ast} \Delta(Y)}{\longrightarrow} \pi_1 \map \eta_{\ast} \Delta(Y) \circ f(X, (Y \wr \Sigma_k)_p) \]
is trivial.
\begin{proof}
According to Proposition 3.8(a) the triangle in the diagram below commutes up to homotopy.
\[
\begin{array}{c}
\map f(X, Y) \xrightarrow{\Delta(Y)} \map \Delta f(X, Y \wr \Sigma_k) \xrightarrow{\eta_{\ast}} \map \eta_{\ast} \Delta f(X, (Y \wr \Sigma_k)_p) \\
\downarrow \quad \downarrow \quad \downarrow \\
\map f(X, Y) \wr \Sigma_k \xrightarrow{\eta} (\map f(X, Y) \wr \Sigma_k)_p
\end{array}
\]
Since \( \map \eta_{\ast} \Delta(Y) \circ f(X, (Y \wr \Sigma_k)_p) \) is \( p \)-complete, the map \( (\eta_{\ast} \circ \gamma)_{\ast} \) gives rise to a choice of a map for the dotted arrow so that the square is homotopy commutative.

We can now apply Lemma 4.2 to the diagonal arrow \( \Delta \) and the bottom arrow \( \eta \).
\end{proof}
5. The wreath product of $p$-local finite groups

Given a finite group $G$, the space $(BG)\wr \Sigma_k$ is the classifying space of the group $G\wr \Sigma_k$ (see 3.6). In this section we prove an analogous result for $p$-local finite groups.

Recall from Remark 2.5 that any $p$-local finite group $(S,F,L)$ is equipped with functions $\delta_{P,Q}: N_S(P,Q) \to L(P,Q)$, where $P,Q$ are $F$-centric. We shall denote $\delta_{P,Q}(s)$ by $\hat{s}$. Thus, an element $s \in S$ permutes the set of all morphisms $L$, by either pre-composition with $s^{-1}$ (i.e. $\varphi \mapsto \varphi \circ s^{-1}$) or by post-composition with $s$ (i.e. $\varphi \mapsto s \circ \varphi$). We obtain an action of $S$ on $L$ by conjugation of the subgroup $P \leq S$ and by conjugation of morphisms $\varphi \mapsto \hat{s} \circ \varphi \circ s^{-1}$.

5.1. Definition. The action of a group $G$ on $S$ is called fusion preserving if the image of $G \xrightarrow{\tau} \text{Aut}(S)$ consists of fusion preserving automorphisms, that is, for every $\varphi \in F(P,Q)$ and every $g \in G$ the composition $\tau_g \circ \varphi \circ \tau_g^{-1}$ belongs to $F(\tau_g(P),\tau_g(Q))$.

In this section we prove Theorem 5.2 which is a variant of [4, Theorem 4.6]. While condition (2) of Theorem 5.2 offers some simplifications, we relax the assumption imposed in [4] that $G$ is a finite $p$-group. The main idea of the proof remains the same but some new arguments were also needed and therefore we decided to present a complete proof of Theorem 5.2.

5.2. Theorem. Let $G$ be a finite group which acts on the centric linking system $L_0$ of a $p$-local finite group $(S_0,F_0,L_0)$. The action of $G$ on $\varphi \in L_0$ is denoted by $\varphi \mapsto g \cdot \varphi \cdot g^{-1}$. Assume that $S_0 \leq G$ and let $S$ be a Sylow $p$-subgroup of $G$. Assume further that:

(1) $\text{Aut}_G(S_0)$ acts via fusion preserving automorphisms.
(2) For any $g \in G$, if $c_g \in F_0(P_0,Q_0)$ for $F_0$-centric subgroups $P_0,Q_0 \leq S_0$, then $g \in S_0$.
(3) The action of $G$ on $L_0$ extends the action of $S_0$ on $L_0$ by conjugation.
(4) The monomorphism $\delta_{S_0}: S_0 \to \text{Aut}_{L_0}(S_0)$ is $G$-equivariant.
(5) The projection $\pi_0: L_0 \to F_0$ is $G$-equivariant, that is $\pi_0(g \cdot \varphi \cdot g^{-1}) = c_g \circ \pi_0(\varphi) \circ c_{g^{-1}}$.
(6) There is a compatible choice of lifts of inclusions in $L_0$ such that for any $g \in G$ and every inclusion of $F_0$-centric subgroups $P_0 \leq Q_0$, we have $g \cdot Q_0 \cdot g^{-1} = \hat{g} Q_0$.

Then, there exists a $p$-local finite group $(S,F,L)$ with the following properties:

(a) There are inclusions $F_0 \subseteq F$, $F_0^c \subseteq F^c$ and $L_0 \subseteq L$ in such a way that the distinguished monomorphisms $\delta_P$ in $L$ extend the ones in $L_0$. The map $i: |L_0| \hookrightarrow |L|$ is induced by the inclusion fits in a homotopy fibre sequence $|L_0| \stackrel{i}{\to} |L| \to B(G/S_0)$.

Moreover, if $S_0$ has a complement $K$ in $G$, that is $G = S_0 \rtimes K$, then:

(b) There is a homotopy equivalence $|L_0|_{hK} \xrightarrow{\sim} |L|$ such that the composition $|L_0| \to |L_0|_{hK} \simeq |L|$ is homotopic to $|L_0| \to |L|$ and such that $\Theta: BS \to |L|$ is homotopic to the composition $BS \xrightarrow{\text{Binc}} BG \xrightarrow{(\Theta_0)_{hK}} |L_0|_{hK} \simeq |L|$.
(c) Up to isomorphism \((S, \mathcal{F}, \mathcal{L})\) is the unique \(p\)-local finite group with the properties in (b).

As a corollary we obtain the proof of Theorem 1.1 in the Introduction.

**Proof of Theorem 1.1.** By Remark 2.18 there is an action of \(\Sigma_n\) on the \(n\)-fold product \((S_0, \mathcal{F}_0, \mathcal{L}_0) = (S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})\) by permuting the factors. The action of \(S_0\) on \(\mathcal{L}_0\) by conjugation clearly extends to an action of \(S_0 \times \Sigma_n\) because \(S_0 = S^{\times n}\) acts on every coordinate of \(\mathcal{L}_0 = \mathcal{L}^{\times n}\) and \(\Sigma_n\) acts by permuting the factors of \(\mathcal{L}_0\) and the factors of \(S_0 = S^{\times n}\). Set \(G = S \wr K = S_0 \times K\). We shall now show that the action of \(G\) on \(\mathcal{L}_0\) satisfies hypotheses (1)-(6) of Theorem 5.2.

Hypothesis (1) is clearly satisfied because \(K\) acts on \(S_0\) by permuting the factors which is an automorphism of \(\mathcal{F}_0 = \mathcal{F}^{\times n}\). Hypothesis (3) holds by the definition of the action of \(G = S_0 \times K\) on \(\mathcal{L}_0\). Hypothesis (4) holds for similar reasons since \(K \leq \Sigma_n\) acts on \(P_0 \leq S_0\) and on \(\text{Aut}_\mathcal{F}(P_0) \leq \prod_i \text{Aut}_\mathcal{F}(P_0^{(i)})\) by permuting the factors (see Definition 2.15) where \(P_0^{(i)}\) is the image of \(P_0\) under the projection \(p^i: S^{\times n} \to S\) to the \(i\)th factor. For hypothesis (5) note that \(\pi: \mathcal{L}_0 \to \mathcal{F}_0 = \Sigma_n\)-equivariant and it is also \(S_0\)-equivariant since \(\pi(s) = c_s\) for any \(s \in S\). Hypothesis (6) holds as we indicated above for the choice of the morphisms \(\{\iota^i_{P_0}\}\) which we described in Remarks 2.16 and 2.18.

It remains to check hypothesis (2). Fix an \(\mathcal{F}_0\)-centric subgroup \(P_0 \leq S_0\) and let \(P_0^{(i)}\) be defined as above (see 2.13). Since \(P_0^{(i)}\) are \(\mathcal{F}\)-centric for \(i = 1, \ldots, n\) by Lemma 2.14 and \(S \neq 1\), it follows that \(P_0^{(i)} \neq 1\) whence \(Z(P_0^{(i)}) \neq 1\) for all \(i = 1, \ldots, n\). Also note that \(\prod_i Z(P_0^{(i)}) = \prod_i C_S(P_0^{(i)}) = C_{S_0}(P_0) \leq P_0\) because \(P_0\) is \(\mathcal{F}_0\)-centric. Fix some \(g = (s_1, \ldots, s_n; \sigma) \in G = S \wr K\) and assume that \(g \notin S_0\), namely \(\sigma \neq 1\). Without loss of generality we can assume that \(\sigma(1) = 2\). Choose \(1 \neq z_i \in C_S(P_0^{(i)})\) and consider \((z_1, 1, \ldots, 1; id) \in \prod_{i=1}^n Z(P_0^{(i)}) \leq P_0\). Then

\[
c_g((z_1, 1, \ldots, 1; id)) = (s_1, \ldots, s_n; \sigma)(z_1, 1, \ldots, 1; id)(s_{\sigma^{-1}(1)}^{-1}, \ldots, s_{\sigma^{-1}(n)}^{-1}; \sigma^{-1}) = (1, s_2z_1s_2^{-1}, 1, \ldots, 1; id).
\]

Therefore \(c_g \notin \mathcal{F}_0(P_0, S_0)\) because it cannot be a restriction of a morphism in \(\prod_i \mathcal{F}\).

Now we apply Theorem 5.2(b) to conclude that there exists a \(p\)-local finite group \((S', \mathcal{F}', \mathcal{L}')\) with \((|\mathcal{L}_0|)_hK \simeq |\mathcal{L}'|\) such that

\[
(1) \quad BS' \xrightarrow{\text{Bind}} BG \simeq (BS_0)_hK \xrightarrow{\iota \eta K} |\mathcal{L}_0|_hK \simeq |\mathcal{L}'|
\]

is homotopic to \(\Theta': BS' \to |\mathcal{L}'|\). Also observe that the horizontal arrows in

\[
\begin{array}{ccc}
(BS)^{\times n} & \xrightarrow{\Theta^{\times n}} & BS_0 \\
\downarrow \Theta^n & & \downarrow \Theta_0 \\
|\mathcal{L}|^{\times n} & \xrightarrow{\sim} & |\mathcal{L}_0|
\end{array}
\]

form a \(\Sigma_n\)-equivariant map of the vertical arrows. It follows that the composite in (1) is homotopic to the map

\[
BS' \xrightarrow{\text{Bind}} BG \simeq (BS)_\wr K \xrightarrow{\Theta \wr K} |\mathcal{L}|_\wr K \simeq |\mathcal{L}'|.
\]

which is therefore homotopic to \(\Theta': BS' \to |\mathcal{L}'|\). Finally, the uniqueness of \((S', \mathcal{F}', \mathcal{L}')\) with this property is guaranteed by part (c) of Theorem 5.2. \(\square\)
5.3. **Remark.** If the $p$-local finite group in Theorem 1.1 is associated with a finite group $G$ then $(S', F', \mathcal{L}')$ satisfies $|\mathcal{L}'|_p \simeq (|\mathcal{L}|_p \wr K)_p \simeq (BG_p \wr K)_p$. Those equivalences follow from the Serre spectral sequence associated to $|\mathcal{L}|^n \times KEK$ and [3, Lemma I.5.5] since the spaces involved are $p$-good ([7, Proposition 1.12]).

In the remainder of this section we will prove Theorem 5.2. From now on, the hypotheses and notation of Theorem 5.2 are in force. The construction of $(S, F, \mathcal{L})$ will be obtained in a sequence of steps which we describe now in 5.4–5.17. These statements will be proved after the proof of Theorem 5.2 which succeeds them.

5.4. **Definition.** Let $\mathcal{H}_0$ denote the set of all the $\mathcal{F}_0$-centric subgroups of $S_0$. Fix once and for all a Sylow $p$-subgroup $S_0$ of $G$ and for every $P \leq S$ let $P_0$ denote $P \cap S_0$.

5.5. **Lemma.** The action of $G$ on the set of all subgroups of $S_0$ by conjugation restricts to an action on the set $\mathcal{H}_0$.

5.6. **Definition.** Let $\mathcal{F}_1$ be the fusion system on $S_0$ generated by $\mathcal{F}_0$ and $\text{Aut}_G(S_0)$. Define a category $\mathcal{L}_1$ whose object set is $\mathcal{H}_0$ and

$$\text{Mor}(\mathcal{L}_1) = \left\{ \prod_{P_0, Q_0 \in \mathcal{H}_0} G \times \mathcal{L}_0(P_0, Q_0) \right/ \left\{ (gs, \varphi) \sim (g, \hat{s} \circ \varphi) \mid s \in S_0 \right\}.$$ 

The morphisms set $\mathcal{L}_1(P_0, Q_0)$ where $P_0, Q_0 \in \mathcal{H}_0$ consists of the equivalence classes $[g : \varphi]$ such that $g \in G$ and $\varphi \in \mathcal{L}_0(P_0, Q_0)$. Composition is given by the formula

$$(g : \varphi) \circ [h : \psi] = [gh : (h^{-1} \varphi h) \circ \psi],$$

and identities are the elements of the form $[1 : \text{id}_{P_0}]$.

Define a functor $\pi_1 : \mathcal{L}_1 \to \mathcal{F}_1$ which is the identity on the set of objects and

$$\pi_1([g : \varphi]) = c_g \circ \pi_0(\varphi).$$

We also define functions $\hat{\delta}_{P_0, Q_0} : \mathcal{L}_0(P_0, Q_0) \to \mathcal{L}_1(P_0, Q_0)$ by $g \mapsto [g : t_{P_0}^{Q_0}]$ and denote the image of $g$ by $\hat{g}$.

After showing that $\mathcal{L}_1$ is well defined we will prove the following properties.

5.7. **Lemma.** The category $\mathcal{L}_1$ satisfies the following properties:

(a) There is an inclusion functor $j : \mathcal{L}_0 \to \mathcal{L}_1$ which is the identity on objects and $\varphi \mapsto [1 : \varphi]$ on morphisms.

(b) Every morphism in $\mathcal{L}_1$ has the form $\hat{g} \circ \varphi$ where $\varphi$ is a morphism in $\mathcal{L}_0 \subseteq \mathcal{L}_1$.

If $\varphi \in \mathcal{L}_0(P_0, Q_0)$ and $x \in N_G(P_0)$, then $\varphi \circ \hat{x} = \hat{x} \circ (x^{-1} \varphi x)$.

(c) There is a homotopy fibre sequence

$$\mathcal{L}_0 \xrightarrow{|\mathcal{L}_1|} \mathcal{L}_1 \xrightarrow{\theta} B(G/S_0).$$

If $S_0$ admits a complement $K$ in $G$ then there is a homotopy equivalence $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ such that the composition $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ is homotopic to the map induced by the inclusion $j$. Moreover, the composite

$$BG \simeq (BS_0)_{hK} \xrightarrow{\theta_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$$

is homotopic to the map $BG \to |\mathcal{L}_1|$ induced by the functor $k : BG \to \mathcal{L}_1$ with $k(\bullet_G) = S_0$ and $k(g) = [g : 1_{S_0}]$. 

16
The next step in our construction is to define the following category.

5.8. **Definition.** Define a category $\mathcal{L}_2$ whose object set is

$$\mathcal{H} = \{P \leq S : P_0 \in \mathcal{H}_0\}$$

and whose morphism sets are defined by

$$L_2(P, Q) = \{\psi \in L_1(P_0, Q_0) : \forall x \in P \exists y \in Q(\psi \circ \hat{x} = \hat{y} \circ \psi)\}.$$  

By construction $L_2(P, Q) \subseteq L_1(P_0, Q_0)$ and composition of morphisms is obtained by composing them in $L_1$. Identities $\text{id}_P$ have the form $[1 : \text{id}_{P_0}]$. Also define maps $\hat{\delta}_{P, Q} : \mathcal{N}_G(P, Q) \to \mathcal{L}_2(P, Q)$ by $g \mapsto [g : \iota^{P_0}_{Q_0}]$ and denote the image of $g$ by $\hat{g}$.

The main properties of the category $\mathcal{L}_2$ and its relation to the previously defined $\mathcal{L}_1$ are contained in next two lemmas.

5.9. **Lemma.** The category $\mathcal{L}_1$ is the full subcategory of $\mathcal{L}_2$ on the objects $\mathcal{H}_0$ and the inclusion $j : \mathcal{L}_1 \to \mathcal{L}_2$ induces a homotopy equivalence on nerves.

5.10. **Lemma.** The category $\mathcal{L}_2$ satisfies the following properties:

(a) For every morphism $\psi \in \mathcal{L}_2(P, Q)$ there exists a unique group monomorphism $\pi_2(\psi) : P \to Q$ which satisfies $\psi \circ \hat{x} = \pi_2(\psi)(x) \circ \psi$ in $\mathcal{L}_2$ for all $x \in P$. Moreover, $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$.

(b) $\pi_2$ carries identities to identities and $\pi_2(\lambda) \circ \pi_2(\psi) = \pi_2(\lambda \circ \psi)$ for every $P \to Q \xrightarrow{\lambda} R$ in $\mathcal{L}_2$.

(c) For every $\hat{g} \in \mathcal{L}_2(P, Q)$ with $g \in \mathcal{N}_G(P, Q)$, we have $\pi_2(\hat{g}) = e_g$.

(d) Given $\psi \in \mathcal{L}_2(P, Q)$, if $\pi_2(\psi)$ is an isomorphism of groups then $\psi$ is an isomorphism in $\mathcal{L}_2$.

Lemma 5.10 justifies the following definition.

5.11. **Definition.** Let $\mathcal{F}_2$ be the category whose object set is $\mathcal{H}$ (see Definition 5.8) and whose morphism sets $\mathcal{F}_2(P, Q)$ are the set of group monomorphisms $\pi_2(\mathcal{L}_2(P, Q))$ defined by Lemma 5.10. By the properties shown in this lemma, there results a projection functor $\pi_2 : \mathcal{L}_2 \to \mathcal{F}_2$ which is the identity on objects.

5.12. **Lemma.** The category $\mathcal{F}_2$ satisfies the following properties:

(a) For every $P, Q \in \mathcal{H}$, $\text{Hom}_G(P, Q) \subseteq \mathcal{F}_2(P, Q)$. In particular, $\mathcal{F}_2$ contains all the inclusions $P \leq Q$ of groups in $\mathcal{H}$.

(b) Every morphism in $\mathcal{F}_2$ factors as an isomorphism in $\mathcal{F}_2$ followed by an inclusion. In particular, every isomorphism of groups $f : P \to Q$ in $\mathcal{F}_2$ is an isomorphism in $\mathcal{F}_2$.

Thus, $\mathcal{F}_2$ falls short of being a fusion system on $S$ only because its set of objects $\mathcal{H}$ need not contain all the subgroups of $S$.

5.13. **Definition.** Let $\mathcal{F}$ denote the fusion system on $S$ generated by $\mathcal{F}_2$.

5.14. **Lemma.** The fusion system $\mathcal{F}$ over $S$ satisfies the following properties:

(a) $\mathcal{F}_2$ is the full subcategory of $\mathcal{F}$ generated by the objects in $\mathcal{H}$.

(b) Every $P \in \mathcal{H}$ is $\mathcal{F}$-centric. In particular, $\mathcal{H}_0 \subseteq \mathcal{F}_0$.

(c) Every morphism $f \in \mathcal{F}(P, Q)$ restricts to a morphism $f|_{P_0} \in \mathcal{F}(P_0, Q_0)$.

5.15. **Lemma.** The functor $\pi_2 : \mathcal{L}_2 \to \mathcal{F}$ satisfies all the axioms of a centric linking system on the object set $\mathcal{H}$.
Finally, the last step in the proof is to show that the fusion system \((S, \mathcal{F})\) defined in 5.13 is saturated and that \(\mathcal{L}_2\) can be extended to a unique centric linking system \(\mathcal{L}\) associated to \(\mathcal{F}\).

5.16. **Lemma.** \(\mathcal{F}\) is a saturated fusion system on \(S\).

5.17. **Lemma.** There exists a \(p\)-local finite group \((S, \mathcal{F}, \mathcal{L})\) such that \(\pi_2: \mathcal{L}_2 \to \mathcal{F}\) is the restriction of \(\pi: \mathcal{L} \to \mathcal{F}\) and moreover \(\hat{\delta}_P: P \to \text{Aut}_{\mathcal{L}_2}(P)\) are the distinguished monomorphisms of \((S, \mathcal{F}, \mathcal{L})\) for all \(P \in \mathcal{H}\). Moreover, \(\mathcal{L}_2\) is a full subcategory of \(\mathcal{L}\) and the inclusion \(\mathcal{L}_2 \subseteq \mathcal{L}\) induces a homotopy equivalence on nerves.

Assuming definitions and lemmas 5.4–5.17, we can now prove Theorem 5.2.

Proof of Theorem 5.2. The \(p\)-local finite group \((S, \mathcal{F}, \mathcal{L})\) is constructed in Lemma 5.17. Together with Lemma 5.9 we obtain inclusions of full subcategories \(\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}\) which induce homotopy equivalences on nerves. By Lemma 5.7(c), there results the homotopy fibre sequence of part (a).

Now assume that \(S_0\) has a complement \(K\) in \(G\) and we prove points (b) and (c). Lemma 5.7(c) shows that there are homotopy equivalences \(|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1| \simeq |\mathcal{L}|\) such that \(|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|\) is homotopic to the map induced by the inclusion \(\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}\). Moreover the map \(\text{BS} \xrightarrow{\text{Binc}} \text{BG} \simeq (\text{BS}_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|\)

is induced by the functor \(\Lambda_0: \text{BS} \to \mathcal{L}\) which sends \(s\) to \(S_0\) and defined on morphisms by \(s \mapsto [s: 1_{S_0}] = \hat{s} \in \text{Aut}_\mathcal{L}(S_0)\) (see Lemmas 5.17, 5.7 and Definition 5.8). The map \(\Theta: \text{BS} \to |\mathcal{L}|\) is the realisation of the functor \(\Lambda_1: \text{BS} \to \text{BAut}_\mathcal{L}(S) \to \mathcal{L}\) where \(s \mapsto \hat{s} \in \text{Aut}_\mathcal{L}(S)\), then the lift of the inclusion \(\iota^2_{S_0} \in \mathcal{L}(S_0, S)\) provides a natural transformation \(\Lambda_0 \to \Lambda_1\) (note that \(\hat{s} \circ \iota^2_{S_0} = \iota^2_{S_0} \circ \hat{s}\) by Remark 2.5). Therefore \(|\mathcal{L}_0|\) and \(|\mathcal{L}_1|\) are homotopic and the proof of point (b) is complete.

Now assume that \((S, \mathcal{F}', \mathcal{L}')\) is another \(p\)-local finite group which satisfies the properties in point (b). Let \(\lambda\) denote the composition \(\text{BS} \to \text{BG} = (\text{BS}_0)_{hK} \to |\mathcal{L}_0|_{hK}\). By assumption there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\text{BS} & \xrightarrow{\lambda} & |\mathcal{L}_0|_{hK} \\
\downarrow{\Theta} & \cong & \downarrow{\Theta'} \\
|\mathcal{L}| & \cong & |\mathcal{L}'|.
\end{array}
\]

The isomorphism of \((S, \mathcal{F}, \mathcal{L})\) and \((S, \mathcal{F}', \mathcal{L}')\) follows from [7, Theorem 7.7] \(\square\)

The rest of the section is devoted to the proof of statements in 5.5–5.17.

Proof of Lemma 5.5. First of all, observe that \(S_0 \triangleleft G\) so for any \(P_0 \in \mathcal{H}_0\) and \(g \in G\) we have \(C_{S_0}(gP_0g^{-1}) = gC_{S_0}(P_0)g^{-1} = Z(gP_0g^{-1})\) because \(P_0\) is \(\mathcal{F}_0\)-centric.

Now fix some \(P_0 \in \mathcal{H}_0\) and \(g \in G\). It follows from hypothesis (1) that every \(R_0 \leq S_0\) which is \(\mathcal{F}_0\)-conjugate to \(gP_0g^{-1}\) has the form \(gQ_0g^{-1}\) for some \(Q_0 \leq S_0\) which is \(\mathcal{F}_0\)-conjugate to \(P_0\). In particular \(Q_0 \in \mathcal{H}_0\). It follows from the calculation above that \(C_{S_0}(gP_0g^{-1}) = Z(gP_0g^{-1})\) and that \(C_{S_0}(R_0) = Z(gQ_0g^{-1}) = Z(R_0)\). This shows that \(gP_0g^{-1}\) is \(\mathcal{F}_0\)-centric, namely \(gP_0g^{-1} \in \mathcal{H}_0\) \(\square\)

5.18. **Lemma.** For every \(\mathcal{F}_0\)-centric \(P_0, Q_0 \leq S_0\), every \(s \in N_{S_0}(P_0, Q_0)\) and every \(g \in G\) we have \(gsg^{-1} = gsg^{-1}\) as morphisms in \(\mathcal{L}_0^*(gP_0, gQ_0)\).
Proof. Set \( R_0 = gQ_0g^{-1} \). It suffices to show that the equality holds after post-composition with \( t_{R_0}^S \) because the latter is a monomorphism in \( L_0 \) (see Remark 2.6). Note that \( t_{R_0}^S = g(t_{Q_0}^S)g^{-1} \) by hypothesis (6), therefore using Remark 2.5, we conclude that \( t_{R_0}^S \circ gsg^{-1} = gsg^{-1} \) and \( t_{R_0}^S \circ gsg^{-1} = gsg^{-1} \) as morphisms in \( L_0(P_0, S_0) \). We may therefore prove the equality needed in this lemma under the assumption that \( Q_0 = S_0 \).

Remark 2.5 shows that \( s \colon P_0 \to S_0 \) is equal to \( \delta_{S_0}(s) \circ t_{P_0}^S \), which together with hypothesis (6) and the fact that \( t_{gP_0g^{-1}} \) is an epimorphism in \( L_0 \) imply that it suffices to prove (5.18) when \( P_0 = S_0 \). But this is hypothesis (4) of Theorem 5.2.

Proof of Definition 5.6. By Lemma 5.5 if \( Q_0 \in H_0 \) then \( Q_0^p \in H_0 \) for any \( q \in G \). This shows that pairs \( [g : \varphi] \) where \( \varphi \in L_0(P_0, Q_0^q) \) are well defined and that, moreover, every element \( [g : \varphi] \) in \( \text{Mor}(L_1) \) has this form. The verification that the formula for composition of morphisms is well defined is identical to the one in \([4, \text{Theorem 4.6}]. Specifically, for any \( g_0, h_0 \in S_0 \)

\[
[gg_0 : \varphi] \circ [hh_0 : \psi] = [gg_0hh_0 : \Phi(h_0^r\varphi h_0) \circ \psi] = \text{by hypothesis (3)}
\]

\[
[gg_0h : h^{-1}\varphi h] \circ \theta_0 \circ \psi] = [gh : h^{-1}g_0h \circ (h^{-1}\varphi h) \circ \theta_0 \circ \psi] = \text{by Lemma 5.18}
\]

\[
[gh : h^{-1}(\theta_0 \circ \varphi)h \circ \theta_0 \circ \psi] = [g : \theta_0 \circ \varphi] \circ [h : \theta_0 \circ \psi].
\]

Associativity is straightforward as well as checking that \([1 : 1P_0] \) are identity morphisms \( P_0 \to P_0 \).

It is evident from the definition that \( \pi_1 \) maps identity morphisms in \( L_1 \) to identities in \( F_1 \). It also respects compositions by the following calculation which uses hypothesis (5) in the third equality

\[
\pi_1([g : \varphi]) \circ \pi_1([h : \psi]) = c_{g_0} \circ \pi_0(\varphi) \circ c_h \circ \pi_0(\psi)
\]

\[
=c_{gh} \circ (c_{h^{-1}} \circ \pi_0(\varphi) \circ c_h) \circ \pi_0(\psi) = c_{gh} \circ \pi_0(h^{-1}\varphi h) \circ \pi_0(\psi)
\]

\[
=c_{gh} \circ \pi_0(h^{-1}\varphi h) \circ \psi) = \pi_1([gh : h^{-1}\varphi h \circ \psi]) = \pi_1([g : \varphi] \circ [h : \psi]).
\]

Proof of Lemma 5.7. (a) By Definition 5.6 we have \([1 : \varphi] \circ [1 : \varphi'] = [1 : \varphi \circ \varphi'] \) so \( j \) is clearly associative and unital. It is an inclusion functor because \([1 : \varphi] = [1 : \varphi'] \) if and only if \( \varphi = \varphi' \) by the definition of morphisms in \( L_1 \).

(b) Clearly, every morphism \( \psi \) in \( L_1 \) has the form \([g : \varphi] = [g : 1] \circ [1 : \varphi] = \hat{g} \circ \varphi \). Given \( \varphi \) and \( x \) as in the statement, by Definition 5.6

\[
\varphi \circ \hat{x} = [1 : \varphi] \circ [x : 1] = [x : x^{-1}\varphi x] = [x : 1Q_0] \circ [1 : x^{-1}\varphi x] = \hat{x} \circ x^{-1}\varphi x.
\]

(c) Set \( \hat{G} = G/S_0 \) and denote its elements by \( \hat{g}_0 = gS_0 \). There is a functor \( L_1 \to B(G) \) which sends every object of \( L_1 \) to \( \bullet_G \) and maps \([g : \varphi] \mapsto \hat{g} \). This assignment is evidently well defined and functorial by the constructions of \( L_1 \) in Definition 5.6.

Now, consider the comma category \( (\bullet_G \downarrow \Pi) \). Its objects are pairs \((\hat{g}, P_0)\) and morphisms \((\hat{g}, P_0) \to (\hat{h}, Q_0)\) are morphisms \([x : \lambda] \in L_1(P_0, Q_0)\) such that \( \hat{x} = h_0^{-1} \). We can easily check that \( \hat{g} : P_0 \to P_0 \) provides an isomorphism \((\hat{e}, P_0) \to (\hat{g}, P_0)\) in \((\bullet_G \downarrow \Pi)\). Therefore, the set of objects of the form \((\hat{e}, P_0)\) form a skeletal full subcategory of \((\bullet_G \downarrow \Pi)\), that is, it contains an element from every isomorphism
class of objects. This subcategory is clearly isomorphic to \( \mathcal{L}_0 \) and moreover the composition \( \mathcal{L}_0 \subseteq (\bullet_G \downarrow \Pi) \to \mathcal{L}_1 \) is the inclusion \( j \) in part (a).

Moreover, any morphism \( \bar{g} \in BG \) clearly induces an automorphism of the category \( (\bullet_G \downarrow \Pi) \). Therefore, Quillen’s theorem B [18] applies in this situation to show that \( (\bullet_G \downarrow \Pi) \to |\mathcal{L}_1| \to |\mathcal{B}(G/S_0)| \) is a homotopy fibre sequence. Finally, using the homotopy equivalence \( |j| \) we obtain the homotopy fibre sequence

\[ |\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \xrightarrow{|\beta|} BG/S_0. \]

Now suppose that \( S_0 \) has a complement \( K \) in \( G \). Recall that \( G \) acts on the category \( \mathcal{L}_0 \) and we view the restriction of this action to \( K \) as a functor \( BK \to \textbf{Cat} \). Let \( Tr_K(\mathcal{L}_0) \) denote the transporter category (or Grothendieck construction) of this functor; see e.g. [22]. The object set of \( Tr_K(\mathcal{L}_0) \) is \( \mathcal{H}_0 \), and the morphisms \( P_0 \to Q_0 \) are pairs \((k, \varphi)\) where \( \varphi \in \mathcal{L}_0(kP_0, Q_0) \). Composition is given by the following formula: \( (k_2, \varphi_2) \circ (k_1, \varphi_1) = (k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1}) \). Define a functor \( \Phi: Tr_K(\mathcal{L}_0) \to \mathcal{L}_1 \) which is the identity on objects and

\[ \Phi: Tr_K(\mathcal{L}_0)(P_0, Q_0) \to \mathcal{L}_1(P_0, Q_0) \]

is defined by \( (k, \varphi) \mapsto [k : k^{-1}\varphi k] \). It is clear that \( \Phi(1, \text{id}) = [1 : \text{id}] \) and for any pair of composable morphisms \( (k_2, \varphi_2) \) and \( (k_2, \varphi_2) \) in \( Tr_K(\mathcal{L}_0) \),

\[ \Phi(k_2, \varphi_2) \circ \Phi(k_1, \varphi_1) = [k_2 : k_2^{-1}\varphi_2k_2] \circ [k_1 : k_1^{-1}\varphi_1k_1] = [k_2k_1 : k_1^{-1}\varphi_2k_2k_1 \circ k_1^{-1}\varphi_1k_1] = \Phi(k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1}). \]

By definition \( \Phi \) is bijective on the object set. We will show now that it is bijective on morphism sets. For any morphism \( \psi = [g : \varphi'] \in \mathcal{L}_1(P_0, Q_0) \) there is a unique \( k \in K \cap gS_0 \), hence \( \psi = [k : \varphi'] \) for a unique \( k \in K \) and a unique \( \varphi' \in \mathcal{L}_0(kP_0, k^{-1}Q_0) \). Then \( (k, k\varphi'k^{-1}) \in Tr_K(\mathcal{L}_0)(P_0, Q_0) \) is a preimage of \( [k : \varphi'] \) under \( \Phi \). In fact, it is unique because \( K \cap S_0 = 1 \).

Thomason [22] constructed a homotopy equivalence \( |\mathcal{L}_0|_{hK} \xrightarrow{\beta} |Tr_K(\mathcal{L}_0)| \) such that \( |\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \) is homotopic to the map induced by the inclusion \( \mathcal{L}_0 \subseteq Tr_K(\mathcal{L}_0) \) via \( \varphi \mapsto [e : \varphi] \). Furthermore, by inspection \( \Phi \) carries the subcategory \( \mathcal{L}_0 \) in \( Tr_K(\mathcal{L}_0) \) onto \( \mathcal{L}_0 \subseteq \mathcal{L}_1 \) via the identity map. We deduce that \( |\Phi| \circ |\beta| \) is a homotopy equivalence \( |\mathcal{L}_0|_{hK} \to |\mathcal{L}_1| \) whose composition with \( |\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \) is homotopic to the map induced by the inclusion \( j: \mathcal{L}_0 \to \mathcal{L}_1 \).

To complete the proof we now consider the subcategory \( BS_0 \) of \( B\text{Aut}_{\mathcal{L}_1}(S_0) \subseteq \mathcal{L}_0 \) via the monomorphism \( \delta_{S_0}: S_0 \to \text{Aut}_{\mathcal{L}_1}(S_0) \) and observe that it is invariant under the action of \( K \) by Lemma 5.18. Thus, there is an inclusion of subcategories \( Tr_K BS_0 \subseteq Tr_K \mathcal{L}_0 \) induced by \( Tr_K(\delta_{S_0}) \). By inspection there is an isomorphism of categories \( Tr_K BS_0 \cong BG \) via the functor \( (k, s) \mapsto sk \) such that he composition

\[ BG \cong Tr_K(\mathcal{L}_0) \subseteq Tr_K(\mathcal{L}_0) \xrightarrow{\Phi} \mathcal{L}_1 \]

is the functor which sends \( \bullet_G \) to \( S_0 \) and \( g \mapsto [g : 1] \in \text{Aut}_{\mathcal{L}_1}(S_0) \).

Here are more properties of \( \mathcal{L}_1 \) that we will need later.

5.19. **Lemma.** The category \( \mathcal{L}_1 \) satisfies the following properties:

(a) For every \( P_0, Q_0, R_0 \in \mathcal{H}_0 \) and every \( g \in N_G(P_0, Q_0) \) and \( h \in N_G(Q_0, R_0) \) the equality \( h \circ \hat{g} = h\hat{g} \) holds in \( \mathcal{L}_1 \).

(b) Fix \( P_0, Q_0 \in \mathcal{H}_0 \) and \( \psi \in \mathcal{L}_1(P_0, Q_0) \). Then, for every \( x \in N_G(P_0) \) there exists at most one \( y \in N_G(Q_0) \) such that \( \psi \circ x = \hat{y} \circ \psi \). In this case
implies that there exists some $s$ suffices to prove the result for $(c)$ By inspection, every $\hat{s} \in L_1(P_0, Q_0)$ is both a monomorphism and an epimorphism.

(d) Fix $\psi \in L_1(P_0, Q_0)$ such that $\pi_1(\psi)(P_0) \leq R_0$ for some $R_0 \leq Q_0$. Then there exists $\lambda \in L_1(P_0, R_0)$ such that $\psi = \iota \circ \lambda$ where $\iota = \hat{e} \in L_1(R_0, Q_0)$.

(e) If $\pi_1(\psi) = \pi_1(\psi')$ where $\psi, \psi' \in L_1(P_0, Q_0)$ then $\psi' = \psi \circ \hat{z}$ for a unique $z \in Z(P_0)$.

(f) Fix $P_0 \in \mathcal{H}_0$ and set $H := \{g \in G \mid gP_0g^{-1} \text{ is } \mathcal{F}_0\text{-conjugate to } P_0\}$. Then $H$ is a subgroup of $G$ which contains $S_0$ and $|\text{Aut}_{L_0}(P_0) : \text{Aut}_{L_0}(P_0)| = |H : S_0|$.

Proof. (a) From Definition 5.6, there are equalities $\hat{h} \circ \hat{g} = [h : \iota^0_{P_0}] \circ [g : \iota^0_{P_0'}] = [hg : \iota^0_{Q_0'} \circ \iota^0_{P_0}] = [hg : \iota^0_{P_0'}] = \hat{h} \circ \hat{g}$.

(b) By Definition 5.6, $\psi$ has the form $[g : \varphi]$ for some $g \in G$ and $\varphi \in L_0(P_0, Q_0')$. If $y$ exists then, again by Definition 5.6,

\[
\hat{y} \circ \psi = [y : 1] \circ [g : \varphi] = [yg : \varphi], \quad \psi \circ \hat{x} = [g : \varphi] \circ [x : 1] = [gx : x^{-1}\varphi x].
\]

Since $\psi \circ \hat{x} = \hat{y} \circ \psi$ in $L_1$, there exists some $s \in S_0$ such that

\[
(i) \ yg = gxs \quad \text{and} \quad (ii) \ \varphi = s^{-1} \circ (x^{-1}\varphi x).
\]

Note that $x^{-1}gx$ is an epimorphism in $L_0$ (Remark 2.6) so the morphism $s^{-1} \in \text{Iso}_{L_0}(Q_0^g, Q_0^g)$ which solves equation (ii) must be unique, hence $s$ is unique. Set $s_0 = gsg^{-1}$. Then $s_0 \in S_0$ because $S_0 \subseteq G$ and $y = gxs = gxs^{-1} = gxs^{-1} \circ s_0$.

If $x \in P_0$ then axiom (C) satisfied by the linking system $L_0$ (see Definition 2.4) implies that

\[
\psi \circ \hat{x} = [g : \varphi] \circ [x : 1] = [gx : x^{-1} \circ \varphi \circ \hat{x}] = [g : \varphi \circ \hat{x}] = [g : \pi_0(\varphi)(x) \circ \varphi] = [c_g(\pi_0(\varphi)(x)) \cdot g : \varphi] = c_g(\pi_0(\varphi)(x)) \cdot g \circ \psi.
\]

(c) By inspection, every $\hat{g} \in L_1(P_0, Q_0)$ has the form $\iota \circ \hat{g}$ where $\hat{g} \in L_1(P_0, \iota^0_{P_0'})$ and $\iota = \hat{e} \in L_1(\iota^0_{P_0}, Q_0)$. Since $\hat{g}$ in this factorisation is clearly an isomorphism, it suffices to prove the result for $\iota$ of the form $\iota = [\varepsilon : \iota_{Q_0'}]$.

Assume that $[h : \varphi], [h' : \varphi'] \in L_1(R_0, P_0)$ satisfy $\iota \circ [h : \varphi] = \iota \circ [h' : \varphi']$. Since

\[
\iota \circ [h : \varphi] = [1 : \iota^0_{P_0}] \circ [h : \varphi] = [h : \iota^0_{P_0} \circ \varphi]
\]

and similarly $\iota \circ [h' : \varphi'] = [h' : \iota^0_{P_0} \circ \varphi']$, we see from the definition that there exists some $s \in S_0$ such that $h' = hs$ and

\[
\iota^0_{P_0} \circ \varphi' = s^{-1} \circ \iota^0_{P_0} = \iota^0_{P_0} \circ s^{-1} \circ \varphi \quad \text{in } L_0.
\]

Since $\iota^0_{P_0}$ is a monomorphism in $L_0$ it follows that $\varphi' = s^{-1} \circ \varphi$ and therefore $[h' : \varphi'] = [hs : s^{-1} \circ \varphi] = [h : \varphi]$. This shows that $\iota$ is a monomorphism.
Now assume that the morphisms $[h : \phi], [h' : \phi'] \in \mathcal{L}_1(Q_0, R_0)$ are such that $[h : \phi] \circ i = [h' : \phi'] \circ i$. Then

$$[h : \phi \circ i_{Q_0}] = [h' : \phi' \circ i_{Q_0}]$$

and it follows from the definition that there exists some $s \in S_0$ such that $h' = hs$ and $\phi' \circ i_{Q_0} = s^{-1} \circ \phi \circ i_{Q_0}$. Since $i_{Q_0}$ is an epimorphism in $\mathcal{L}_0$ we obtain that $[h' : \phi'] = [hs : s^{-1} \circ \phi] = [h : \phi]$. Therefore $i$ is an epimorphism.

(d) Write $\psi = [g : \phi]$ for some $\psi \in \mathcal{L}_0(P_0, Q_0)$. Note that $\pi_1(\psi) = c_g \circ \pi_0(\phi)$ so $\pi_0(\psi)(P_0) = R_0^g$. Since $\mathcal{L}_0$ is a linking system, [7, Lemma 1.10] implies that we can factor $\psi$ as $P_0 \xrightarrow{\zeta} R_0^g \xrightarrow{i_{Q_0}} Q_0^g$. We shall now consider $\lambda \in \mathcal{L}_1(P_0, R_0)$ defined by $\lambda = [g : \phi]$. By hypothesis (6)

$$\iota \circ \lambda = [e : i_{Q_0}] \circ [g : \phi] = [g : i_{R_0} \circ \phi] = [g : \phi] = \psi.$$

(e) Write $\psi = [g : \phi]$ and $\psi' = [g' : \phi']$ in $\mathcal{L}_1(P_0, Q_0)$. By assumption and Definition 5.6 we see that $c_g \circ \pi_0(\phi) = c_{g'} \circ \pi_0(\phi')$, whence $\pi_0(\phi) = c_{g' \circ \phi} \circ \pi_0(\phi')$. Since $\pi_0(\phi), \pi_0(\phi') \in \mathcal{F}_0$, we obtain that $c_{g' \circ \phi} \circ \pi_0(\phi') \in \mathcal{F}_0$. Then hypothesis (2) implies that $g^{-1}g' \in S_0$.

Denote $\phi'' = gg^{-1} \circ \phi$ and $s = g^{-1}g' \in S_0$. Then $\psi' = [gs : \phi''] = [g : \phi'']$ and $\pi_1(\psi') = \pi_1(\psi'')$ reads $c_g \circ \pi_0(\phi) = c_{g'} \circ \pi_0(\phi'')$. In particular $\pi_0(\phi) = \pi_0(\phi'')$ and the axioms of $\mathcal{L}_0$ guarantee the existence of a unique $z \in Z(P_0)$ such that $\phi'' = \phi \circ z$. It now follows that $\psi' = [g : \phi''] = [g : \phi \circ \zeta] = \psi \circ \zeta$. Finally, the element $z \in Z(P_0)$ is unique because

$$\psi \circ \zeta = [g : \phi] \circ [z : 1] = [g : \phi] \circ [1 : z] = [g : \phi \circ \zeta],$$

That is, if $\psi \circ \zeta = \psi' \circ \zeta'$ then by Definition 5.6 we see that $\phi \circ \zeta = \phi \circ \zeta'$ and therefore $z = z'$ because $\phi$ is a monomorphism in $\mathcal{L}_0$ and $\delta_{P_0} : P_0 \to \text{Aut}_{\mathcal{L}_0}(P_0)$ is a monomorphism of groups.

(f) By hypothesis (1) if $Q_0$ is $\mathcal{F}_0$-conjugate to $Q'_0$ then $gQ_0g^{-1}$ is $\mathcal{F}_0$-conjugate to $gQ'_0g^{-1}$ for any $g \in G$. This implies that $H$ is a subgroup of $G$ and it contains $S_0$ because $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$.

Let $g_1, \ldots, g_n$ be representatives for the cosets of $S_0$ in $H$. By Definition 5.6 every element $\psi \in \text{Aut}_{\mathcal{L}_1}(P_0)$ can be described as $\psi = [g_i : \phi]$ by a unique pair $(g_i, \phi)$ for some $i = 1, \ldots, n$ where $\phi \in \mathcal{L}_0(P_0, g_i P_0)$. Also note that $|\mathcal{L}_0(P_0, g_i P_0)| = |\text{Aut}_{\mathcal{L}_0}(P_0)|$ because $g_i P_0$ is $\mathcal{F}_0$-conjugate to $P_0$. This shows that $|\text{Aut}_{\mathcal{L}_1}(P_0)| = n \cdot |\text{Aut}_{\mathcal{L}_0}(P_0)| = |H : S_0| \cdot |\text{Aut}_{\mathcal{L}_0}(P_0)|$. \hfill $\square$

We now turn to the study of the properties of the category $\mathcal{L}_2$.

Proof of Definition 5.8. If $\psi \in \mathcal{L}_2(P, Q)$ and $\rho \in \mathcal{L}_2(Q, R)$, we leave it as an easy exercise for the reader to check that $\rho \circ \psi \in \mathcal{L}_1(P_0, R_0)$ belongs to $\mathcal{L}_2(P, R)$. Thus, composition of morphisms in $\mathcal{L}_2$ is well defined. It is easily seen to be unital and associative because this is the case in $\mathcal{L}_1$.

Since $S_0 \triangleleft G$ it follows that $N_G(P, Q) \subseteq N_G(P_0, Q_0)$, $N_G(P) \subseteq N_G(P_0)$ and $N_G(Q) \subseteq N_G(Q_0)$. Now fix some $g \in N_G(P, Q)$ and $x \in P$ and set $y = gxg^{-1} \in Q$. It follows from Lemma 5.19(a) that $\tilde{g} \circ \tilde{x} = \tilde{g} \tilde{x} = \tilde{y} \tilde{g} = \tilde{g} \circ \tilde{y}$. Therefore $\tilde{g} \in \mathcal{L}_2(P, Q)$. \hfill $\square$
Proof of Lemma 5.9. By construction \( L_2(P_0, Q_0) \subseteq L_1(P_0, Q_0) \) for any \( P_0, Q_0 \in \mathcal{H}_0 \). For every \( x \in P_0 \) and every \( \psi = [g : \phi] \in L_1(P_0, Q_0) \) it follows from Lemma 5.19(b) that \( \psi \circ \hat{x} = \hat{y} \circ \psi \) in \( L_1 \) where \( y = \pi_1(\psi)(x) \in Q_0 \). Therefore \( \psi \in L_2(P_0, Q_0) \) and we conclude that \( L_1(P_0, Q_0) = L_2(P_0, Q_0) \).

The inclusion functor \( j : L_1 \to L_2 \) has a left inverse \( r : L_2 \to L_1 \) which maps an object \( P \) to \( P_0 \) and maps morphisms via the inclusions \( L_2(P, Q) \subseteq L_1(P, Q_0) \). Observe that \( r \circ j = \text{Id}_{L_1} \) because \( L_2(P_0, Q_0) = L_1(P_0, Q_0) \).

By Lemma 5.19(b) we see that \( L_2(P_0, P) \) contains \([e : 1_{P_0}] = \hat{e}\). These morphisms define a natural transformation \( j \circ r \to \text{Id} \). This is because we recall that \([e : 1_{P_0}] \) and \([e : 1_{Q_0}] \) are the identities of \( P_0 \) and \( Q_0 \) in \( L_1 \) and for any \( \psi \in L_2(P, Q) \subseteq L_1(P_0, Q_0) \)

\[
\psi \circ [e : 1_{P_0}] = [e : 1_{Q_0}] \circ \psi.
\]

Then it follows that \( j \) and \( r \) yield homotopy equivalences on nerves. □

Proof of Lemma 5.10. (a) By Definition 5.8, for every \( x \in P \) there exists some \( y \in Q \) such that \( \psi \circ \hat{x} = \hat{y} \circ \psi \). Since \( P \leq N_G(P_0) \) and \( Q \leq N_G(Q_0) \), Lemma 5.19(b) implies that \( y \) is unique. There results a well defined function \( \tau_2(\psi) : P \to Q \). In addition, since \( \hat{x} \) and \( \hat{y} = \tau_2(\psi)(x) \) are morphisms in \( L_2 \) (see Definition 5.8) and \( L_2(P, Q) \subseteq L_1(P_0, Q_0) \) we deduce that the equation \( \psi \circ \hat{x} = \tau_2(\psi)(x) \circ \psi \) holds in \( L_2 \) and moreover \( \tau_2(\psi) : P \to Q \) is the unique function that satisfies this equality for all \( x \in P \). The fact that \( \tau_2(\psi)|_{P_0} = \pi_1(\psi) \) follows from the last assertion in Lemma 5.19(b).

We claim that \( \tau_2(\psi) : P \to Q \) is a group monomorphism. For \( x, x' \in P \), let \( y = \tau_2(\psi)(x) \) and \( y' = \tau_2(\psi)(x') \). Then, in \( L_1 \),

\[
\psi \circ \hat{x} = \psi \circ \hat{x}' = \hat{y} \circ \psi = \psi \circ \hat{y}' = y \circ \psi = \hat{y} \circ \psi = \hat{y} \circ \psi = \hat{y} \circ \psi.
\]

This shows that \( \tau_2(\psi) \) is a homomorphism. If \( x \in \ker \pi_2(\phi) \) then \( \psi \circ \hat{x} = 1 \circ \psi \) so \( \tau_2(\psi)(x) = 1 \). Clearly \( \tau_2(\psi)(x) = 1 \) for any \( x \in P \cap S_0 = P_0 \).

(b) Clearly \( \tau_2(\psi)(x) = 1 \). Now given \( \psi : P \to Q \) and \( z = \tau_2(\psi)(x) \) and \( z = \tau_4(\psi)(y) \). Then \( \psi \circ \hat{x} = \hat{y} \circ \psi \) and \( \lambda \circ \hat{y} = \hat{z} \circ \lambda \psi \) whence, by the uniqueness statement in Lemma 5.19(b), we conclude that \( z = \tau_2(\psi)(x) \).

(c) This follows from Lemma 5.19(a) because for any \( x \in P \) we have \( \hat{y} \circ \hat{x} = \hat{y} \circ \hat{x} = \hat{c}_y(x) \).

(d) Observe that \( \tau_2(\psi)(P_0) = \pi_1(\psi)(P_0) \leq Q_0 \) by part (a). Since \( \tau_2(\psi) : P \to Q \) is an isomorphism, for every \( y_0 \in Q_0 \leq Q \) there exists some \( x \in P \) such that \( \tau_2(\psi)(x) = y_0 \), namely \( \psi \circ \hat{x} = y_0 \circ \psi \). By Lemma 5.19(b) we know that \( y_0 = gxg^{-1} \mod S_0 \) and since \( S_0 < G \) we deduce that \( x \in S_0 \cap P = P_0 \). This shows that \( \tau_2(\psi)(P_0) = Q_0 \) and therefore \( \pi_1(\psi) \) is an isomorphism of groups.

Write \( \psi = [g : \phi] \). Since \( \pi_1(\psi) \) is an isomorphism, \( \phi \in L_0(P_0, Q_0) \) is an isomorphism and therefore \( \psi \) is an isomorphism in \( L_1 \) whose inverse \( \psi^{-1} = \psi \) in \( L_1 \). Since these morphisms are invertible in \( L_1 \) we see that \( \psi^{-1} = \psi \circ \hat{y} \).

This shows that \( \psi^{-1} \) is an inverse to \( \psi \) in \( L_2 \).

For later use we also need the following technical lemma.
5.20. **Lemma.** Fix some $P \in \mathcal{H}$ and consider $N_S(P_0)$ as a subgroup of $\text{Aut}_\mathcal{L}(P_0)$ via $\delta_{P_0,P}: x \mapsto \hat{x}$. Let $Q$ be a subgroup of $N_S(P_0)$ and assume that $Q = \psi P \psi^{-1}$ for some $\psi \in \text{Aut}_\mathcal{L}(P_0)$. Then $P_0 = Q_0$ and $\psi$ is an isomorphism in $\mathcal{L}_2$ from $P$ to $Q$.

**Proof.** Recall from Lemma 5.9 that $\text{Aut}_\mathcal{L}(P_0) = \text{Aut}_\mathcal{L}(P_0)$. For $x \in P_0$ set $y = \psi x \psi^{-1} \in Q$. Thus $\psi \circ \hat{x} = y \circ \psi$ and by Definition 5.11, $y = \pi_2(\psi)(x) \in P_0$. This shows that $P_0 = \psi P_0 \psi^{-1}$ and, in particular, $P_0 \leq Q_0$. Moreover $P_0 \triangleleft Q$ because $P_0 \triangleleft P$.

Since $P_0 \leq Q_0$ we may consider $\iota := \hat{e} \in \mathcal{L}_1(P_0,Q_0)$ where $e \in G$ is the identity element, and define $\lambda = \iota \circ \psi \in \mathcal{L}_1(P_0,Q_0)$. For every $x \in P$ set $y = \psi x \psi^{-1}$. By definition $y \in Q$ which normalises $Q_0$ and $P_0$ so Lemma 5.19(a) implies

$$\lambda \circ \hat{x} = \iota \circ \psi \circ \hat{x} = y \circ \psi = \hat{y} \circ \iota \circ \psi = \hat{y} \circ \psi.$$

We conclude from Definition 5.8 that $\lambda \in \mathcal{L}_2(P,Q)$. Furthermore, $\pi_2(\lambda)$ is an isomorphism because it is a monomorphism by Lemma 5.10(a) and $|P| = |Q|$. Lemma 5.10(d) now shows that $\lambda$ is an isomorphism in $\mathcal{L}_2$ and, in particular, it is an isomorphism of the objects $P_0$ and $Q_0$ in $\mathcal{L}_1$. In particular $|P_0| = |Q_0|$ and therefore $\lambda = \psi$.

**Proof of Lemma 5.12.** (a) This is immediate from Lemma 5.10(c). By taking $e \in N_G(P,Q)$ for any inclusion $P \leq Q$ in $\mathcal{H}$ we obtain $\text{incl}_Q^P \subset \mathcal{F}_2(P,Q)$.

(b) Fix a homomorphism $f: P \to Q$ in $\mathcal{F}_2$ and set $R = f(P)$. Note that by Lemma 5.10(a)

$$f(P_0) = \pi_2(\psi)_{|P_0}(P_0) = \pi_1(\psi)(P_0) \leq Q_0.$$

Therefore $f(P_0) \leq Q_0 \cap R \leq S_0 \cap R = R_0$. Also $R_0 = S_0 \cap R \leq S_0 \cap Q = Q_0$. Now, by definition $\psi \in \mathcal{L}_1(P_0,Q_0)$ and Lemma 5.19(d) asserts that in $\mathcal{L}_1$ we can write $\psi = \iota \circ \lambda$ where $\lambda \in \mathcal{L}_1(P_0,R_0)$ and $\iota = \hat{e} \in \mathcal{L}_1(R_0,Q_0)$.

We now claim that $\lambda \in \mathcal{L}_2(P,R)$. To check this, we fix some $x \in P$. By definition $y = f(x) \in R$ satisfies $\psi \circ \hat{x} = y \circ \psi$ in $\mathcal{L}_1$. Equivalently $\iota \circ \lambda \circ \hat{x} = y \circ \iota \circ \lambda$. Now, $y \in R \leq N_G(R_0)$ and also $y \in Q \leq N_G(Q_0)$, so Lemma 5.19(a) implies that $\iota \circ \lambda \circ \hat{x} = \iota \circ \hat{y} \circ \lambda$.

Lemma 5.19(c) implies that $\iota$ is a monomorphism in $\mathcal{L}_1$ so $\lambda \circ \hat{x} = \hat{y} \circ \lambda$ in $\mathcal{L}_1$. This shows that $\lambda \in \mathcal{L}_2(P,R)$ as needed, and that moreover $\psi = \iota \circ \lambda$ in $\mathcal{L}_2$ because $\iota$ is in $\mathcal{L}_2$ as well. In particular, by parts (b) and (c) of Lemma 5.10, we obtain that

$$f = \pi_2(\psi) = \text{incl}_R^Q \circ \pi_2(\lambda).$$

From this equality it follows that $\pi_2(\lambda)$ is an isomorphism of groups because $|P| = |R|$. Moreover, Lemma 5.10(d) implies that $\lambda$ is an isomorphism in $\mathcal{L}_2$ and therefore $\pi_2(\lambda)$ is an isomorphism in $\mathcal{F}_2$. This completes the proof. □

5.21. **Lemma.** Consider $P \leq S$ such that $P_0 \in \mathcal{H}_0$. Then $C_G(P) = C_{S_0}(P) = Z(P_0)^P$ where $P$ acts on $Z(P_0)$ by conjugation.

**Proof.** If $g \in C_G(P)$ then $g_{|P_0} = \text{id}_{P_0} \in \text{Aut}_\mathcal{F}_0(P_0)$. By hypothesis (2), $g \in S_0$, and it follows that $C_G(P) = C_{S_0}(P)$. Now, $C_{S_0}(P) \leq C_{S_0}(P_0) = Z(P_0)$ because $P_0$ is $\mathcal{F}_0$-centric. Therefore, $C_G(P) = C_{Z(P_0)}(P) = Z(P_0)^P$. □
Proof of Lemma 5.14. (a) Clearly \( H \) is closed to taking supergroups because \( H_0 \) is closed to taking supergroups in \( S_0 \). Since \( F \) is generated by inclusions and restriction of homomorphisms in \( F_2 \), Lemma 5.12 shows that for any \( P, Q \in H \) the inclusion \( \mathcal{F}_2(P, Q) \subseteq \mathcal{F}(P, Q) \) is an equality.

(b) By definition \( P_0 \in H_0 \). By Lemma 5.21, \( C_S(P) = Z(P_0)^P \leq P \). Assume that \( Q \) is \( F \)-conjugated to \( P \). By part (a) there exists some \( \psi \in L_2(P, Q) \) such that \( \pi_2(\psi)(P) = Q \). Parts (a) and (d) of Lemma 5.10 imply that \( \psi \) is an isomorphism in \( L_2 \). From Definition 5.8 it is clear that \( \psi \) is an isomorphism in \( L_1(P_0, Q_0) \) and in particular \( Q_0 \in H_0 \), namely \( Q_0 \) is \( F_0 \)-centric. It follows from Lemma 5.21 that \( C_S(Q) = Z(Q_0)^Q \cong Z(P_0)^P \), whence \( P \) is \( F \)-centric.

(c) For any \( f \in \mathcal{F}(P, Q) \) where \( P, Q \in H \), part (a) implies that \( f = \pi_2(\psi) \) for some \( \psi \in L_2(P, Q) \subseteq L_2(P_0, Q_0) \). The result follows from Lemma 5.10(a) which shows that \( f|_{P_0} = \pi_1(\psi) \) whose image is contained in \( Q_0 \) by Definition 5.6. \qed

Proof of Lemma 5.15. The monomorphisms \( \delta_P : P \to \text{Aut}_{L_2}(P) \) are the restrictions of the maps \( \delta_{P_0} : N_G(P_0, Q_0) \to L_2(P_0, Q_0) \), i.e., \( \delta_P(g) = [g : 1_{P_0}] \).

To verify axiom (A) in [7, Definition 1.7], see also 2.4, we need to show that for any \( P, Q \in H \) the set \( \pi_2^{-1}(f) \) where \( f \in \mathcal{F}(P, Q) \) admit a transitive free action of \( C_S(P) \) via \( \delta_P : N_G(P) \to \text{Aut}_{L_2}(P) \). Note that \( \mathcal{F}(P, Q) = \mathcal{F}_2(P, Q) \) by Lemma 5.14. Consider \( \psi, \psi' \in L_2(P, Q) \) such that \( \pi_2(\psi) = \pi_2(\psi') \) and recall that \( \psi, \psi' \in L_1(P_0, Q_0) \). By restriction to \( P_0 \), Lemma 5.10(a) shows that \( \pi_1(\psi) = \pi_1(\psi') \). Lemma 5.19(f) shows that there exists \( z \in Z(P_0) \) such that \( \psi' = \psi \circ z \in L_1 \). Note that \( \hat{z} \in \text{Aut}_{L_2}(P_0) \) by Definition 5.6 so the equality \( \psi' = \psi \circ \hat{z} \) also holds in \( L_2 \). Furthermore, Lemma 5.19(c) implies that

\[
\pi_2(\psi) = \pi_2(\psi') = \pi_2(\psi \circ \hat{z}) = \pi_2(\psi) \circ c_z.
\]

As a consequence \( \psi \in C_2(P) \) and we conclude that \( C_S(P) \) acts transitively on the fibres of \( \pi_2 : L_2(P, Q) \to \mathcal{F}(P, Q) \). The action is free by Lemma 5.21 and the uniqueness assertion in Lemma 5.19(f).

Axiom (B) holds by Lemma 5.10(e). To verify axiom (C) we fix a morphism \( \psi \in L_2(P, Q) \) and an element \( g \in P \). Set \( f = \pi_2(\psi) \in \mathcal{F}(P, Q) \). By the definition of the morphisms in \( L_2 \), see Lemma 5.10(a) we have \( \psi \circ \hat{g} = f(\hat{g}) \circ \psi \), which is what we need. \qed

Notation. We shall write \( P \simeq_{\mathcal{F}} Q \) for the statement that \( P, Q \leq S \) are \( \mathcal{F} \)-conjugate.

Clearly \( S_0 \) acts on \( H_0 \) by conjugation and \([P_0]_{S_0} \) denotes the orbit of \( P_0 \), i.e. the conjugacy class. By Lemma 5.5, \( G \) acts on \( H_0 \) as well. Since \( G \) acts via fusion preserving automorphisms, it also acts on the set \( H_0 / \mathcal{F}_0 \) of the \( \mathcal{F}_0 \)-conjugacy classes of the subgroups \( P_0 \in H_0 \) which we denote \([P_0]_{\mathcal{F}_0} \). The stabiliser of \([P_0]_{\mathcal{F}_0} \) under this action of \( G \) is denoted, as usual, by \( G_{[P_0]_{\mathcal{F}_0}} \). Now, \( G_{[P_0]_{\mathcal{F}_0}} \) acts on the set \([P_0]_{\mathcal{F}_0} \). Clearly, \( S_0 \leq G_{[P_0]_{\mathcal{F}_0}} \) because \( \mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0 \). Moreover, since \( S_0 < G \), this action induces an action of \( G_{[P_0]_{\mathcal{F}_0}} \) on the set \( P \) of all the \( S_0 \)-conjugacy classes of the subgroups of \( S_0 \) that are \( \mathcal{F}_0 \)-conjugate to \( P_0 \).

5.22. Lemma. For every \( P \in H \) there exist \( \hat{P}, P' \in H \) such that

(a) \( \hat{P} = P \) for some \( a \in G \) and \( \hat{P} \simeq_{\mathcal{F}} P' \), whence \( P \simeq_{\mathcal{F}} P' \), and

(b) \( P'_0 \) is fully \( \mathcal{F}_0 \)-normalised and \( P'_0 \simeq_{\mathcal{F}_0} P_0 \).
In addition, $\overline{S} := N_{P}(P_{0})S_{0}$ is a Sylow $p$-subgroup of $G_{[P_{0}]}x_{0}$ and $\overline{S}/S_{0}$ fixes the $S_{0}$-conjugacy class $[P_{0}]/S_{0}$.

Proof. The argument follows the one in the proof of step 3 in [4, Theorem 4.6].

Clearly $S_{0} \cdot P \leq G_{[P_{0}]}x_{0}$ because $P \leq N_{G}(P_{0})$ and $F_{S_{0}}(S_{0}) \subseteq F_{0}$. Choose $S' \in \text{Syl}_{p}(G_{[P_{0}]}x_{0})$ which contains $S_{0} \cdot P$. By Sylow's theorems, there exists some $a \in G$ such that $S' = G_{[P_{0}]}x_{0} \cap S^{a}$. Set $\bar{P} = aP$ and observe that

$$P = aP \leq (G_{[P_{0}]}x_{0} \cap S^{a}) \leq S.$$ 

Also $\bar{P}_{0} = aP_{0} \in H_{0}$ by Lemma 5.5, so $\bar{P} \in H$. In addition, $G_{[P_{0}]}x_{0} = a(G_{[P_{0}]}x_{0})$. It follows that

$$\bar{S} := S \cap G_{[P_{0}]}x_{0} = a(S') \in \text{Syl}_{p}(G_{[P_{0}]}x_{0}).$$

Consider now the set $\mathcal{P}_{f_{n}}$ of all the $S_{0}$-conjugacy classes of the fully $F_{0}$-normalised subgroups $R \leq S_{0}$ which are $F_{0}$-conjugate to $P_{0}$. Since $G$ normalises $S_{0}$ and it is fusion preserving, it carries fully $F_{0}$-normalised subgroups of $S_{0}$ to ones, and therefore $G_{[P_{0}]}x_{0}$ acts on $\mathcal{P}_{f_{n}}$.

We now restrict the action of $G_{[P_{0}]}x_{0}$ on $\mathcal{P}_{f_{n}}$ to $\bar{S}$. By [4, Proposition 1.16] we know that $|\mathcal{P}_{f_{n}}| \neq 0 \mod p$. Therefore $\overline{S}/S_{0}$ must have some fixed point $[R_{0}]_{S_{0}}$. Thus, $R_{0}$ is fully $F_{0}$-normalised and is $F_{0}$-conjugate to $P_{0}$. Recall that $\overline{S} \leq S$. For every $g \in \overline{S}$ we have $gR_{0}g^{-1} \simeq_{S_{0}} R_{0}$ so $\overline{S} \leq N_{S}(R_{0})S_{0}$. On the other hand $S_{0}N_{S}(R_{0}) \leq G_{[R_{0}]}x_{0} = G_{[P_{0}]}x_{0}$ and $\bar{S}$ is a Sylow $p$-subgroup of the latter group, hence

$$\bar{S} = S_{0} \cdot N_{S}(R_{0}).$$

It remains to find some $P' \in H$ such that $P' \simeq_{\mathcal{L}} \bar{P}$ and such that $P'_{0} = R_{0}$. Now, since $\bar{P} \leq \bar{S}$, it must stabilise $[R_{0}]_{S_{0}}$. We conclude that $P/P_{0}$ acts on

$$X := \{[f] \in \text{Rep}_{\mathcal{L}_{0}}(P_{0}, S_{0}) : \text{Im } f \text{ is } S_{0}\text{-conjugate to } R_{0}\}$$

via $[f_{0}] \mapsto [c_{y} \circ f_{0} \circ c_{y^{-1}}]$. Clearly $X$ is not empty because by construction $P_{0} \simeq_{\mathcal{L}_{0}} R_{0}$. Choose some $f \in \mathcal{F}_{0}(P_{0}, R_{0})$. Then every element of $X$ has the form $[\alpha \circ f]$ for some $\alpha \in \text{Aut}_{\mathcal{L}_{0}}(R_{0})$. Moreover $[\alpha \circ f] = [\beta \circ f]$ if and only if $\alpha^{-1}\beta \in \text{Aut}_{S_{0}}(R_{0})$. Therefore

$$|X| = \frac{[\text{Aut}_{\mathcal{L}_{0}}(R_{0})]}{[\text{Aut}_{S_{0}}(R_{0})]} \neq 0 \mod p$$

because $R_{0}$ is fully $F_{0}$-normalised. Since $\bar{P}$ is a finite $p$-group, there is some $[f_{0}] \in X' \subseteq X$ where $f_{0} \in \mathcal{F}_{0}(P_{0}, S_{0})$ and $\text{Im } f_{0} = R_{0}$. Let $\psi_{0} \in \mathcal{L}_{0}(P_{0}, S_{0})$ be a lift of $f_{0}$.

Recall from Lemma 5.7(a) that we may consider $\psi_{0}$ as a morphism in $\mathcal{L}_{1}(P_{0}, S_{0})$ via an inclusion $\mathcal{L}_{0} \subseteq \mathcal{L}_{1}$. Fix some $x \in \bar{P}$. Since $\bar{P}$ fixes $[f_{0}]$, there exists some $s \in S_{0}$ such that

$$c_{x}^{-1} \circ f_{0} \circ c_{x} = c_{s} \circ f_{0}.$$ 

Lifting to $\mathcal{L}_{0}$ and using hypothesis (5), we see that there exists a unique $z \in C_{S_{0}}(P_{0}) = Z(P)_{0}$ such that

$$x^{-1} \psi_{0} x = s \circ \psi_{0} \circ z = s \circ f_{0}(z) \circ \psi_{0} \text{ in } \mathcal{L}_{0}.$$ 

Set $y := xsf_{0}(z)$ and note that $y \in \bar{P} \cdot S_{0} \cdot Z(R_{0}) \subseteq S$. Lemma 5.7(c), equation (1) and Remark 2.5 imply that

$$\psi_{0} \circ z = \hat{x} \circ (x^{-1} \psi_{0} x) = \hat{x} \circ s \circ f_{0}(z) \psi_{0} = \hat{y} \circ \psi_{0}.$$
Therefore, by definition, $\psi_0 \in \mathcal{L}_2(\bar{P}, S)$. Consider $f = \pi_2(\psi_0) \in \mathcal{F}(P, S)$ and set $P' = f(\bar{P})$. By Lemmas 5.14(a) and 5.12(b), $f$ restricts to an isomorphism $f: \bar{P} \to P'$ in $\mathcal{F}$. By Lemma 5.14(b) we see that $f\mid_{\bar{P}_0} = \bar{\sigma}_0(\psi_0) = f_0 \in \mathcal{F}_0(\bar{P}_0, R_0)$. Since $f \in \mathcal{F}(\bar{P}, P')$ is an isomorphism we deduce from Lemma 5.14(c) that $P_0' = f(\bar{P}_0) = R_0$. This completes the proof since $f$ is an $\mathcal{F}$-isomorphism between $\bar{P}$ and $\bar{P}$ which restricts to an $\mathcal{F}_0$-isomorphism $f_0$ between $\bar{P}_0$ and $R_0 = P_0'$.

5.23. Lemma. [4, Step 4] If $P \leq S$ is $\mathcal{F}$-centric but $P \notin \mathcal{H}$, then there exists $P' \leq S$ which is $\mathcal{F}$-conjugate to $P$ such that

$$\text{Out}_{\mathcal{S}}(P') \cap O_p(\text{Out}_{\mathcal{F}}(P')) \neq 1.$$ 

Proof. The argument is almost repeated from step 4 in the proof of [4, Theorem 4.6], but we include it for completeness. Consider $P$ and $P'$ as in Lemma 5.22.

Note that $P \notin \mathcal{H}$ because $P \notin \mathcal{H}$, namely $P_0 \notin \mathcal{H}_0$, so $P_0 \notin \mathcal{H}_0$ by Lemma 5.5.

the action of $G$ is $\mathcal{F}_0$-preserving. As a consequence $P_0' \notin \mathcal{H}_0$ because $P_0 \simeq P_0'$. Since $P_0'$ is fully $\mathcal{F}_0$-normalised, it is fully $\mathcal{F}_0$-centralised and since it is not $\mathcal{F}_0$-centric, we deduce that $C_{\mathcal{F}_0}(P_0') \notin P_0'$.

Since $P'$ normalises $S_0$ and $P_0'$ it acts on $C_{\mathcal{F}_0}(P_0')P_0'/P_0'$ by conjugation leaving a non-identity subgroup $Q'P_0'/P_0'$ fixed where $Q \leq C_{\mathcal{F}_0}(P_0')$ and $Q \notin P_0'$. Thus, $[P', Q] \leq P_0'$ and in particular $Q \leq N_S(P')$. If $x \in Q \setminus P_0'$ then $1 \neq [c_x] \in \text{Out}(P')$ because $P'$ is $\mathcal{F}$-centric so $C_S(P) \leq P'$ and $Q \setminus P_0' = Q \setminus P_0'$. Lemma 5.14(c) shows that restriction $\varphi \mapsto \varphi|_{P_0'}$ induces a homomorphism $\text{Aut}_{\mathcal{F}}(P') \xrightarrow{\text{res}} \text{Aut}_{\mathcal{F}}(P'_0)$.

Let $\text{Aut}_{\mathcal{F}}(P'; P'_0)$ denote its kernel and observe that it contains $c_x$ because $Q$ centralises $P_0'$. Also observe that $c_x$ induces a trivial homomorphism on $P'/P_0'$ because $[P', Q] \leq P_0'$. Thus, $c_x$ is a non-trivial element in the kernel of

$$\text{Aut}_{\mathcal{F}}(P'; P'_0) \xrightarrow{\text{proj}} \text{Aut}(P'/P_0')$$

which is a $p$-group by [4, Proposition 1.15]. This shows that $c_x$ is an element of $O_p(\text{Aut}_{\mathcal{F}}(P'; P'_0))$ which is a characteristic subgroup of $\text{Aut}_{\mathcal{F}}(P'; P'_0) \leq \text{Aut}_{\mathcal{F}}(P')$. Hence, $c_x \in O_p(\text{Aut}_{\mathcal{F}}(P'))$. Since $\text{Aut}_{\mathcal{F}}(P') \rightarrow \text{Out}_{\mathcal{F}}(P')$ is an epimorphism and $[c_x] \neq 1$, we see that $O_p(\text{Out}_{\mathcal{F}}(P')) \cap \text{Out}_{\mathcal{S}}(P') \neq 1$.

Proof of 5.16. We will apply [5, Theorem 2.2] to the collection $\mathcal{H}$ of objects in $\mathcal{F}$. The condition (*) in that theorem has been verified in Lemma 5.23 so, for the proof of the saturation of $\mathcal{F}$ it remains to check conditions (I) and (II) of saturation in [7, Definition 1.2], see also 2.2 for the elements of $\mathcal{H}$. The argument is again present in [4] with some changes.

Condition 1. Fix $P \in \mathcal{H}$ which is fully $\mathcal{F}$-normalised. We have to show that it is fully $\mathcal{F}$-centralised and that $\text{Aut}_G(P)$ is a Sylow $p$-subgroup of $\text{Aut}_{\mathcal{F}}(P)$. By Lemma 5.14(b) we know that $P$ is $\mathcal{F}$-centric and in particular fully $\mathcal{F}$-centralised.

Consider $P$ and $P'$ as in Lemma 5.22. Recall that $\bar{S} = N_S(P_0)S_0$ is a Sylow $p$-subgroup of $G|_{\bar{P}_0,S_0}$. Lemma 5.7(a) shows that $\text{Aut}_{\mathcal{L}_0}(P_0) \leq \text{Aut}_{\mathcal{L}_0}(P_0)$ and by Lemma 5.19(g)

\[ (1) \quad |\text{Aut}_{\mathcal{L}_0}(P_0') : \text{Aut}_{\mathcal{L}_0}(P_0)| = |G|_{\bar{P}_0,S_0} : S_0|. \]
By definition $N_{S_0}(P_0') = S_0 \cap N_S(P_0')$ so

$$|N_S(P_0')/N_{S_0}(P_0')| = |N_S(P_0')S_0/S_0| = |\bar{S}/S_0|.$$  

Now, $P_0'$ is fully $F_0$-normalised and is $F_0$-centric so

$$|\text{Aut}_{\mathcal{L}_1}(P_0') : N_{S_0}(P_0')| \neq 0 \mod p. \tag{3}$$

Since $|G_{P_0'} : \bar{S}| \neq 0 \mod p$, we deduce from (1), (2) and (3) that

$$|\text{Aut}_{\mathcal{L}_1}(P_0') : N_{S_0}(P_0')| = |\text{Aut}_{\mathcal{L}_1}(P_0')|/|\text{Aut}_{\mathcal{L}_1}(P_0')| \cdot |N_{S_0}(P_0')| \neq 0 \mod p,$$

namely $N_S(P_0') \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_1}(P_0'))$.

Fix $\psi \in \text{Aut}_{\mathcal{L}_1}(P_0')$ such that $\psi^{-1}N_S(P_0')\psi \supseteq R \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_1}(P_0')(P''))$ and set

$$P'' = \psi P' \psi^{-1} \subseteq N_S(P_0').$$

Lemma 5.20 shows that $P_0'' = P_0'$ and that $\psi \in \mathcal{L}_2(P', P'')$ is an isomorphism. In particular, $P''$ is $F$-conjugate to $P'$, hence also to $P$ because $P' = \alpha P$ for some $\alpha \in G$ and $\hat{\alpha} \in \mathcal{L}_2(P, P')$ is an isomorphism. We now claim that

(i) $\text{Aut}_{\mathcal{L}_1}(P'') = N_{\text{Aut}_{\mathcal{L}_1}(P_0')}(P'')$ \quad and \quad (ii) $N_S(P'') = N_{S_0}(P_0')(P'')$.

Clearly (i) follows from the definition of the morphisms in $\mathcal{L}_2$ because

$$\lambda \in \text{Aut}_{\mathcal{L}_1}(P'') \iff \forall x \in P'' \exists y \in P''(\lambda \circ \tilde{x} \circ \lambda^{-1} = \tilde{y}) \iff \lambda \in N_{\text{Aut}_{\mathcal{L}_1}(P_0')}(P'').$$

For (ii), note that $P'' \subseteq N_{S_0}(P_0') \subseteq \text{Aut}_{\mathcal{L}_1}(P_0')$ so by the choice of $\psi$ in equation (4),

$$N_{S_0}(P_0')(P'') = N_{S_0}(P_0') \cap N_{\text{Aut}_{\mathcal{L}_1}(P_0')}(P'') \in \text{Syl}_p(\text{N}_{\text{Aut}_{\mathcal{L}_1}(P_0')}(P'')).$$

On the other hand

$$N_{S_0}(P_0')(P'') \subseteq N_S(P'') \subseteq N_{\text{Aut}_{\mathcal{L}_1}(P_0')}(P''),$$

hence $N_S(P'') = N_{S_0}(P_0')(P'')$. We deduce that $N_S(P'') \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_1}(P''))$.

Finally, $\text{Aut}_{\mathcal{L}_1}(P) \cong \text{Aut}_{\mathcal{L}_1}(P'')$ because $P''$ and $P$ are isomorphic in $\mathcal{L}_2$ (via $\psi \circ \hat{\alpha}$). Also, $|N_S(P)| \geq |N_S(P'')|$ because $P$ is fully $F$-normalised. Therefore $N_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_1}(P))$ and Lemma 5.15 implies that $\text{Aut}_S(P)$ is a Sylow $p$-subgroup of $\text{Aut}_F(P)$.

**Condition II.** Fix $P \in \mathcal{H}$ and $\varphi \in \mathcal{F}(P, S)$. Definition 5.11 and part (a) of Lemma 5.14 show that $\varphi(P) \in \mathcal{H}$ and part (b) of this lemma shows that $\varphi(P)$ is $F$-centric and in particular it is fully $F$-centralised. We have to prove that $\varphi$ extends to some $\psi \in \mathcal{F}(N_\varphi, S)$ where

$$N_\varphi = \{ g \in N_S(P) : \varphi \circ c_g = c_s \circ \varphi \text{ for some } s \in S \}.$$  

Note that $s \in N_S(\text{Im} \varphi)$ in this definition. Set, for convenience $Q = N_\varphi$. We observe that

$$Q \subseteq N_S(Q_0) \quad \text{and} \quad Q \subseteq N_S(P) \subseteq N_S(P_0). \tag{5}$$

Let $\tilde{\varphi} \in \mathcal{L}_2(P, S)$ be a lift for $\varphi$, that is $\varphi = \pi_2(\tilde{\varphi})$. By definition, for every $q \in Q$ there exists some $s_q \in S$ such that $\varphi \circ c_{s_q} = c_{s_q} \circ \varphi$. Lifting to $\mathcal{L}_2$, we see from
Lemma 5.15 that there exists some \( z \in C_q(P) = Z(P) \) such that \( \tilde{\varphi} \circ \tilde{q} = \delta_q \circ \tilde{\varphi} \circ \tilde{z} = \delta_q \circ \varphi(z) \). Set \( y_q = s_q \varphi(z) \), then \( y_q \in S \) and

\[
(6) \quad \tilde{\varphi} \circ \tilde{q} = \tilde{y}_q \circ \tilde{\varphi}
\]

By Definition 5.8 the morphism \( \tilde{\varphi} \) is an element in \( L_1(P_0, S_0) \). By Lemma 5.7(c) we see that \( \tilde{\varphi} = \tilde{g} \circ \tilde{\lambda} \) where \( g \in G \) and \( \tilde{\lambda} \in L_0(P_0, S_0) \). Set \( \lambda = \pi_0(\tilde{\lambda}) \in F_0(P_0, S_0) \).

From parts (a) and (c) of Lemma 5.10 we see that \( \varphi|_{P_0} = \pi_1(\varphi) = \pi_1(\tilde{g} \circ \tilde{\lambda}) = c_g \circ \lambda \).

By definition, for every \( x \in Q_0 \) there exists some \( s \in S \) such that
\( \varphi \circ c_x = c_s \circ \varphi \) in \( F \).

By restriction to \( P_0 \) we obtain an equality of homomorphisms \( P_0 \rightarrow S \)
\( (7) \quad c_g \circ \lambda \circ c_x = c_s \circ c_y \circ \lambda \).

By restriction of \( \lambda \) to an isomorphism onto its image we see that
\( c_g^{-1} c_s \lambda = \lambda \circ c_x \in F_0 \quad \because x \in Q_0 \leq S_0 \).

Hypothesis (2) implies that \( g^{-1} c_s \lambda \in S_0 \) and therefore \( s \in S_0 \). We can therefore rewrite equation (7) as \( \lambda \circ c_x = c_g^{-1} c_s \lambda \circ \lambda \) where \( g^{-1} c_s \lambda \in S_0 \). Together with equation (5), this shows that \( x \in N_\lambda \) where
\( N_\lambda = \{ x \in N_{S_0}(P_0) : \lambda \circ c_x = c_y \circ \lambda \text{ for some } y \in S_0 \} \).

We deduce that \( Q_0 \leq N_\lambda \).

Since \( P_0 \) is \( F_0 \)-centric, so is \( \lambda(P_0) \) and in particular it is fully \( F_0 \)-centralised. Axiom (II) in \( F_0 \) enables us to extend \( \lambda \in F_0(P_0, S_0) \) to some \( \rho \in F_0(Q_0, S_0) \). Let \( \tilde{\rho} \) be a lift for \( \rho \) in \( L_0 \). Now, \( \lambda = \rho \circ \pi_0 Q_0 \circ \lambda = \pi_0(\tilde{\rho} \circ \lambda Q_0) \), so there exists some \( z \in Z(P_0) \leq P_0 \leq Q_0 \) such that
\( \tilde{\lambda} = \tilde{\rho} \circ \lambda Q_0 \circ \tilde{\lambda} = \tilde{\rho} \circ \tilde{z} \circ \lambda Q_0 \).

Set \( \tilde{\rho} = \tilde{\rho} \circ \tilde{\lambda} \) and \( \theta = \pi_0(\tilde{\rho}) \). Thus, \( \tilde{\theta} \in L_0(Q_0, S_0) \) and \( \theta \in F_0(Q_0, S_0) \) satisfy
\( \tilde{\lambda} = \tilde{\theta} \circ \lambda Q_0 \) and \( \theta|_{P_0} = \lambda \)

because \( \pi_0(\tilde{\theta})|_{P_0} = \pi_0(\tilde{\rho} \circ \tilde{\lambda})|_{P_0} = \rho \circ c_x|_{P_0} = \rho|_{P_0} = \lambda \).

Recall that we started with a lift \( \tilde{\varphi} = \tilde{g} \circ \tilde{\lambda} \) for \( \varphi \). By Lemma 5.7(a) we view \( \tilde{\theta} \) as a morphism in \( L_1 \) and define
\( \tilde{\psi} := \tilde{g} \circ \tilde{\theta} \in L_1(Q_0, S_0) \).

We now prove that for every \( q \in Q \), the element \( y_q \in S \) defined in equation (6) satisfies
\( (8) \quad \tilde{\psi} \circ \tilde{q} = \tilde{y}_q \circ \tilde{\psi} \) in \( L_1 \).

Observe that \( Q = N_\lambda \) so \( P \leq Q \) and in particular \( P_0 \leq Q_0 \). We shall now consider \( \iota := \iota \in L_1(P_0, Q_0) \) where \( \iota \in N_{G_0}(P_0, Q_0) \) is the identity of \( G \). Note that under the inclusion \( L_0 \subseteq L_1 \) in Lemma 5.7(a) we have \( \iota = \iota Q_0 \). Therefore
\( \tilde{\psi} \circ \iota = \tilde{g} \circ \tilde{\theta} \circ \iota Q_0 = \tilde{g} \circ \tilde{\lambda} = \tilde{\varphi} \) in \( L_1 \).

Equation (5), Lemma 5.19(a) and equation (6) imply that in \( L_1 \)
\( \tilde{\psi} \circ \tilde{q} \circ \iota = \tilde{\psi} \circ \tilde{q} \circ \iota \circ \tilde{\psi} = \tilde{\psi} \circ \tilde{\psi} \circ \tilde{q} = \tilde{\varphi} \circ \tilde{q} = \tilde{y}_q \circ \tilde{\varphi} = \tilde{y}_q \circ \tilde{\psi} \circ \iota \).

We deduce that equation (8) holds because \( \iota \) is an epimorphism in \( L_1 \) by Lemma 5.19(d). By Definition 5.8 we see that \( \psi \in L_2(Q, S) \). Set \( \psi := \pi_2(\tilde{\psi}) \). Then
\[ \psi \in \mathcal{F}_2(Q,S) = \mathcal{F}(Q,S) \] and by Lemma 5.10(c) we see that \( \psi|_P = \pi_2(\psi \circ \iota) = \pi_2(\tilde{\varphi}) = \varphi. \) This completes the proof. \( \Box \)

**Proof of Lemma 5.17.** Our notation was chosen in such a way that the argument in [4, Theorem 4.6, Step 7] can be read verbatim and we shall therefore avoid reproducing it. \( \Box \)

### 6. Maps from a homotopy colimit

Let \( \mathcal{C} \) be a small category, and \( X: \mathcal{C} \to \text{Top} \) be a diagram of spaces over \( \mathcal{C} \). The values taken by the functor will be denoted by \( X(c) \) and \( X(\varphi) \) where \( c \in \mathcal{C} \), \( \varphi \in \text{Mor}_{\mathcal{C}}(c,c') \). The homotopy colimit of the diagram \( X \) is the space

\[
\hocolim_{\mathcal{C}} X = \left( \coprod_{n \geq 0} \prod_{c_0 \to \cdots \to c_n} X(c_0) \times \Delta^n \right) / \sim
\]

where we divide by the usual face and degeneracy identifications [3, Ch. XII].

We filter the homotopy colimit by using the skeleta of the nerve of \( \mathcal{C} \), and we define \( F_n X \) to be the image of the union of \( X(c) \times \Delta^m \) in \( \hocolim_{\mathcal{C}} X \) for all \( m \leq n \). Notice that \( F_0 X \) is just \( \coprod_{c \in \mathcal{C}} X(c) \) and \( F_1 X \) is the union of the mapping cylinders of all \( \varphi \in \text{Mor}(\mathcal{C}) \). Observe that a map \( f_1: F_1 X \to Y \) is the same as a set of maps \( f_1(c): X(c) \to Y \) together with homotopies \( f_1(c') \circ X(\varphi) \simeq f_1(c) \) for every \( \varphi \in \text{Mor}(c,c') \). A set of maps \( X(-) \xrightarrow{f(-)} Y \) which admits such homotopies is called a system of homotopy compatible maps and it gives rise to an element in the set \( \lim_{\mathcal{C}} \pi(X(c),Y) \).

Fix a system of homotopy compatible maps \( X(-) \xrightarrow{f(-)} Y \). By the remark above it gives rise to a map \( f_1: F_1 X \to Y \) where \( f_1|_{X(c)} = f(c) \). Wojtkowiak [24] addressed the question whether \( f_1 \) can be extended, up to homotopy, to a map \( f: \hocolim_{\mathcal{C}} X \to Y \). The method is to extend \( f_1 \) by induction on the spaces \( F_n X \).

Given a map \( f_n: F_n X \to Y \) whose restriction to \( X(c) \) is homotopic to \( f(c) \), Wojtkowiak developed an obstruction theory for extending it to \( F_{n+1} X \) without changing it on \( F_{n-1} X \). The existence of such an extension depends on the vanishing of a certain obstruction class in \( \lim^{n+1} \pi_n(\text{map} f(c)(X(c),Y)) \). The extension from \( F_1 X \) to \( F_2 X \) involves in general a functor of non-abelian groups, into the category of groups and representations, whose \( \lim^2 \) term is described in Wojtkowiak's work. Fortunately, if these groups are abelian then the Wojtkowiak's definition of \( \lim^2 \) coincides with the usual one from homological algebra. Once the map has been extended to \( F_2 X \), a choice of homotopies allow to define well-defined functors \( \pi_n(\text{map} f(c)(X(c),Y)) \) into abelian groups for \( n > 1 \).

Given two maps \( f_1, f_2: \hocolim_{\mathcal{C}} X \to Y \) whose restrictions to \( X(c) \) are homotopic to \( f(c) \), Wojtkowiak also studies an obstruction theory for the construction of a homotopy \( f_1 \simeq f_2 \). Clearly, \( f_1 \) and \( f_2 \) give rise to a homotopy \( f_1|_{F_{n} X} \xrightarrow{H_0} f_2|_{F_{n} X} \). The idea is to extend the homotopy \( H_0 \) inductively to \( I \times F_{n} X \). Given a homotopy \( f_1|_{F_{n} X} \xrightarrow{H_{n-1}} f_2|_{F_{n} X} \), the possibility of extending it to a homotopy between the restrictions of \( f_1 \) and \( f_2 \) to \( F_{n-2} X \) without changing its values on \( F_{n-2} X \) depends on the vanishing of an obstruction class in \( \lim^n \pi_n(\text{map} f(c)(X(c),Y)) \).
6.1. **Definition** ([7, Definition 3.3]). Fix a prime $p$. We say that a small category $\mathcal{C}$ has bounded limits at $p$ if there exists $d \geq 0$ such that every functor $F: \mathcal{C} \to \mathbb{Z}_p\text{-mod}$ has the property that $\lim_{\mathcal{C}}^d F = 0$. We call $d$ the **height** of $\mathcal{C}$.

6.2. **Theorem.** Let $\mathcal{C}$ be a finite category with bounded limits at $p$ of height $d$ and consider a sequence of maps $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1}$ with partial composites $g_i = g_i \circ \cdots \circ g_0: Y_0 \to Y_{i+1}$. Given a functor $X: \mathcal{C} \to \text{Top}$ and a system of homotopy compatible maps $f(-): X(-) \to Y_0$, define new systems of homotopy compatible maps $f_i(-) = g_i \circ f(-): X(-) \to Y_{i+1}$ for all $i = 0, \ldots, d$. Assume that

(i) For every $c \in \mathcal{C}$ and every $i = 1, \ldots, d$ the induced map

$$\pi_i \text{map}^{f_i-1(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \pi_i \text{map}^{f_i(c)}(X(c), Y_{i+1})$$

is the trivial homomorphism between abelian groups.

(ii) The groups $\pi_{>0} \text{map}^{f_i(c)}(X(c), Y_i)$ are $\mathbb{Z}_p$-modules for all $c \in \mathcal{C}$ and all $i$.

Then

(a) There exists map $\tilde{f}: \hocolim_{\mathcal{C}} X \to Y_d$ which renders the following square homotopy commutative for all $c \in \mathcal{C}$,

$$\begin{array}{ccc}
X(c) & \xrightarrow{f(c)} & Y_0 \\
\downarrow{\iota(c)} & & \downarrow{g_{d-1}} \\
\hocolim_{\mathcal{C}} X & \xrightarrow{\tilde{f}} & Y_d.
\end{array}$$

(b) If $\tilde{f}_1, \tilde{f}_2: \hocolim_{\mathcal{C}} X \to Y_0$ satisfy $\tilde{f}_1|_{X(c)} \simeq \tilde{f}_2|_{X(c)} \simeq f(c)$ for all $c \in \mathcal{C}$ then the compositions $\hocolim_{\mathcal{C}} X \xrightarrow{\tilde{f}_1, \tilde{f}_2} Y_0 \xrightarrow{g_d} Y_{d+1}$ are homotopic.

**Proof.** (a) We shall define by induction maps $\tilde{f}_i: F_i X \to Y_i$ for all $i = 1, \ldots, d$ such that $\tilde{f}_i|_{X(c)} \simeq f_i(-)(c)$ for all $c \in \mathcal{C}$.

Note that, by definition of a system of homotopy compatible maps, we can construct a map $\tilde{f}_1: F_1 X \to Y_1$. Assume by induction that $\tilde{f}_i: F_i X \to Y_i$ with $\tilde{f}_i|_{X(c)} \simeq f_{i-1}(-)$ has been constructed for some $1 \leq i < d$. The obstruction class $\Theta_{i+1}$ for the extension of $\tilde{f}_i$ to $F_{i+1} X$ is mapped by the homomorphism

$$
\lim_{\mathcal{C}_{op}}^i \pi_{i+1} \text{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \lim_{\mathcal{C}_{op}}^i \pi_{i+1} \text{map}^{f_{i}(c)}(X(c), Y_{i+1})
$$

to the obstruction class $\Theta_{i+1}$ for the extension of $g_i \circ \tilde{f}_i$ to $F_{i+1} X$. When $i \geq 1$, by hypothesis (i) the groups are abelian and this homomorphism is trivial, whence $\Theta_{i+1} = 0$. Wojtkowiak’s obstruction theory guarantees the existence of a map $\tilde{f}_{i+1}: F_{i+1} X \to Y_{i+1}$ which agrees with $g_i \circ \tilde{f}_i$ on $F_{i-1} X$ and such that $\tilde{f}_{i+1}|_{X(c)} \simeq g_i \circ f_{i-1}(c) = f_i(c)$. This completes the induction step.

Hypothesis (ii) and the assumption on $\mathcal{C}$ imply that the groups

$$\lim_{\mathcal{C}_{op}}^i \pi_{i-1} \text{map}^{f_{i-1}(c)}(X(c), Y_d)$$

are trivial for all $i \geq d + 1$. Thus, the obstructions to the extension of $\tilde{f}_d$ to $F_i X$ where $i > d$ must all vanish. We can therefore construct by induction on $i \geq d + 1$ maps $\tilde{f}_i: F_i X \to Y_d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{d-1}(c)$ for all $c \in \mathcal{C}$ and such that $\tilde{f}_{i+1}$
agrees with \( f_i \) on \( F_{i-1}X \). We can finally define \( \tilde{f} : \text{hocolim}_C X = \bigcup_{\mathcal{C}} F_i X \to Y_d \) with the required properties. In fact, \( \tilde{f}|_{F_n X} = \tilde{f}_{n+1}|_{F_n X} \) for all \( n > d \).

(b) First, we construct by induction homotopies \( y_i \circ \tilde{f}_1|_{F_i X} \simeq y_i \circ \tilde{f}_2|_{F_i X} \) for all \( i = 0, \ldots, d \). Recall that \( F_0 X = \coprod_{c \in \mathcal{C}} X(c) \) and we define \( H_0 \) as the sum of the homotopies \( y_0 \circ \tilde{f}_1|_{X(c)} \simeq y_0 \circ \tilde{f}_2|_{X(c)} \).

Assume by induction that \( H_i : y_i \circ \tilde{f}_1|_{F_i X} \simeq y_i \circ \tilde{f}_2|_{F_i X} \) has been constructed where \( 0 \leq i < d \). The obstruction \( \Upsilon_i \) for the extension of \( H_i \) to a homotopy \( y_i \circ \tilde{f}_1|_{F_{i+1}X} \simeq y_i \circ \tilde{f}_2|_{F_{i+1}X} \) is mapped by the homomorphism

\[
\lim_{C^{op}} \pi_{i+1} \text{map}^f c(X(c), Y_{i+1}) \to \lim_{C^{op}} \pi_{i+1} \text{map}^{f+1} c(X(c), Y_{i+2})
\]
to the obstruction class \( \Upsilon_i \) for the extension of \( g_{i+1} \circ H_i : I \times F_i X \to Y_{i+2} \) to \( I \times F_{i+1}X \). This homomorphism is trivial by hypothesis (i). Therefore \( \Upsilon_i = 0 \), and by Wojtkowiak’s theory there is a homotopy \( y_{i+1} \circ \tilde{f}_1|_{F_{i+1}X} \simeq y_{i+1} \circ \tilde{f}_2|_{F_{i+1}X} \). This completes the induction step.

Now, the hypothesis on \( \mathcal{C} \) together with (ii) imply that the groups

\[
\lim_{C^{op}} \pi_i \text{map}^{f+1} c(X(c), Y_{d+1})
\]
are trivial for all \( i \geq d + 1 \). We can therefore construct by induction on \( i \geq d + 1 \) homotopies \( y_d \circ \tilde{f}_1|_{F_{i}X} \simeq y_d \circ \tilde{f}_2|_{F_{i}X} \) such that \( H_{i+1} \) and \( H_i \) agree on \( I \times F_{i-1}X \). There results a homotopy \( y_d \circ \tilde{f}_1 \simeq y_d \circ \tilde{f}_2 \).

\[ \square \]

7. Maps between \( p \)-local finite groups

7.1. Definition. Let \((S, F)\) be a fusion system. A map \( f : BS \to X \) is called \( F \)-invariant, if for every \( \varphi \in F(P, S) \) the composition \( BP \xrightarrow{B\varphi} BS \xrightarrow{f} X \) is homotopic to \( f|_{BP} = f \circ \operatorname{Bin}^P \).

7.2. Example. Let \((S, F, \mathcal{L})\) be a \( p \)-local finite group. The map \( \Theta : BS \to |\mathcal{L}| \) of 2.8 is \( F \)-invariant by Proposition 2.9.

Given a \( p \)-local finite group \((S, F, \mathcal{L})\), the question we address in this section is when an \( F \)-invariant map \( f : BS \to X \) can be extended to a map \( |\mathcal{L}| \to X \). Here is the main result of this section which uses the constructions in \( \S 3 \).

7.3. Theorem. Let \((S, F, \mathcal{L})\) and \((S', F', \mathcal{L}')\) be \( p \)-local finite groups and consider an \( F \)-invariant map \( f : BS \to |\mathcal{L}'|_p \). Then:

\[ (a) \text{ There exists } m > 0 \text{ and a map } \tilde{f} : |\mathcal{L}| \to (|\mathcal{L}'| \wr \Sigma_{pm})_p^\wedge \text{ which renders the following square homotopy commutative} \]

\[
\begin{array}{ccc}
BS & \xrightarrow{f} & |\mathcal{L}'|_p^\wedge \\
\Theta & & \downarrow \Delta_p^\wedge \\
|\mathcal{L}| & \xrightarrow{\tilde{f}} & (|\mathcal{L}'| \wr \Sigma_{pm})_p^\wedge 
\end{array}
\]

32
7.4. Example. If \( f = \Theta : BS \to |\mathcal{L}| \) then \( \tilde{f} \) can be chosen as the identity on \(|\mathcal{L}|^p\).

For a finite abelian group \( A \), set \( A_{(p)} = A \otimes \mathbb{Z}_{(p)} \); this is the set of \( p \)-power order elements in \( A \). The abelianisation of a group \( G \) is denoted \( G_{ab} \). The subgroup \( O^p(G) \) of a finite group \( G \) is the subgroup generated by all the elements of order \( p \); it is the smallest normal subgroup of \( G \) whose quotient is a \( p \)-group.

7.5. Proposition. Let \( H = G \wr \Sigma_k \) where \( G \) is a finite group. If \( p > 2 \) and \( k \geq 2 \) then \( H/O^p(H) \) is a factor group of \( (G_{ab})_{(p)} \). If \( p = 2 \) and \( k \geq 3 \) then \( H/O^p(H) \) is a factor group of \( (G_{ab})_{(2)} \times \mathbb{Z} \).

Proof. Write \( \tilde{H} = H/O^p(H) \) and consider the quotient homomorphism \( \pi : H \to \tilde{H} \).

Denote by \( G_i \) the \( i \)th copy of \( G \) in \( G^{	imes k} \). For any \( x \in G \) we shall denote by \( x_i \) the image of \( x \in G_i \) in \( H \) via the inclusion \( G^{	imes k} \subseteq H \). Note that \( x_i \) and \( y_j \), where \( x, y \in G \), commute in \( H \) if \( i \neq j \).

Assume that \( p > 2 \) and that \( k = 2 \). Since \( \Sigma_k \) is generated by involutions then \( \Sigma_k \leq O^p(H) \). Also note that \( H \) is generated by \( \Sigma_k \) and any one of \( G_i \), hence \( \tilde{H} \) is generated by any one of the images of \( G_i \) under \( \pi \). Let \( \tau \) denote \((1,2) \in \Sigma_2 \) (note that \( k \geq 2 \)). Since \( \tau \in O^p(H) \) we see that for any \( x \in G \) we have \( \pi(x_1) = \pi(x_1 \tau) = \pi(x_2) = \pi(x_2) \). Thus, given elements \( \bar{x}, \bar{y} \in H \) we can choose preimages \( x_1 \) and \( y_2 \) and observe that \( \bar{x} \bar{y} = \pi(x_1) \pi(y_2) = \pi(x_1 y_2) = \pi(y_2 x_1) = \bar{y} \bar{x} \).

This shows that \( \tilde{H} \) is a commutative factor group of \( G \) and since it is a \( p \)-group it must be a factor of \( (G_{ab})_{(p)} \).

Now assume that \( p = 2 \) and that \( k \geq 3 \). Clearly \( A_k \leq O^2(H) \) because \( A_k \) is generated by elements of odd order. Since \( H \) is generated by \( \Sigma_k \) and any one of the \( G_i \)’s, it follows that \( \tilde{H} \) is generated by the image of \( \tau = (1,2) \in \Sigma_k \) and by the images of any one of the \( G_i \)’s. Let \( \sigma \) denote the cycle \((1,2,3) \in A_k \) (note that \( k \geq 3 \)). Note that \( \sigma \in O^2(H) \) and that \( \sigma^{-1} x_1 \sigma = x_2 \) for any \( x \in G \). Therefore

\[
\pi(x_1) = \pi(x_2).
\]

(1)

Let \( \bar{\tau} \) denote \( \pi(\tau) \). Then \( \bar{\tau} \) and the element \( \bar{x} = \pi(x_1) \) commute in \( \tilde{H} \) because \( \bar{x} \bar{\tau} = \pi(x_1) \pi(\tau) = \pi(x_1 \tau) = \pi(x_2) = \bar{\pi}(\pi(x_1)) = \bar{\pi}(\pi(x_2)) = \bar{\pi}(\pi(x_2)) = \bar{\pi}(x_2) = \bar{x} \).

This shows that \( \bar{\tau} \in Z(\tilde{H}) \) and that \( \tilde{H} \) is a factor group of \( G \times C_2 \) because \( \tilde{H} \) is generated by \( \bar{\tau} \) and \( \bar{x} \) for all \( x \in G \). Now consider \( \bar{x}, \bar{y} \in \tilde{H} \) where \( \bar{x} = \pi(x_1) \) and \( \bar{y} = \pi(y_1) \). For some \( x, y \in G \) since \( \pi(y_1) = \pi(y_2) \) by (1), we conclude that \( \bar{x} \bar{y} = \pi(x_1) \pi(y_2) = \pi(x_1 y_2) = \pi(y_2 x_1) = \pi(y_2) \pi(x_1) = \bar{y} \bar{x} \).

It follows that \( \tilde{H} \) is an abelian 2-group hence it is a factor group of \( (G_{ab})_{(2)} \times C_2 \).

7.6. Lemma. For any \( p \)-local finite group \( (S, \mathcal{F}, \mathcal{L}) \), \( \pi_i(|\mathcal{L}|^p) \) are finite \( p \)-groups for all \( i \geq 1 \).

Proof. The fundamental group \( \pi_1(|\mathcal{L}|^p) \) is a finite \( p \)-group by [4, Theorem B]. Using a Serre class argument (see [21, Ch 9.6, Theorem 15]), we only need to show that the integral homology is finite at each degree. In [19], it is proven that the suspension spectrum \( \Sigma^\infty|\mathcal{L}|^p \) is a retract of \( \Sigma^\infty BS \) all of whose integral homology groups are finite abelian \( p \)-groups.
7.7. Proposition. Fix an integer $k \geq 3$ and let $(S, F, \mathcal{L})$ be a $p$-local finite group. Given a map $f: BP \to \hat{\mathcal{L}}_F$, let $g$ denote the composition

$$BP \xrightarrow{f} \mathcal{L}_F^\wedge \xrightarrow{\Delta} \mathcal{L}_F^\wedge \wr \Sigma_k \xrightarrow{\eta} (\mathcal{L}_F^\wedge \wr \Sigma_k)_F^\wedge.$$ 

Then all the homotopy groups of $\text{map}^q(BP, (\mathcal{L}_F^\wedge \wr \Sigma_k)_F^\wedge)$ are finite abelian $p$-groups.

Proof. If $S = 1$ then $|\mathcal{L}| = *$ hence $(|\mathcal{L}_F^\wedge \wr \Sigma_k)_F^\wedge \simeq (BS\Sigma_k)_F^\wedge$ and $g$ is null-homotopic. Dwyer-Zabrodsky’s result [12] shows that the space under study is homotopy equivalent to $(BS\Sigma_k)_F^\wedge$ and the result follows from Proposition 7.5 together with [6, Proposition A.2] and Lemma 7.6.

We shall therefore assume that $S \neq 1$. By [7, Theorem 4.4(a)] $f$ is homotopic to

$$BP \xrightarrow{\Theta(BP)} |\mathcal{L}| \xrightarrow{\eta} \hat{\mathcal{L}}_F^\wedge$$

for some $\rho: P \to S$. There results a diagram in which the bottom row is $g$, the first square commutes up to homotopy and the other squares commute on the nose

$$\begin{array}{ccc}
BP & \xrightarrow{\Theta(BP)} & |\mathcal{L}| \\
\downarrow & & \downarrow \\
BP & \xrightarrow{f} & \mathcal{L}_F^\wedge \\
\downarrow & & \downarrow \\
BP & \xrightarrow{\Delta} & \mathcal{L}_F^\wedge \wr \Sigma_k \\
\downarrow & & \downarrow \\
BP & \xrightarrow{\eta} \mathcal{L}_F^\wedge \wr \Sigma_k & \xrightarrow{\eta \Sigma_k} (\mathcal{L}_F^\wedge \wr \Sigma_k)_F^\wedge.
\end{array}$$

Since $|\mathcal{L}|$ is $p$-good by [7, Proposition 1.12], a Serre spectral sequence argument and [3, Lemma I.5.5] show that the vertical arrow on the right of the diagram is a homotopy equivalence. It follows that

$$\text{map}^q(BP, (\mathcal{L}_F^\wedge \wr \Sigma_k)_F^\wedge) \simeq \text{map}^q(BP, (\mathcal{L}_F^\wedge \wr \Sigma_k)_F^\wedge).$$

By Theorem 1.1 there exists a $p$-local finite group $(S', F', \mathcal{L}')$ where $S'$ is a Sylow $p$-subgroup of $S \wr \Sigma_k$ such that there is a homotopy equivalence $\omega: |\mathcal{L}| \wr \Sigma_k \xrightarrow{\simeq} |\mathcal{L}'|$ and the composition

$$BS' \xrightarrow{B(\Sigma_k)\wr |\mathcal{L}|} B(S \wr \Sigma_k) \simeq (BS) \wr \Sigma_k \xrightarrow{\Theta \wr \Sigma_k} |\mathcal{L}| \wr \Sigma_k \xrightarrow{\omega} |\mathcal{L}'|$$

is homotopic to $\Theta': BS' \to |\mathcal{L}'|$. Moreover, $\Delta: BS \to (BS) \wr \Sigma_k$ is induced by the diagonal inclusion $S \leq S \wr \Sigma_k$ which factors through the Sylow subgroup $S'$, and it is therefore homotopic to $BS \xrightarrow{B(\Sigma_k)\wr |\mathcal{L}|} BS \xrightarrow{B(\Sigma_k)} B(S \wr \Sigma_k) \simeq (BS) \wr \Sigma_k$. We therefore have the following homotopy commutative diagram

$$\begin{array}{ccc}
BS & \xrightarrow{B(\Sigma_k)} & BS \\
\downarrow & & \downarrow \\
BS' & \xrightarrow{B(\Sigma_k)\wr |\mathcal{L}|} & (BS) \wr \Sigma_k \\
\downarrow & & \downarrow \\
BS & \xrightarrow{\Theta \wr \Sigma_k} & |\mathcal{L}| \wr \Sigma_k \\
\downarrow & & \downarrow \\
BS & \xrightarrow{\omega \wr \Delta} & |\mathcal{L}'|,
\end{array}$$

from which it follows that

$$BS \xrightarrow{\Theta \wr \Sigma_k} |\mathcal{L}| \wr \Sigma_k \xrightarrow{\omega} |\mathcal{L}'|$$

is homotopic to

$$BS \xrightarrow{B(\Sigma_k) \wr |\mathcal{L}|} BS \xrightarrow{\Theta'} |\mathcal{L}'|.$$
Since \( w^\wedge_p \) is a homotopy equivalence and \( w^\wedge_p \circ \eta = \eta \circ w \), Proposition 2.11(a) and (3) imply that the mapping space in (2) is homotopy equivalent to

\[ \text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BP, |\mathcal{L}|^\wedge) \simeq \text{map}^{\rho}(|\mathcal{L}|^\wedge) \]

where \( Q = \rho(P) \leq S' \). Part (b) of Proposition 2.11 shows that the map obtained by applying the \( p \)-completion functor to

\[ \text{map}^\Theta\mid \mathcal{B}Q(BQ, |\mathcal{L}|^\wedge) \]

induces split surjections on homotopy groups. Since \( Q \leq S \leq S' \) then (3) implies that \( \Theta \mid \mathcal{B}Q \simeq w \circ \Delta \circ \Theta \mid \mathcal{B}Q \) and therefore, after \( p \)-completion

\[ \text{map}^{\Delta \circ \Theta \mid \mathcal{B}Q}(BQ, |\mathcal{L}|^\wedge) \eta \rightarrow \text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}| \Sigma_k)^\wedge) \]

induces split surjections on homotopy groups where by (4) the space on the right is homotopy equivalent to (2). Diagram (1) shows that (6) factors up to homotopy through

\[ \text{map}^{\Delta \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}| \Sigma_k)^\wedge) \eta \rightarrow \text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}| \Sigma_k)^\wedge) \]

which in addition must also be surjective on homotopy groups. It remains to show that the homotopy groups of the space on the left are finite abelian \( p \)-groups.

Proposition 3.8(b) implies that

\[ \text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}| \Sigma_k)^\wedge) \simeq \text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}| \Sigma_k)^\wedge) \]

By Proposition 2.11(a) the space \( \text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}|)^\wedge) \) is homotopy equivalent to the \( p \)-completed classifying space of a \( p \)-local finite group. It is therefore \( p \)-complete by [7, Proposition 1.12] and its homotopy groups are finite \( p \)-groups by Proposition 7.6, albeit the fundamental group is not necessarily abelian. By Remark 3.6, the homotopy groups of the mapping space in (8) are

\[ \pi_1(\text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}|)^\wedge)) \Sigma_k, \quad \text{and} \]

\[ \oplus_k \pi_i(\text{map}^{\rho \circ \Theta \mid \mathcal{B}Q}(BQ, (|\mathcal{L}|)^\wedge)) \quad \text{for } i > 1. \]

Now [3, Proposition VII.4.3] shows that the homotopy groups of the \( p \)-completion of (8) are finite \( p \)-groups. The fundamental group is abelian by Proposition 7.5 together with [6, Proposition A.2].

\[ \square \]

**Proof of Theorem 7.3.** First, we assume that \( S \neq 1 \), or else the result is a triviality. Set \( \mathcal{C} = \mathcal{O}(\mathcal{F}^c) \) and recall from [7, Corollary 3.4] that \( \mathcal{C} \) is a finite category which has bounded limits at \( p \) of height \( d \geq 1 \).

We shall now construct inductively a sequence of spaces and maps

\[ |\mathcal{L}|^\wedge_i = Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1} \]

together with integers \( m_0, m_1, \ldots, m_{d+1} \), where \( m_i \geq 2 \), with the following properties. First, \( Y_0 = |\mathcal{L}|^\wedge_p \). Set \( f_i = g_i \circ \cdots \circ g_0 \circ f : BS \rightarrow Y_{i+1} \) and set \( G_i = \Sigma_{m^0} \Sigma_{m^1} \cdots \Sigma_{m^i} \). Then the following holds for all \( i = 0, \ldots, d \).

(i) There are homotopy equivalences \( \omega_{i+1} : Y_{i+1} \simeq (|\mathcal{L}|^\wedge_p \cap G_{i+1})^\wedge \)

such that

\[ |\mathcal{L}|^\wedge_p = Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1} \xrightarrow{\omega_{i+1}} (|\mathcal{L}|^\wedge_p \cap G_{i+1})^\wedge \]

is homotopic to \( |\mathcal{L}|^\wedge_p \xrightarrow{\Delta} |\mathcal{L}|^\wedge_p \cap G_{i+1} \xrightarrow{\eta} (|\mathcal{L}|^\wedge_p \cap G_{i+1})^\wedge \).
(ii) $\pi_i(\text{map}^f_{\text{fr}}(BP,Y_{i+1}))$ are finite abelian groups for all $P \leq S$.

(iii) If $i \geq 1$ then for all $P \leq S$ the homomorphism induced by $g_i$

\[ \pi_i(\text{map}^f_{\text{fr}}(BP,Y_i)) \xrightarrow{[g_i]^*} \pi_i(\text{map}^f_{\text{fr}}(BP,Y_{i+1})) \]

is trivial.

Let $L_0 = L'$ and $Y_0 = |L_0|^\wedge$. We now define by induction on $i \geq 1$ the integers $m_{i-1}$ and maps $Y_{i-1} \xrightarrow{\varphi_{i-1}} Y_i$ with the properties (i)-(iii) above. To begin the induction set $m_0 = 2$ and $Y_1 = (Y_0 \wedge \Sigma_{i^p})^\wedge$ and set $g_0 = \eta \circ \Delta(Y_0)$. Condition (i) holds directly from this definition, condition (ii) follows from Proposition 7.7 since $p^2 \geq 3$ and condition (iii) holds vacuously since $g_0$ is not required to satisfy it.

Assume by induction that $m_{i-1}$ and $g_{i-1} : Y_{i-1} \to Y_i$ have been defined for some $1 \leq i < d+1$ such that (i)-(iii) hold. We construct the next pair $(g_i : Y_i \to Y_{i+1}, m_i)$ as follows. Let $p^{m_i}$ be the maximum of $p^2$ and the exponent of the finite abelian $p$-group

\[ \bigoplus_{P \in \mathcal{O}(F')} \pi_i(\text{map}^f_{\text{fr}}(BP,Y_i)). \]

Define $Y_{i+1} = (Y_i \wedge \Sigma_{p^{m_i}})^\wedge$ and let $g_i : Y_i \to Y_{i+1}$ be the composition

\[ Y_i \xrightarrow{\Delta(Y_i)} Y_i \wedge \Sigma_{p^{m_i}} \xrightarrow{\eta} (Y_i \wedge \Sigma_{p^{m_i}})^\wedge. \]

Since $|L'|$ is $p$-good by [7, Proposition 1.12], the induction hypothesis (i) on $Y_i$, a Serre spectral sequence argument together with [3, I.5.5] and Theorem 1.1 show that

\[ Y_i \cong (|L'|_p \wedge G_i)^\wedge \cong (|L'|_p \wedge G_i)^\wedge \cong |L'|_p \wedge G_i \]

for some $p$-local finite group $(S_i, F_i, L_i)$. Condition (ii) for $g_i$ holds by Proposition 7.7 because $Y_{i+1} \cong (|L'|_p \wedge \Sigma_{p^{m_i}})^\wedge$.

Furthermore, all the homotopy groups of $|L'|_p \wedge \Sigma_{p^{m_i}}$ are finite by Proposition 7.6 and Remark 3.6, whence this space is $p$-good by [3, Ch. VII.4.3]. It follows that $Y_{i+1}$ is $p$-complete. Condition (iii) holds for $g_i : Y_i \to Y_{i+1}$ by Proposition 4.3 and the way that $m_i$ was chosen.

By induction hypothesis there is a homotopy equivalence $w_i : Y_i \to (|L'|_p \wedge G_i)^\wedge$ which renders the top-left square in the following diagram homotopy commutative.

\[ \begin{array}{ccc}
|L'|_p \wedge G_i & \xrightarrow{g_{i-1} \circ \cdots \circ g_0} & Y_i \\
\Delta & \cong & w_i \\
|L'|_p \wedge G_i & \xrightarrow{\eta \Sigma_{p^{m_i}}} & (|L'|_p \wedge G_i)^\wedge \\
\Delta & \cong & \eta \Sigma_{p^{m_i}} \\
|L'|_p \wedge G_i & \xrightarrow{\eta \Sigma_{p^{m_i}}} & (|L'|_p \wedge G_i)^\wedge \\
\end{array} \]

The remainder of the diagram commutes and the composition $\eta \circ \Delta(Y_i)$ in the first row is by definition $g_i$. By Theorem 1.1, [7, Proposition 1.12] and [3, Lemma 1.5.5], the arrows on the right are homotopy equivalences. Define the equivalence $w_{i+1} : Y_{i+1} \to (|L'|_p \wedge G_{i+1})^\wedge$ as the composition of the equivalences in the right
column. Now property (i) follows from this diagram and Proposition 3.5. Also, the
diagram above shows that
\[
\text{map}^{1,3}_{1,5}(BP, Y_{i+1}) \simeq \text{map}^{3,6}_{1,5}(BP, (|\mathcal{L}'|_p \wr G_{i+1})^\wedge)
\]
and property (ii) for \( f \) holds by Proposition 7.7.

We now consider the functor \( B: \mathcal{C} \to \text{Top} \) recalled in 2.7. Clearly \( f: BS \to |\mathcal{L}'|_p \)
gives rise to a system of homotopy compatible maps \( f_0: \mathcal{C}(-) \to |\mathcal{L}'|_p \) in the sense
described in Section 56. By applying part (a) of Theorem 6.2 to the compositions
\[
BS \xrightarrow{f_0} Y_0 \xrightarrow{g_0} \cdots \xrightarrow{g_d} Y_{d+1}
\]
we conclude that there exists a map \( f_0: \mathcal{C} \to Y_d \simeq (|\mathcal{L}'| \wr G_d)_p^\wedge \) whose restriction to \( BS \) is homotopic to
\[
(1) \quad BS \xrightarrow{f} |\mathcal{L}'|_p^\wedge \xrightarrow{\eta \circ \Delta} (|\mathcal{L}'|_p \wr G_d)_p^\wedge.
\]
Since \(|\mathcal{L}'|\) is \( p \)-good by [7, Proposition 1.12], we have the following commutative
diagram in which the vertical right arrow is a homotopy equivalence
\[
\begin{array}{ccc}
|\mathcal{L}'| & \xrightarrow{\Delta} & |\mathcal{L}'| \wr G_d \\
\eta \downarrow & & \eta \downarrow \\
|\mathcal{L}'|_p^\wedge & \xrightarrow{\Delta} & (|\mathcal{L}'| \wr G_d)_p^\wedge
\end{array}
\]
Therefore \( Y_d \simeq (|\mathcal{L}'| \wr G_d)_p^\wedge \). From Theorem 1.1 we also see that the spaces on the
right of this diagram are \( p \)-complete. Applying [3, Proposition II.2.8] we deduce that
\( \eta \circ \Delta \) in (1) is homotopic to \(|\mathcal{L}'|_p^\wedge \xrightarrow{\Delta} (|\mathcal{L}'| \wr G_d)_p^\wedge \) composed with the equivalence
in the right of the diagram. Part (a) of this theorem follows by composition with the
map induced by the inclusion \( G_d \leq \prod_{p^{m_d+m_{d+1}}} \).

To prove part (b), we analogously apply part (b) of Theorem 6.2 to deduce that
\[
|\mathcal{L}'|_p^\wedge \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1} \simeq (|\mathcal{L}'| \wr G_{d+1})_p^\wedge
\]
are homotopic. The result now follows by composition with the map induced by the
inclusion \( G_{d+1} \leq \prod_{p^{m_d+m_{d+1}}} \). \( \square \)

**Proof of Theorem 1.3.** The induced map \( BS \xrightarrow{Bp} BS' \xrightarrow{\eta \circ \varphi} |\mathcal{L}'|_p^\wedge \) is clearly \( \mathcal{F} \)-
invariant because \( BS' \to |\mathcal{L}'|_p^\wedge \) is \( \mathcal{F} \)-invariant by 7.2 and \( p \) is fusion preserving.
The result is now a direct consequence of Theorem 7.3 and Theorem 1.1. \( \square \)

We say that \( \rho: S \to \Sigma_n \) is \( \mathcal{F} \)-invariant if \( \rho|_\mathcal{F} \) and \( \rho \circ \varphi \) are equivalent representations
for every \( P \leq S \) and \( \varphi \in \mathcal{F}(P, S) \).

**7.8. Proposition.** Let \((S, \mathcal{F}, \mathcal{L})\) be a \( p \)-local finite group and let \( \rho: S \to \Sigma_n \) be a
homomorphism. Then the following statements are equivalent:

1. \( \rho \) is \( \mathcal{F} \)-invariant.
2. \( Bp: BS \to B\Sigma_n \) is an \( \mathcal{F} \)-invariant map.
3. \( \eta \circ \rho: BS \to (B\Sigma_n)_p^\wedge \) is an \( \mathcal{F} \)-invariant map.

**Proof.** It follows immediately from Dwyer-Zabrodsky’s result [12] which gives rise to
bijections \( \text{Rep}(P, \Sigma_n) \approx [BP, \Sigma_n] \xrightarrow{\text{iso}} [BP, (B\Sigma_n)_p^\wedge] \) for all \( P \leq S \). \( \square \)
7.9. Proposition. The regular permutation representation of a finite $p$-group $S$ induces an $\mathcal{F}$-invariant map $B\text{reg}_S$: $BS \to B\Sigma_{|S|}$ for any fusion system $\mathcal{F}$ on $S$.

Proof. By Proposition 7.8, it is enough to check that $\text{reg}_S: S \to \Sigma_{|S|}$ is $\mathcal{F}$-invariant. Note that $S$ acts freely on $S$ via $\text{reg}_S: S \to \Sigma_{|S|}$, that is all the isotropy subgroups are trivial. In particular, any group monomorphism $\varphi: P \to S$ where $P \leq S$ renders $S$ a free $P$-set via $\text{reg}_S \circ \varphi$. Since any two free $P$-sets of the same cardinality are equivalent, it follows that $\text{reg}_S |_P$ and $\text{reg}_S \circ \varphi$ are conjugate in $\Sigma_n$.

By Example 7.2 and Proposition 7.8, every map $f: [L] \to (BS_n)^\wedge$ gives rise to an $\mathcal{F}$-invariant representation $\rho$ of $S$ of rank $n$ where $B\rho \simeq f|_{BS}$. Not every $\mathcal{F}$-invariant representation of $S$ arises necessarily in this way. However, next proposition gives a partial answer to that question.

7.10. Proposition. Let $(S, \mathcal{F}, L)$ be a $p$-local finite group.

(a) Given $\rho \in \text{Rep}_n(\mathcal{F})$, there exists some $k \geq 0$ and an element $\tilde{f} \in \text{Rep}_{p^k n}(L)$ such that $f|_{BS}$ is homotopic to $BS \xrightarrow{\text{B}(p^k \rho)} BS_{p^k n} \xrightarrow{\eta_n} (BS_{p^k n})^\wedge$.

(b) Consider $f_1, f_2 \in \text{Rep}_n(L)$ such that $f_1|_{BS} \simeq f_2|_{BS}$. Then there exists some $c \geq 0$ such that $p^c \cdot f_1 = p^c \cdot f_2$ in $\text{Rep}_{p^c n}(L)$.

Proof. Let $(S, \mathcal{F}, L)$ be the $p$-local finite group associated with $S_n$. Then [7, Proposition 1.12] with a standard Serre spectral sequence argument show that

\[(B\Sigma_n)^\wedge \simeq [L]^\wedge \xrightarrow{\Delta^\wedge_p} ([L]^\wedge \otimes \Sigma_k)^\wedge \simeq ((B\Sigma_n)^\wedge \otimes \Sigma_k)^\wedge \xrightarrow{\text{B}\text{inl}^\wedge_p} (BS_{nk})^\wedge\]

\[(B\Sigma_n)^\wedge \xrightarrow{(B\Delta\Sigma)^\wedge_p} (B\Sigma_{nk})^\wedge\]

where $\Delta: \Sigma_n \to \Sigma_{nk}$ is the diagonal inclusion, are homotopic. Both (a) and (b) follow directly from Proposition 7.8, Theorem 7.3 and (1) taking into account the definition of the operation $+$ in $\bigoplus_{n \geq 0} \text{Rep}_n(\mathcal{F})$ and $\bigoplus_{n \geq 0} \text{Rep}_n(L)$.

Proof of Theorem 1.5. Apply Propositions 7.9 and 7.10(a) to obtain some $f \in \text{Rep}_{p^k n}(L)$ such that $f|_{BS}$ is homotopic to $\eta B(p^k \cdot \text{reg}_S)$, that is, $\Phi(f) = p^k \cdot \text{reg}_S$.

By [6, Lemma 2.3], $H^*(S; F_p)$ is a finitely generated module over the Noetherian $F_p$-algebra $H^*\left(B\Sigma_{p^k n}^{\mathcal{F}}; F_p\right)$ via the algebra map $(p^k \cdot \text{reg}_S)^*$. Finally, $H^*([L]; F_p)$ is a submodule of $H^*\left(S; F_p^\wedge\right)$ by [7, Theorem B] and it is therefore finitely generated. Now apply [6, Lemma 2.3] again to deduce that $f$ is a homotopy monomorphism.

8. The index of the Sylow subgroup

Let $(S, \mathcal{F}, L)$ be a $p$-local finite group and let $f: [L] \to (BS_n)^\wedge$ be a map. The restriction $f|_{BS} = f \circ \Theta$ is $\mathcal{F}$-invariant by Example 7.2 and is homotopic to $(B\rho)^\wedge$ for a unique $\rho \in \text{Rep}(S, \Sigma_n)$ which is $\mathcal{F}$-invariant by Proposition 7.8 and [12]. There results maps $\text{Rep}_n(L) \to \text{Rep}_n(\mathcal{F})$ which are compatible with the operations $+$ and $\times$ defined in the introduction. They give rise to a ring homomorphism

$\Phi: \text{Rep}(L) \to \text{Rep}(\mathcal{F})$.

8.1. Proposition. The abelian groups underlying $\ker(\Phi)$ and $\text{coker}(\Phi)$ are $p$-torsion.
Proof. An element in $\ker(\Phi)$ has the form $f_1 - f_2$ where $f_1, f_2 \in \text{Rep}_n(\mathcal{L})$ for some $n$ and $f_1|_{BS} \simeq f_2|_{BS}$. Proposition 7.10 implies that $\rho^p \cdot (f_1 - f_2) = 0$ in $\text{Rep}(\mathcal{L})$ and it follows that $\ker(\Phi)$ is $p$-torsion.

An element of $\text{Rep}({\mathcal{F}})$ has the form $\rho_1 - \rho_2$ for some $\rho_1 \in \text{Rep}_{n_1}(\mathcal{F})$ and $\rho_2 \in \text{Rep}_{n_2}(\mathcal{F})$. By Proposition 7.10, the definition of $\Phi$ and the definition of the operations $+$ in $\text{Rep}(\mathcal{F})$ and $\text{Rep}(\mathcal{L})$, we see that there exist integers $k_1, k_2 \geq 0$ and representations $f_1 \in \text{Rep}_{p^{k_1}n_1}(\mathcal{L})$ and $f_2 \in \text{Rep}_{p^{k_2}n_2}(\mathcal{L})$ such that $\Phi(f_1) = p^{k_1} \cdot \rho_1$ and $\Phi(f_2) = p^{k_2} \cdot \rho_2$. Then $\omega = p^{k_2} \cdot f_1 - p^{k_1} \cdot f_2$ is an element of $\text{Rep}(\mathcal{L})$ such that $\Phi(\omega) = p^{k_1+k_2} \cdot (\rho_1 - \rho_2)$. It follows that $\text{coker}(\Phi)$ is $p$-torsion. □

By Propositions 7.9 the ring $\text{Rep}(\mathcal{F})$ contains $\text{reg}_S: S \to \Sigma|S|$ which generates an (additive) infinite cyclic group $\text{Rep}^{\text{reg}}(\mathcal{F}) := \{n \cdot \text{reg}_S\}_{n \in \mathbb{Z}}$ in $\text{Rep}(\mathcal{F})$. Similarly let $\text{Rep}^{\text{reg}}(\mathcal{L})$ denote the additive subgroup of the ring $\text{Rep}(\mathcal{L})$ generated by all the $S$-regular representations of $(S, \mathcal{F}, \mathcal{L})$; See Definition 1.4.

It follows directly from the definitions that $\Phi$ restricts to a group homomorphism

$$\Phi^{\text{reg}}: \text{Rep}^{\text{reg}}(\mathcal{L}) \to \text{Rep}^{\text{reg}}(\mathcal{F}).$$

8.2. Corollary. The cokernel of $\Phi^{\text{reg}}$ is a cyclic $p$-group. The kernel of $\Phi^{\text{reg}}$ is an abelian torsion $p$-group and $\text{Rep}^{\text{reg}}(\mathcal{L}) \cong \mathbb{Z} \oplus \text{abelian } p\text{-torsion group}.$

Proof. This follows from Proposition 8.1 which in particular implies that the image of $\Phi^{\text{reg}}$ is isomorphic to $\mathbb{Z}$, whence it splits off from $\text{Rep}^{\text{reg}}(\mathcal{L})$. □

Given a finite group $G$ there is a natural one-to-one correspondence between equivalence classes of permutation representations $G \to \Sigma_n$ and equivalence classes of $G$-sets of cardinality $n$. Sum and products of representations (as described in the introduction) correspond to disjoint unions and products of the associated $G$-sets. Note that $\text{reg}_G$ corresponds to a free $G$-set with one orbit.

Let us return to discuss $\text{Rep}(\mathcal{F})$. Since the product of a free $S$-set with any other $S$-set is again a free set, it follows that $\text{Rep}^{\text{reg}}(\mathcal{F})$ and $\text{Rep}^{\text{reg}}(\mathcal{L})$ are in fact ideals in $\text{Rep}(\mathcal{F})$ and $\text{Rep}(\mathcal{L})$ and that $\Phi^{\text{reg}}$ is a ring homomorphism.

8.3. Example. Let $(S, \mathcal{F}, \mathcal{L})$ be the $p$-local finite group of a finite group $G$. The restriction of $(B \text{reg}_G)_{p}^\wedge: |\mathcal{L}|^\wedge_{p} \to (B \Sigma|G|)^\wedge_{p}$ to $BS$ is homotopic to $n \cdot (B \text{reg}_S)_{p}^\wedge$ where $n = |G: S|$ because $\text{reg}_G: G \to \Sigma|G|$ renders $G$ a free $G$-set, whence a free $S$-set. In particular $(B \text{reg}_G)_{p}^\wedge \circ \Theta$ is an element in $\text{Rep}^{\text{reg}}(\mathcal{L})$ which is mapped by $\Phi$ to $n \cdot \text{reg}_S$. It follows that $|G: S| \in \text{Im}(\Phi^{\text{reg}})$, whence $|\text{coker}(\Phi^{\text{reg}})|$ divides $|G: S|$.

8.4. Definition. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Define the upper and lower index of $S$ in $\mathcal{L}$ by

$$\text{Uind}(\mathcal{L}: S) = |\text{coker}(\Phi^{\text{reg}})|$$

$$\text{Lind}(\mathcal{L}: S) = |\text{Rep}^{\text{reg}}(\mathcal{F}): \text{Rep}^{\text{reg}}(\mathcal{F}) \cap \text{Im}(\Phi)|.$$

Clearly $\text{Lind}(\mathcal{L}: S)$ divides $\text{Uind}(\mathcal{L}: S)$ because $\text{Im}(\Phi^{\text{reg}}) \leq \text{Im}(\Phi) \cap \text{Rep}^{\text{reg}}(\mathcal{F})$.

8.5. Lemma. Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then $\text{Uind}(\mathcal{L}: S)$ is a $p$-power. If there exists a permutation representation $\rho: |\mathcal{L}| \to (B \Sigma_n|_p^\wedge)$ such that $\rho|_{BS} \simeq B(n \cdot \text{reg}_S)$ with $n \geq 1$ prime to $p$, then $\text{Uind}(\mathcal{L}: S) = 1$, and in particular also $\text{Lind}(\mathcal{L}: S) = 1$.

Proof. The first statement follows from Corollary 8.2. The existence of $\rho$ shows that $n \in \text{Im}(\Phi^{\text{reg}})$ hence, $\text{Uind}(\mathcal{L}: S) = 1$. □
We shall now prove Theorem 1.8. In fact we prove the following stronger result.

8.6. **Theorem.** Under the hypotheses of Theorem 1.8 we have \( \text{Uind}(\mathcal{L}; S) = 1 \).

**Proof.** (1) This follows from Lemma 8.5 and Example 8.3.

(2) Let \( C_n \) be the poset \( \{c_0, c_1, c_2\} \) whose only relations are defined by \( c_1 \prec c_0 \) and \( c_2 \prec c_0 \) for all \( i = 1, \ldots, n \). View \( C_n \) as a small category where \( x \prec y \) corresponds to an arrow \( x \to y \).

In [16, Section 7], the authors prove that if the longest chain of proper inclusions of \( \mathcal{F} \)-centric \( \mathcal{F} \)-radical subgroups of \( S \) has length \( \leq 2 \), then \( |L| \simeq \text{hocolim}_{C_n} F \) where the functor \( F: C_n \to \text{Top} \) has the following properties. The values of \( F \) are the classifying spaces of finite groups \( G \) and \( c_0, c_1, c_2 \) for \( i = 1, \ldots, n \) and the maps \( F(c_i) \to F(c_0) \) and \( F(c_i) \to F(c_j) \) are induced by inclusion of groups \( G_0 \leq G_1 \leq G_2 \). In addition, \( k_i = |G_i| \) are prime to \( p \), and \( S \) is a subgroup of \( G_0 \) of index prime to \( p \). Also, the map \( \Theta: BS \to |\mathcal{L}| \) factors up to homotopy through \( BG_0 \simeq F(c_0) \to \text{hocolim}_{C_n} F \simeq |\mathcal{L}| \).

Set \( k = \prod_{i=1}^{n} k_i \) and \( k_0 = |G_0| \cdot k \). Note that \( k_0 \) is divisible by \( |G_i| \) and \( |G_j| \) for all \( i \) because \( k_0 = k \cdot |G_0| = k \cdot |G_1| \cdot |G_0| \cdot |G_1| \) and \( k_i \) divides \( k_0 \). Set \( \ell_i = k_0 / |G_i| \) and \( m_i = k_0 / |G_2| \). Consider the following permutation representations for \( i = 1, \ldots, n \)

\[
\begin{align*}
k \cdot \text{reg}_{G_0}: G_0 &\to \Sigma_{k_0}, \\
\ell_i \cdot \text{reg}_{G_1}: G_1 &\to \Sigma_{\ell_i}, \\
m_i \cdot \text{reg}_{G_2}: G_2 &\to \Sigma_{m_i}.
\end{align*}
\]

Note that \( (k \cdot \text{reg}_{G_0})|_{G_1} \) and \( (m_i \cdot \text{reg}_{G_2})|_{G_1} \) are equivalent to \( \ell_i \cdot \text{reg}_{G_1} \) because all of them render the set \( \{1, \ldots, k_0\} \) a free \( G_1 \)-set with \( \ell_i \) orbits. By taking classifying spaces there results a system of homotopy compatible maps \( F \to B\Sigma_{k_0} \). It can be rectified to a system of compatible maps \( F \to B\Sigma_{k_0} \) as follows. First, set the maps \( F(c_i) \to B\Sigma_{k_0} \) to be the composition of \( F(c_i) \to F(c_0) \to B\Sigma_{k_0} \). Next, replace the maps \( F(c_i) \to B\Sigma_{k_0} \) by cofibrations and change the maps \( F(c_i) \to B\Sigma_{k_0} \) to homotopy to obtain a system of compatible maps \( F \to B\Sigma_{k_0} \).

There results a map \( f: |\mathcal{L}| \simeq \text{hocolim} F \to B\Sigma_{k_0} \) such that \( f|_{BS} = f \circ B_t G_0 \simeq k \cdot |G_0| \cdot S \cdot B\text{reg}_S \) where \( k \cdot |G_0| \cdot S \) is prime to \( p \). By applying Lemma 8.5 we deduce that \( \text{Uind}(\mathcal{L}; S) = 1 \).

Now, all the exotic examples in [7, Examples 9.3, 9.4], [8] and [11] satisfy the condition of [16, Section 7] that chains of proper inclusions of \( \mathcal{F} \)-centric \( \mathcal{F} \)-radical subgroups of \( S \) have length \( \leq 2 \). \( \square \)

8.7. **Conjecture.** For all \( p \)-local finite groups \( \text{Uind}(\mathcal{L}; S) = 1 \).

**References**


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