Combinatorial Dyson–Schwinger equations and inductive data types

Joachim Kock *

Departament de matemàtiques
Universitat Autònoma de Barcelona

Abstract

The goal of this contribution is to explain the analogy between combinatorial Dyson–Schwinger equations and inductive data types to a readership of mathematical physicists. The connection relies on an interpretation of combinatorial Dyson–Schwinger equations as fix-point equations for polynomial functors (established elsewhere by the author, and summarised here), combined with the now-classical fact that polynomial functors provide semantics for inductive types. The paper is expository, and comprises also a brief introduction to type theory.

Introduction

The aim of this contribution is to point out and explain some connections between Dyson–Schwinger equations, as employed in quantum field theory (QFT), and inductive data types as they appear in constructive type theory, at the foundations of mathematics. The paper is expository and targets a readership of mathematical physicists with no background in category theory or type theory, attempting to explain the required background knowledge along the way.

Briefly, the combinatorial Dyson–Schwinger equations are regarded as syntactic specifications of inductive types: the B_+-operators play the role of

*Email address: `kock@mat.uab.cat`

constructors, whereas the equations themselves, which are manifestly fixpoint equations, express the inductive character of the type, which in type theory is given by eliminators. While this connection is, to some extent, a banality — the inductive character of Feynman graphs is plain, and the only access we have to the solutions of the Dyson–Schwinger equations is through induction in some form or another — nevertheless, I believe it is potentially useful to formalise this analogy and to develop it further. I would like to suggest that the 'natively inductive' methods of type theory could, perhaps, be useful for structuring computations in quantum field theory. On the other hand, there are several algebraic structures related to Dyson–Schwinger equations, such as Hopf algebras and pre-Lie algebras, which could possibly be of interest also in type theory. I should admit from the outset that I am relatively ignorant of the finer details of both quantum field theory and type theory, and that the potential consequences of the presented connections are speculative at present. My path into these questions originated in experience with polynomial functors, which are, in my view, an important vehicle for establishing the connection.

The polynomial-functor approach to combinatorial Dyson–Schwinger equations is developed in detail elsewhere [13]. Its main point, explained in the first part of this paper, is to lift the equations from the Hopf-algebraic level to the objective combinatorial level, dealing directly with the combinatorial objects themselves through explicit bijections and categorical methods. The second part interprets the same ideas in type theory, exploiting a well-known fact in the categorical semantics of type theory [22], namely that least fixpoints for polynomial functors correspond precisely to W-types, a certain class of inductive types.

1 Combinatorial Dyson–Schwinger equations

The Dyson–Schwinger equations are the quantum equations of motion, and are comprised of infinite hierarchies of functional equations. Solving these equations is a significant challenge, especially in the field of quantum chromodynamics, where perturbative methods are impossible below the confinement scale [24]. While there is, of course, an essential analytic aspect to the Dyson–Schwinger equations, which involves Feynman integrals, there is also a structural aspect that is essentially related to combinatorics. For a long period of time, this combinatorial aspect was characteristic for *pertur-*

bative QFT, beginning with Feynman, progressing via Bogoliubov, Parasiuk, Hepp, and Zimmerman, and culminating in the work of Kreimer [15] and his collaborators around the turn of the millennium, when this combinatorics was distilled into clear-cut algebraic structures with numerous connections to many fields of mathematics [3]. In particular, Kreimer [15] discovered that the combinatorics of perturbative renormalisation is encoded in a Hopf algebra of trees, now called the Connes–Kreimer Hopf algebra. However, it is gradually becoming clear [2], [16] that this combinatorial and algebraic insight is also valuable in the *non-perturbative* regime.

The solution of the full Dyson–Schwinger equations can be expressed in the form of an infinite sum of integrals. In that case, the solution to the so-called *combinatorial* Dyson–Schwinger equations, introduced by Bergbauer and Kreimer [2] and recalled below, requires determination of this sum's index. The remaining task essentially involves application of the Feynman rules. The combinatorial Dyson–Schwinger equations are formulated inside a preexisting combinatorial Hopf algebra, typically the Connes–Kreimer Hopf algebra, which we begin by reviewing. While these Hopf algebras of graphs or trees belong to the perturbative regime, they contain smaller Hopf algebras spanned by the solutions to the combinatorial Dyson–Schwinger equations, which also have a non-perturbative meaning.

1.1 The Connes–Kreimer Hopf algebra of (rooted) trees. For discussions of the notions of bialgebra and Hopf algebra, see the contribution of Weinzierl [28] in the present volume. The *Connes–Kreimer Hopf algebra* of (rooted) trees (also called the *Butcher–Connes–Kreimer Hopf algebra*) is the free algebra $\mathscr{H}_{\mathrm{CK}}$ on the set of isomorphism classes of *combinatorial trees*, such as $\bullet$, , . ('Combinatorial' as opposed to the operadic trees (2.7) that will play an important role in what follows.)

The comultiplication is given on generators by

$$\begin{array}{rcl}\Delta : \mathscr{H}_{\mathrm{CK}} & \longrightarrow & \mathscr{H}_{\mathrm{CK}} \otimes \mathscr{H}_{\mathrm{CK}} \\ T & \longmapsto & \displaystyle\sum_c P_c \otimes S_c,\end{array}$$

where the sum is over all admissible cuts of T; the left-hand factor P_c is the forest (interpreted as a monomial) found above the cut, and S_c is the subtree found below the cut (or the empty forest, in the case where the cut is below the root). Admissible cut means: either a subtree containing the root, or the empty set. $\mathscr{H}_{\mathrm{CK}}$ is a connected bialgebra: the grading is by the number of nodes, and $(\mathscr{H}_{\mathrm{CK}})_0$ is spanned by the unit. Therefore, by general principles (see for example [5]), it acquires an antipode and becomes a Hopf algebra.

1.2 Combinatorial Dyson–Schwinger equations. The *combinatorial Dyson–Schwinger equations* of Bergbauer and Kreimer [2] refer to an ambient combinatorial Hopf algebra $\mathscr{H}$ and a collection of Hochschild 1-cocycles. By *Hochschild* 1-*cocycle* is meant a linear operator B_+ satisfying the equation

$$\Delta \circ B_+ = \big((\mathrm{Id} \otimes B_+) + (B_+ \otimes \eta\varepsilon)\big) \circ \Delta,$$

where ε is the counit and η is the algebra unit. The general form of the Dyson–Schwinger equation considered by Bergbauer and Kreimer [2] is

$$X = 1 + \sum_{n \geq 1} w_n \alpha^n \, B_+^n(X^{n+1}). \tag{1}$$

Here B_+^n is a sequence of 1-cocycles, w_n are scalars, and the parameter α is a coupling constant. The solution X is a formal series, an element in $\mathscr{H}[[\alpha]]$. By making the ansatz $X = \sum_{k \geq 0} c_k \alpha^k$, substituting it into the equation, and solving for powers of α, it is easy to see that there exists a unique solution, which can be calculated explicitly up to any given order, as exemplified below.

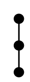
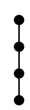
1.3 Theorem. (Bergbauer and Kreimer [2]) *The c_k span a Hopf subalgebra of $\mathscr{H}$, which is isomorphic to the Faà di Bruno Hopf algebra.*

The importance of this result is that while $\mathscr{H}$ is inherently of perturbative nature, the Faà di Bruno Hopf subalgebra spanned by the solution has a meaning also non-perturbatively. In practice, $\mathscr{H}$ is a Hopf algebra of Feynman graphs, but for many purposes one can reduce to $\mathscr{H}_{\mathrm{CK}}$, the Hopf algebra of combinatorial trees. In this case, there is only one B_+-operator, namely the one that receives as input a forest and grafts the trees in onto a new root node to produce a single tree. The following three examples refer to this Hopf algebra.

1.4 Example: A linear Dyson–Schwinger equation. Consider the equation

$$X = 1 + \alpha B_+(X),$$

but note that the exponent of α does not adhere to the general form in (1). With $X = \sum_{k \geq 0} c_k \alpha^k$, one readily finds

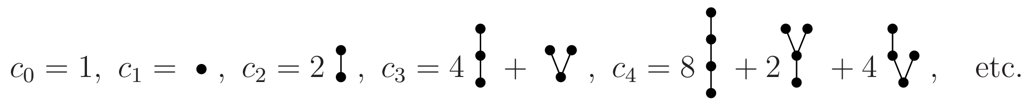
$c_0 = 1,$ $c_1 =$, $c_2 =$  , $c_3 =$  , $c_4 =$  , etc.

This is the ladder case; although it is of relevance in QFT [21], our interest here is mainly that it is, in a precise sense, an expression of recursion in its purest form, as we shall see in Section 4.

1.5 Example: A quadratic Dyson–Schwinger equation [2]. In the example

$$X = 1 + \alpha B_+(X^2),$$

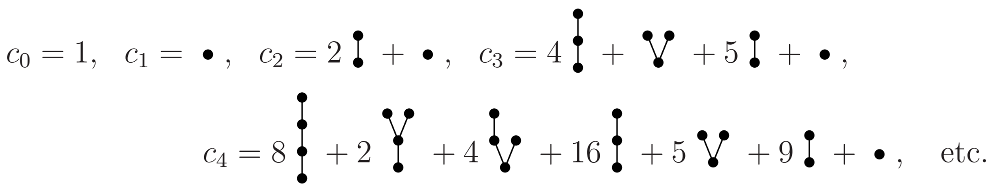
substituting the ansatz $X = \sum_{k\geq 0} c_k \alpha^k$ into the equation and solving the result for powers of α readily yields



1.6 Example: More complex trees. We finally consider the following 'infinite' example:

$$X = 1 + \sum_{n\geq 1} \alpha^n \, B_+(X^{n+1}).$$

With $X = \sum_{k\geq 0} c_k \alpha^k$ again, one finds



The pattern here may not be obvious. The actual mechanism will only become clear once we move to the setting of polynomial functors (see in particular 2.13).

2 Polynomial functors and initial algebras

While the Dyson–Schwinger equations are traditionally formulated inside a preexisting combinatorial Hopf algebra, typically in the style of Connes–Kreimer, the following abstraction steps were taken in [13]. One begins with an abstract polynomial fixpoint equation formulated in sets (or groupoids), without reference to Hopf algebras or B_+-operators. The solution is always a set (or groupoid) of certain operadic trees, and these trees automatically form a Connes–Kreimer-like bialgebra. Inside this bialgebra, there are canonical B_+-operators (although they are *not* Hochschild 1-cocycles), in terms of which the original equation can be internalised to the bialgebra. The solution, a groupoid of trees, always spans a sub-bialgebra isomorphic to the Faà di Bruno bialgebra. In this way, each equation defines its own bialgebra, a canonical home for it. However, these bialgebras are related: every cartesian natural transformation between polynomial functors yields a homomorphism between the associated bialgebras, along which the solutions to the Dyson–Schwinger equations are preserved. A special case of this is the cartesian subfunctors: these correspond to truncations which automatically produce sub-bialgebras. Finally, every such bialgebra of P-trees comes with a canonical bialgebra homomorphism to the Connes–Kreimer Hopf algebra, which therefore also receives a plethora of different Faà di Bruno sub-bialgebras.

In this section, we briefly elaborate on thses mathematical constructs, referring the reader to [13] for all details.

2.1 Categories and functors. The setting for the material in this section is category theory, but very little is required for the level of detail presented here. For further background information on category theory, see Leinster [18] for a short and concise introduction, and Spivak [25] for an account targeted at non-mathematicians.

A *category* has objects and morphisms, and morphisms can be composed. A basic example is the category of sets, denoted **Set**, where the objects are sets and the morphisms are maps of sets. Other examples are the category **Vect** of vector spaces and linear maps, or the category **Bialg** of bialgebras and bialgebra homomorphisms. A *functor* is a 'morphism of categories,' i.e., it sends objects to objects and morphisms to morphisms, in such a way as to preserve composition. For example, there is a functor $F : \mathbf{Set} \to \mathbf{Vect}$ sending a set S to the vector space spanned by S and sending a set map $f : S \to T$ to the linear map induced by its value on the basis vectors.

We work in the category **Set** of sets and set maps.

2.2 Polynomial functors. The theory of polynomial functors has roots in many different fields of mathematics and computer science, but the work of unifying these developments is more recent [8]. The basic notion required here are elementary:

Given a map of sets $p : E \to B$, we define a *polynomial functor* as

$$\begin{aligned} P : \mathbf{Set} &\longrightarrow \mathbf{Set} \\ X &\longmapsto \sum_{b \in B} X^{E_b}. \end{aligned} \tag{2}$$

In the formula, the sum sign denotes disjoint union of sets, and $E_b = p^{-1}(b)$ denotes the inverse image of an element $b \in B$. The exponential notation X^A stands for the set of maps from A to X; this notation (standard in category theory) is justified by the fact that if X is an n-element set and A is a k-element set, then X^A is an n^k-element set. We see that the role played by the map $p : E \to B$ is to deliver a family of sets indexed by B, namely

$$(E_b \mid b \in B). \tag{3}$$

(Note that B may be an infinite set, in which case the sum in (2) is, accordingly, infinite. Therefore, in a sense, polynomial functors are more like power series than polynomials.)

To say that P is a *functor* means that it operates not just on sets, but also on maps: given a map of sets $a : X \to Y$, there is induced a map

$$\sum_{b \in B} X^{E_b} \to \sum_{b \in B} Y^{E_b}$$

termwise given by

$$\begin{aligned} X^{E_b} &\longrightarrow Y^{E_b} \\ f &\longmapsto a \circ f. \end{aligned}$$

2.3 Polynomial fixpoint equations. The abstract combinatorial 'Dyson–Schwinger equations' we consider here are equations of the form

$$X \xleftarrow{\sim} 1 + P(X), \tag{4}$$

where P is a polynomial functor and 1 denotes a singleton set. This is an equation of sets, and to solve this equation means to find a set X together with a specific bijection with $1 + P(X)$, as indicated by the symbol $\xleftarrow{\sim}$. In fact, we are not satisfied with finding just *any* solution, rather, we require the *best* solution, the *least fixpoint*. Making this notion precise requires some further concepts from category theory:

2.4 Initial objects. An object I in a category $\mathscr{C}$ is called *initial* if for every object C there is a unique morphism in $\mathscr{C}$ from I to C. (See the Appendix for further discussion.) It is easy to show that an initial object, if it exists, is unique (up to isomorphism). For example, the category of sets has an initial object, namely the empty set $\emptyset$: for any set X there is a unique set map $\emptyset \to X$. (As another example, the category of rings has an initial object, namely the ring $\mathbb{Z}$.)

2.5 P-algebras and initial algebras. A P-*algebra* is by definition a pair (A, a) where A is a set and $a : P(A) \to A$ is a set map. A *homomorphism* of P-algebras from (A, a) to (B, b) is a set map $f : A \to B$ compatible with the structure maps a and b, i.e. such that this square commutes:

$$\begin{tikzcd} P(A) \arrow[r, "a"] \arrow[d, "P(f)"'] & A \arrow[d, "f"] \\ P(B) \arrow[r, "b"'] & B. \end{tikzcd}$$

Note that the functoriality of P is necessary even for this compatibility to be stated: we require the ability to evaluate P not just on sets, but also on maps. Altogether, there is a category P-**alg** of P-algebras and P-algebra homomorphisms.

Lambek's lemma says that if the category of P-algebras has an initial object (A, a), then the structure map a is invertible. (This is not a difficult result.) This states precisely that an initial P-algebra (A, a) is a solution to the equation $X \xleftarrow{\sim} P(X)$: the underlying set A is X and the structure map a is the required bijection. Initiality is the technical condition that justifies reference to this as the *least fixpoint*.

Now, the equation we wish to solve is not $X \xleftarrow{\sim} P(X)$, but rather

$$X \xleftarrow{\sim} 1 + P(X),$$

so what we are looking for is not the initial P-algebra, but rather the initial $(1 + P)$-algebra. This is a subtle point; let us simply remark that the 1 appears in the Dyson–Schwinger equations (1), and also that the 1 in the polynomial fixpoint equations (4) has some strong motivations in category theory. Its presence ensures that the solution yields a nice class of trees, as discussed below.

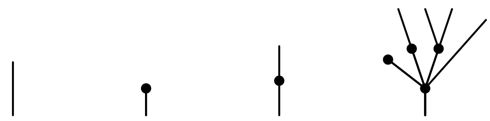
2.6 Theorem. *If P is a polynomial functor, then the fixpoint equation*

$$X \stackrel{\sim}{\leftarrow} 1 + P(X)$$

has a least solution, that is, the category of $(1 + P)$*-algebras has an initial object. This solution is the set of (isomorphism classes of) P-trees, now to be defined.*

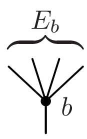
This result is mostly folklore. The explicit characterisation of the solution is from [9].

2.7 Operadic trees. By *operadic trees* we mean rooted trees admitting open-ended edges (leaves and root), such as the following:



They are called operadic because each node is regarded as an operation, with the incoming edges (reading from top to bottom) as input slots and the outgoing edge as output slot. Note the difference between a leaf (an open-ended edge) and a nullary node.

2.8 P-trees. (Cf. [9]) For P a polynomial functor represented by a set map $p : E \to B$, we think of the elements in B as operations. The most important aspect of b is its *arity*, which is not just a number, but rather the set E_b itself, interpreted as the *set* of input slots of the operation b. For example, if E_b is a 2-element set, b is a binary operation. It is sensible to picture the elements in B as corollas:



(5)

A *P-tree* is an operadic tree with nodes decorated by elements in B, and for each node x decorated by b a specified bijection between the incoming edges of x and the set E_b. In other words, each node is decorated with an operation of matching arity, and hence a P-tree can also be regarded as a tree configuration of operations from P.

2.9 Example: Binary trees. Consider the polynomial functor P defined by the set map $p : \{\texttt{left}, \texttt{right}\} \to 1$. It is the functor

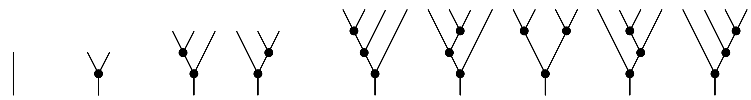
$$\begin{array}{rcl} \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ X & \longmapsto & X^2. \end{array}$$

For this P, a P-tree is precisely a (planar) binary tree. Indeed, since in this case the set B is just singleton, to P-decorate a tree amounts to specifying a bijection, for each node, between the set of incoming edges and the set $\{\texttt{left}, \texttt{right}\}$. For this bijection to be possible, each node must have precisely two incoming edges, and the bijection says which is the left branch and which the right.

The relevant fixpoint equation $X \stackrel{\sim}{\leftarrow} 1 + P(X)$ is now

$$X \stackrel{\sim}{\leftarrow} 1 + X^2,$$

and the theorem thus says that the solution, the initial $(1+P)$-algebra, is the set of planar binary trees. Indeed, the fixpoint equation can be read as saying: a planar binary tree is either the trivial tree, or it is given by a pair of planar binary trees. This is precisely the recursive characterisation of binary trees. Here are the first few elements:



(6)

2.10 Example: Planar trees. Take $P(X) = X^0 + X^1 + X^2 + X^3 + \cdots$ This is the list endofunctor, which sends a set X to the set of lists of elements in X. Then P-trees are planar trees. The fixpoint equation

$$X \stackrel{\sim}{\leftarrow} 1 + X^0 + X^1 + X^2 + X^3 + \cdots$$

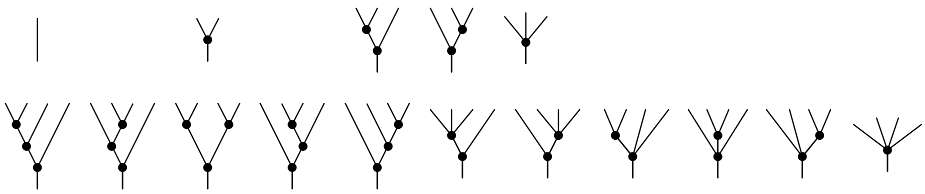
says that a planar tree is either the trivial tree or a list of planar trees.

Since we allow nullary and unary nodes (corresponding to the terms X^0 and X^1 in the polynomial functor), for each fixed number of leaves, there are infinitely many trees. For the sake of comparison with 1.2, it is interesting to tweak this functor slightly in order to avoid this infinity:

2.11 Example: Stable planar trees. Consider instead the polynomial functor

$$P(X) = X^2 + X^3 + X^4 + \cdots$$

for which P-trees are *stable* planar trees, meaning they have no nullary or unary nodes. The exclusion of nullary and unary nodes implies that, for each fixed number k, there is now only a finite number of trees with k leaves. These are the Hipparchus–Schröder numbers, $1, 1, 3, 11, 45, 197, 903, \ldots$ Here are pictures of all the stable planar trees with up to 4 leaves:



(7)

We briefly list some further facts to illustrate the workings of P-trees, and highlight a few of the features of this approach to Dyson–Schwinger equations. See [13] for all details.

2.12 Bialgebra of P-trees. P-trees form a Connes–Kreimer-style bialgebra [11]. Note however, that cut edges are really cut rather than removed, as exemplified by

This is an essential point, as otherwise the decorations would be spoiled: removing an edge rather than cutting it would break the given arity bijections in the decorations (not rendered in the drawing). Note also that this bialgebra is not connected: the degree-zero piece is spanned by the nodeless trees and forests.

Each $b \in B$ defines a B_+-operator (although *not* a Hochschild 1-cocycle), in terms of which the original equation (4) can be internalised to this bialgebra.

2.13 Core. There is a canonical bialgebra homomorphism from any such bialgebra of P-trees to the Connes–Kreimer bialgebra $\mathscr{H}_{\mathrm{CK}}$, given by taking core [11]: this amounts to forgetting the P-decorations and shaving off all leaf edges and the root edge. In other words, the core of a P-tree is the combinatorial tree given by its inner edges.

Consider the binary trees constituting the least fixpoint for $X \mapsto 1 + X^2$, the first few of which are listed in (6). Taking core transforms these binary trees into sub-binary combinatorial trees (i.e., they have at most two incoming edges at each node), and where the planar structure has been lost. We see that the coefficients c_k appearing in the solution of the quadratic Dyson–Schwinger equation of 1.5 are precisely the numbers of binary trees with a given core. (This interpretation of the coefficients c_k has been given already by Bergbauer and Kreimer [2], in some form.)

Similarly, consider the stable planar trees from 2.11, the first few of which are listed in (7). Taking core yields exactly the combinatorial trees found in the solution to the Dyson–Schwinger equation in 1.6, and again the coefficients c_k in the solution are precisely the numbers of trees with $k+1$ leaves and a given core.

2.14 Faà di Bruno sub-bialgebras. While taking core immediately places us in the realm of the familiar Connes–Kreimer Hopf algebra, it is important that the information contained in the leaves and root is not discarded: the strict type obedience characteristic for P-trees (respect for arities) allows for meaningful automorphism groups and the existence of meaningful Green functions

$$G = \sum_T \frac{T}{|\mathrm{Aut}(T)|},$$

where the sum is over iso-classes of P-trees. (To actually observe any automorphisms, one must work with groupoids instead of sets, cf. [10] and [7], but that is beyond the scope of the present exposition.) This sum can be split into summands given by trees with n leaves,

$$G = \sum_{n \geq 0} g_n.$$

Further, there is now a Faà di Bruno formula [7]

$$\Delta(G) = \sum_{n \geq 0} G^n \otimes g_n,$$

in the style of van Suijlekom [27]. The point here is that the exponent n on the left-hand tensor factor counts n trees, each with a root, precisely matching the subscript n in the right-hand tensor factor, which is the number of leaves on the trees in g_n. This kind of information cannot be observed at the level of combinatorial trees.

Concern may arise that P-trees form bialgebras and not Hopf algebras, as used in renormalization. However, this is not a problem, as it can be shown [12] that Hopf-algebra renormalisation also works for bialgebras of P-trees. In fact it is insinuated in [12] that the bialgebras are closer to actual physics than the Hopf algebras that can be derived from them.

3 Type theory

We now proceed to interpret the polynomial fixpoint equations in type theory, our first task being to explain what type theory is about. Doing this in just a few pages will necessarily be a superficial account. Specifically, we mostly neglect the important notion of *identity types*, which is playing an increasingly central role in modern research [26] (see 6.3). In particular, we use the equality symbol $=$ in the most naive manner.

A classic reference on this subject is Nordström–Petersson–Smith [23]. A more modern account, which is highly recommended, is the book *Homotopy Type Theory—Univalent Foundations of Mathematics* [26].

Type theory provides a foundation for mathematics. Before explaining its primary concepts, we first take a very brief look at the most common foundation for mathematics, set theory.

3.1 Set theory. The standard foundation for mathematics is set theory, and more specifically what is called Zermelo–Fraenkel set theory. This theory begins with first-order logic, the language written with operators

$\wedge$	conjunction (`AND`)
$\vee$	disjunction (`OR`)
$\top$	`TRUE`
$\perp$	`FALSE`
$\Rightarrow$	implication
$\forall$	universal quantifier
$\exists$	existential quantifier

On top of this, Zermelo–Fraenkel set theory is defined as a one-sorted theory with one binary relation, namely the membership relation $\in$, as in $a \in A$, expressing that a is a member of A. Here, both a and A are sets — the only kind of object there is (that's what 'one-sorted' means).

Depending on how it is formulated, there are about ten axioms, all written in first-order logic. Most of these axioms express what is needed for the elementary theory of sets, as used in everyday mathematics, and say for example that there exists the 'empty set' $\emptyset = \{\}$, that one can add a new element to a given set, that one can form the union of two sets, that the subsets of a given set form a set again, that there exist infinite sets, and so on. In addition, there are some further, more technical axioms, which are primarily useful for the application of set theory to encode all of mathematics. For many purposes, the axiom of choice is also added in some form.

Zermelo–Fraenkel set theory serves as a foundation for mathematics in the sense that all of mathematics can be encoded in it, beginning with the natural numbers and progressing to the rationals, the reals, all of algebra, analysis, geometry, and so on. However, although this encoding can be achieved, there is no really canonical way of doing it. For example, here are two different ways of encoding the natural numbers. Von Neumann did it in this way: define $0 = \{\}$, the empty set, and define successively each new natural number to be the set of all the previously defined natural numbers:

$$1 = \{0\} = \{\{\}\}, 2 = \{0, 1\} = \{\{\}, \{\{\}\}\}, 3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}, \dots$$

Zermelo did like this: define $0 = \{\}$, the empty set, and successively define inductively each new natural number to be the set containing only the previous natural number:

$$0 = \{\}, 1 = \{0\} = \{\{\}\}, 2 = \{1\} = \{\{\{\}\}\}, 3 = \{2\} = \{\{\{\{\}\}\}\}, \dots$$

Everything in mathematics *can* be encoded as sets, but since this encoding is somewhat arbitrary or artificial, it is also possible to write things that actually make no sense mathematically. For example, $3 \in 17$. This is a proposition in Zermelo-Fraenkel set theory, which is to say that it is either true or false. However, its truth value depends on how the natural numbers are encoded. For example, this proposition is true according to von Neumann's encoding and false according to the Zermelo encoding. (This is a famous example due to Benacerraf (1965).) This is regarded by some as a conceptual problem with Zermelo-Fraenkel set theory: although it can encode all of mathematics, its structure does not faithfully reflect the structure of mathematics.

3.2 Type theory. Type theory is an alternative to set theory as a foundation for mathematics. Its development was initiated by Russell in the 1930s, and found its modern form mainly in the work of Martin-Löf in the 1970s and 1980s. What we loosely refer to as type theory is more precisely called *dependent type theory* or *Martin-Löf type theory*, but we completely gloss over the finer details that distinguish its many variants.

The basic ingredients in type theory are type declarations (called judgements), such as

$$a : A,$$

akin to programming languages where one might write `n:INT` to mean that n is a variable of integer type. Here it reads a is a term of type A. Before writing it, one should actually declare that A is a type:

$$A : \mathrm{Type}$$

(This can be regarded as a special kind of the basic type declaration, provided one assumes a universe type Type of all types.) (The word 'declaration' has the advantage that it reminds us that $a : A$ is not a proposition: for example, it cannot be negated. In this way, it differs from the operator $\in$ in Zermelo–Fraenkel set theory, where $a \in A$ is a proposition, i.e., it may or may not be true.)

Type theory was developed to provide a *constructive* foundation for mathematics, and as a result, it has proved very useful as a foundation for computer science. In fact, type theory can be regarded as a programming language, and everything written in type theory can be verified using proof assistants such as Coq [4] or Agda [1]. With the recent advent of Homotopy Type Theory and Univalent Foundations [26] (whose main points are outside the scope of this exposition), the aim of providing a practical foundation for mathematics is coming nearer fulfilment, with a closer interaction between humans and computers as an important bonus feature.

3.3 Sets as types, and some basic type formers. At a first level of understanding, a judgement $a : A$ can be read as a membership, $a \in A$. There are constructions and rules in type theory allowing one to do elementary set theory in this way.

For example, one can form product types: given that A : Type and B : Type then there is inferred a new type $A \times B$: Type. The inference rules in type formation like this are traditionally written with a horizontal bar:

$$\frac{A : \mathrm{Type},\ \ B : \mathrm{Type}}{A \times B : \mathrm{Type}}$$

meaning that if the judgements above the line are assumed, then the judgements below it can be inferred. (A more informal writing idiom, more akin to prose, is advocated in the book [26].) There is also a singleton type $1 : \mathrm{Type}$.

Similarly, there is a function type, denoted $A \to B$:

$$\frac{A : \mathrm{Type},\ B : \mathrm{Type}}{A \to B : \mathrm{Type}}$$

A term of this type, $f : A \to B$, is the analogue of an actual function or set-mapping in set theory. There are further rules expressing how to construct terms of these new types, and how to compute with them.

3.4 Dependent types. An important aspect is *dependent types*. These are families of types indexed by a given type. With the universe interpretation of Type, a dependent type can be written as

$$B : A \to \mathrm{Type}$$

It says that for each term $a : A$, there is a type $B(a)$. The analogue in set theory is a family of sets $\{B(a) \mid a \in A\}$ indexed by a set A, as in (3). For what follows, it is important to see this also as being encoded as a single map of sets $B \to A$, often thought of as a fibration: then each family member $B(a)$ is defined to be the fibre over a point $a \in A$. Conversely, given a family of sets like this, one can form the map $\sum_{a \in A} B(a) \to A$, where the sum sign signifies a disjoint union of sets, and the map is projection onto the indexing set of the sum.

3.5 Dependent products. We have already mentioned function types: a term of type $A \to B$ is interpreted as an assignment that to every term in A associates a term in B. Here A and B are fixed types. There is a dependent version, which will be crucial in what follows. Suppose $B : A \to \mathrm{Type}$ is a dependent type. Then there is a new type $\prod_{a:A} B(a)$ called *dependent product*:

$$\frac{B : A \to \mathrm{Type}}{\prod_{a:A} B(a)}$$

A term of this type is again interpreted as an assignment that to every term $a : A$ associates a term, but this time in a 'codomain' that depends on a itself. Under the interpretation of the dependent type $B : A \to \mathrm{Type}$ as a fibration $p : B \to A$, the terms in $\prod_{a:A} B(a)$ are sections to p.

3.6 Propositions as types. We have seen that type theory can emulate elementary set theory by interpreting types as sets and terms as elements; then type declaration is read as membership. However, there is another very useful interpretation of type theory, which shows that type theory *contains* all of first-order logic, rather than depending on it as a meta language. In this interpretation, a type is interpreted as a proposition and a term is interpreted as a proof. Hence the type declaration $a : A$ says that a is a proof of A. This ties in with the constructive aspect of type theory: to establish the truth of a proposition, one must explicitly *construct* a proof term using the rules of type theory.

The basic type formers (not all of which were explained here) then have the following logical interpretation:

$\times$	$\wedge$
$+$	$\vee$
1	$\top$
0	$\bot$
$\rightarrow$	$\Rightarrow$
$\prod$	$\forall$
$\sum$	$\exists$

It should be noted that the strict interpretation of 1 as 'true' actually means rather 'true with a unique proof.' More loosely, when a type is interpreted as a proposition, any term of that type will be interpreted as a proof of the corresponding proposition, so a looser notion of true is 'inhabited,' or as most people would put it, 'non-empty.' This is the notion used in what follows.

3.7 Predicates as dependent types. In logic, a *predicate* is a proposition that depends on a variable. Under the propositions-as-types interpretation, the corresponding notion is precisely dependent types. This interpretation can be very helpful for the reading of complicated formulae, particularly when it comes to dependent products, which are then read as universal quantifiers. Recall that we had previously viewed the dependent product $\prod_{a:A} B(a)$ as a kind of function space, a space of sections: for *every* a in A we assign some $f(a)$ in $B(a)$. If now instead we interpret $B(a)$ as a predicate, then a term in $\prod_{a:A} B(a)$ is a proof that *for all* a in A the predicate $B(a)$ holds. Similarly, although we shall not really need it here, the notion of dependent sum corresponds to the existential quantifier.

3.8 Type formation rules. We have, very superficially, mentioned many type formers: cartesian product, singleton type, function types, dependent products, and dependent sums. Each time, we sketched what they are, but neglected to list the further rules governing them. In fact, one beauty of type theory is that these rules follow a very strict pattern, common for all type formers. We explain this pattern here, and then exemplify it in the following sections, where we exploit it in more detail to describe inductive types.

Every type former is given by four rules (or four groups of rules).

The first is a *formation rule*, which stipulates the new type. For the product type, this just reads

$$\frac{A : \text{Type},\ B : \text{Type}}{A \times B : \text{Type}}$$

(So far in the text, we have only seen formation rules.)

The second rule, or group of rules, is called the *introduction rule*: it populates the new type with terms that characterises it. In the case of the product, it says that the new terms are pairs of terms:

$$\frac{a : A,\ b : B}{\mathsf{pair}(a, b) : A \times B}$$

The individual introduction rules are referred to as *constructors*. The product type is thus characterised by a single constructor, pair.

Next comes the *elimination rule*, which tells how terms in the type are used; that is, lists their characteristic properties by stipulating an *eliminator*, as we shall see in the examples below. Finally there is the *computation rule* which stipulates how introduction and elimination interact. The elimination and computation rules tend to appear a bit unwieldy, but we shall see in detail in the next sections how they work.

4 Classical induction: The natural numbers

Since type theory is constructive, practically the only way of getting hold of infinite structures is through induction principles. They include first of all the natural numbers, the most basic inductive type, but also trees of many kinds, as we shall see in Section 5.

4.1 Dedekind–Peano natural numbers. Dedekind (1888) and Peano (1889) defined (or rather characterised) the natural numbers as a set $\mathbb{N}$ with a distinguished element $0 \in \mathbb{N}$ and a successor function $s : \mathbb{N} \to \mathbb{N}$ satisfying

(i) 0 is not a successor;
(ii) Every element $x \neq 0$ is a successor;
(iii) The successor function is injective;
(iv) If a subset $U \subset \mathbb{N}$ contains 0 and is stable under the successor function, then $U = \mathbb{N}$.

Note that (i)+(ii)+(iii) amount to saying that the map

$$\{*\} + \mathbb{N} \xrightarrow{\langle 0,s\rangle} \mathbb{N}$$

is a bijection. Axiom (iv) is called the *induction axiom*.

Note that unlike Zermelo's and von Neumann's definitions briefly mentioned in 3.1 above, the Dedekind–Peano definition does not actually define a concrete set in terms of its elements. Rather, the set is defined *structurally*, meaning that it is characterised by how it works, through its relationship with other sets. In fact, each of Zermelo's and von Neumann's definitions can be seen as a specific implementation of the Dedekind–Peano axioms.

4.2 Lawvere: Natural numbers as an initial algebra. Lawvere (1964) observed that the Dedekind–Peano definition can be reformulated categorically as an example of an initial algebra, as follows. The set of natural numbers is a set $\mathbb{N}$ together with an element $0 \in \mathbb{N}$ and a map $s : \mathbb{N} \to \mathbb{N}$, with the following property: whenever A is a set with $a \in A$ and $r : A \to A$, there is a unique function $u : \mathbb{N} \to A$ such that $u(0) = a$ and $u(s(n)) = r(u(n)), \forall n \in \mathbb{N}$. Phrased in the terminology of Section 2, the data given (the zero and the successor function) amount precisely to saying that $\mathbb{N}$ is an algebra for the polynomial functor $X \mapsto 1 + X$. And the characterising property says precisely that for any other such algebra A, there is a unique homomorphism of algebras $\mathbb{N} \to A$. Put differently, the Peano–Dedekind axioms for the natural numbers say precisely that $\mathbb{N}$ is the initial algebra for the polynomial functor $X \mapsto 1 + X$! (In other words, the natural numbers are P-trees, for P the identity functor.)

(Lawvere's observation was made in the context of his *Elementary Theory of the Category of Sets* (see [17]), which is a *structural* alternative to Zermelo–Fraenkel set theory: instead of starting from the membership relation, it takes the notion of maps of sets as primitive. A set is then no longer characterised by its elements, but rather by its relationship to other sets. While this may seem abstract at first sight, it is actually much closer than Zermelo–Fraenkel set theory to mathematical practice. The Dedekind–Peano–Lawvere definition of the natural numbers gives a hint of the flavour of this structural set theory.)

4.3 Natural numbers in type theory. In type theory, the natural numbers are introduced as a type, as follows.

The formation rule simply stipulates that there is a type $\mathbb{N}$:

$$\frac{}{\mathbb{N} : \text{Type}}$$

The introduction rule specifies the two constructors

$$\frac{}{\mathsf{zero} : \mathbb{N}} \qquad \frac{n : \mathbb{N}}{\mathsf{succ}(n) : \mathbb{N}}$$

(The second can be thought of as a B_+-operator, with reference to Example 1.4: it takes a 'forest' consisting of a single (unary) tree and returns a new (unary) tree by grafting that tree onto a new root node.)

The elimination rule is where the induction principle is encoded. The correct version involves *dependent elimination.* Before coming to it, it is worth giving a simplified version, the non-dependent elimination rule, which is easier to grasp, but which is not quite sufficient. Trying to mimic what the initiality means, we are led to write

$$\frac{A : \text{Type}, \quad z : A, \quad s : A \to A}{\mathsf{rec} : \mathbb{N} \to A}$$

This is meant to say: given another such structure (i.e. another type A with the same constructors), there exists a map to it from $\mathbb{N}$. And then finally write the computation rule, which states that this function $\mathsf{rec} : \mathbb{N} \to A$ of course must be required to be compatible with the structure:

$$\frac{\texttt{"}}{\mathsf{rec}(\mathsf{zero}) = z, \quad \mathsf{rec}(\mathsf{succ}(n)) = s(\mathsf{rec}(n))}$$

(The symbol ", here and in what follows, is meant to say 'same hypotheses as in the elimination rule,' as this is always the case.) The essence of all this is that $\mathbb{N}$ is designed so that we can define functions out of it by recursion (hence the symbol rec).

The missing bit in order to faithfully render the initial-algebra idea is that we need to say that rec is *unique* with these properties. This is not something one can say directly in type theory, for reasons related to its constructive nature. What turns out to work much better is the following *dependent elimination rule*, which we first write down, then explain in detail:

$$\frac{C : \mathbb{N} \to \mathrm{Type}, \quad z : C(0), \quad s : \prod_{n:\mathbb{N}} C(n) \to C(\mathsf{succ}(n))}{\mathsf{rec} : \prod_{n:\mathbb{N}} C(n)} \tag{8}$$

And finally the computation rule:

$$\frac{\text{"}}{\mathsf{rec}(\mathsf{zero}) = z \ , \quad \mathsf{rec}(\mathsf{succ}(n)) = s(n, \mathsf{rec}(n))}$$

4.4 How to read (and write) elimination rules. The non-dependent elimination rule says that under certain hypotheses, we can get a function out of our new type, in this case $\mathbb{N}$. The dependent elimination rule gives instead a dependent product. (Recall that a function type can be viewed as a 'constant' dependent product.) A recommended way to read (and write) an elimination rule is to start with the following question:

$$\frac{C : \mathbb{N} \to \mathrm{Type}, \quad \boxed{\qquad ??? \qquad}}{\mathsf{rec} : \prod_{n:\mathbb{N}} C(n)}$$

The question asks: given a dependent type $C : \mathbb{N} \to \mathrm{Type}$, what is the data needed $\boxed{???}$ in order to obtain a term in $\prod_{n:\mathbb{N}} C(n)$? If we think of the dependent type as a 'fibration,' the question is: what is needed to get a section?

$$\begin{tikzcd} C \arrow[d] \\ \mathbb{N} \arrow[u, bend left, dotted, "\mathsf{rec}"] \end{tikzcd}$$

If instead we think of C as a predicate, then the question is: what is needed to prove $C(n)$ for all n?

Either way, once the question has been formulated, we proceed to fill the answer into the template to get the final rule (8), repeated here to stare at:

$$\frac{C : \mathbb{N} \to \mathrm{Type}, \quad z : C(0), \quad s : \prod_{n:\mathbb{N}} C(n) \to C(\mathsf{succ}(n))}{\mathsf{rec} : \prod_{n:\mathbb{N}} C(n)}$$

The rule says: in order to construct a section rec we need: a point z in the fibre over 0, and a way of passing from one fibre to 'the next': more precisely, for each n, we need a term of the function type $C(n) \to C(n+1)$. Alternatively, in terms of predicates: in order to prove $C(n)$ for all n, we must first prove $C(0)$ and then prove the 'induction step': 'assuming $C(n)$, prove $C(n+1)$.' If we have these ingredients, then we can deduce $C(n)$ for all n.

Finally, there should be a computation rule, of course, telling us that rec must be compatible with the original data given:

$$\frac{''}{\mathsf{rec}(\mathsf{zero}) = z \ , \quad \mathsf{rec}(\mathsf{succ}(n)) = s(n, \mathsf{rec}(n))}$$

(Notice that the function s takes two arguments: the first is $n : \mathbb{N}$ (indexing the dependent product), the second is from $C(n)$, depending on the first argument.)

Note that the non-dependent eliminator is like defining functions recursively, while the dependent eliminator is like proof by induction. The former is a special case of the latter: given the abstract type A, one can always form the dependent type $n \mapsto A$ (the constant dependent type). So given the dependent eliminator, we can emulate the non-dependent eliminator as a special case. The dependent case is stronger, though: it actually implies the uniqueness. For this we refer to the Appendix.

4.5 Relationship with initial $(1 + P)$-algebra (in this case, $P = \mathrm{Id}$). Interpreting the dependent type as a 'fibration,' the hypothesis of the elimination rule (the part above the bar) says precisely that there is a map $C \to \mathbb{N}$ and that this map is an $(1 + P)$-algebra homomorphism. The outcome of the elimination rule says that this map has a section, and finally the whole computation rule says that this section is in fact a $(1 + P)$-algebra homomorphism. So altogether, algebraically, the rules say that there is an algebra such that any algebra map into it has a section. It is a general result in category theory (see Appendix) that this condition is equivalent to being an initial algebra.

The link with combinatorial Dyson–Schwinger equations goes through the polynomial fixpoint equation. There are two constructors: one nullary constructor stipulating that there exists a special element zero, and one unary constructor which takes as input one element and produces another (the successor). The first corresponds to the 1 in the Dyson–Schwinger equation, the second corresponds to the B_+-operator, and more precisely to the polynomial X in its argument. The corresponding combinatorial Dyson–Schwinger equation is

$$X = 1 + \alpha B_+(X),$$

the ladder case (1.4).

5 Inductive types: W-types

There is a quite general class of inductive types called *W-types*, W for *well-founded trees*, which as we shall see is closely related to Dyson–Schwinger equations. Before embarking on the general case, we go through the example of binary trees.

5.1 Binary trees. We form the type of (planar) binary trees, here denoted W, following the same pattern as for the natural numbers, but with a binary constructor instead of a unary constructor.

Formation rule:

$$\frac{}{W : \text{Type}}$$

Introduction rule:

$$\frac{}{\mathsf{nil} : W} \qquad \frac{(t_1, t_2) : W \times W}{\mathsf{sup}(t_1, t_2) : W}$$

The second says that whenever we are given an ordered pair of trees, we can construct a new tree. It is traditionally called sup because it is in some manner the supremum of the two trees t_1 and t_2, namely the smallest tree containing t_1 and t_2 as subtrees. It corresponds precisely to the B_+-operator in Example 1.5.

The crucial rule is the elimination rule. (For the beginner, this is the most difficult rule to write down, but for inductive types there is actually a mechanical way of deriving it from the introduction rule. For example, for inductive types in the proof assistant Agda [1], it is enough to write the formation and introduction rules, then the computer figures out by itself what the elimination and computation rules should be.)

Elimination rule:

$$\frac{C : W \to \text{Type},\ s_{\mathsf{nil}} : C(\mathsf{nil}),\ s_{\mathsf{sup}} : \prod_{(t_1,t_2):W\times W} C(t_1) \times C(t_2) \to C(\mathsf{sup}(t_1, t_2))}{\mathsf{rec} : \prod_{w:W} C(w)}$$

Let's go through it, in the interpretation of $C : W \to \mathrm{Type}$ as a predicate, i.e. a proposition about (binary) trees. The elimination rule then says: given a predicate on trees (that's the left-hand part of above-the-line) in order to prove this predicate for all trees (that's what's below the line), it is enough to be able to prove it for the trivial tree, and know how to derive from the statement about any two trees the statement about the tree obtained by grafting the two trees onto a new root node.

And finally there is the computation rule:

$$\frac{''}{\mathsf{rec}(\mathsf{nil}) = s_{\mathsf{nil}}, \quad \mathsf{rec}(\mathsf{sup}(t_1, t_2)) = s_{\mathsf{sup}}(t_1, t_2, \mathsf{rec}(t_1), \mathsf{rec}(t_2))}$$

5.2 W-types, general case. See also [26], §5.3. The notion of W-type is due to Martin-Löf himself [20]; it covers the two previous examples. The interpretation as initial algebras for polynomial functors is due to Moerdijk and Palmgren [22]. A W-type refers to a general dependent type $E : B \to \mathrm{Type}$, which plays the role of the polynomial (or power series) inside the B_+-operator in a combinatorial Dyson–Schwinger equation as in 1.2. Here B is the set of possible branching types, and for fixed $b : B$, the type $E(b)$ is the arity of that node type. In the first example, that of natural numbers, we had $B = 1$ (only one kind of node) and $E(1) = 1$ (that node is unary). In the second example, that of binary trees, again $B = 1$ (only one kind of node) and now with $E(1)$ a 2-element set (that node is binary).

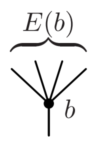
Relative to a given dependent type $E : B \to \mathrm{Type}$, we shall now define the W-type, to be thought of as the type of all trees of a certain kind. It ought to be denoted $W_{B,E}$ (or even $\mathrm{W}_{b:B}E(b)$) to express this dependency, but to lighten the notation we shall denote it just W. Formation rule:

$$\frac{E : B \to \mathrm{Type}}{W : \mathrm{Type}}$$

Introduction rule:

$$\frac{}{\mathsf{nil} : W} \qquad \frac{b : B, \quad t : E(b) \to W}{\mathsf{sup}_b(t) : W}$$

In this case, it is appropriate to think of one B_+-operator for each element b in B. Each b can be pictured as a small corolla



The rule says that given an $E(b)$-indexed family of trees, we can glue all those trees onto the corresponding leaves of the corolla b to obtain a new tree (with b as its root node).

The elimination rule now reads

$$\frac{C : W \to \text{Type},\ \ s_{\mathsf{nil}} : C(\mathsf{nil}),\ \ s_{\mathsf{sup}} : \prod_{b:B}\prod_{t:E(b)\to W}\prod_{e:E(b)} C(t(e)) \to C(\mathsf{sup}_b(t))}{\mathsf{rec} : \prod_{w:W} C(w)}$$

That's a mouthful, but the principle is exactly the same as for binary trees. It says that in order to prove something C for this kind of trees, it is necessary to prove it for the trivial tree, and also establish the following induction step: for any kind of node b (that's the 'universal quantification' $\prod_{b:B}$), and for any $E(b)$-indexed family of trees, assuming C holds for each of the trees in that family (that's the $\prod_{e:E(b)} C(t(e))$ part), we can deduce C for the tree obtained by grafting onto b. Under these hypotheses we can then deduce that C holds for all trees.

And finally the computation rule:

$$\frac{\text{"}}{\mathsf{rec}(\mathsf{nil}) = s_{\mathsf{nil}}, \quad \mathsf{rec}(\mathsf{sup}_b(t)) = s_{\mathsf{sup}}(b, t, e, \mathsf{rec}(t(e)))}$$

5.3 Remark. In type theory, the nil constructor for the trivial tree is usually not listed separately, but is rather subsumed as an extra nullary member of the dependent family. For the present purposes it is preferable always to have this separate nil constructor, because on one hand it fits better into the polynomial formalism (where the initial $(1 + P)$-algebra is the set of operations of the free monad on P), and also since the special term 1 is included in the Dyson–Schwinger equations (1). Foissy [6] has studied Dyson–Schwinger equations without this special term. For a thorough analysis of the differences implied, see [13].

6 Feynman graphs and outlook

6.1 Combinatorial trees versus graphs. Ultimately, for the purposes of quantum field theory, the combinatorial Dyson–Schwinger equations should take place in Hopf algebras of graphs, not of trees. The interest in the Hopf algebra of combinatorial trees stems from the fact that it enjoys a universal property, which allows transfer of knowledge to the graph case. Precisely, $\mathscr{H}_{\mathrm{CK}}$ together with its canonical Hochschild 1-cocycle B_+ can be shown to be an initial object in the category of commutative combinatorial Hopf algebras equipped with a Hochschild 1-cocycle [3]. In other words, for any other such Hopf algebra with a 1-cocycle, for example a Hopf algebra of Feynman graphs $\mathscr{H}$, there is a unique Hopf algebra homomorphism $\mathscr{H}_{\mathrm{CK}} \to \mathscr{H}$ compatible with the Hochschild 1-cocycles. In this way, $\mathscr{H}_{\mathrm{CK}}$ serves as a universal approximation.

6.2 ***P*-trees versus graphs.** P-trees provide more faithful approximations. Given a class of graphs (belonging to some quantum field theory), it is possible to encode each graph as a tree decorated with primitive graphs, according to how the graph is built from nesting of primitive graphs. It is shown in [14] that these trees can be described formally as P-trees for a suitable polynomial functor P, in such a way that the automorphism group of the P-tree agrees with the automorphism group of the graph with its nestings. The polynomial functors P here are considerably more complicated than those considered in the present paper. For one thing, since graph insertions must match residues with fertilities, the corresponding trees must have decorations also on the edges to keep track of such typing constraints. This corresponds precisely to considering polynomial functors in many variables as in [9]. Secondly, because of the existence of symmetries of graphs, it is necessary to upgrade the theory from polynomial functors over sets to polynomial functors over groupoids, as in [10]. (A groupoid is a category all of whose morphisms are invertible.) Glossing over many details, the polynomial functor corresponding to a given quantum field theory is represented by the map $p : E \to B$, where B is the groupoid of all primitive graphs of the theory (and their isomorphisms), and E is the groupoid of all such graphs with a marked vertex (and their isomorphisms). There is now a bialgebra homomorphism $\mathscr{H} \to \mathscr{B}_P$ from the bialgebra of graphs to the bialgebra of P-trees. It sends a graph to the sum of all the P-trees that are recipes for building it (that's a finite sum). In other words, this bialgebra homomorphism precisely resolves overlapping divergences, and it does so in the gentlest possible way. This bialgebra homomorphism is compatible with Green functions. The P-trees form initial algebras — they are genuine W-types, with all the good properties that entails. The graphs themselves are *not* W-types, due precisely to the fact that some graphs can be constructed from primitive graphs in more than one way.

It should be mentioned that some issues remain with regard to the polynomial-functor approach, which have not been sorted out satisfactorily yet (cf. [13] for further discussion). The main issue is with edge insertions: the combinatorial Dyson–Schwinger equations in Hopf algebras of graphs typically involve denominators corresponding to propagators, and each edge represents an ordered infinity of insertion places. This feature is difficult to render in a strictly operadic setting, otherwise than insisting that all mass and kinetic terms be marked explicitly with crosses on the graphs. This problem is under active investigation.

6.3 Homotopy type theory: identity types as path spaces. The above review of type theory is very crude. One glaring omission is that of identity types. In type theory, because of its strictly constructive nature, one cannot say that two terms of a type are equal without actually providing a proof, a *construction* of the equality, so to speak. This is handled as follows. Given two terms a and b of a type A there is a new type $\mathrm{Id}_A(a, b)$ called the *identity type*, whose terms can be thought of as proofs that a and b are identical. Formally there is a formation rule

$$\frac{A : \mathrm{Type}, \quad a : A, \ b : A}{\mathrm{Id}_A(a, b) : \mathrm{Type}}$$

Now, two terms in the type $\mathrm{Id}_A(a, b)$ may or may not be equal, and if they are equal, that needs proof again, so there is a identity type of the identity type. And so on. *Homotopy type theory* [26] exploits the significant discovery that this structure is intimately analogous to homotopies and higher homotopies in topology, so that a valid interpretation of types is to regard them as topological spaces (up to homotopy), terms are regarded as points, identity types are path spaces, and so on. In this manner, type theory can serve as a formal language for homotopy theory.

6.4 Higher inductive types. A fundamental ingredient in homotopy theory is the based circle S^1 (by which is meant any closed curve, from a point to itself). For example, homotopy groups are defined by mapping based circles into spaces. The circle can be rendered in homotopy type theory too [26], and is revealed to be of inductive nature as well. The circle is a basic example of a *higher inductive type*. This means that it is specified in a way similar to W-types, but with constructors allowed to be terms not just in the type itself (called 0-constructors) but also in its identity type (called 1-constructors). Precisely, the circle S^1 is given by two constructors, namely a basepoint and a 1-constructor which is a path from the basepoint to itself. Formally, here is the introduction rule:

$$\overline{\mathsf{base} : S^1} \qquad \overline{\mathsf{loop} : \mathrm{Id}_{S^1}(\mathsf{base}, \mathsf{base})}$$

The elimination and computation rules essentially follow the same pattern as we have seen for W-types, but they are slightly more involved. The upshot is that the circle, as well as many other types of topological origin, which are also higher inductive types, can be manipulated 'by induction.'

It is worth mentioning here that even the notion of identity type itself is in some sense an inductive type, whose elimination and computation rules are analogous to those for W-types. A remarkable range of structures are now susceptible to inductive methods. For an outsider, the arguments employed may look like magic, but for the computer verifying them and to the seasoned programmer, they are simply a natural and absolutely fundamental principle taken to the next level of generality.

6.5 Feynman graphs as higher inductive types? While topologically a term in an identity type is a path, logically it is an equality, or an equation imposed. While conventional W-types parametrise operations of free algebraic structures (technically, they are the operations of the free monad on P [9]), higher inductive types parametrise operations of free algebraic structures quotiented by certain equations.

As explained above, graphs-with-a-nesting are precisely certain P-trees (and hence a conventional W-type), and graphs themselves are obtained by quotienting the set of P-trees by identifying two P-trees if they build the same graph. On these grounds, it is not unreasonable to speculate that Feynman graphs may form a higher inductive type. Actually establishing this will require further structural insight into Feynman graphs.

Higher inductive types are a rather recent addition to type theory, and their formalisation has not yet completely crystallised. In particular, the appropriate freedom in imposing equations has not yet been fully determined. It transpires from work of Lumsdaine and Shulman [19] that the specification of the equations (1-constructors) should be 'polynomial' in nature, in a certain sense, just as the 0-constructors in W-types.

Establishing that Feynman graphs form higher inductive types involves giving some uniform description of overlapping divergences, perhaps in terms of rewrite systems, and showing that the governing patterns are given by polynomial data in a sense compatible with the formalism developed in [19].

Hopes that such advances are achievable stem from the fact that practitioners of quantum field theory have already garnered an extremely large body of experience with Feynman graphs, and attention has already been given to the subtleties of overlapping divergences. I speculate that it could be of some importance to sort out these questions.

Acknowledgments. This contribution is a write-up of my talk at the workshop in Trento on *Dyson-Schwinger Equations in Modern Mathematics & Physics*. I wish to thank the organisers, Mario Pitschmann, Craig Roberts, and Wolfgang Lucha, for the wonderful workshop, for the invitation to speak there, and for the opportunity to learn a lot about the subject area. I am grateful to Kurusch Ebrahimi-Fard for his patience, over the years, in explaining me some rudiments of quantum field theory, and to Nicola Gambino and Egbert Rijke for helping me learn type theory. This work was funded by grant number MTM2013-42293-P of Spain.

Appendix: initiality in terms of dependent elimination

In category theory, an object I of a category $\mathscr{C}$ is called *initial* when, for any other object A, there is a unique morphism $I \to A$ in $\mathscr{C}$.

Lemma. *Let $\mathscr{C}$ be a category, assumed to admit finite limits. Then an object I of $\mathscr{C}$ is initial if and only if every morphism $C \to I$ has a section.*

Proof. Suppose I is initial, and suppose given $p : C \to I$. Since I is initial, there exists a morphism $f : I \to C$. This morphism f is in fact a section to p, because both the composite $p \circ f$ and the identity morphism are morphisms $I \to I$, so by initiality of I, they must coincide.

Conversely, suppose every $C \to I$ has a section. For an arbitrary object A, we need to establish that there is precisely one morphism $I \to A$. Consider the product $I \times A$; by assumption, this first projection $I \times A \to I$ has a section, i.e. a morphism $I \to I \times A$. Composed with the second projection this gives the existence of $a : I \to A$. To see that it is unique, suppose we have also another, $b : I \to A$. The two morphisms together constitute a morphism $(a, b) : I \to A \times A$. We wish to show that this morphism factors through the diagonal $A \to A \times A$, because that is precisely to say that they coincide. But to find that factorisation

$$\begin{tikzcd} & A \arrow[d, "\mathrm{diag}"] \\ I \arrow[ur, dotted] \arrow[r, "(a,b)"'] & A \times A \end{tikzcd}$$

is equivalent to finding a section to the pullback (fibre product)

$$\begin{tikzcd} I \times_{A\times A} A \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{diag}"] \\ I \arrow[r, "(a,b)"'] \arrow[u, dotted, bend left] & A \times A. \end{tikzcd}$$

But this section exists by assumption. □

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