Weak identity arrows in higher categories

JOACHIM KOCK

Abstract

There are a dozen definitions of weak higher categories, all of which loosen the notion of composition of arrows. A new approach is presented here, where instead the notion of identity arrow is weakened — these are tentatively called fair categories. The approach is simplicial in spirit, but the usual simplicial category $\Delta$ is replaced by a certain ‘fat’ delta of ‘coloured ordinals’, where the degeneracy maps are only up to homotopy. The first part of this exposition is aimed at a broad mathematical readership and contains also a brief introduction to simplicial viewpoints on higher categories in general. It is explained how the definition of fair $n$-category is almost forced upon us by three standard ideas.

The second part states some basic results about fair categories, and give examples, including Moore path spaces and cobordism categories. The category of fair 2-categories is shown to be equivalent to the category of bicategories with strict composition laws. Fair 3-categories correspond to tricategories with strict composition laws. The main motivation for the theory is Simpson’s weak-unit conjecture according to which $n$-groupoids with strict composition laws and weak units should model all homotopy $n$-types. A proof of a version of this conjecture in dimension 3 is announced, obtained in joint work with A. Joyal. Technical details and a fuller treatment of the applications will appear elsewhere.

Contents

0 Introduction 2
1 Categories as simplicial sets 6
2 Homotopy and coloured categories 9
3 Semi-categories, coloured semi-categories, and the fat delta 11
4 Definition of fair categories and fair $n$-categories 14
5 Fair 2-categories 21
6 Fair 3-categories 25
7 $n$-groupoids, homotopy $n$-types, and Simpson’s conjecture 27
8 A couple of examples 30
Appendix: Discrete objects 37
0 Introduction

Higher categories. While conventional category theory encompasses mathematics in the paradigm of objects and arrows-between-objects, higher category theory considers also 2-arrows between arrows, 3-arrows between 2-arrows, and so on. Higher categorical structures were first discovered in algebraic geometry and homotopy theory, and are now becoming increasingly important in many areas of mathematics, as well as in theoretical physics and computer science. It is not difficult to define strict higher categories, but the crucial point for applicability of higher categories is weakening, and the theory is inherently of homotopical nature. The theory of weak higher categories is young, with about a dozen competing definitions (cf. Leinster’s survey [24]). All these approaches emphasise weakened composition laws.

Weak identity arrows. The present paper introduces a new approach to the problematic, with a systematic theory of weak identity arrows. While strict identity arrows appear naturally whenever the arrows represent some sort of mappings, there are higher-dimensional contexts where they seem less nature-given and sometimes problematic:
- identity arrows are degenerate things, and tend to collapse other things as well (cf. the Eckmann-Hilton argument);
- identity arrows tend to be of a ‘wrong geometrical type’ (for instance: not cofibrant, or of defective dimension (e.g., identity $n$-cobordisms are ‘cylinders’ of height zero)).

One would like instead to speak of merely ‘up-to-homotopy’ identity arrows, so some sort of homotopical context is needed, i.e. a category with a notion of equivalence $\simeq$.

In a usual category the identity arrows appear as the image of a degeneracy map

$$O \rightarrow A$$

where $O$ is the set of objects, and $A$ the space of arrows. (And this map is subject to the well-known identity axioms relating to the composition map.)

There are essentially two possible ways one can try to define weak identity arrows. Either one can stick with the map $O \rightarrow A$, but weaken the axioms. This amounts to making non-canonical choices for the identity arrows, and handling the ensuing coherence issues. Such approaches to weak higher categories are usually called ‘algebraic’, since the operations (in this case the ‘nullary’ operation $O \rightarrow A$) remain well-defined.

The other approach, which is the route taken in this work, is to weaken the very shape of the diagram above: the idea is to avoid the artificial choices and instead relax the structure by specifying an acyclic space of weak identity arrows $U$, sitting between $O$ and $A$ like this:

$$\begin{array}{c}
O \\
\approx \\
U \\
\end{array} \rightarrow A$$
(and satisfying certain axioms relative to the composition law). Thus the theory remains purely diagrammatic, and becomes homotopical rather than algebraic, in the sense that there is no longer a well-defined ‘nullary’ operation — it has been replaced by an up-to-homotopy-only operation. Given the space $U$, it is sometimes possible to revert to an algebraic description by merely choosing a pseudo-section $u : O \to U$; the axioms satisfied by $U$ then automatically equips $u$ with the extra structure needed to serve as weak identity arrows and with coherence constraints built in. Conversely, given some algebraic notion of weak identity arrows, ideally their existence should be a property, not a structure, in an up-to-homotopy sense, meaning that the space $U(x)$ of all possible weak identity structures on an object $x$ should be contractible.

One example to have in mind (treated in detail in Section 8) is the following. A Moore path in a topological space $X$ is a continuous map $I \to X$ where $I$ is an interval of any positive length. Moore paths can be composed in a strictly associative way (by concatenation of intervals, hence without need of passing to homotopy classes of paths), but since we exclude the interval of zero length there are no units for this composition. Composing a path with its ‘inverse’ suggests that the weak units should be the null-homotopic loops, but we need to specify in which way each such null-homotopic loop is a weak unit: a weak unit at a point $x$ is a null-homotopic loop at $x$ together with a null-homotopy. The space $U(x)$ of such pairs is contractible, and our $U$ is the disjoint union $\coprod U(x)$ where $x$ runs over the points of $X$.

**Tamsamani categories and fair categories.** The diagrammatic weakening just outlined is analogous to the weakening of the composition laws in the theory of Tamsamani higher categories [34], which seems to be the most developed theory of higher categories. While the strict composition law is a diagram of shape

$$A \leftarrow A \times O A,$$

in a Tamsamani category this shape has been replaced by

$$
\begin{array}{ccc}
    A & \longrightarrow & A \times O A \\
    \downarrow \cong & & \downarrow \cong \\
    B & \longrightarrow & \end{array}
$$

Tamsamani’s theory is iterated simplicial, and the weakening of the composition laws is obtained by relaxing the nerve condition (see 1.5 below). It is not possible to obtain a similar diagrammatic weakening of the identity arrow structure in a purely simplicial setting, because the identity structure does not come from an external condition like the nerve condition; it is inherent from the degeneracies in $\Delta$. Instead, the usual simplicial category $\Delta$ is replaced by a larger category $\Delta$ which is introduced in this paper. Still the theory is simplicial in spirit, and it goes hand in hand with Tamsamani’s theory: one version of it combines the two weakenings.

In order to emphasise the weak-identity-arrow aspects of the theory, we will mostly concentrate on the case of strict composition laws. Such categories (strict composition laws but weak identity arrows) are tentatively called *fair categories*. Strict composition laws are not usually what is encountered in nature, but according to a strong form of Simpson’s conjecture, the homotopy category of weak $n$-categories is equivalent to
the homotopy category of $n$-categories with strict composition laws and only weak identity arrows. A proof of a weak version of Simpson’s conjecture in dimension 3 is announced here. This result has been obtained jointly with André Joyal and will appear separately [15]. The emphasis on strict composition laws is justified by this result. Section 7 contains some background for Simpson’s conjecture and motivation for the theory.

**The fat delta.** The fat delta $\Delta$, which is the basis for the theory, has the following concise description as a subcategory of the category of arrows in $\Delta$: its objects are the epimorphisms in $\Delta$, and the arrows are the monomorphisms in $\text{Arr}(\Delta)$; these are the commutative diagrams in $\Delta$ whose vertical arrows are epimorphisms and whose top arrow is a monomorphism:

```
· ⊂ ·
      ·
```

However, this description does not convey much intuition. Instead, a less concise but more conceptual definition is given, and it is shown how the fat delta and the definition of fair category follow quite naturally from three standard ideas: the simplicial idea, the homotopy idea, and the idea of ‘semi-’. The fat delta is the category of coloured semi-ordinals.

**Organisation of the paper and overview of the results**

The first three sections are aimed at a broad mathematical readership and serve to introduce the simplicial viewpoint on higher categories, establish some terminology, and to motivate and introduce the fat delta. With these preparations, the definition of fair category 4.5 is a one-liner: *A fair category in $S$ is a functor $\Delta^{\text{op}} \to S$ that preserves equimorphisms, discrete objects, and fibre product over discrete objects.*

The second part of the paper, containing the results, is a bit more advanced, and some of the details of proofs are deferred to forthcoming papers.

Section for section:

§1 **Nerves and simplicial enrichment.** The definition of fair categories is simplicial in spirit, and the first section recalls the basic viewpoint of categories as special simplicial sets. $n$-categories are defined inductively as certain special simplicial objects in the category of $(n-1)$-categories. The fundamental combinatorial structure is $\Delta$, the category of non-empty finite ordinals.

§2 **Homotopy and coloured categories.** In order to be able to talk of weakness and ‘up-to-homotopy’, we must work in a category $S$ with a notion of equivalence — these are called *coloured categories* in this work. Weakening the composition laws in the simplicial viewpoint leads to Tamsamani’s notion of higher category, cf. [34].

Those two ideas: ‘simplicial’ and ‘up-to-homotopy’, lead directly to the concept of coloured ordinals. The category of (finite, nonempty) coloured ordinals plays the same rôle for coloured categories as the usual $\Delta$ does for plain categories.
§3 Semi-categories, coloured semi-categories, and the ‘fat delta’ $\Delta$. To arrive at fair categories, a third idea is involved, namely semi-categories (categories without identities). Combining this idea with the previous two ideas yields the (coloured) category $\Delta$ of (finite, nonempty) coloured semi-ordinals. This category captures the shape of fair categories, in the sense that fair categories will be certain $\Delta^{\text{op}}$-diagrams in $S$, just as usual categories are seen as $\Delta^{\text{op}}$-diagrams.

§4 Definition of fair categories and fair $n$-categories. A fair category in $S$ is by definition a functor $X : \Delta^{\text{op}} \to S$ that preserves equimorphisms, discrete objects, and fibre products over discrete objects. That $X$ preserves equimorphisms expresses the up-to-homotopy identity arrow condition. Preserving fibre products over discrete objects expresses the condition of strict composition laws. (At this point the reader may wish to have a glance at the examples given in Section 8.)

Inductively, a fair $n$-category is obtained with $S = (n-1)\text{Cat}$, the category of fair $(n-1)$-categories. For this to make sense we must define what an equivalence of fair $S$-categories is. Some technicalities on discrete objects needed in the definition are relegated to an appendix — the reader can safely think of the usual notion of discrete objects in $\text{Top}$ or in $\text{Cat}$.

One basic result, also needed in the induction, is that every fair 1-category is isomorphic to a usual category.

§5 Fair 2-categories. The following is the main result of §5:

Proposition 5.2. The category of fair 2-categories is equivalent to the category of bicategories with strict composition laws.

The fair 2-category viewpoint on such a bicategory encodes all the possible unit structures on the underlying semi-bicategory.

§6 Fair 3-categories. Morally, fair 3-categories should correspond to tricategories with strict composition laws. A special semi-strict case of particular interest is worked out:

Proposition 6.3. Fair monoidal strict 2-categories correspond to monoidal strict 2-categories with weak units in the sense of Gordon-Power-Street [8].

§7 $n$-groupoids, homotopy $n$-types, and Simpson’s conjecture. This section serves as a second introduction, explaining the main motivation for considering weak units, and the justification for considering only strict composition laws: Simpson [31] has conjectured that every homotopy $n$-type arises as the geometric realisation of a strict $n$-groupoid with weak identity arrows — for a suitable notion of weak identity arrows. There is a straightforward notion of fair $n$-groupoid, and it is conjectured that this notion will fulfil Simpson’s conjecture. A proof of a version of the conjecture in dimension 3 is announced here, obtained jointly with André Joyal. The key result is this:

Theorem 7.8. (Joyal-Kock [15].) Every braided monoidal category arises as $\text{End}(I)$, where $I$ is a weak unit in an otherwise completely strict monoidal 2-category.

Corollary 7.9. (Cf. [15].) Strict 2-groupoids with invertible tensor product and weak units can model all 1-connected homotopy 3-types.
§8 Examples. Weak identity arrows arise as ‘honest’ replacements for ‘artificial’ identity arrows or approximation to non-existent identity arrows. Three examples of this are given, the first in some detail, the other two more succinctly.

The first example concerns Moore paths in a topological space. The weak identity arrows are null-homotopic paths together with a null-homotopy.

The second example concerns a monoidal model category in the sense of Hovey [13]. If the unit is cofibrant then the full subcategory of all cofibrant objects is a genuine monoidal coloured category. If the unit is not cofibrant one gets instead a fair monoidal coloured category, the space of weak units being the cofibrant replacements of the unit.

The final example concerns cobordism categories, and is perhaps of particular interest. In summary:

**Proposition 8.13.** Oriented $n$-cobordisms naturally assemble into a fair Tamsamani 2-category, for which the straight cylinders are weak identity arrows.

Acknowledgements. The bulk of this paper was written during the Summer of 2003, and preliminary versions of it were circulated in connection with three talks: in Aarhus in April 2003, at the CATS2 conference in Nice, May 2003, and at the Workshop on Higher-order Geometry and Categorification in Lisbon, July 2003. In Lisbon, I presented an example of a fair 2-monoid in $\mathbf{Cat}$ and claimed that it was equivalent to the braid category. This statement is wrong — I am grateful to André Enríques for pointing out the problems in my argument. This led to the discovery that all fair 2-fold monoidal categories collapse to symmetric ones [20], and for a long time it seemed that Simpson’s conjecture would be false; for this reason the present paper was shelved. Decisive new insight resulted from many conversations with André Joyal during my year in Montréal, culminating with Theorem 7.8.

This research was supported by a Marie Curie Fellowship 2001–2002, by the University of Nice (2003), and by a CIRGET postdoc grant at the University of Québec at Montréal, 2004. I am very much indebted to André Hirschowitz, Bertrand Toën, and André Joyal for their help and encouragement, and I would like also to acknowledge fruitful conversations and e-mail correspondence with Anders Kock, Bill Lawvere, Carlos Simpson, Clemens Berger, Markus Spitzweck, and Tom Leinster.

1 Categories as simplicial sets

We shall briefly review Tamsamani’s definition of higher categories, and settle on some basic terminology. The fundamental viewpoint is that a category is a simplicial set satisfying certain conditions.

1.1 Simplicial sets, categories, and the (strict) Segal condition. Let $\Delta$ be the category whose objects are the nonempty finite ordinals, (i.e. the linearly ordered nonempty finite sets

$$0 = \{0\}, \quad 1 = \{0 \leq 1\}, \quad 2 = \{0 \leq 1 \leq 2\}, \quad \ldots, \quad n = \{0 \leq 1 \leq 2 \leq \ldots \leq n\}, \quad \ldots$$

and whose arrows are the order-preserving maps, i.e., functions $f : m \to n$ such that $f(i) \leq f(j)$ in $n$ whenever $i \leq j$ in $m$. It is convenient to interpret each $n$ as a category
by viewing each inequality \( i \leq j \) as an arrow from \( i \) to \( j \). Then \( \Delta \) becomes the full subcategory of \( \textbf{Cat} \) formed by the categories \( 0, 1, 2, \ldots \).

A simplicial set is a functor \( \Delta^{\text{op}} \to \textbf{Set} \); a simplicial map is a natural transformation of such functors. We denote by \( \text{sSet} \) the category of simplicial sets and maps.

The \textit{nerve} of a (small) category \( \mathcal{C} \) is by definition the simplicial set
\[
\mathcal{C} : \Delta^{\text{op}} \longrightarrow \textbf{Set}
\]
\[
\mathbf{n} \longmapsto \text{Hom}_{\text{Cat}}(\mathbf{n}, \mathcal{C}).
\]

There are natural identifications
\[
\mathcal{C}_0 = \text{Ob}(\mathcal{C}), \quad \mathcal{C}_1 = \bigsqcup_{x, y \in \mathcal{C}_0} \text{Hom}_{\mathcal{C}}(x, y),
\]
and more generally \( \mathcal{C}_k \) is interpreted as the set of strings of \( k \) composable arrows in \( \mathcal{C} \). Functors between categories turn into simplicial maps between their nerves, and the whole construction defines a functor from categories to simplicial sets. This functor is fully faithful. The simplicial sets that arise as nerves of categories are characterised by \textit{strict Segal condition}: the natural maps
\[
\mathcal{C}_{p+q} \to \mathcal{C}_p \times_{\mathcal{C}_0} \mathcal{C}_q, \quad \forall p, q,
\]
are isomorphisms. (Here the fibre product refers to the maps \( \mathcal{C}_p \to \mathcal{C}_0 \leftarrow \mathcal{C}_q \), ‘target of last arrow’ and ‘source of first arrow’, respectively.) Hence it makes sense to say that \( \mathcal{C} \) is a category if it is a simplicial set satisfying the \textit{strict Segal condition}.

(This viewpoint and the terminology \textit{nerve} go back to Grothendieck [9], 1959. The (non-strict) Segal condition played a fundamental rôle in Segal [29] (1974), and was named after him by Tamsamani [34] in 1996.)

It will be helpful to have the following graphical description in mind: The category \( \mathbf{n} \) is pictured with a dot for each object; the arrows are not drawn explicitly; instead the order is expressed by having the dots arranged in a column, with \( 0 \) at the bottom and \( n \) at the top. Now a functor \( \mathbf{m} \to \mathbf{n} \) (i.e., an order preserving map) is represented by linking each dot in \( \mathbf{m} \) to its image dot in \( \mathbf{n} \); the order preservation then corresponds to the requirement that these strands do not cross over each other. Here is a picture of the most fundamental arrows in \( \Delta \):

\[
\begin{array}{cccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
1 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 1 \rightarrow 2
\end{array}
\]

If \( \mathcal{C} : \Delta^{\text{op}} \to \textbf{Set} \) is (the nerve of) a category, then the four arrows in the picture are mapped to the following four maps: the map \( \mathcal{C}_0 \to \mathcal{C}_1 \) that sends an object to its identity arrow; the source map \( \mathcal{C}_1 \to \mathcal{C}_0 \); the target map \( \mathcal{C}_1 \to \mathcal{C}_0 \); and the composition of arrows \( \mathcal{C}_2 \to \mathcal{C}_1 \).

The Segal condition is an exactness condition: In \( \Delta \), two arrows \( \mathbf{m} \leftarrow 0 \rightarrow \mathbf{n} \) admit a pushout if and only if the dot of \( 0 \) is included in \( \mathbf{m} \) as the last dot, and in \( \mathbf{n} \) as the first dot (or conversely), like in this example:
These amalgamated sums in $\Delta$ are fibre products in $\Delta^{op}$, and the Segal condition amounts precisely to saying that $X : \Delta^{op} \to \mathbf{Set}$ preserves fibre products over 0 (i.e., those fibre products over 0 that happen to exist).

1.2 Strict higher categories. A (strict) 2-category is defined to be a category enriched over $\mathbf{Cat}$. This means that the definition of category is repeated, but the hom sets (and the structure maps between them) are replaced by hom categories $\text{Hom}_C(x, y)$ (and the structure maps by functors). There is a category $\mathbf{2Cat}$ whose objects are 2-categories, and whose arrows are 2-functors. Inductively then, a (strict) $n$-category is a category enriched over $(\mathbf{n-1} \mathbf{Cat})$. Unwinding this recursive definition, one finds that an $n$-category has objects, arrows between objects, 2-arrows between arrows, and so on up to $n$-arrows between $(n - 1)$-arrows. $k$-arrows can be composed in $k$ compatible ways, provided their lower-dimensional cells match appropriately, and for each $k$-arrow ($k < n$) there is an identity $(k + 1)$-arrow.

It is important to note that throughout this induction, the objects of an $n$-category always form a set, not an $(n - 1)$-category. Now a set can in a natural way be considered as a category, namely a category whose only arrows are the identity arrows. Similarly a category can be considered as a 2-category, etc., and in particular a set can be considered as an $n$-category for any $n$; these are just the $n$-categories whose only $k$-arrows are the identity arrows ($0 < k \leq n$). Such $n$-categories are called discrete. With this remark, the iterated enrichment can be conveniently formulated in terms of nerves, and this formulation is the key to the generalisation and weakening we will come to in the next section: A 0-category is just a set, and a 1-category is just a usual category. An $n$-category is a functor

$$C : \Delta^{op} \to (\mathbf{n-1} \mathbf{Cat})$$

such that $C_0$ is discrete and such that the strict Segal condition (1) holds.

1.3 $S$-categories. The definition of higher categories involves considering simplicial objects in many different categories and imposing the ‘category condition’. Hence the natural need for an abstract notion of $S$-category: this should be a simplicial object $X : \Delta^{op} \to S$ in a category $S$, such that $X_0$ is discrete and such that $X_{p+q} \simeq X_p \times_{X_0} X_q$. For this to make sense, $S$ must be have a notion of discrete objects, and it should possess fibre products over discrete objects. These two notions are unavoidable, and in fact the category condition is nothing but preservation of these two notions: Namely, define 0 to be the only discrete object in $\Delta$ (or in $\Delta^{op}$). (Thinking of discreteness as dual to connectedness, clearly this is the only reasonable way to define discrete objects in $\Delta$.)

**Definition.** Let $S$ be a category with a notion of discrete objects, and admitting fibre products over discrete objects. An $S$-category is a functor $X : \Delta^{op} \to S$ that
preserves discrete objects and fibre products over discrete objects. Let $S$-$\text{Cat}$ denote the category whose objects are $S$-categories $X: \Delta^{\text{op}} \to S$ and whose morphisms are the natural transformations between such.

1.4 Weakening. An important lesson from category theory is summarised in the following two remarks. On level 0: when comparing two sets (i.e., 0-categories), it is more important to say if they are isomorphic than if they are actually equal. On level 1: given two categories it is more important to say if they are equivalent than if they are actually isomorphic. Recall that an equivalence of categories is a functor that is fully faithful and essentially surjective. Henceforth such a functor will rather be called an equimorphism. According to this lesson, the above definition of 2-category is not the ‘correct’ one, since the strict Segal condition refers to isomorphism of categories. A weaker notion of higher category is obtained by requiring the Segal maps $C_{p+q} \to C_p \times C_0 C_q$ to be merely equimorphisms, not isomorphisms:

1.5 Tamsamani higher categories. Define a Tamsamani 0-category to be a set. A weak $n$-category in the sense of Tamsamani [34] is defined inductively as a functor

$$C: \Delta^{\text{op}} \to (n-1)\text{wCat}$$

such that $C_0$ is discrete, and satisfying the (non-strict) Segal condition, namely that the morphisms $C_{p+q} \to C_p \times C_0 C_q$ should be equimorphisms in $(n-1)\text{wCat}$, the category of weak $(n-1)$-categories. ‘Equimorphism’ means ‘fully faithful’ and ‘essentially surjective’, notions which are also defined inductively. These definitions rely crucially on properties of the notion of discrete object, which we come to in a minute.

Note that an equimorphism is not in general invertible, so there is no longer any well-defined composition like this:

$$C_1 \leftarrow C_1 \times_{C_0} C_1$$

This map is now defined only ‘up to homotopy’: it exists only inasmuch as we regard the equimorphism $C_2 \simeq C_1 \times_{C_0} C_1$ as invertible. The new structure is rather this:

$$C_1 \leftarrow C_1 \times_{C_0} C_1 \leftarrow C_2$$

The theory of Tamsamani higher categories seems to be the most developed among the theories of higher categories, thanks in particular to the work of Simpson and his collaborators (see for example [30], [32], [12], [27], [37]). The main reason is that it is a simplicial theory, and huge bodies of simplicial methods in homotopical algebra can be applied.

2 Homotopy and coloured categories

2.1 Coloured categories. Weakening makes sense in categories equipped with a notion of equivalence. Then a property can be said to hold up to equivalence (or up to
homotopy), meaning that it only holds inasmuch as equivalences are considered as equalities. The following terminology is handy, but not standard in the literature. A **coloured category** is a category $C$ with a specified subcategory $W$ comprising all the objects. The arrows in $W$ are called **coloured arrows** or **equimorphisms** or **equiarrows**. An **equivalence** is a zigzag of equimorphisms, and two objects in $C$ are equivalent if there is an equivalence between them. (This relation is clearly reflexive, symmetric, and transitive.) The homotopy category $\text{Ho}(C) = C[W^{-1}]$ of a coloured category $(C, W)$ is the category obtained by formally inverting the equiarrows (cf. [7]). Let $\mathbf{CCat}$ denote the category of coloured categories and colour-preserving functors (i.e. functors preserving coloured arrows).

Key examples of coloured categories are $\mathbf{Top}$ and $\mathbf{sSet}$, colouring the weak homotopy equivalences, and also $\mathbf{Cat}$ and $\mathbf{nCat}$, with the appropriate notions of fully faithful and essentially surjective functors as equimorphisms. We shall also have good use of the coloured category $\mathbf{Set}$ in which the equimorphisms are the bijections. (Note that any category admits three ‘trivial’ colourings: colouring the identity arrows only, colouring the isomorphisms only, or colouring all arrows.)

### 2.2 Tamsamani $S$-categories.

Given a coloured category $S$ with discrete objects and admitting fibre products over discrete objects, it makes sense to define a **Tamsamani $S$-category** to be a functor $X : \Delta^{op} \to S$ subject to the weak Segal condition and with $X_0$ discrete. An important example of this generalisation is given by taking $S = \mathbf{sSet}$: these are called **Segal categories** (cf. [5]), and in many context they play the rôle of certain $\infty$-categories, cf. Hirschowitz-Simpson [12], Toën-Vezzosi [37], and Toën [35].

### 2.3 Getting the colours right, and discrete objects.

Returning to the inductive definition of Tamsamani $n$-category (1.5), in order to make sense of the induction step, we must define what an equimorphism of $S$-categories is, and the subtle point is to get this definition right. The subtlety can be observed already in the definition of equimorphism in $\mathbf{Cat}$. The notion of equimorphism is weaker than the pointwise one (because an equimorphism of categories is not necessarily a bijection on objects), and it is stronger than the notion induced from weak homotopy equivalence in $\mathbf{sSet}$ (e.g., as categories $0$ and $1$ are not equivalent, but their nerves are weakly equivalent). The condition ‘admitting a quasi-inverse’ happens to be the correct notion for ordinary categories, but already for 2-categories it is too strong. The good general notion of equimorphism turns out to be ‘essentially surjective and fully faithful’, provided these notions are well chosen. Tamsamani’s definition — which is good, in view of his main theorem, quoted in 7.5 below — relies on certain properties of discrete objects. The axiomatisation of these properties must take into account that they should reproduce themselves under the induction step; the details are given in the Appendix.

**Coloured ordinals**

### 2.4 Coloured ordinals.

Since nonempty finite ordinals are fundamental categories, it seems worthwhile to look at the corresponding coloured notion (cf. [18]). Recall that ordinals can be seen as free categories on linearly ordered graphs (strings of arrows). Consider the coloured version of these three notions:
A coloured graph is a graph some of whose edges have been singled out as coloured. The free coloured category on a coloured graph is defined by taking $C$ to be the free category on the whole graph and taking $W$ to be the free category on the coloured part of the graph (including all vertices). This means that in a free coloured category, the composite of two arrows is an equimorphism if and only if both arrows are equimorphisms. Finally a (finite) coloured ordinal is the free coloured category on a (finite) linearly ordered coloured graph.

So what it all boils down to is to take finite strings of arrows, some of which are coloured. Let $T$ denote the full subcategory of $\mathbf{CCat}$ consisting of the (finite and non-empty) coloured ordinals. There are many other descriptions of this category (cf. [18]): as the category of epimorphisms in $\Delta$; as the category of planar trees of height 2; as opposite to the category of subdivided finite strict intervals; as a Grothendieck construction or moduli space for all coherent colourings of finite ordinals, etc.

The category $T$ plays the same rôle for coloured categories as $\Delta$ does for plain categories. For example, a coloured category $C$ can be described in terms of its coloured nerve,

$$
\mathbf{T}^{\mathbf{op}} \to \mathbf{Set} \\
K \mapsto \text{Hom}_{\mathbf{CCat}}(K, C).
$$

### 2.5 Graphical interpretation.

We represent the objects of a coloured ordinal $K$ as dots arranged in a column (just like the drawing for ordinals on page 1.1). The ordinary arrows are not drawn (the order expressed by the column indicates everything). The equimorphisms are drawn as a link (the direction of the arrow being expressed by the order in the column: the arrows go upwards).

The graphical expression for functors between coloured ordinals is just as for usual ordinals what the dots are concerned, and for the links the rule is that a link can be set but may not be broken. Here is a list of the most basic arrows in $T$ (not mentioning the identity arrows):

If $C : \mathbf{T}^{\mathbf{op}} \to \mathbf{Set}$ is (the nerve of) a coloured category $(C, W)$, then the images of these basic arrows have precise interpretation as structure maps, just as remarked in 1.1: the first one associates the identity arrow to an object; ditto for the second but it specifies that this arrow is an equimorphism; the third is the inclusion of $W$ into $C$, and so on. The second-to-last is the composition of an arbitrary arrow with an equimorphism.

### 2.6 The projection $T \to \Delta$ and colour structure on $T$.

Consider the natural projection functor $\pi : T \to \Delta$ given by ‘taking equi-connected components’, i.e., contracting all links. Now $T$ has a natural colour structure, given by taking the equimorphisms $T$ to be those arrows mapping to identity arrows in $\Delta$.

### 3 Semi-categories, coloured semi-categories, and the fat delta

We want to weaken the identity arrow axiom, but there is no way to do that within the purely simplicial viewpoint: the identity arrows arise inherently as a consequence
of the degeneracy maps in $\Delta$. Plainly removing degeneracy maps leads to $\Delta_{\text{mono}}$-diagrams, as we proceed to explain. $\Delta_{\text{mono}}$-diagrams satisfying the Segal condition are semi-categories — note that the Segal condition relates only to face maps. Passing from categories to semi-categories is too drastic a reduction however, and there are many constructions with categories that fail for semi-categories. The fat delta $\Delta$ will be a sort of intermediate between $\Delta_{\text{mono}}$ and $\Delta$.

3.1 Semi-categories and ‘semi-ordinals’. We use the prefix semi- consistently to mean ‘non-unital’: A semi-category is just like a category, except that identity arrows are not required. A semi-functor is a map compatible with the composition law. Note that the identity semi-functor exists for any semi-category, so semi-categories and semi-functors form a genuine category $\mathcal{C}at$, not just a semi-category. A (finite) ‘semi-ordinal’ is the semi-category associated to a (finite) total strict order relation $<$. Since $<$ is not reflexive, a given element is not related to itself, so there are no identity arrows. As a consequence, all morphisms between semi-ordinals are injective, and the category of finite non-empty semi-ordinals is naturally identified with $\Delta_{\text{mono}}$. The (semi-)nerve of a semi-category is a functor $\Delta_{\text{mono}}^{\text{op}} \to \text{Set}$ satisfying the (strict) Segal condition.

It is straightforward to copy over the definition of higher categories to the case of semi-categories. But while the basic definitions of semi-categories are easy, their theory is quite different from that of categories. For instance, in the theory of categories constant diagrams always exist, and one can define limits and colimits as adjoints of the constant diagram functor [26]. For semi-categories, constant diagrams do not in general exist, because the supporting object is not required to have an identity arrow.

3.2 Coloured semi-ordinals. Combining all the previous notions we finally come to the promised fat delta: the category $\Delta$ of coloured finite non-empty semi-ordinals. The definitions should be obvious: a (finite) coloured semi-category is a semi-category with a sub-semi-category comprising all objects, and a morphism between coloured semi-categories is a semi-functor required to preserve colour. There is a (genuine) category $C\mathcal{C}at$ of coloured semi-categories and their morphisms (and in fact this category can be considered a coloured category).

The coloured (finite) semi-ordinals are the free coloured semi-categories on (finite) linearly ordered coloured graphs. So it boils down to giving a (finite) string of arrows, some of which are coloured; there are no identity arrows at the vertices.

3.3 The fat delta $\Delta$. The fat delta $\Delta$ is by definition the full subcategory of $C\mathcal{C}at$ consisting of all finite non-empty coloured semi-ordinals. It is naturally identified with the category of monomorphisms in $T$:

$$\Delta = T_{\text{mono}}.$$  

The various characterisations of $T$ yield alternative descriptions of $\Delta$. In particular we get the following concise description of $\Delta$ as a subcategory of the category of arrows in $\Delta$: its objects are the epimorphisms in $\Delta$, and the arrows are the monomorphisms in $\text{Arr}(\Delta)$; these are the commutative diagrams in $\Delta$ whose downward arrows are epi-
morphisms and whose top arrow is a monomorphism:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) {\cdot};
\node (B) at (1,1) {\cdot};
\node (C) at (1,0) {\cdot};
\node (D) at (1,-1) {\cdot};
\draw (A) -- (B);
\draw (B) -- (C);
\end{tikzpicture}}
\end{array}
\]

The drawings of objects and arrows in \( \Delta \) are the same as for \( T \), except that the maps are required to be ‘injective on dots’. Thus the first two figures listed on page 11 are not arrows in \( \Delta \).

3.4 The projection \( \Delta \to \Delta \) and colours in \( \Delta \). An important rôle is played by the projection functor \( \pi : \Delta \to \Delta \), sending a coloured (semi-) ordinal to the ordinal of equi-connected components; let \( V \subset \Delta \) denote the subcategory of vertical arrows for \( \pi \) (i.e. whose image in \( \Delta \) is an identity arrow), then the pair \((\Delta, V)\) is a coloured category. From the drawing on page 11, these are equimorphisms:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) {\cdot};
\node (B) at (1,1) {\cdot};
\node (C) at (1,0) {\cdot};
\node (D) at (1,-1) {\cdot};
\draw (A) -- (B);
\draw (B) -- (C);
\end{tikzpicture}}
\end{array}
\]

and these are not:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) {\cdot};
\node (B) at (1,1) {\cdot};
\node (C) at (1,0) {\cdot};
\node (D) at (1,-1) {\cdot};
\draw (A) -- (B);
\draw (B) -- (C);
\end{tikzpicture}}
\end{array}
\]

These last four arrows should be compared to the four arrows in \( \Delta \) drawn on page 7. The important thing to note, compared to the situation in \( \Delta \) is that the figure \( \Rightarrow \) (which was responsible for the existence of identity arrows), does not exist in \( \Delta \). Instead we have \( \Leftarrow \).

Here is a picture of the first few arrows in \( \Delta \):

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) {\cdot};
\node (B) at (1,1) {\cdot};
\node (C) at (1,0) {\cdot};
\node (D) at (1,-1) {\cdot};
\draw (A) -- (B);
\draw (B) -- (C);
\end{tikzpicture}}
\end{array}
\]

and their images in \( \Delta \):

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) {\cdot};
\node (B) at (1,1) {\cdot};
\node (C) at (1,0) {\cdot};
\node (D) at (1,-1) {\cdot};
\draw (A) -- (B);
\draw (B) -- (C);
\end{tikzpicture}}
\end{array}
\]

3.5 The two inclusions \( \Delta_{\text{mono}} \subset \Delta \). In the triangle diagram above we see the beginning of two copies of \( \Delta_{\text{mono}} \). There is the ‘horizontal’ inclusion \( i : \Delta_{\text{mono}} \to \Delta \), interpreting a semi-ordinal as a coloured semi-ordinal with nothing coloured:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\node (A) at (0,0) {\cdot};
\node (B) at (1,1) {\cdot};
\node (C) at (1,0) {\cdot};
\node (D) at (1,-1) {\cdot};
\draw (A) -- (B);
\draw (B) -- (C);
\end{tikzpicture}}
\end{array}
\]

The composite functor

\[
\Delta_{\text{mono}} \to \Delta \to \Delta
\]
is just the standard inclusion of the monos in $\Delta$. This is the sense in which the fat delta is intermediate between $\Delta_{\text{mono}}$ and $\Delta$.

There is also the ‘vertical’ inclusion $\zeta: \Delta_{\text{mono}} \hookrightarrow \Delta$, interpreting a semi-ordinal as a coloured semi-ordinal with everything coloured. The image is

$$
\bullet \rightarrow \circ \rightarrow \bullet \rightarrow \circ \rightarrow \ldots
$$

(for typographical reasons drawn horizontally, but to fit into the drawing above it should be vertical).

3.6 Discrete object structure on $\Delta$ (and $\Delta^{\text{op}}$). Just like for $\Delta$ (and $\Delta^{\text{op}}$), we declare the single dot to be the only discrete object in $\Delta$ (or in $\Delta^{\text{op}}$).

4 Definition of fair categories and fair $n$-categories

Fair $S$-categories

Henceforth, let $S$ denote a coloured category with a notion of discrete objects. (It is reasonable to assume the existence of fibre products over discrete objects, and require this operation to preserve equiarrows, but in fact the definition makes sense without these requirements.)

4.1 The idea. A fair category in $S$ is going to be a $\Delta^{\text{op}}$-diagram

$$
X: \Delta^{\text{op}} \rightarrow S
$$

satisfying three obvious axioms. In order to exhibit the axioms as obvious, let us right away explain how $X$ is to be interpreted as a nerve: put

$$
O := X_\bullet, \quad A := X_\circ, \quad U := X_{\bullet^\circ},
$$

and think of these as the spaces of objects, arrows, and (weak) identity arrows, respectively. This notation will be used throughout.

The images of the first few maps (i.e., the beginning of the $\Delta^{\text{op}}$-diagram) looks like this:

$$
\begin{array}{ccc}
O & \rightarrow & A \\
\downarrow & & \uparrow \\
U & \rightarrow & A
\end{array}
$$

(2)

The maps $s, t$ are source and target, and $u$ is thought of as the inclusion of the space of identity arrows into the space of all arrows.

4.2 Discreteness. Now the first axiom is clear:

The object $O$ (image of the single dot $\bullet$) is discrete.
Taking \( \bullet \) as the only discrete object in \( \Delta^{\text{op}} \), the first axiom says that \( X : \Delta^{\text{op}} \to S \) preserves discrete objects.

### 4.3 Segal condition
Second, we want a Segal condition which should induce the usual Segal condition on each \( \Delta_{\text{mono}} \subset \Delta \). Just like in \( \Delta \) (and in \( \Delta_{\text{mono}} \)), pushouts over the single dot exists in \( \Delta \) if and only if the dot is included as last dot in the first summand and as the first dot in the second (or vice versa). Let the pushout of \( m \quad \bullet \quad n \) be denoted by \( m \downarrow n \). The generalised Segal condition requires the generalised Segal maps

\[
X_{m \downarrow n} \longrightarrow X_m \times_O X_n
\]

to be isomorphisms. This formulation presupposes existence of fibre products over discrete objects, but we might as well just say that for all \( m \) and \( n \), the square

\[
\begin{array}{ccc}
X_{m \downarrow n} & \longrightarrow & X_m \\
\downarrow & & \downarrow \\
X_m & \longrightarrow & O
\end{array}
\]

should be a pullback square. Hence our second axiom is:

\( X \) preserves fibre products over discrete objects.

In practice, this means two things. First of all, the restriction to either copy of \( \Delta_{\text{mono}} \subset \Delta \) is a \( \Delta_{\text{mono}}^{\text{op}} \)-diagram which satisfies the Segal condition. Hence \( A \) and \( U \) are each semi-categories, i.e. carry associative composition operations over \( O \). The \( \Delta_{\text{mono}}^{\text{op}} \)-diagram \( \Delta_{\text{mono}}^{\text{op}} \xrightarrow{\iota} \Delta^{\text{op}} \xrightarrow{X} S \) is the underlying semi-category of \( X \).

Second, the Segal condition means that that the rest of the diagram can be constructed from the above triangle diagram (2), provided \( U \to A \) is a semi-functor (i.e. is compatible with composition in \( U \) and in \( A \)).

### 4.4 Weak identity arrows
Finally we want the points in \( U \) to act as identity arrows. For each object there should be a contractible space of weak identity arrows, so we want equimorphisms for the two maps \( U \longrightarrow O \), images of

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\end{array}
\]

The identity condition is that composing with a weak identity arrow should be neutral up to homotopy. So we want the composition maps \( U \times_O A \longrightarrow A \longrightarrow A \times_O U \) and \( U \times_O U \longrightarrow U \) to be equimorphisms — these are the image of

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\end{array}
\]

respectively. Note that the five maps mentioned here are all images of vertical maps in \( \Delta \). In fact, these five maps generate the category of vertical arrows in \( \Delta \) under the \( \downarrow \) operation (see [18] for details), so in view of the Segal condition and the assumption that equimorphisms in \( S \) are stable under fibre products over discrete objects, the requirement that these five maps be equimorphisms is equivalent to requiring every vertical map in \( \Delta \) to be sent to an equimorphism in \( S \). This is of course a more uniform and conceptual condition (and it makes sense without the extra assumptions), so the third axiom is:
X: Δ^op → S preserves colours.

In short, the definition is simply this:

4.5 Definition of fair category. A fair S-category is a colour-preserving functor Δ^op → S, preserving also discrete objects and fibre products over discrete objects.

A morphism of fair S-categories is just a natural transformation. Let S-FairCat denote the category of fair S-categories and their morphisms. Note that these morphisms are strict, and in some situations a weaker notion is needed. For example, in order to get the correct inner homs, one should consider derived morphisms, just like in the theory of Tamsamani categories, cf. [30]. See Remark 5.8 below for a particular case of a weaker notion of morphism.

4.6 Strict categories, fair categories, and semi-categories. There is an obvious forgetful functor S-FairCat → S-½Cat, defined by restriction to Δmono via the horizontal inclusion ι: Δmono → Δ. Similarly every strict S-category is canonically interpreted as a fair S-category: pre-composition with π: Δ → Δ yields a full embedding π*: S-Cat → S-FairCat. Non-strict examples of fair S-categories are given in Section 8.

Equimorphisms of fair S-categories

4.7 Standard discrete objects. Just as in the case of Tamsamani S-categories, in order to get a good notion of equimorphism, further axioms must be imposed on the notion of discrete objects and their relation to the subcategory of equiarrows in S. From now on,

We assume that S has standard discrete objects with compatible colours,

in the sense of the Appendix. In particular there is an adjunction π0⊣δ:

$$\begin{array}{c}
S \\
\xleftarrow{\delta}
\xrightarrow{\pi_0}
Set;
\end{array}$$

the objects in the image of δ are the discrete objects, and the left adjoint π0 is the ‘components functor’. Both these functors preserve discrete objects, and fibre products over discrete objects. The compatibility of colours with respect to the discrete objects means that δ and π0 are required to preserve equimorphisms. Here and throughout, Set is considered a coloured category by taking the bijections as equimorphisms.

4.8 Hom spaces. It follows from the decomposition property (A.4) that in a fair S-category X = (O, A, U) we have a decomposition of the space of arrows

$$A = \coprod_{x,y \in O} A(x,y)$$

according to source and target. Here A(x,y) is the subspace of A consisting of all arrows whose source is x and whose target is y.
4.9 Lemma. In a fair $S$-category $X = (O, A, U)$, the two maps $O \xrightarrow{=} U$ coincide. In other words, weak identity arrows are endomorphisms.

Proof. The composite

$$
\Delta^{op} \xrightarrow{X} S \xrightarrow{\pi_0} \text{Set}
$$

is a fair category in $\text{Set}$, since $\pi_0$ preserves discrete objects, fibre products over discrete objects, and equimorphisms (all the notions involved in the definition of fair category). Since $O$ is discrete and $\pi_0$ is the reflector, we have a bijection

$$S(U, O) \leftrightarrow \text{Set}(\pi_0 U, \pi_0 O),$$

so it is enough to establish the result in the case of a fair category in $\text{Set}$. The two maps $O \xrightarrow{=} U$ are the beginning of a $\Delta^{op}_{\text{mono}}$-diagram

$$
O \xrightarrow{s} U \xrightarrow{a} U \times_O U \ldots
$$

and since we are in $\text{Set}$, all these maps are bijections. The face map identities read $s \circ a = s \circ b$, $t \circ b = t \circ c$, and $s \circ c = t \circ a$. Since $s$ and $t$ are invertible, the first two equations imply $a = b = c$. Now the third equation can be written $s \circ b = t \circ b$, and since $b$ is invertible we conclude $s = t$.

The definition of fair $n$-category will be inductive: a fair $n$-category is a fair category in the coloured category of fair $(n-1)$-categories. The base for this induction, and the key to understanding $\Delta^{op}$-diagrams as categories, is the case where $S$ is the category of sets, with bijections as coloured arrows.

4.10 Fair $\mathbf{Set}$-categories. We consider $\mathbf{Set}$ a coloured category by taking the bijections as equiarrows. Given a fair category $\Delta^{op} \to \mathbf{Set}$, denoted $X = (O, A, U)$ as in 4.1, then $U \to O$ is a bijection, and therefore $U \to A$ is an injection, as seen in the diagram in 4.1. The restriction of $X$ to $\iota : \Delta^{op}_{\text{mono}} \hookrightarrow \Delta^{op}$ is a semi-category $O \xrightarrow{=} A$ with the property that every object has an identity arrow. This follows because the inverse of $U \to O$, followed by $U \to A$, provides the degeneracy map $O \xrightarrow{=} A$ (and the remaining degeneracy maps are provided by the Segal condition); the commutativity of the $\Delta^{op}$-diagram immediately implies that the degeneracy map identities are satisfied. This construction defines a functor

$$\theta : \text{Set-FairCat} \longrightarrow \text{Cat}.$$

4.11 Fair nerve of a category. In the other direction, starting from any category $\mathcal{C}$, there is a natural ‘fair nerve’ associated with it: let $\mathcal{C}$ be coloured by taking the identity arrows as equiarrows, then the fair nerve functor

$$\rho : \text{Cat} \to \text{Set-FairCat}$$

is defined by sending a category $\mathcal{C}$ to

$$\Delta^{op} \longrightarrow \text{Set}$$

$$K \longmapsto \text{Hom}_{\text{CCat}}(K, \mathcal{C}),$$
which is readily seen to be a fair category in $\mathbf{Set}$. If we denote it $\mathcal{C} = (O, A, U)$ like in 4.1, then $O$ is the set of all objects in $\mathcal{C}$, $A$ is the set of all arrows in $\mathcal{C}$, and $U$ is the set of all identity arrows in $\mathcal{C}$. In this interpretation, the image of the fair nerve functor consists of those fair $\mathbf{Set}$-categories $(O, A, U)$ for which $U \to A$ is an inclusion, not just an injection. This is a full reflective subcategory; the reflector is $\theta$.

Starting with a fair $\mathbf{Set}$-category $(O, A, U)$, applying $\theta$ gives the semi-category $O \subseteq A$ which happens to be a category, i.e. has identity arrows $u(U)$ (and the morphisms happen to preserve these), and then applying $\rho$ gives the fair category $(O, A, u(U))$, so the unit for the adjunction is the isomorphism $(O, A, U) \to (O, A, u(U))$ given as the identity functor on the $A$-semi-nerve, and the isomorphism $u : U \to u(U)$ on the $U$-semi-nerve.

Conversely starting with a category $\mathcal{C} = (O \subseteq A)$, applying $\rho$ we first get the fair category $(O, A, U = \text{Id}(\mathcal{C}))$, and then we throw the $U$-part away and observe that the resulting semi-category is just the category $\mathcal{C}$ again.

In summary:

**4.12 Proposition.** There is an adjoint equivalence of categories

$$\mathbf{Set}\text{-FairCat} \xrightarrow{\cong} \mathbf{Cat}.$$  

In the above back-and-forth construction we took a category viewpoint, focusing on the start of the $\Delta^{\text{op}}$-diagram $(O, A, U)$ and letting the Segal condition take care of the rest. But in fact the same result holds without requiring the Segal condition:

**4.13 Proposition.** There is an adjoint equivalence of categories

$$\mathbf{CCat}(\Delta^{\text{op}}, \mathbf{Set}) \simeq \mathbf{Cat}(\Delta^{\text{op}}, \mathbf{Set}) = \mathbf{sSet}.$$  

For the fine points of the appendix, it is important to observe that the functors of these two propositions preserve sums and finite products, as well as discrete objects.

We shall give $\mathbf{S}\text{-FairCat}$ the structure of a coloured category, and then the above equivalence is an equivalence of coloured categories. Just as for ordinary categories, a functor $F : A \to B$ of fair categories in $\mathbf{S}$ will be called an equimorphism if it is fully faithful and essentially surjective. These notions must now be defined.

**4.14 Fully faithfulness.** Given a fair category $X = (O, A, U)$, write $A = \coprod_{x,y \in O} A(x,y)$ and $U = \coprod_{x \in O} U(x)$, as in Remark 4.8. Now a fair functor $F : X \Rightarrow X'$ is said to be fully faithful if for each pair of objects $x, y$ the map $A(x,y) \to A'(Fx,Fy)$ is an equimorphism in $\mathbf{S}$, and for each object $x$ the map $U(x) \to U'(Fx)$ is an equimorphism. (Note that this last condition is automatic if the two-out-of-three property holds for equimorphisms in $\mathbf{S}$.)

The notion of essential surjectivity is subtler, and depends on a notion of truncation. We take our clue from usual categories: consider the truncation functor $\tau_0 : \mathbf{Cat} \to \mathbf{Set}$ which to a category associates the set of isomorphism classes of its objects. Note that this functor is different from the components functor $\pi_0$, but that the two functors
agree on the discrete categories. A morphism in \textbf{Cat} is essentially surjective if its truncation is a surjection of sets. The crucial features of \(\tau_0\) are that it comes with a natural transformation \(\tau_0 \Rightarrow \pi_0\), and that it preserves finite products and equimorphisms. We take these as axioms for the truncation functor:

4.15 **Truncation.** A truncation functor on a coloured category \(S\) with standard discrete objects is a colour-preserving functor \(\tau_0 : S \to \textbf{Set}\), equipped with a natural transformation \(\tau_0 \Rightarrow \pi_0\), required to preserve arbitrary sums and finite products. Since sums are preserved, \(\tau_0\) and \(\pi_0\) agree on discrete objects. It follows from the decomposition property (A.4) that \(\tau_0\) also preserves fibre products over discrete objects.

4.16 **Essential surjectivity.** For a fixed truncation functor \(\tau_0 : S \to \textbf{Set}\), there is induced a functor

\[
\textbf{S-FairCat} \xrightarrow{\tau_0*} \textbf{Set-FairCat} \simeq \textbf{Cat}.
\]

by sending a \(\Delta^{op} \to S\) to \(\Delta^{op} \to S \xrightarrow{\tau_0} \textbf{Set}\) and invoking Proposition 4.12. This works because \(\tau_0\) preserves the notions involved in the definition of fair category.

Now a morphism \(F : X \to X'\) in \(\textbf{S-FairCat}\) is said to be essentially surjective (relative to \(\tau_0\)) if \(\tau_0*F\) is essentially surjective in the ordinary sense (it is a functor between ordinary categories). It should be noted that this notion of essentially surjective might not be the correct one, unless combined with fully faithful. But that situation is all we care about:

4.17 **Equivalences of fair categories.** Let \(S\) be a coloured category with standard discrete objects and a truncation functor \(\tau_0 : S \to \textbf{Set}\). A morphism of fair \(S\)-categories is called an equimorphism if it is fully faithful and essentially surjective (with respect to \(\tau_0\)). Henceforth this is the notion referred to when talking about the coloured category \(\textbf{S-FairCat}\). Note that equimorphisms between strict \(S\)-categories (considered as fair \(S\)-categories via \(\pi^* : \textbf{S-Cat} \to \textbf{S-FairCat}\), cf. 4.6) are precisely the usual equivalences of \(S\)-categories.

4.18 **Lemma.** Equimorphisms in \(\textbf{S-FairCat}\) are stable under sums and finite products.

\textbf{Proof.} In fact this is true independently for ‘fully faithful’ and ‘essential surjective’. For ‘fully faithful’ it follows because the hom ‘sets’ of a sum (resp. a finite product) is the sum (resp. the product) of the hom ‘sets’, and equimorphisms in \(S\) are stable under sums (resp. finite products). For ‘essentially surjective’ it follows because \(\tau_0\) preserves sums and finite products. \(\square\)

4.19 **Proposition.** \(\textbf{S-FairCat}\) has standard discrete objects.

This is Proposition A.14. The discrete-objects adjunction is given by

\[
\textbf{S-FairCat} \xrightarrow{\delta_*} \textbf{Set-FairCat} \simeq \textbf{Cat} \xleftarrow{\pi_0*} \textbf{Set},
\]

where \(\pi_0*\) is defined by sending a fair \(S\)-category \(\Delta^{op} \to S\) to \(\Delta^{op} \to S \xrightarrow{\pi_0} \textbf{Set}\).
4.20 Lemma. The components functor $S\text{-FairCat} \to \text{Set}$ preserves equimorphisms as defined in 4.17.

Proof. Note first that since $\pi_0 : S \to \text{Set}$ preserves equimorphisms, and since the definitions of fully faithful in $S\text{-FairCat}$ and $\text{Set-FairCat}$ are both defined in terms of hom sets (in $S$ in $\text{Set}$ respectively), it follows that $\pi_{0*}$ preserve fully faithfulness.

Seeing that $\pi_{0*}$ preserves essential surjectivity relies on the natural transformation $u : \tau_0 \Rightarrow \pi_0$. For a fair $S$-category $X : \Delta^{\text{op}} \to S$, the two categories $\tau_0* X$ and $\pi_{0*} X$ have the same object set, since both $\tau_0$ and $\pi_0$ are the identity on discrete objects. To say that a morphism $F : X \to X'$ of fair $S$-categories is essentially surjective means that $\tau_{0*} F$ is an essentially surjective functor of categories. This in turn means that for every object $x'$ of $\tau_{0*} X'$ there exists an object $x$ of $\tau_{0*} X$ and an isomorphism $\phi \in \tau_{0*} X'(Fx,x')$. But then $u(\phi)$ is an isomorphism in $\pi_{0*} X'(Fx,x')$ witnessing that $\pi_{0*} F$ is essentially surjective too, as required.

Combining the three previous results we find that:

4.21 Proposition. $S\text{-FairCat}$ has standard discrete objects with compatible colours.

4.22 Truncation for $S\text{-FairCat}$. Given a truncation functor $\tau_0 : S \to \text{Set}$, define a truncation functor $S\text{-FairCat} \to \text{Set}$ as the composite

$$ S\text{-FairCat} \xrightarrow{\tau_{0*}} \text{Set-FairCat} \simeq \text{Cat} \xrightarrow{\tau_{0*} \text{Cat}} \text{Set}. $$

It follows readily that this new truncation functor preserves sum and finite products (and it preserves equimorphisms by construction). The natural transformation to the components functor is the horizontal composition of $u_* : \tau_{0*} \Rightarrow \pi_{0*}$ and the natural transformation $\tau_{0*} \text{Cat} \Rightarrow \pi_{0*} \text{Cat}$.

Now $S\text{-FairCat}$ has been equipped with the same type of structure as $S$, and the induction works:

4.23 Fair $n$-categories. A fair 0-category is defined to be just a set. Assuming we have already defined the coloured category $(n-1)\text{FairCat}$ of all fair $(n-1)$-categories, we define a fair $n$-category to be a colour-preserving functor

$$ \Delta^{\text{op}} \to (n-1)\text{FairCat} $$

preserving discrete objects and fibre products over discrete objects (this is the strict Segal condition). Let $n\text{FairCat}$ be the coloured category whose objects are the fair $n$-categories, whose morphisms are the natural transformations, and the equimorphisms are as defined in 4.17.

Notice that a strict $n$-category can be regarded as a fair $n$-category in a canonical way. This follows by induction from 4.6 and the observation that the embedding $(n-1)\text{Cat} \to (n-1)\text{FairCat}$ preserves discrete objects, fibre products over discrete objects, and equimorphisms (cf. 4.17).
4.24 Fair Tamsamani $n$-categories. Repeating the definition of fair category (and the other definitions involved), but with weak Segal condition instead of the strict one, yields a notion of fair Tamsamani $n$-categories. An interesting example of a fair Tamsamani 2-category is given on page 35. Fair Tamsamani $n$-categories should be the context for comparing Tamsamani $n$-categories with fair $n$-categories, since both can be regarded as special cases of this notion. It is expected that given such a fair Tamsamani $n$-category then there exists an equivalent Tamsamani $n$-category (strict identities). In other words, weak identities can be strictified in the context of fair Tamsamani categories. Also there should exist an equivalent fair $n$-category, i.e., weak composition laws can be strictified in the context of fair Tamsamani categories. This last statement is one version of Simpson’s conjecture 7.6.

4.25 Fair monoids. A fair monoid in $S$ is just a fair $S$-category such that $O$ is a singleton set $\ast$. Let $S$-FairMon denote the full subcategory of $S$-FairCat consisting of fair monoids. There is a forgetful functor $S$-FairMon $\to S$ which sends $X = (\ast, A, U)$ to its underlying space $A$. Given two fair monoids $X = (\ast, A, U)$ and $X' = (\ast, A', U')$, it is easy to see that a morphism $F : X \to X'$ is an equimorphism in $S$-FairMon if and only if $A \to A'$ and $U \to U'$ are equimorphisms in $S$ — essential surjectivity is an empty condition. In view of the Segal condition and the assumption that equimorphisms in $S$ are stable under products, this in turn is equivalent to requiring all components of $F$ to be equimorphisms, so the notion of equivalence of monoids is level-wise.

To give a fair monoid in $S$ amounts to giving a pair of semi-monoids $(A, U)$ in $(S, \times)$ and a semi-monoid homomorphism $U \to A$, such that $U$ is contractible and such that the arrows $U \times A \to A \leftarrow A \times U$ are equiarrows in in $S$. We will refer to a fair monoid by the notation $(U \to C)$.

We will use the term fair monoidal category for a fair monoid in $Cat$.

4.26 Fair monoids in non-cartesian enriched contexts. The above characterisation of fair monoids, in terms of a pair of semi-monoids with one of them contractible and so on, might also be useful for the reason that it makes sense in monoidal coloured categories (or monoidal model categories) in which the monoidal structure is not the cartesian product. For example, define a fair monoid in the monoidal category of chain complexes $(Ch, \otimes, \mathbb{Z})$ to be a pair of semi-monoids $A$ and $U$, with $U$ contractible (in the usual sense of homotopies of chain complexes), together with a semi-monoid homomorphism $U \to A$, such that the arrows $U \otimes A \to A \leftarrow A \otimes U$ are weak equivalences in $Ch$. I have not investigated this definition further.

5 Fair 2-categories

In this section we work out the case of dimension 2, i.e. fair categories in $Cat$. In view of the canonical equivalence $Cat \simeq 1$FairCat we will abuse of notation and let 2FairCat denote the category of fair categories in $Cat$, not in 1FairCat. In the same vein, but still
more abusively, by *semi-2-category* we shall mean a semi-category enriched over *Cat*, i.e. a semi-2-category which happens to have identity 2-cells.

The main result is that a fair 2-category is essentially the same thing as a bicategory with strict composition law, just as one would expect, both notions being semi-2-categories (in the abusive sense) with some extra unit structure. The fair category viewpoint on such a bicategory $\mathcal{C}$ encodes *all* possible unit structures on $\mathcal{C}$ instead of favouring one of them.

### 5.1 Bicategories with strict composition law.

A bicategory with strict composition law is just like a strict 2-category, except that each object is not required to have a strict identity 1-cell, but only a specified weak one. In order to encourage the interpretation of bicategories as ‘many-object monoidal categories’, we denote the composition law by $\otimes$ and compose from the left to the right. Objects will be omitted from the notation whenever possible, and arrows are denoted by uppercase letters. A weak identity arrow for an object $o$ is a triple $(I_o, \lambda, \rho)$ consisting of an arrow $I_o : o \to o$, a left constraint $\lambda$ and a right constraint $\rho$: these are invertible 2-cells $\lambda_Y : I_o \otimes Y \simeq Y$ and $\rho_X : X \otimes I_o \simeq X$, natural in arrows $Y : o \to \cdot$ and $X : \cdot \to o$ respectively. The left and right constraints are subject to the condition

$$
\begin{align*}
\begin{array}{c}
\lambda_Y \\
\downarrow \\
Y
\end{array}
\end{align*}
\xymatrix{
\cdot & \ar[rr]^X & & o & \ar[ll]_{\lambda_Y} & \cdot \\
& & \ar[ll] \\
Y
\end{array}
\quad =
\begin{array}{c}
\rho_X \\
\downarrow \\
X
\end{array}
\xymatrix{
\cdot & \ar[rr]^X \otimes I_o & & o & \ar[ll]_{\rho_X} & \cdot \\
& & \ar[ll] \\
X
\end{array}
$$

(4)

A homomorphism of bicategories with strict composition law is a bifunctor $F : \mathcal{C} \to \mathcal{C}'$ which preserves the composition strictly. The only non-strict part of $F$ is the comparison of identity arrows: for each object $o$ there is specified an invertible 2-cell $\phi_o : I_{F(o)} \simeq F(I_o)$, compatible with the respective left and right constraints like this:

$$
\begin{align*}
\xymatrix{
F(I_o) \otimes F(X) & F(X) \ar[ll]_{\lambda_{FX}} \\
& \ar[ll]_{\phi_o \otimes F_X} \\
F(I_o) \otimes F(X) = F(I_o \otimes X)
\end{align*}
\quad \quad \quad
\begin{align*}
\xymatrix{
F(X) & \ar[ll]_{\rho_{FX}} \\
& \ar[ll]_{F \otimes \phi_o} \\
F(X) \otimes I_o = F(X \otimes I_o)
\end{align*}
$$

Let $\mathbb{B}$ denote the category of bicategories with strict composition law and bifunctors preserving the composition strictly.

### 5.2 Proposition.

There is an equivalence of categories

$$2\text{FairCat} \simeq \mathbb{B}.$$ 

The equivalence is described below. The functor $\mathbb{B} \to 2\text{FairCat}$ is canonical; the pseudo-inverse depends on a choice. The construction relies on a couple of basic observations about identity arrows in bicategories which are not widely known. Further details can be found in [19], for the one-object case.
5.3 Identity-arrow structures. Just as identity arrows in a category are uniquely determined by the unit axioms, in a bicategory the identity arrows are unique up to unique isomorphism. To be precise, define an identity arrow in a semi-2-category to be a triple \((I_o, \lambda, \rho)\) where \(I_o : o \rightarrow o\) is an endo-arrow of an object \(o\), and \(\lambda\) and \(\rho\) are left and right constraints satisfying the axioms above. A morphism of identity arrows is given by a 2-cell \(I \rightarrow J\) compatible with the left and right constraints. Clearly the category of identity arrows and their morphisms is the disjoint union

\[
\text{Id}_\mathcal{C} = \bigsqcup_{o \in \text{Ob}(\mathcal{C})} \text{Id}_\mathcal{C}(o)
\]

where \(\text{Id}_\mathcal{C}(o)\) is the category of identity arrows of the object \(o\).

The category of identity arrows has a natural composition law lifting the composition law on \(\mathcal{C}\) (and in particular is object-wise): the composition of \((I, \lambda, \rho)\) with \((I', \lambda', \rho')\) is the composite \(I \otimes I'\) equipped with left and right constraints

\[
\begin{align*}
I \otimes I' \otimes X & \xrightarrow{I \otimes \lambda'} I \otimes X \xrightarrow{\lambda} X \\
X \otimes I \otimes I' & \xrightarrow{\rho \otimes I'} X \otimes I' \xrightarrow{\rho'} X
\end{align*}
\]

It is straightforward to check that these constraints satisfy the identity arrow axiom (4), and it also follows readily that this composition law is strictly associative if the original composition law is so.

Altogether, we have a semi-category \(\text{Id}_\mathcal{C}\) enriched in \(\text{Cat}\), (which is the disjoint union of the semi-monoidal categories \(\text{Id}_\mathcal{C}(o), o \in \text{Ob}(\mathcal{C})\)).

5.4 Lemma. The category \(\text{Id}_\mathcal{C}\) of identity arrows in a bicategory \(\mathcal{C}\) is equivalent to the discrete category \(\text{Ob}(\mathcal{C})\), i.e. \(\text{Id}_\mathcal{C}(o)\) is contractible for each object \(o\).

Proof. Given units \((I, \lambda, \rho)\) and \((I', \lambda', \rho')\) of an object \(o\), one checks that the isomorphism

\[
\begin{align*}
I' & \xrightarrow{\rho' I} I' I \xrightarrow{\lambda I} I
\end{align*}
\]

is compatible with the left and right constraint of \(I\) and \(I'\), hence constitutes a morphism of identity arrows, hence the category of identity arrows of \(o\) is iso-connected. Compatibility with the left and right constraints also implies that there can be at most one connecting arrow. \(\square\)

5.5 From bicategories to fair 2-categories. With the above observations it is easy to define the functor \(\mathbb{B} \rightarrow 2\text{FairCat}\). In analogy with the fair nerve described in 4.11, given a bicategory \(\mathcal{C}\) with strict composition we define a \(\Delta^{op}\)-diagram \(X = (O, A, U)\) in \(\text{Cat}\) like this:

\[
\begin{align*}
\Delta^{op} & \rightarrow \text{Cat} \\
\bullet & \mapsto O := \text{Ob} \mathcal{C} \\
\ast & \mapsto A := \bigsqcup_{x, y \in \text{Ob} \mathcal{C}} \text{Hom}(x, y) \\
! & \mapsto U := \bigsqcup_{x \in \text{Ob} \mathcal{C}} \text{Id}_\mathcal{C}(x)
\end{align*}
\]
The map \( U \to A \) is the forgetful functor sending an identity triple \((I, \lambda, \rho)\) to its supporting arrow. The rest of the \( \Delta^{op}\)-diagram is defined by invoking the strict Segal condition and observing that \( U \to A \) is a \( \otimes \)-functor.

By definition of the morphisms in \( \text{Id}_\mathcal{C} \), the collection of all left constraints at \( o \) assembles into an invertible natural transformation

\[
\text{Id}_\mathcal{C}(o) \times \text{Hom}_\mathcal{C}(o, \cdot) \xrightarrow{\otimes} \text{Hom}_\mathcal{C}(o, \cdot).
\]

The component on a unit \((I, \lambda, \rho)\) and an arrow \( X \) is nothing but \( \lambda_X : I \otimes X \to X \). Since \( \text{Id}_\mathcal{C}(o) \) is contractible, the projection map is an equimorphism, and hence the composition map is too. Summing over all the objects we get an invertible 2-cell

\[
U \times_O A \xrightarrow{\otimes} A
\]

hence the composition map is an equimorphism as required. Similarly for the right constraints.

The same argument works for any vertical arrow in \( \Delta \): it is the dot-sum (\( \downarrow \)) of identity arrows and the case just treated.

One can check that this construction is functorial (cf. [19]): the unit part \( \phi_o \) of a bifunctor \( F : \mathcal{C} \to \mathcal{D} \) amounts precisely to a lift of \( F \) to \( \text{Id}_\mathcal{C} \to \text{Id}_\mathcal{D} \), which in turn is equivalent to extending the natural transformation between the corresponding \( \Delta^{op}_{\text{mono}} \)-diagrams to a natural transformation of \( \Delta^{op}_\mathcal{C} \)-diagrams. This finishes the construction of the functor \( \mathbb{B} \to 2\text{FairCat} \).

### 5.6 From fair 2-categories to bicategories

Given a fair category \((O, A, U)\) in \( \text{Cat} \), the \( A \)-part already constitutes a semi-2-category \( A \), whose composition we denote by \( \otimes \). It remains to use the \( U \)-part of the diagram to provide weak identity arrows for the semi-2-category \( A \). Note that \( U \) also constitutes a semi-category in \( \text{Cat} \); again we denote the composition law by \( \otimes \); then the functor \( U \to A \) is a \( \otimes \)-functor. The equivalence \( U \simeq O \) has a pseudo-section (since we are just talking plain categories), i.e., for each element \( o \in O \) we can pick an object \( I_o \) in \( U \), a chosen weak unit. The functor we are constructing depends on this choice, but different choices will yield canonically isomorphic results.

Since \( U \) is contractible,

(S1) we have an isomorphism \( \alpha : I_o \otimes I_o \simeq I_o \),

and since the composition functors \( U \times_O A \xrightarrow{\otimes} A \xleftarrow{\otimes} A \times_O U \) are equimorphisms of categories, we see that

(S2) composition with \( I_o \) defines equimorphisms \( A(o, \cdot) \to A(o, \cdot) \) and \( A(\cdot, o) \to A(\cdot, o) \).
This means that for each arrow \( X : o \rightarrow \cdot \) there is a bijection
\[
2\text{Cell}_C(I \otimes X, X) \simeq 2\text{Cell}_C(I \otimes I \otimes X, I \otimes X).
\]
On the right-hand side we have the canonical 2-cell \( \alpha \otimes X \), so take the inverse image and call it \( \lambda_X \), the required left constraint. The right constraint is constructed analogously. It is easy to see that these constraints are natural, and one can also check that they satisfy the coherence condition (4). Both these claims follow from the fact that \( \lambda \) and \( \rho \) are defined in terms of the isomorphism \( \alpha \) in \( U \), which is automatically coherent since it lives in a contractible category. (See [19] for further details.)

5.7 Remark. Conditions (S1) and (S2) provide in fact a useful definition of unit. This viewpoint goes back to Saavedra [28] in the case of monoidal categories; it is exploited further in [19]. The relevance of this viewpoint in higher dimensions was first suggested by Simpson [31]. The basic 3-dimensional theory is worked out in Joyal-Kock [14].

5.8 Non-strict bifunctors. Even for strict 2-categories it is sometimes necessary to consider weaker notions of 2-functors (strong or lax), which do not respect composition strictly, but only up to specified comparison arrows (isomorphisms or arbitrary morphisms), cf. Bénabou [2]. Such 2-functors can also be captured in the setting of fair \textbf{Cat}-categories: they correspond precisely to weaker natural transformations (strong or lax) between \( \Delta^p \)-diagrams \( u : F \Rightarrow G \), satisfying a strict ‘Segal condition’: \( u_{m+n} = u_m \times_{u_0} u_n \). (Note that any weak natural transformation satisfies this condition up to isomorphism, and is equivalent to one satisfying it strictly.)

6 Fair 3-categories

For more details on the results and constructions in this section, see Joyal-Kock [14].

Gordon, Power, and Street [8] introduced the notion of tricategories, designed to be the weakest possible definition of 3-dimensional category. In analogy with the 2-dimensional case, fair 3-categories should correspond to tricategories with strict composition laws. Carrying out the comparison in the style of the previous section would seem to be somewhat involved, though. What we undertake here is a comparison in the semi-strict situation of fair categories in \textbf{2Cat} (interpreted as a semi-strict sort of fair 3-categories in the canonical way, cf. 4.23), which we compare to locally strict tricategories with strict composition laws. Furthermore for the sake of clarity, we restrict to the one-object version.

6.1 Strictly monoidal strict 2-categories with weak units. Starting with the definition in [8], we specialise to the locally strict case (i.e. every hom bicategory is a strict 2-category). Then we restrict to the one-object case, and finally we require the composition law to be strict. The resulting notion is a strict 2-category \( C \) with a strict tensor product with a weak unit. The notion of weak unit in this situation is a specified triple \( (I, \lambda, \rho) \) where \( I \) is an object in \( C \), \( \lambda \) is a natural family of equimorphisms (i.e. admitting a pseudo-inverse) \( \lambda_X : I \otimes X \rightarrow X \), and \( \rho \) is a natural family of equimorphisms \( \rho_X : X \otimes I \rightarrow X \). The naturality is only up to isomorphism, so specifying the lambdas
and rhos involves specifying certain invertible 2-cells. Finally there is specified a natural family of invertible 2-cells (a modification)

\[ K : \rho_X \otimes Y \Rightarrow X \otimes \lambda_Y \]

subject to two coherence axioms called left and right normalised 4-cocycle conditions ([8], p.11 and p.12), which we do not reproduce here.

6.2 Fair monoids in 2Cat. Let \((U \to C)\) denote a fair monoid in 2Cat, cf. 4.25. This means that

- \(U\) and \(C\) are strict semi-monoidal 2-categories,
- there is a strict semi-monoidal 2-functor \(U \to C\),
- \(U\) is contractible, and the multiplication maps

\[ U \times C \to C \leftarrow C \times U \]

are equi-2-functors.

6.3 Proposition. Fair monoids in 2Cat correspond to strict 2-categories with strict tensor product and weak unit.

The proof is comprised by the following two subsections.

6.4 From fair monoid in 2Cat to monoidal 2-category. Given a fair monoid \((U \to C)\), since \(U\) is contractible, for each object \(I\), there exists an arrow \(\alpha : I \otimes I \sim \to I\) in \(U\); this arrow is an equiarrow, and any two such are uniquely isomorphic. We fix \(I\) and \(\alpha\), and we use the same symbols for their images in \(C\). It also follows from contractibility of \(U\) that there is an associator modification \(A : I \otimes \alpha \sim \to \alpha \otimes I\) which satisfies the pentagon equation.

As in the 2-dimensional case, the fact that the multiplication map \(U \times C \to C\) is an equi-2-functor means that the same is true for the 2-functor ‘tensoring with \(I\) from the left’ (and similarly with tensoring from the right). Therefore, for each object \(X\) in \(C\) there is an equimorphism of categories

\[ \text{Hom}(I \otimes X, X) \sim \to \text{Hom}(I \otimes I \otimes X, I \otimes X). \]

In the second category we have the canonical object \(\alpha \otimes X\). Hence there exists a pseudo pre-image \(\lambda_X\), together with an isomorphism \(L : I \otimes \lambda_X \Rightarrow \alpha \otimes X\). For chosen \(\lambda_X\) and \(L\), there is a unique way to assemble the lambdas into a natural transformation (this involves specifying some 2-cells), in such a way that \(L\) becomes natural in \(X\). The pair \((\lambda, L)\) is not unique, but any two such are uniquely isomorphic.

Similarly there is a natural transformation \(\rho\) with components \(\rho_X : X \otimes I \to X\) equipped with a natural modification \(R : \rho_X \otimes I \Rightarrow X \otimes \alpha\) (and this data is unique up to unique isomorphism).

The lambdas and rhos are the required left and right constraints. Finally, using \(L\), \(R\), and \(A\) one can construct a modification \(K : \rho_X \otimes Y \Rightarrow X \otimes \lambda_Y\) which satisfies the normalised 4-cocycle conditions of [8] — this is a consequence of the pentagon equation for \(A\), and hence ultimately a consequence of contractibility of \(U\). See [14] for details.
There were many choices involved in the construction: first the choice of \(I\) and \(\alpha\), then the choices of \(\lambda\) and \(\rho\), together with the auxiliary 2-cells \(L\) and \(R\). But all these choices lead to equivalent monoidal 2-categories.

**6.5 From monoidal 2-category to fair monoid in \(2\text{Cat}\)**. Starting from a monoidal 2-category \(\mathcal{C}\) like specified in 6.1, ideally one would construct the 2-category of all unit structures on \(\mathcal{C}\). The objects are quadruples \((I, \lambda, \rho, K)\) like in 6.1, and arrows and 2-cells are defined to come equipped with compatibility data with respect to these structures. This category is contractible as required, but unfortunately its tensor product is not strict as required in order to play the role of \(U\) in a fair monoid \((U \rightarrow C)\).

An alternative construction is used. Define \(\alpha : I \otimes I \cong I\) by taking \(\alpha : = \lambda I\). (We could equally well have used \(\rho I\): there is a canonical modification \(\lambda I \rightleftharpoons \rho I\) constructed from \(K\) and the naturality data specified with \(\lambda\) and \(\rho\).) Now the 2-functors \(\mathcal{C} \rightarrow \mathcal{C}\) defined by multiplying with \(I\) from the left or the right are equi-2-functors. Hence the pair \((I, \alpha)\) satisfies the 2-dimensional version of (S1) and (S2) on page 24. It is shown in [14] that \(\alpha : I \otimes I \rightarrow I\) is associative up to a canonical associator modification \(A : I \otimes \alpha \cong \alpha \otimes I\), which satisfies the pentagon equation. Hence \((I, \alpha)\) is a weak semi-monoid. The set of such pairs \((I, \alpha)\) is the object set of a contractible 2-category, the arrows being weak-semi-monoid equimorphisms, and the 2-cells being invertible 2-cells in \(\mathcal{C}\). Again, however, the tensor structure on this 2-category is not strict. But in this case there is a strict alternative: it is the free strictly semi-monoidal 2-category whose objects are the positive powers of \(I\), whose generating arrows are \(\alpha\) and a fixed right adjoint \(\beta\), and whose generating 2-cells are \(A\) together with unit and counit for \(\alpha \dashv \beta\). This 2-category is contractible and does the job as \(U\). The canonical 2-functor consisting in interpreting this category in \(\mathcal{C}\) respects the tensor product. Now we have got semi-monoidal 2-categories \(U\) and \(C\) and a semi-monoidal 2-functor between them, so the rest of the \(\Delta^\text{op}\)-diagram is determined by the strict Segal condition.

Just like in the 2-dimensional case, the left and right constraints provide natural 2-cells between multiplication-with-a-unit and projection, showing that \(U \times A \rightarrow A \leftarrow A \times U\) are equivalences. Hence the \(\Delta^\text{op}\)-diagram is a fair monoid.

**7 \(n\)-groupoids, homotopy \(n\)-types, and Simpson’s conjecture**

**7.1 Motivation.** Important motivation for higher category theory is the desire of giving a completely algebraic account of homotopy theory (cf. Grothendieck [10]). In particular, every topological space should have associated a higher fundamental groupoid. This should be an \(\infty\)-groupoid, but truncated homotopy \(n\)-types should be described by \(n\)-groupoids. Ideally this description should be a pair of (weakly) adjoint functors

\[
R : \infty\text{Grpd} \rightleftarrows \text{Top} : \Pi
\]

where \(\Pi\) is the fundamental groupoid mentioned above, and \(R\) is a geometric realisation functor, and for each \(n \geq 0\) this pair of functors should induce an equivalence between the homotopy category of \(n\)-groupoids and that of \(n\)-truncated topological spaces.
7.2 Completely strict \(n\)-groupoids. The strictest possible definition of \(n\)-groupoid is as a strict \(n\)-category such that every \(k\)-cell is strictly invertible, for \(k \leq n\). It was shown quite early (Brown-Higgins [3]) that such strict \(n\)-groupoids cannot realise all homotopy \(n\)-types. The problem occurs already with the 3-type of \(S^2\): this space has a non-trivial Whitehead operation \(\pi_2 \otimes \pi_2 \rightarrow \pi_3\), but a version of the Eckmann-Hilton argument shows that every strict \(n\)-groupoid gives trivial Whitehead brackets. (See Simpson [31] for a detailed account of these arguments.)

A slightly weaker notion of strict \(n\)-groupoid is obtained by requiring the \(k\)-cells to be invertible only up to a \((k + 1)\)-cell, which in turn should be weakly invertible in the same sense, and so on, up to the \(n\)-cells which should be strictly invertible. The following formulation of this idea can be interpreted in either strict \(n\)-categories, Tamsamani \(n\)-categories, or fair \(n\)-categories:

7.3 \(n\)-groupoids. An \(n\)-category \(X\) is called an \(n\)-groupoid if the category \(\tau_0 X\) is a groupoid, and if for each pair of objects \(x, y\), the \((n - 1)\)-category \(X(x, y)\) is an \((n - 1)\)-groupoid.

7.4 Kapranov-Voevodsky \(n\)-groupoids. If in the above definition, ‘\(n\)-category’ is taken to mean ‘strict \(n\)-category’ then the \(n\)-groupoid notion is that of Kapranov-Voevodsky [16]. Since the composition laws as well as the identity cells are strict, the Eckmann-Hilton argument still applies, showing that the standard geometric realisation of such an \(n\)-groupoid will have trivial Whitehead brackets. Kapranov and Voevodsky constructed a new realisation functor with certain properties (preservation of homotopy groups), and claimed that with this, their \(n\)-groupoids could realise all homotopy \(n\)-types. Simpson [31] showed instead that any realisation functor satisfying those properties will yield only homotopy types with trivial Whitehead brackets, and concluded that there must be an error in either [16] or in [31].

7.5 Tamsamani \(n\)-groupoids. If in Definition 7.3, ‘\(n\)-category’ is taken in the sense of Tamsamani, the situation is different: Tamsamani [34] has constructed the fundamental \(n\)-groupoid of an \((n\)-truncated\) topological space, and a left adjoint realisation functor, and he has shown that this adjunction induces an equivalence of homotopy categories between \(n\)-groupoids in his sense and \(n\)-truncated topological spaces. (The existence of this construction is one of the main advantages of his definition over many of the other existing definitions of weak higher categories.)

7.6 Simpsons conjecture(s). Although Tamsamani’s theorem shows that weakening the notion of composition suffices to capture all homotopy \(n\)-types, it is a puzzling idea that the real problem with strictness are the strict identity arrows: the Whitehead operations are trivial because of the Eckmann-Hilton argument, which in turn relies crucially on strict identities. A detailed analysis of these issues led Simpson [31] to formulate the following conjecture(s):

*There exists a notion of strict \(n\)-groupoid with weak identity arrows and a notion of geometric realisation such that every homotopy \(n\)-type appears in this way.*

*In the other direction there should be a fundamental \(n\)-groupoid (with strict composition and weak identities) associated to every topological space, and these two functors should induce*
an equivalence between the homotopy categories of \(n\)-groupoids with weak identities on one side and \(n\)-truncated topological spaces on the other side.

More generally, the homotopy theory of strict \(n\)-categories with weak identities should be equivalent to the homotopy theory of Tamsamani \(n\)-categories.

An ad hoc definition of weak identities was sketched in the preprint (based on (S1) and (S2) from page 24), but it was acknowledged that it might not be the correct definition to turn the conjecture true, and in fact, the details of this definition were never worked out. Simpson’s conjecture was one starting point for the present work, and the notion of fair category emerged gradually from an attempt to understand his ideas.

The conjecture in its strong form has startling consequences, defying all trends in higher category theory: every weak higher category should be equivalent to one with strict composition!

As an annex to Simpson’s conjecture, and at the same time a concretisation of the objects of its assertion, I want to propose that the notion of fair \(n\)-groupoid is appropriate for fulfilling the conjecture. For emphasis, here is the definition:

\[7.7 \text{ Fair groupoids. A fair } n\text{-category } X : \Delta^{op} \to (n-1)\text{FairCat is called a fair } n\text{-groupoid if the category } \tau_0 X \text{ is a groupoid and if for each pair of objects } x, y, \text{ the fair } (n-1)\text{-category } X(x, y) \text{ is a fair } (n-1)\text{-groupoid.}
\]

(Note that this definition does not explicitly refer to the weak identity arrows, but that it relies on 4.10 as base for the induction.)

Simpson’s conjecture in dimension 3

The crucial test for the conjecture is dimension 3. If \(*\) is an object in a (strict or weak) 3-groupoid \(G\), and \(I\) is a (weak) identity arrow of \(*\), then \(\text{End}(I)\) is a braided monoidal category (in fact a braided categorical group). The braiding corresponds to the Whitehead operation \(\pi_2 \otimes \pi_2 \to \pi_3\) (relative to the base point corresponding to \(*\)). This Whitehead bracket is zero if and only if the braiding collapses to a symmetry. So the failure of strict 3-groupoids to realise all 3-types is equivalent to the fact that if \(G\) is strict then the braiding of \(\text{End}(I)\) is a symmetry (in fact, is equivalent to a commutativity). There is no essential generality lost in treating only the one-object case, so we consider 1-object 3-groupoids, which is the same thing as monoidal 2-groupoids such that every object has a weak tensor inverse; let \(I\) denote the (weak) unit for the tensor product. Simpson’s conjecture in this case says that such monoidal 2-groupoids with strict composition laws and strict tensor can realise all pointed 3-types. Let us further restrict attention to simply connected 3-types: this corresponds to having a contractible space of objects, so all objects are equivalent to \(I\). The following result is a form of Simpson’s conjecture in this case:

\[7.8 \text{ Theorem. (Joyal-Kock [15].) Every braided monoidal category arises as } \text{End}(I), \text{ where } I \text{ is a weak unit in an otherwise completely strict monoidal 2-category (as treated in Proposition 6.3).}
\]

1-connected homotopy 3-types correspond to braided categorical groups. Under the correspondence of the theorem, these correspond to strict 2-groupoids with invert-
ible tensor product and weak units (which in turn are one-object fair 3-groupoids in the sense of 7.7). Hence we get the following version of Simpson’s conjecture in dimension 3:

**7.9 Corollary.** (Cf. [15].) Strict 2-groupoids with invertible tensor product and weak units can model all 1-connected homotopy 3-types.

The idea of the proof of the theorem is this: the braiding from $g \circ f$ to $f \circ g$ relies on the arrow $\alpha : I \otimes I \to I$, cf. the proof of 6.3, and a chosen quasi-inverse. In graphical notation (reading the string diagrams from the bottom to the top), we picture $\alpha$ as

Then the braiding is this:

The picture suggests that going left past each other is not the same as going right past each other, and hence the braiding should not be a symmetry. The proof that every braided monoidal category arises as such an $\text{End}(I)$ consists in taking these diagrams seriously, relating them (in an up-to-homotopy sense) to the geometry of labelled configuration spaces.

Theorem 7.8 provides a direct check of the first non-trivial case of Simpson’s conjecture, and by expressing the arguments in terms of familiar mathematical objects like braided monoidal categories it also provides good intuitive insight to the problem. However it comes short in providing the notions and tools necessary for generalisation to higher dimension: at present, the appropriate notions of geometric realisation of fair categories have not been worked out, and there is no general construction of fundamental fair groupoid to provide a functor in the other direction (but see however the first example in the next section).

**8 A couple of examples**

The first example, somewhat detailed, concerns Moore path spaces. It is a fair category in $\text{Top}$. The second and third examples are more succinct: the second example is about cofibrant objects in a monoidal model category; the final example is a fair Tamsamani 2-category of oriented cobordisms.
Moore path spaces

This first example is a fair category in $S = \text{Top}$, where for simplicity we take the equimorphisms to be the homotopy equivalences.

8.1 Moore paths. Let $X$ be a topological space. A Moore path in $X$ is a continuous map $[0, r] \to X$, where $r > 0$. When composing paths by concatenation of intervals (see below), the length of the domain interval increases; since there is no reparametrisation involved, the composition is strictly associative. In this way, taking the set of points of $X$ as object set $O$, and taking as arrows the space $A$ of all Moore paths (with the compact-open topology), we have got a topological semi-category $O \subseteq A$.

Since the length of the domain interval increases strictly in every composition, there can be no strict unit paths. One could of course just allow the zero-length interval as domain for a path, then these paths would be strict units. But note that for each point $x \in X$ the space of Moore loops at $x$ of length $r > 0$ is isomorphic to the classical loop space $\Omega_x$, while for $r = 0$ we get just a point. Furthermore, since we want to think of the path space as a groupoid, the null-homotopic loops should actually also be considered unit paths — weak unit paths, that is.

8.2 A topological fair category of Moore paths. We would like to consider all null-homotopic loops at a point $x \in X$ as weak identity arrows of $x$. However, the space of null-homotopic loops at $x$ is not contractible, as required by the colour axiom 4.4. What is missing is of course to specify in precisely which sense such a null-homotopic loop serves as weak unit: as $U(x)$ we need to take the space of all null-homotopic loops based at $x$, together with a null-homotopy. Put $U := \bigsqcup_{x \in O} U(x)$.

8.3 Proposition. The triple $(O, A, U)$ of points, Moore paths, and null-homotopic-Moore-loops-with-given-null-homotopy, as defined above constitutes a fair category in $\text{Top}$.

Proof. We first give an explicit description of the semi-categories $A$ and $U$ and the semi-functor $U \to A$; then we show that $U(x)$ is contractible for each $x$, and finally that the maps $U \times_O A \cong A \cong A \times_O U$ are equivalences. These verifications are pretty straightforward, but it is instructive to see in detail how the homotopies built into the individual weak units assemble into the global equivalences required by the axioms of fair category.

8.4 Specific description of the semi-categories $A$ and $U$. Given two Moore paths $\gamma_1 : [0, r_1] \to X$ and $\gamma_2 : [0, r_2] \to X$ with $\gamma_1(r_1) = \gamma_2(0)$, their composite is the Moore path $\gamma_1 \otimes \gamma_2$ defined by

$$\gamma_1 \otimes \gamma_2 : [0, r_1 + r_2] \to X$$

$$t \mapsto \begin{cases} \gamma_1(t) & \text{for } t \leq r_1 \\ \gamma_2(t - r_1) & \text{for } t \geq r_1. \end{cases}$$

It is clear that this composition law is strictly associative.

Anticipating the definition of $U$, define the interval $[0, r]$ to be the segment of the $(y=1)$-line in $\mathbb{R}^2$ starting at $(\frac{r}{2})$ and ending at $(\frac{r}{2})$: 
A weak unit loop is by definition a null-homotopic loop together with a specified null-homotopy. To be explicit, it is a continuous map $\omega : T_r \to X$, where $T_r$ is the triangle

Such that the fat sides of the triangle are mapped to the same point $x$; then the top side describes the null-homotopic loop.

The map $U \to A$ simply sends $\omega$ to its restriction to the top side of the triangle.

The composition of two such triangle maps, $\omega_1 : T_{r_1} \to X$ followed by $\omega_2 : T_{r_2} \to X$, is defined by skewing the second triangle by the linear transformation

$$v \mapsto \begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} v$$

and placing it next to the first triangle like this:

In other words, the composite is the map

$$\omega_1 \otimes \omega_2 : T_{r_1+r_2} \to X$$

$$v \mapsto \begin{cases} 
\omega_1(v) & \text{for } v \in T_{r_1} \\
\omega_2((1-r_1) v) & \text{for } v \notin T_{r_1}.
\end{cases}$$

It is clear that this composition law is strictly associative — if you wish it is because the semigroup of matrices $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}_{r > 0}$ is isomorphic to $\mathbb{R}_+$. For the same reason, $U \to A$ is clearly compatible with composition, i.e. is a semi-functor.

8.5 For each $x \in X$, $U(x)$ is contractible. Consider the two continuous maps

$$U(x) \xrightarrow{\ell} \mathbb{R}_+$$

where $\ell : U(x) \to \mathbb{R}_+$ is the length function (length of the top side of the domain triangle), and $\Gamma x^\gamma$ is the section that sends $r \in \mathbb{R}_+$ to the constant map on $T_r$ with value $x$. We claim that $\Gamma x^\gamma \circ \ell$ is homotopic to the identity map on $U(x)$, and hence $U(x)$ is contractible. Indeed, for each map $\omega : T_r \to X$, and for each $t \in [0,1]$, let $\omega_t$ denote the composite given by precomposing $\omega$ with scaling by a factor $t$.
\[
T_r \longrightarrow T_r \longrightarrow X
\]

Now the required homotopy from \( ΓX \circ \ell \) to the identity of \( U(x) \) is given by
\[
[0,1] \times U(x) \longrightarrow U(x)
(t, ω) \longrightarrow ω_t.
\]

**8.6 The maps \( U ×_O A \longrightarrow A \) are homotopy equivalences.** The projection map is an equivalence because \( U(x) \) is contractible for each \( x ∈ O \). We shall construct a homotopy between the projection map and the composition map, and hence the composition map is also an equivalence.

Each Moore path \( γ : [0,r_2] → X \) has a unique extension \( \overline{γ} \) to the rectangle
\[
\begin{array}{c}
1 \\
r_2
\end{array}
\]
constant in the vertical direction. Given a Moore loop \( ω : T_{r_1} → X \) based at \( γ(0) \), define the composite of \( ω \) with \( \overline{γ} \) to be the map \( ω \boxtimes \overline{γ} \) with domain
\[
\begin{array}{c}
1 \\
r_1 \quad r_2 \quad r_1+r_2
\end{array}
\]
given by
\[
v \longmapsto \begin{cases}
ω(v) & \text{for } v ∈ T_{r_1} \\
\overline{γ}(\frac{1}{1-r_1}v) & \text{for } v \notin T_{r_1}
\end{cases}
\]
This defines a continuous map
\[
U ×_O A → B
\]
where \( B \) is the space of maps to \( X \) from such trapezia. Now the composition map \( U ×_O A → A \) is given by postcomposing this map with ‘restriction to the \( (y=1) \)-line’
\[
U ×_O A \longrightarrow B \longrightarrow A
(ω, γ) \longrightarrow ω \boxtimes \overline{γ} \longrightarrow (ω \boxtimes \overline{γ})|_{y=1} = ω \otimes γ.
\]
The required homotopy from projection \( (t = 0) \) to composition \( (t = 1) \) is now given by
\[
[0,1] × U ×_O A \longrightarrow B \longrightarrow A
(t, ω, γ) \longrightarrow ω \boxtimes \overline{γ} \longrightarrow (ω \boxtimes \overline{γ})|_{y=t}
\]
consisting in sliding up the support line \( y = t \):
To be pedantic with the domain, we should say that the path \((\omega \boxtimes \gamma)|_{y=t}\) is defined to be the map

\[
[0, tr_1 + r_2] \to X \\
\quad s \mapsto (\omega \boxtimes \gamma)(\frac{s}{t})
\]

This finishes the proof of the proposition. \(\Box\)

### Monoidal model categories

The next example is of a **fair monoidal coloured category**, i.e., a fair monoid in \(\mathsf{CCat}\) (cf. 2.1). To make sense of this it must be specified in which sense the category \(\mathsf{CCat}\) is a coloured category. There are several possible colour structures — the crucial desired property is that the Dwyer-Kan simplicial localisation functor \(L: \mathsf{CCat} \to \mathsf{sCat}\) [4] should be colour-preserving. For simplicity we take this as the definition: a (colour-preserving) functor \(F: (C, W) \to (C', W')\) between coloured categories is called an **equifunctor** if \(LF\) is an equivalence of simplicial categories. This means that \(\pi_0 LF : \pi_0 LC \to \pi_0 LC'\) is an equivalence of categories and for each pair of objects \(x, y \in C\) the map \(LC(x, y) \to LC'(Fx, Fy)\) is a weak equivalence of simplicial sets.

#### 8.7 Cofibrant objects of a monoidal model category.

Let \((M, \otimes, I)\) be a monoidal model category in the sense of Hovey [13]. This means that \(M\) is at the same time a model category and a monoidal category and that the following compatibility conditions are met:

(i) \(\otimes\) preserves weak equivalences between cofibrant objects;
(ii) for every weak equivalence \(Z \simeq I\) with \(Z\) cofibrant, and for every cofibrant object \(X\), the composite \(Z \otimes X \to I \otimes X \to X\) is a weak equivalence.

Now if \(I\) happens to be cofibrant then the subcategory of cofibrant objects \((M^c, \otimes, I)\) is a genuine monoidal coloured category with unit (i.e., a monoid in \(\mathsf{CCat}\)), and induces a monoidal structure on the homotopy category \(\mathsf{Ho}(M)\). If \(I\) is not cofibrant then \((M^c, \otimes)\) will not have a unit object. Instead:

#### 8.8 Lemma. The category of cofibrant objects \(M^c\) is naturally a fair monoidal coloured category (i.e., a fair monoid in \(\mathsf{CCat}\)). The weak units are the equivalences \(Z \simeq I\) with \(Z\) cofibrant.

The functor \(\Delta^{op} \to \mathsf{CCat}\) is given by taking \(O\) to be singleton, \(A\) to be the coloured category \(M^c\), and \(U\) the coloured category whose objects are weak equivalences \(Z \simeq I\) with \(Z\) cofibrant, and whose arrows are triangles \(Z' \simeq Z \simeq I\). The multiplication functor \(\otimes\) clearly turns \(A\) and \(U\) into semi-monoids as required, hence the Segal condition is satisfied. To check the weak identity arrow axiom, it must first be shown that \(U\) is contractible; this is done by comparing it with the category \(M^c/I'\) of cofibrant objects over a fixed cofibrant replacement \(I' \simeq I\). This category has a terminal object and is
therefore contractible. Second, it must be shown that the two functors $U \times A \xrightarrow{m} A$ are equifunctors. The projection $p$ is an equifunctor because $U$ is contractible, and $m$ is an equifunctor because the equivalences of (ii) assemble into a natural transformation $m \Rightarrow p$.

8.9 Spitzweck’s monoidal model categories with pseudo-unit. The fair monoid structure on $M^c$ does not depend on the fact that $I$ is a genuine unit: it is enough to have maps $I \otimes X \to X$ such that for every $Z \sim I$ with $Z$ cofibrant, the composite $Z \otimes X \to I \otimes X \to X$ is a weak equivalence, for cofibrant $X$. Such model categories have been studied by Spitzweck [33] under the name ‘monoidal model categories with pseudo-unit’. The lemma holds also in this situation.

Such structures arise in connection with modules and algebras over $E_\infty$-operads. Let $L$ denote the linear isometries operad — it is an unital $E_\infty$-operad (see [6] for more information). Consider the category $S$-$\text{Mod}$ of modules over the differential graded algebra $S = L(1)$. It was shown in Kříž-May [22] that $S$-$\text{Mod}$ can be equipped with an associative tensor product (analogous to the one constructed in [6] for spectra), $X \boxtimes Y := \mathcal{L}(2) \otimes_{S \otimes S} X \otimes Y$, but there is no unit for $\boxtimes$. A similar construction works for modules over a given $S$-algebra.

Now if $M$ is a model category satisfying some mild technical conditions, all these notions and constructions make sense in $M$, and Spitzweck [33] shows that the category of $S$-modules in $M$ is a monoidal model category with pseudo-unit, and that if $A$ is a cofibrant $S$-algebra the same is true for the category of $A$-modules in $M$. Hence, in each case the category of cofibrant objects becomes a fair monoidal coloured category.

Cobordism categories

The final example in this exposition is an example of a fair Tamsamani 2-category. That is, a $\Delta^{op}$-diagram in $\text{Cat}$ whose Segal maps are only equimorphisms (and satisfying the other axioms for a fair category).

8.10 Classical cobordism categories. (See [17] for all details.) Classically, the category $n\text{Cob}$ of oriented $n$-dimensional cobordism is the category whose objects are closed oriented $(n - 1)$-manifolds, and whose morphisms from $\Sigma_0$ to $\Sigma_1$ are equivalence classes of oriented cobordisms whose ‘in-boundary’ is $\Sigma_0$ and whose ‘out-boundary’ is $\Sigma_1$. Here two cobordisms are considered equivalent if there is a diffeomorphism between them that induces the identity map on the boundaries. It is necessary to pass to this quotient because given two manifolds with matching boundaries the gluing is not canonical: the gluing does exist and all possible gluings are diffeomorphic, but there is no universal property and no unique comparison diffeomorphism. Note also that identity cobordisms cannot exist before dividing out by diffeomorphisms, since the only true identities would be the cylinders of height zero, and they are not $n$-manifolds.

8.11 Non-algebraic composition law. Instead of choosing a specific composition for two given cobordisms, which as explained would be an artificial choice anyway, we
can indicate the space of all possible compositions. In other words, we will define a
functor $X : \Delta_{\text{mono}}^{\text{op}} \to \mathbf{Cat}$ and argue that it satisfies the weak Segal condition. The
set of objects $X_0$ is the set of all closed oriented $(n - 1)$-manifolds. The category $X_1$
has as objects the oriented cobordisms; the two maps $X_1 \xrightarrow{\sim} X_0$ associate to a given
cobordism its in-boundary and its out-boundary. The arrows of the category $X_1$ are the homotopy classes of diffeomorphisms
that restrict to the identity on the boundaries. To be more precise, a cobordism from $\Sigma_0$ to $\Sigma_1$ is a quintuple $(M; \Sigma_0, \Sigma_1; \sigma_0, \sigma_1)$ where $\sigma_0$ is a diffeomorphism from $\Sigma_0$ onto the in-boundary of the $n$-manifold $M$, and $\sigma_1$ is a diffeomorphism from $\Sigma_1$ onto its out-boundary. The arrows in the category between $(M; \Sigma_0, \Sigma_1; \sigma_0, \sigma_1)$ and $(M'; \Sigma_0, \Sigma_1; \sigma'_0, \sigma'_1)$ are then the homotopy classes of diffeomorphisms $M \cong M'$ compatible with the $\sigma_i$.

So far the description is classical. Now instead of having a composition map, we
specify the category $X_2$: its objects are cobordisms with a specified decomposition. Precisely they are septuples $(M; \Sigma_0, \Sigma_1, \Sigma_2; \sigma_0, \sigma_1, \sigma_2)$, where $\sigma_0$ is a diffeomorphism from $\Sigma_0$ onto the in-boundary of $M$; $\sigma_2$ is a diffeomorphism from $\Sigma_2$ onto the out-boundary of $M$, and $\sigma_1$ is an embedding of $\Sigma_1$ into $M$ splitting it into two cobordisms, one $M'$ from $\Sigma_0$ to $\Sigma_1$, and another $M''$ from $\Sigma_1$ to $\Sigma_2$. A morphism between two such objects is just a homotopy class of diffeomorphisms of the cobordism compatible with the sigmas.

Now we have to specify the three face maps $X_2 \xrightarrow{\sim} X_1$: these are: return $M''$, return $M'$, or return the whole $M$ forgetting the splitting data $(\Sigma_1, \sigma_1)$. In general, the category $X_k$ is defined to be the category of tuplets $(M; \Sigma_i, \sigma_i)_{0 \leq i \leq k}$ and diffeomorphisms preserving the submanifolds as explained for $k = 2$. It is clear that all these categories form a functor $\Delta_{\text{mono}}^{\text{op}} \to \mathbf{Cat}$. To verify that the Segal holds condition we must show that such a subdivided cobordism is determined up to diffeomorphism by the pieces it is made up of. This statement follows from the fact that it is possible to glue up to diffeomorphism.

### 8.12 Cylinders as weak units

There is no way of extending this functor $X : \Delta_{\text{mono}}^{\text{op}} \to \mathbf{Cat}$ to $\Delta$ to get a simplicial category. But there is a natural extension to $\Delta$. To begin with, send $I$ to the category of straight cylinders: the objects are the cobordisms $\Sigma \times I$ (and more generally, cobordisms $\Sigma \rightarrow \Sigma$ equipped with a diffeomorphism to such a straight cylinder), and the arrows are homotopy classes of boundary preserving diffeomorphisms induced from diffeomorphisms of intervals. Clearly this category is contractible. There is a functor from this category to $X_1$ consisting in forgetting the straight structure.

More generally, an object $K \in \Delta$ is sent to the category of all split cobordisms such that the pieces corresponding to links in $K$ are straight cylinders. Again the vertical maps are seen to be equivalences. Altogether we have defined a colour-preserving functor

$$X : \Delta^{\text{op}} \to \mathbf{Cat}$$

with $X_*$ discrete, and satisfying the weak Segal condition, and hence a fair Tamsamani category in $\mathbf{Cat}$. In fact the hom cats are all groupoids.

### 8.13 Proposition

Oriented $n$-cobordisms naturally assemble into a fair Tamsamani 2-category, for which the straight cylinders are weak identity arrows.
8.14 Remark. The above construction is clearly 2-truncated. It is possible to avoid this truncation: instead of getting a fair Tamsamani category in groupoids, the result is a fair Tamsamani category in simplicial sets, i.e. a fair Segal category, reflecting one level of cobordism structure and reflecting homotopy for all higher levels. Simplicial representations of such fair Segal categories of cobordisms might be an interesting alternative approach to the idea of extended topological quantum field theories advocated by Lawrence [23] and Baez-Dolan [1].

Appendix: Discrete objects

In the main text, three crucial notions are ‘discrete objects’, ‘fibre products over discrete objects’, and ‘equimorphisms’, and the important functors are those that preserve these notions. For flexibility, ‘discrete objects’ is considered a structure to be specified, not a property; the main reason is that we want to say that \( \bullet \) is the only discrete object in \( \Delta \) (or in \( \Delta \)), whereas this case is not covered by the notion of standard discrete objects described below.

Just for the notions of \( S \)-category and fair \( S \)-category to make sense, the only requirement on the discrete objects in \( S \) is that \( S \) should admit fibre products over discrete objects. However, in order to get a reasonable theory, and in particular to get a good notion of equimorphisms in the categories of \( S \)-categories and fair \( S \)-categories, certain features of the discrete objects are needed, requiring in turn certain closure properties of \( S \), and finally the notion of equimorphism in \( S \) must be compatible with these notions.

A.1 Closure properties. We require of \( S \) that it has all sums, and these should be disjoint and universal (cf. SGA 4.1 [11], Exp. II, Def. 4.5). Recall that a sum \( A = \bigsqcup_{i \in I} A_i \) is called disjoint when each structure map \( A_i \to A \) is a monomorphism and admits all pullbacks, and for each \( i \neq j \) the pullback \( A_i \times_A A_j \) is an initial object in \( S \). Universal means that the pullback of a sum diagram is again a sum diagram.

Next we require \( S \) to have all finite products. It follows automatically from the sum requirements that finite products distribute over sums,

\[
A \times \bigsqcup_{i \in I} B_i \simeq \bigsqcup_{i \in I} (A \times B_i) .
\]

A.2 Standard discrete objects. For a category \( S \) with the closure properties of A.1, the standard discrete objects structure is the case where the discrete objects are the sums of terminal objects \( \ast \), and the following two conditions are satisfied:

- **DO1** The discrete objects form a full reflective subcategory. That is, the discrete-objects functor

\[
\delta : \text{Set} \longrightarrow S \\
I \longmapsto \bigsqcup_{i \in I} \ast
\]

is fully faithful and has a left adjoint denoted \( \pi_0 \) (the components functor). (Note that there is always a right adjoint: \( A \mapsto \Hom_s(\ast, A) \).)

- **DO2** The components functor \( \pi_0 \) preserves finite products.
The second axiom simply expresses full compatibility of the adjunction with the stipulated sums and products — indeed it is already automatic that \( \pi_0 \) preserves sums and that \( \delta \) preserves products, and \( \delta \) also preserves sums by construction.

**A.3 Remark.** The description of the discrete-objects functor \( \delta : \textbf{Set} \to \mathbf{S} \) should not be taken too literally as a specific functor defined in terms of an arbitrary choice of \( \coprod \) and \( * \) in \( \mathbf{S} \); rather it is characterised by a universal property (preservation of sums and finite products), and any object isomorphic to a discrete object should be discrete again. In fact the category \( \textbf{Set} \) could be regarded as a mere placeholder for a genuine subcategory-of-discrete-objects-in-\( \mathbf{S} \), i.e., derived from \( \mathbf{S} \).

**A.4 Decomposition.** Given an arrow \( A \to \delta I \) to a discrete object; for each \( i \in I \) define \( A_i \) to be the fibre product

\[
\begin{array}{ccc}
A_i & \to & A \\
\downarrow & & \downarrow \\
* & \to & \delta I.
\end{array}
\]

It follows from the universality of sums that there is a natural isomorphism

\[
A \simeq \coprod_{i \in I} A_i.
\]

**A.5 Fibre products over discrete objects.** More generally, we get the existence of all fibre products over discrete objects:

\[
A \times_{\delta I} B \simeq \coprod_{i \in I} A_i \times B_i.
\]

It then follows from axiom DO2 that \( \pi_0 \) preserves fibre products over discrete objects.

Also, sums commute with fibre products over discrete objects:

\[
\coprod_{i \in I} A_i \times_{\delta X_i} B_i \simeq \coprod_{i \in I} A_i \times \coprod_{i \in I} B_i.
\]

Let us see how this goes by a quick example computation which also shows the importance of the rule: the Segal condition is preserved under sums. Given two fibre products

\[
A_r \simeq A_p \times_{A_0} A_q \quad B_r \simeq B_p \times_{B_0} B_q
\]

with \( A_0 \) and \( B_0 \) discrete (e.g. two Segal conditions!), put \( C_i := A_i \coprod B_i \) \((i = 0, p, q, r)\). Then the statement is that \( C_r \simeq C_p \times_{C_0} C_q \). Indeed, in the fibre product \( C_p \times_{C_0} C_q \simeq (A_p \coprod B_p) \times_{A_0 \coprod B_0} (A_q \coprod B_q) \), decompose over the discrete object \( A_0 \coprod B_0 \) to get

\[
\simeq \coprod_{x \in A_0 \coprod B_0} (A_p \coprod B_p)_x \times (A_q \coprod B_q)_x.
\]

Now write this as the sum of two sums, and use the disjointness axiom to remove half of the summands

\[
\simeq \coprod_{x \in A_0} ((A_p)_x \times (A_q)_x) \coprod_{x \in B_0} ((B_p)_x \times (B_q)_x) \simeq A_r \coprod B_r \simeq C_r.
\]
**A.6 Examples.** The archetypical example is \( \text{Top} \). The discrete spaces are standard discrete objects, and \( \pi_0 \) is the usual components functor. In \( \text{Set} \), every object is discrete, and \( \delta \) and \( \pi_0 \) are the identity functors. The key examples for the present purposes are \( \text{sSet} \) and \( \text{Cat} \), which we analyse a little further.

**A.7 Simplicial sets.** If \( D \) is a small category, the presheaf category \( \text{Cat}(D^{\text{op}}, \text{Set}) \) satisfies the closure properties in A.1 as well as axiom DO1 in A.2 (here \( \delta \) is the constant-presheaf functor, and \( \pi_0 \) returns the colimit of a given \( D^{\text{op}} \)-diagram). However, axiom DO2 is not in general satisfied. Satisfying axiom DO2 is a crucial property of a category \( D \) supposed to serve as base category for combinatorial topology. (This was observed by Grothendieck [10] and has also been stressed by Lawvere.) The category \( \Delta \) has this property, which is to say that \( \text{sSet} \) has standard discrete objects. To see this, note that if \( X \) is a simplicial set then \( \pi_0(X) \) is described as the quotient of \( X_0 \) by the equivalence relation defined by identifying two 0-cells if they can be connected by ‘zigzags’ of 1-cells. The key point here is the existence of degeneracy maps, i.e. reflexivity of the equivalence relation. (In contrast \( \Delta_{\text{mono}} \) does not have the property: consider the \( \Delta_{\text{mono}} \)-diagram \( \overset{X_0}{\cdots} \overset{X_1}{\cdots} \) where \( X_0 = \{0,1\} \), \( X_1 = \{1\} \) are the two inclusions, and \( X_n = \emptyset \) for \( n \geq 2 \). Then \( \pi_0(X \times X) \not\simeq \pi_0(X) \times \pi_0(X) \). The fat delta \( \Delta \) does not have the property either (the previous example can be used again), but as we shall see, in the setting of coloured categories, it does have the property.)

**A.8 Categories.** The full subcategory \( \text{Cat} \subset \text{sSet} \) is closed under sums and finite products, and enjoys the closure properties of A.1. Indeed, the Segal condition is given in terms of fibre products over discrete objects, and these commute with sums and finite products.

The importance of axiom DO2 is that it furthermore allows for the discrete objects structure to descend from \( \text{sSet} \) to \( \text{Cat} \). Indeed, since \( \pi_0^{\text{sSet}} \) (and also \( \delta^{\text{sSet}} \)) preserves fibre products over discrete objects, the discrete-objects adjunction for \( \text{sSet} \) restricts to an adjunction for \( \text{Cat} \):

\[
\begin{array}{ccc}
\text{sSet} & & \overset{\delta^{\text{sSet}}}{\longrightarrow} & & \text{Set} \\
\cup & & \quad & & \downarrow \\
\text{Cat} & & \quad & & \overset{\delta^{\text{Cat}}}{\longrightarrow} \quad \text{Set}
\end{array}
\]

This describes the standard discrete objects structure on \( \text{Cat} \).

These two examples readily generalise to the case where \( \text{Set} \) is replaced by a general category \( S \) with standard discrete objects:

**A.9 Lemma.** If \( S \) has standard discrete objects then so has \( \text{Cat}(\Delta^{\text{op}}, S) \).

*Proof.* The standard-discrete-objects adjunction \( \pi_0 \dashv \delta \) for \( S \) induces an adjunction

\[
\begin{array}{ccc}
\text{Cat}(\Delta^{\text{op}}, S) & & \overset{\delta_*}{\longrightarrow} & & \text{Cat}(\Delta^{\text{op}}, \text{Set}) = \text{sSet}
\end{array}
\]
by postcomposition with \( \delta \) and \( \pi_0 \). Since \( \delta \) and \( \pi_0 \) preserve sums and finite products, and since sums and finite products are computed point-wise, the induced functors \( \delta_* \) and \( \pi_{0*} \) again preserve sums and finite products. The composite adjunction

\[
\text{Cat}(\Delta^{\text{op}}, S) \rightleftharpoons \text{sSet} \rightleftharpoons \text{Set}
\]

describes the standard discrete objects structure on \( \text{Cat}(\Delta^{\text{op}}, S) \) (the discrete objects are the constant presheaves with discrete value).

\[\square\]

A.10 Lemma. If \( S \) has standard discrete objects, then \( S\text{-Cat} \) has standard discrete objects.

Proof. First notice that the full subcategory \( S\text{-Cat} \subseteq \text{Cat}(\Delta^{\text{op}}, S) \) inherits sums and finite products from the ambient category, and hence satisfies the closure properties of A.1. Now the two adjoint functors \( \text{Cat}(\Delta^{\text{op}}, S) \rightleftharpoons \text{Set} \) of the previous lemma preserve discrete objects and fibre products over discrete objects. Hence they restrict to two functors \( S\text{-Cat} \rightleftharpoons \text{Set} \), and since \( S\text{-Cat} \subseteq \text{Cat}(\Delta^{\text{op}}, S) \) is full this pair of functors forms again an adjunction, and since sums and finite products of categories are computed as simplicial sets, these two functors also preserve sums and finite products.

\[\square\]

Discrete objects and colours

A.11 Colours and standard discrete objects. Assume \( S \) has standard discrete objects. We shall now define what it means for a colour structure on \( S \) to be compatible with the standard discrete objects. There are compatibility conditions with respect to the sums and finite products, and a compatibility condition with respect to the discrete-objects adjunction:

— Stability under sums and finite products: If \( (f_i)_{i \in I} \) is a family of equimorphisms then \( \coprod_{i \in I} f_i \) is again an equimorphism, and given equimorphisms \( f_1, \ldots, f_n \), then the product \( \prod_{i=1}^n f_i \) is again an equimorphism.

— Preservation under \( \pi_0 \): If \( f \) is an equimorphism in \( S \) then \( \pi_0(f) \) is an equimorphism in \( \text{Set} \), i.e. a bijection. Note that it is automatic that \( \delta \) preserves equimorphisms. The condition on \( \pi_0 \) implies that the restriction of \( \delta \) to the subcategory \( \text{Bij} \) of sets and bijections is again a full embedding \( \text{Bij} \rightarrow W \).

A.12 Examples. The categories \( \text{sSet}, \text{Top}, \text{Set}, \) and \( \text{Cat} \), with the usual notions of discrete objects and equimorphisms, all fit into the framework above. Here \( \text{Set} \) has bijections as equimorphisms, and it follows that in each case the functors \( \pi_0 \) and \( \delta \) themselves preserve sums, finite products, and equimorphisms. The geometric realisation functor \( \text{sSet} \rightarrow \text{Top} \) as well as the nerve functor \( \text{Cat} \rightarrow \text{sSet} \) are also examples of functors preserving sums, finite products, and equimorphisms.

The components functor \( \pi_0 : \text{Cat}(\Delta^{\text{op}}, \text{Set}) \rightarrow \text{Set} \) does not preserve finite products, for the same reason as mentioned for \( \Delta_{\text{mono}} \) in A.7. However, \( \Delta \) was not designed with presheaves in mind, but rather restricted presheaves with respect to preservation of colours: we are interested in the full subcategories \( \text{CCat}(\Delta^{\text{op}}, \text{Set}) \subseteq \text{Cat}(\Delta^{\text{op}}, \text{Set}) \) and \( \text{CCat}(\Delta^{\text{op}}, S) \subseteq \text{Cat}(\Delta^{\text{op}}, S) \), and in here the full subcategories of fair \( S \)-categories and fair \( \text{Set} \)-categories.
A.13 Lemma. If $S$ has standard discrete objects with compatible colours, then $\mathbb{CCat}(\Delta^{op}, S)$ has standard discrete objects.

Proof. Since the equimorphisms in $S$ are assumed to be stable under sums and finite products, it follows that the full subcategory $\mathbb{CCat}(\Delta^{op}, S) \subset \mathbb{Cat}(\Delta^{op}, S)$ satisfies the closure properties of A.1.

Just like in A.9, the standard-discrete-objects adjunction $\pi_0 \dashv \delta$ for $S$ induces an adjunction

\[
\mathbb{CCat}(\Delta^{op}, S) \xleftrightarrow{\delta_* \pi_0^*} \mathbb{CCat}(\Delta^{op}, \text{Set})
\]

by postcomposition with $\delta$ and $\pi_0$. This time, in order for this to work it is crucial that $\delta$ and $\pi_0$ preserve equimorphisms. Since $\delta$ and $\pi_0$ also preserve sums and finite products, the induced functors $\delta_*$ and $\pi_0*$ again preserve sums and finite products.

In Proposition 4.13 we established an adjoint equivalence $\mathbb{CCat}(\Delta^{op}, \text{Set}) \simeq s\text{Set}$, and it is easy to check that each of these two adjoint functors preserves sums and finite products. Now the standard discrete objects in $\mathbb{CCat}(\Delta^{op}, S)$ are described by the composite adjunction

\[
\mathbb{CCat}(\Delta^{op}, S) \longleftarrow \mathbb{CCat}(\Delta^{op}, \text{Set}) \longleftarrow s\text{Set} \longleftarrow \text{Set}.
\]

\[\square\]

A.14 Proposition. If $S$ has standard discrete objects with compatible colours, then $S\text{-FairCat}$ has standard discrete objects.

Proof. Just like in A.10, the full subcategory $S\text{-FairCat} \subset \mathbb{CCat}(\Delta^{op}, S) \subset \mathbb{Cat}(\Delta^{op}, S)$ inherits sums and finite products from the ambient category, and hence satisfies the closure properties of A.1. Now the adjoint functors of the previous lemma all preserve discrete objects, sums and finite products, and hence fibre products over discrete objects, so they restrict to adjoint functors

\[
S\text{-FairCat} \longleftarrow \text{Set-FairCat} \longleftarrow \text{Cat} \longleftarrow \text{Set}
\]

(which again preserve sums and finite products). This describes the standard discrete objects structure on $S\text{-FairCat}$.

\[\square\]

A.15 Remark. In 4.17, equimorphisms of fair $S$-categories are defined, and in 4.21 it is shown that the notion is compatible with the standard discrete objects.

References


Address: Dept. de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Espanya
E-mail: kock@mat.uab.es